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Decay estimate of solutions for a semi-linear wave equation

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Abstract

In this paper, we investigate the initial value problem for a semi-linear wave equation in *n*-dimensional space. Under a smallness condition on the initial value, the global existence and decay estimates of the solutions are established. Furthermore, time decay estimates for the spatial derivatives of the solution are provided. The proof is carried out by means of the decay property of the solution operator and a fixed point-contraction mapping argument.

MSC: 35L30; 35L75

Keywords: semi-linear wave equation; global existence; asymptotic decay

1 Introduction

In this paper, we investigate the initial value problem for the following semi-linear wave equation:

$$u_{tt} - \Delta u - \Delta u_{tt} + u_t = \Psi(u) \tag{1.1}$$

with the initial value

$$t = 0 : u = u_0(x), \qquad u_t = u_1(x), \tag{1.2}$$

where $u = u(x,t)$ is the unknown function of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t > 0$. The term u_t represents a frictional dissipation, and the term Δu_{tt} corresponds to the rotational inertia effects. $\Psi(u)$ is a smooth function of u and satisfies $\Psi(u) = O(|u|^\theta)$ for $u \to 0$.

Equation (1.1) is an inertial model characterized by the term Δu_{tt}. Without this inertial term Δu_{tt}, (1.1) is reduced to the semi-linear wave equation with a dissipative term,

$$u_{tt} - \Delta u + u_t = \Psi(u). \tag{1.3}$$

The initial value problem for (1.3) has been extensively studied, we refer to [1–6], and [7]. The analysis for (1.3) will be much easier than for (1.1) since the associated fundamental solutions to (1.3) are similar to the heat kernel and exponential decay in the high frequency region. In fact, the decay structure of (1.3) is characterized by

$$\operatorname{Re} \lambda(\xi) \le -c|\xi|^2 / \bigl(1 + |\xi|^2\bigr). \tag{1.4}$$

The decay structure of (1.1) is characterized by

$$\operatorname{Re}\lambda(\xi) \le -c|\xi|^2/\bigl(1+|\xi|^2\bigr)^2. \tag{1.5}$$

The linear term u_t is relatively weak compared with the one given by the damping term $-\Delta u_t$. For the latter case, the equation becomes

$$u_{tt} - \Delta u - \Delta u_{tt} - \Delta u_t = \Psi(u).$$

In the case, the decay structure of in this case is characterized by (1.4). The dissipative structure that is characterized by (1.5) is called the 'regularity-loss' type. This dissipative structure is very weak in the high frequency region, so that even in a bounded domain case it does not give an exponential decay but a polynomial decay of the energy.

If the nonlinear term $\Psi(u)$ is replaced by $\Psi(u_t)$, then (1.1) becomes

$$u_{tt} - \Delta u - \Delta u_{tt} + u_t = \Psi(u_t). \tag{1.6}$$

The global existence and the asymptotic behavior of the solutions to the problem (1.6), (1.2) with L^1 data were established by Wang *et al.* [8]. Later, Wang and Wang [9] proved the global existence and the asymptotic behavior of the solutions to the problem (1.6), (1.2) with L^2 data. The analysis for the problem (1.1), (1.2) is much harder than the problem (1.6), (1.2), since the decay estimate of u_t is better than that of u.

The main purpose of this paper is to establish the global existence and decay estimates of solutions to (1.1), (1.2). We prove the global existence and the following decay estimates of the solutions to the problem (1.1), (1.2) for $n \ge 1$:

$$\bigl\|\partial_x^k u(t)\bigr\|_{H^{s-2k-[\frac{n+1}{2}]}} \le CE_0(1+t)^{-\frac{k}{2}-\frac{n}{4}} \tag{1.7}$$

for $k \ge 0$, $2k + [\frac{n+1}{2}] \le s$, and $s \ge [\frac{n+1}{2}] + 2$. Here $E_0 := \|u_0\|_{H^s\cap L^1} + \|u_1\|_{H^s\cap L^1}$ is assumed to be small. Our proof is carried out by a fixed point-contraction mapping argument, relying on the decay estimates for the linear problem.

The study of the global existence and asymptotic behavior of solutions to dissipative hyperbolic-type equations has a long history. We refer to [2–4, 6, 7] for the damped wave equation. Also, we refer to [10, 11] and [8, 9, 12–16] for various aspects of dissipation of the plate equation.

We give some notations which are used in this paper. Let $F[u]$ denote the Fourier transform of u defined by

$$\hat{u}(\xi) = F[u] = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\,dx,$$

and we denote its inverse transform by F^{-1}. For a nonnegative integer k, ∂_x^k denotes the totality of all the kth order derivatives with respect to $x \in \mathbb{R}^n$.

For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. Let s be a nonnegative integer, $H^s = H^s(\mathbb{R}^n)$ denotes the Sobolev space of L^2 functions, equipped with the norm $\|\cdot\|_{H^s}$. Also, $C^k(I;H^s)$ denotes the space of k-times continuously differentiable functions on the interval I with values in the Sobolev space $H^s = H^s(\mathbb{R}^n)$.

Finally, in this paper, we denote every positive constant by the same symbol C or c without confusion. $[\cdot]$ is the Gauss symbol.

The paper is organized as follows. The decay property of the solution operators to (1.1) is in Section 2. Then, in Section 3, we discuss the linear problem and show the decay estimates. Finally, we prove the global existence and the decay estimates of solutions for the initial value problem (1.1), (1.2) in Section 4.

2 Decay property of solution operator

The aim of this section is to establish decay estimates of the solution operators to (1.1). Firstly, we derive the solution formula for the problem (1.1), (1.2). We investigate the linear equation of (1.1):

$$u_{tt} - \Delta u - \Delta u_{tt} + u_t = 0, \tag{2.1}$$

with the initial data (1.2).

We apply the Fourier transform to (2.1), (1.2); it yields

$$\bigl(1 + |\xi|^2\bigr)\hat{u}_{tt} + \hat{u}_t + |\xi|^2\hat{u} = 0, \tag{2.2}$$

$$t = 0 : \hat{u} = \hat{u}_0(\xi), \qquad \hat{u}_t = \hat{u}_1(\xi). \tag{2.3}$$

The corresponding characteristic equation of (2.2) is

$$\bigl(1 + |\xi|^2\bigr)\lambda^2 + \lambda + |\xi|^2 = 0. \tag{2.4}$$

Let $\lambda = \lambda_\pm(\xi)$ be the solutions to (2.4). It is not difficult to find

$$\lambda_\pm(\xi) = \frac{-1 \pm \sqrt{1 - 4|\xi|^2(1 + |\xi|^2)}}{2(1 + |\xi|^2)}. \tag{2.5}$$

The solution to (2.2), (2.3) in the Fourier space is then given explicitly in the form

$$\hat{u}(\xi, t) = \hat{G}(\xi, t)\bigl(\hat{u}_0(\xi) + \hat{u}_1(\xi)\bigr) + \hat{H}(\xi, t)\hat{u}_0(\xi), \tag{2.6}$$

where

$$\hat{G}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)}\bigl(e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}\bigr) \tag{2.7}$$

and

$$\hat{H}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)}\bigl(\bigl(1 + \lambda_+(\xi)\bigr)e^{\lambda_-(\xi)t} - \bigl(1 + \lambda_-(\xi)\bigr)e^{\lambda_+(\xi)t}\bigr). \tag{2.8}$$

Let

$$G(x, t) = F^{-1}\bigl[\hat{G}(\xi, t)\bigr](x)$$

and

$$H(x,t) = F^{-1}\big[\hat{H}(\xi,t)\big](x),$$

where F^{-1} denotes the inverse Fourier transform. Then we apply F^{-1} to (2.6), and it yields

$$u(t) = G(t) * (u_0 + u_1) + H(t) * u_0. \tag{2.9}$$

Now we return to our nonlinear problem (1.1), (1.2). By the Duhamel principle, the problem (1.1), (1.2) is equivalent to

$$u(t) = G(t) * (u_0 + u_1) + H(t) * u_0 + \int_0^t G(t-\tau) * (1-\Delta)^{-1}\Psi(u)(\tau)\,d\tau. \tag{2.10}$$

Next, we consider the linearized problem (2.1), (1.2) and derive the pointwise estimates of solutions in the Fourier space, which were already obtained in [8]. For the reader's convenience, we give a detailed proof.

Lemma 2.1 *Let u be the solution to the linearized problem* (2.1), (1.2). *Then its Fourier image $\hat{u}$ verifies the pointwise estimate*

$$\big|\hat{u}_t(\xi,t)\big|^2 + \big|\hat{u}(\xi,t)\big|^2 \le Ce^{-c\omega(\xi)t}\big(\big|\hat{u}_1(\xi)\big|^2 + \big|\hat{u}_0(\xi)\big|^2\big), \tag{2.11}$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$, where $\omega(\xi) = \frac{|\xi|^2}{(1+|\xi|^2)^2}$.

Proof By multiplying (2.2) by $\bar{\hat{u}}_t$ and taking the real part, we deduce that

$$\frac{1}{2}\frac{d}{dt}\big\{\big(1+|\xi|^2\big)|\hat{u}_t|^2 + |\xi|^2|\hat{u}|^2\big\} + |\hat{u}_t|^2 = 0. \tag{2.12}$$

We multiply (2.2) by $\bar{\hat{u}}$ and take the real part. This yields

$$\frac{1}{2}\frac{d}{dt}\big\{|\hat{u}|^2 + 2\big(1+|\xi|^2\big)\operatorname{Re}(\hat{u}_t\bar{\hat{u}})\big\} + |\xi|^2|\hat{u}|^2 - \big(1+|\xi|^2\big)|\hat{u}_t|^2 = 0. \tag{2.13}$$

Then by (2.12) and (2.13), we have

$$\frac{d}{dt}E + F = 0, \tag{2.14}$$

where

$$E = \big(1+|\xi|^2\big)^2|\hat{u}_t|^2 + \left[\frac{1}{2} + \big(1+|\xi|^2\big)|\xi|^2\right]|\hat{u}|^2 + \big(1+|\xi|^2\big)\operatorname{Re}(\hat{u}_t\bar{\hat{u}})$$

and

$$F = \big(1+|\xi|^2\big)|\hat{u}_t|^2 + |\xi|^2|\hat{u}|^2.$$

Let

$$E_0 = |\hat{u}_t|^2 + |\hat{u}|^2.$$

It is easy to see that

$$C\big(1 + |\xi|^2\big)^2 E_0 \le E \le C\big(1 + |\xi|^2\big)^2 E_0. \tag{2.15}$$

Noting that $F \ge |\xi|^2 E_0$ and with (2.15), we obtain

$$F \ge c\omega(\xi)E, \tag{2.16}$$

where

$$\omega(\xi) = \frac{|\xi|^2}{(1 + |\xi|^2)^2}.$$

Combining (2.14) and (2.16) yields

$$\frac{d}{dt}E + c\omega(\xi)E \le 0.$$

Thus

$$E(\xi, t) \le e^{-cw(\xi)t} E(\xi, 0),$$

which together with (2.15) proves the desired estimate (2.11). Then we have completed the proof of the lemma. □

Lemma 2.2 *Let $\hat{G}(\xi, t)$ and $\hat{H}(\xi, t)$ be the fundamental solution of* (2.1) *in the Fourier space, which are given in* (2.7) *and* (2.8), *respectively. Then we have the estimates*

$$\big|\hat{G}(\xi, t)\big|^2 + \big|\hat{G}_t(\xi, t)\big|^2 \le Ce^{-c\omega(\xi)t} \tag{2.17}$$

and

$$\big|\hat{H}(\xi, t)\big|^2 + \big|\hat{H}_t(\xi, t)\big|^2 \le Ce^{-c\omega(\xi)t} \tag{2.18}$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$, where $\omega(\xi) = \frac{|\xi|^2}{(1+|\xi|^2)^2}$.

Proof If $\hat{u}_0(\xi) = 0$, from (2.5), we obtain

$$\hat{u}(\xi, t) = \hat{G}(\xi, t)\hat{u}_1(\xi), \qquad \hat{u}_t(\xi, t) = \hat{G}_t(\xi, t)\hat{u}_1(\xi).$$

Substituting the equalities into (2.11) with $\hat{u}_0(\xi) = 0$, we get (2.17).

In the following, we consider $\hat{u}_1(\xi) = 0$; it follows from (2.5) that

$$\hat{u}(\xi, t) = \big(\hat{G}(\xi, t) + \hat{H}(\xi, t)\big)\hat{u}_0(\xi), \qquad \hat{u}_t(\xi, t) = \big(\hat{G}_t(\xi, t) + \hat{H}_t(\xi, t)\big)\hat{u}_0(\xi).$$

Substituting the equalities into (2.11) with $\hat{u}_1(\xi) = 0$, we get

$$\left|\hat{G}(\xi,t) + \hat{H}(\xi,t)\right|^2 + \left|\hat{G}_t(\xi,t) + \hat{H}_t(\xi,t)\right|^2 \le Ce^{-c\omega(\xi)t},$$

which together with (2.17) proves the desired estimate (2.18). The lemma is proved. □

Using the Taylor formula and (2.7), (2.8), we immediately obtain the following.

Lemma 2.3 *Let* $\hat{G}(\xi,t)$ *and* $\hat{H}(\xi,t)$ *be the fundamental solutions of* (2.1) *in the Fourier space, which are given in* (2.7) *and* (2.8), *respectively. Then there is a small positive number* R_0 *such that if* $|\xi| \le R_0$ *and* $t \ge 0$, *we have the following estimates*:

$$\left|\hat{H}(\xi,t)\right| \le C|\xi|^2 e^{-c|\xi|^2 t} + Ce^{-ct}, \tag{2.19}$$

$$\left|\hat{G}_t(\xi,t)\right| \le C|\xi|^2 e^{-c|\xi|^2 t} + Ce^{-ct}, \tag{2.20}$$

and

$$\left|\hat{H}_t(\xi,t)\right| \le C|\xi|^4 e^{-c|\xi|^2 t} + Ce^{-ct}. \tag{2.21}$$

Lemma 2.4 *Let* $1 \le p \le 2$ *and* $k \ge 0$. *Then for* $l \ge 0$ *we have*

$$\left\|\partial_x^k G(t) * \phi\right\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\phi\|_{L^p} + C(1+t)^{-\frac{l}{2}}\left\|\partial_x^{k+l}\phi\right\|_{L^2} \tag{2.22}$$

and

$$\left\|\partial_x^k H(t) * \psi\right\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}-1}\|\psi\|_{L^p} + C(1+t)^{-\frac{l}{2}}\left\|\partial_x^{k+l}\psi\right\|_{L^2}. \tag{2.23}$$

Moreover, we have

$$\left\|\partial_x^k G_t(t) * \phi\right\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}-1}\|\phi\|_{L^p} + C(1+t)^{-\frac{l}{2}}\left\|\partial_x^{k+l}\phi\right\|_{L^2} \tag{2.24}$$

and

$$\left\|\partial_x^k H_t(t) * \psi\right\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}-2}\|\psi\|_{L^p} + C(1+t)^{-\frac{l}{2}}\left\|\partial_x^{k+l}\psi\right\|_{L^2}. \tag{2.25}$$

Proof The proof method of Lemma 2.4 has been used in many papers, we refer to [8] and [17].

We only prove (2.23). It follows from the Plancherel theorem that

$$\begin{aligned}
\left\|\partial_x^k H(t) * \psi\right\|_{L^2}^2 &= \int_{\mathbb{R}^n} |\xi|^{2k}\left|\hat{H}(\xi,t)\right|^2\left|\hat{\psi}(\xi)\right|^2 d\xi \\
&= \int_{|\xi|\le R_0} |\xi|^{2k}\left|\hat{H}(\xi,t)\right|^2\left|\hat{\psi}(\xi)\right|^2 d\xi \\
&\quad + \int_{|\xi|\ge R_0} |\xi|^{2k}\left|\hat{H}(\xi,t)\right|^2\left|\hat{\psi}(\xi)\right|^2 d\xi \\
&\triangleq I_1 + I_2,
\end{aligned} \tag{2.26}$$

where R_0 is a positive constant in Lemma 2.3.

In the following, we estimate I_1. Using (2.19) and the Hölder inequality, we obtain

$$\begin{aligned} I_1 &\le C\int_{|\xi|\le R_0} |\xi|^{2k+4}e^{-c|\xi|^2 t}\big|\hat{\psi}(\xi)\big|^2\,d\xi + C\int_{|\xi|\le R_0} |\xi|^{2k}e^{-ct}\big|\hat{\psi}(\xi)\big|^2\,d\xi \\ &\le C\big\||\xi|^{2k+4}e^{-c|\xi|^2 t}\big\|_{L^q(|\xi|\le R_0)}\big\|\hat{\psi}(\xi)\big\|_{L^{p'}}^2 + C\big\||\xi|^{2k}e^{-ct}\big\|_{L^q(|\xi|\le R_0)}\big\|\hat{\psi}(\xi)\big\|_{L^{p'}}^2, \end{aligned}$$

where $\frac{1}{q}+\frac{2}{p'}=1$ and $\frac{1}{p}+\frac{1}{p'}=1$.

By a straightforward computation, we get

$$\big\||\xi|^{2k+4}e^{-c|\xi|^2 t}\big\|_{L^q(|\xi|\le R_0)} \le C(1+t)^{-\frac{n}{2q}-k-2}$$

and

$$\big\||\xi|^{2k}e^{-c|\xi|^2 t}\big\|_{L^q(|\xi|\le R_0)} \le Ce^{-ct}.$$

The Hausdorff-Young inequality gives

$$\big\|\hat{\psi}(\xi)\big\|_{L^{p'}} \le C\|\psi\|_{L^p}.$$

Thus, we have

$$\begin{aligned} I_1 &\le C(1+t)^{-\frac{n}{2q}-k-2}\|\psi\|_{L^p}^2 \\ &= C(1+t)^{-\frac{n}{2}(\frac{2}{p}-1)-k-2}\|\psi\|_{L^p}^2. \end{aligned} \tag{2.27}$$

Note that $\omega(\xi) \ge c|\xi|^{-2}$ when $|\xi| \ge R_0$. By (2.17), we arrive at

$$\begin{aligned} I_2 &\le C\int_{|\xi|\ge R_0} |\xi|^{2k}e^{-c|\xi|^{-2}t}\big|\hat{\psi}(\xi)\big|^2\,d\xi \\ &\le C\sup_{|\xi|\ge R_0}\big(|\xi|^{-2l}e^{-c|\xi|^{-2}t}\big)\int_{|\xi|\ge R_0} |\xi|^{2(k+l)}\big|\hat{\psi}(\xi)\big|^2\,d\xi \\ &\le C(1+t)^{-l}\big\|\partial_x^{k+l}\psi\big\|_{L^2}^2. \end{aligned} \tag{2.28}$$

Combining (2.27) and (2.28) yields (2.23). Thus we have completed the proof of the lemma. □

3 Decay estimates of solutions to (2.1), (1.2)

In the previous section, we observed that the decay structure of the linear equation (2.1) is of the regularity-loss type. The purpose of this section is to show the decay estimates of the solutions to (2.1), (1.2) when the initial data are in $H^s \cap L^1$.

Theorem 3.1 *Let $s \ge 0$ and suppose that $u_0, u_1 \in H^s(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$. Put*

$$E_0 = \|u_0\|_{H^s\cap L^1} + \|u_1\|_{H^s\cap L^1}.$$

Then the solution $u(x,t)$ of the linear problem (2.1), (1.2), *which is given by* (2.9), *satisfies the decay estimate*

$$\big\|\partial_x^k u(t)\big\|_{H^{s-2k-[\frac{n+1}{2}]}} \le CE_0(1+t)^{-\frac{n}{4}-\frac{k}{2}} \tag{3.1}$$

for $k \geq 0$ and $2k + [\frac{n+1}{2}] \leq s$. Moreover, for each j with $0 \leq j \leq 1$, we have

$$\left\| \partial_x^k u_t(t) \right\|_{H^{s-2k-[\frac{n+1}{2}]-2j}} \leq CE_0(1+t)^{-\frac{n}{4}-\frac{k}{2}-j} \tag{3.2}$$

for $k \geq 0$ and $2k + [\frac{n+1}{2}] + 2j \leq s$.

Remark 3.1 In addition to the above decay estimates, we also have

$$\left\| \partial_x^k u(t) \right\|_{H^{s-2k-2j}} \leq CE_0(1+t)^{-\frac{k}{2}-j}, \quad 0 \leq j \leq \left[\frac{n}{4}\right], \tag{3.3}$$

$$\left\| \partial_x^k u_t(t) \right\|_{H^{s-2k-2j}} \leq CE_0(1+t)^{-\frac{k}{2}-j}, \quad 0 \leq j \leq \left[\frac{n}{4}\right] + 1. \tag{3.4}$$

Proof The proof of (3.1)-(3.4) is similar. We only prove (3.1). Owing to (2.9), to prove (3.1), it suffices to prove the following estimates:

$$\left\| \partial_x^k G(t) * u_1 \right\|_{H^{s-2k-[\frac{n+1}{2}]}} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|u_1\|_{H^s \cap L^1}, \tag{3.5}$$

$$\left\| \partial_x^k G(t) * u_0 \right\|_{H^{s-2k-[\frac{n+1}{2}]}} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|u_0\|_{H^s \cap L^1}, \tag{3.6}$$

and

$$\left\| \partial_x^k H(t) * u_0 \right\|_{H^{s-2k-[\frac{n+1}{2}]-2j}} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}-j} \|u_0\|_{H^s \cap L^1}, \quad 0 \leq j \leq 1. \tag{3.7}$$

Firstly, we prove (3.5). Let $k \geq 0$ and $h \geq 0$, we have from (2.22) with k replaced by $k+h$ and with $p = 1$

$$\left\| \partial_x^{k+h} G(t) * u_1 \right\|_{L^2} \leq C(1+t)^{-\frac{n}{4}-\frac{k+h}{2}} \|u_1\|_{L^1} + C(1+t)^{-\frac{l}{2}} \left\| \partial_x^{k+h+l} u_1 \right\|_{L^2},$$

where $l \geq 0$ and $k+h+l \leq s$. We choose l as the smallest integer satisfying $\frac{l}{2} \geq \frac{n}{4} + \frac{k}{2}$, *i.e.* $l \geq \frac{n}{2} + k$. Thus, we take $l = [\frac{n+1}{2}] + k$. It follows from $k+h+l \leq s$ that $h \leq s - 2k - [\frac{n+1}{2}]$. Then we have

$$\left\| \partial_x^{k+h} G(t) * u_1 \right\|_{L^2} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \|u_1\|_{H^s \cap L^1}$$

for $0 \leq h \leq s - 2k - [\frac{n+1}{2}]$. Then (3.5) is proved.

Similarly, we can prove (3.6).

Finally, we prove (3.7). Let $k \geq 0$ and $h \geq 0$. It follows from (2.23) that

$$\left\| \partial_x^{k+h} H(t) * u_0 \right\|_{L^2} \leq c(1+t)^{-\frac{n}{4}-\frac{k+h}{2}-1} \|u_0\|_{L^1} + C(1+t)^{-\frac{l}{2}} \left\| \partial_x^{k+h+l} u_0 \right\|_{L^2},$$

where $l \geq 0$ and $k+h+l \leq s$.

Let $0 \leq j \leq 1$. We choose l as the smallest integer satisfying $\frac{l}{2} \geq \frac{n}{4} + \frac{k}{2} + j$. Thus we take $l = [\frac{n+1}{2}] + k + 2j$. From $k+h+l \leq s$, we obtain

$$0 \leq h \leq s - 2k - \left[\frac{n+1}{2}\right] - 2j.$$

Then we have

$$\left\|\partial_x^{k+h}H(t)*u_0\right\|_{L^2}\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}-j}\|u_0\|_{H^s}$$

for $0\le h\le s-2k-[\frac{n+1}{2}]-2j$ and $0\le j\le 1$. This proves (3.7). The theorem is proved. □

4 Global existence and decay estimates

The purpose of this section is to prove the global existence and the decay estimates of solutions to the initial value problem (1.1), (1.2). Based on the existence and the decay estimates of the solutions to the linear problem (2.1), (1.2), we define

$$\mathcal{E}=\left\{u\in C\left([0,\infty);H^s\left(\mathbb{R}^n\right)\right):\|u\|_{\mathcal{E}}<\infty\right\},$$

where

$$\|u\|_{\mathcal{E}}=\sum_{\varrho(k,n)\le s}\sup_{t\ge 0}(1+t)^{\frac{n}{4}+\frac{k}{2}}\left\|\partial_x^k u(t)\right\|_{H^{s-\varrho(k,n)}}+\sum_{j=0}^{[\frac{n}{4}]}\sum_{2k+2j\le s}\sup_{t\ge 0}(1+t)^{\frac{k}{2}+j}\left\|\partial_x^k u(t)\right\|_{H^{s-2k-2j}},$$

where

$$\varrho(k,n)=2k+\left[\frac{n+1}{2}\right].$$

For $\mathfrak{R}>0$, we define

$$\mathcal{E}_{\mathfrak{R}}=\left\{u\in X:\|u\|_{\mathcal{E}}\le\mathfrak{R}\right\},$$

where $\mathfrak{R}$ depends on the initial value, which is chosen in the proof of the main result.

Theorem 4.1 *Let $n\ge 1$, $\theta>1+\frac{2}{n}$, and $s\ge[\frac{n+1}{2}]+2$. Assume that $u_0,u_1\in H^s(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$. Put*

$$E_0=\|u_0\|_{H^s\cap L^1}+\|u_1\|_{H^s\cap L^1}.$$

If E_0 is suitably small, the initial value problem (1.1), (1.2) *has a unique global solution*

$$u\in C^0\left([0,\infty);H^s\left(\mathbb{R}^n\right)\right).$$

Moreover, the solution satisfies the decay estimate

$$\left\|\partial_x^k u(t)\right\|_{H^{s-2k-2j}}\le CE_0(1+t)^{-\frac{k}{2}-j}\tag{4.1}$$

for $k\ge 0$, $0\le j\le[\frac{n}{4}]$, and $2k+2j\le s$. Also, we have

$$\left\|\partial_x^k u(t)\right\|_{H^{s-\varrho(k,n)}}\le CE_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}\tag{4.2}$$

for $k\ge 0$ and $\varrho(k,n)\le s$.

Proof We shall prove Theorem 4.1 by the Banach fixed point theorem. Thus we define the mapping

$$\mathcal{M}[u](t) = G(t) * (u_0 + u_1) + H(t) * u_0 + \int_0^t G(t-\tau) * (1-\Delta)^{-1}\Psi(u)(\tau)\,d\tau$$

and

$$\mathcal{M}_0(t) = G(t) * (u_0 + u_1) + H(t) * u_0.$$

For $\forall v, w \in \mathcal{E}$, we arrive at

$$\mathcal{M}[v](t) - \mathcal{M}[w](t) = \int_0^t G(t-\tau) * (1-\Delta)^{-1}\bigl(\Psi(v) - \Psi(w)\bigr)(\tau)\,d\tau.$$

For $\Psi(u) = O(|u|^\theta)$, it is not difficult to get the following nonlinear estimates (see [18]):

$$\begin{aligned}&\bigl\|\partial_x^k\bigl(\Psi(v) - \Psi(w)\bigr)\bigr\|_{L^1}\\ &\quad\le C\bigl\|(v,w)\bigr\|_{L^\infty}^{\theta-2}\bigl(\bigl\|(v,w)\bigr\|_{L^2}\bigl\|\partial_x^k(v-w)\bigr\|_{L^2} + \bigl\|\partial_x^k(v,w)\bigr\|_{L^2}\|v-w\|_{L^2}\bigr)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\bigl\|\partial_x^k\bigl(\Psi(v) - \Psi(w)\bigr)\bigr\|_{L^2}\\ &\quad\le C\bigl\|(v,w)\bigr\|_{L^\infty}^{\theta-2}\bigl(\bigl\|(v,w)\bigr\|_{L^\infty}\bigl\|\partial_x^k(v-w)\bigr\|_{L^2} + \bigl\|\partial_x^k(v,w)\bigr\|_{L^2}\|v-w\|_{L^\infty}\bigr).\end{aligned} \tag{4.4}$$

If $u \in X$, the Gagliardo-Nirenberg inequality gives

$$\bigl\|u(\tau)\bigr\|_{L^\infty} \le C\|u\|_{\mathcal{E}}(1+\tau)^{-\frac{n}{2}}. \tag{4.5}$$

Firstly, we shall prove

$$\bigl\|\partial_x^k\bigl(\mathcal{M}[v] - \mathcal{M}[w]\bigr)\bigr\|_{H^{s-2k-2j}} \le C(1+t)^{-\frac{k}{2}-j}\bigl\|(v,w)\bigr\|_{\mathcal{E}}^{\theta-1}\|v-w\|_{\mathcal{E}}, \tag{4.6}$$

where $0 \le j \le [\frac{n}{4}]$, $k \ge 0$, and $2k + 2j \le s$.

Assume that k, j, m are non-positive integers. Let $j \le [\frac{n}{4}]$ and $2k + 2j \le s$. We apply ∂_x^{k+m} to $\mathcal{M}[v] - \mathcal{M}[w]$. This yields

$$\begin{aligned}&\bigl\|\partial_x^{k+m}\bigl(\mathcal{M}[v] - \mathcal{M}[w]\bigr)(t)\bigr\|_{L^2}\\ &\quad\le \int_0^{\frac{t}{2}}\bigl\|\partial_x^{k+m}G(t-\tau) * (1-\Delta)^{-1}\bigl(\Psi(v) - \Psi(w)\bigr)(\tau)\bigr\|_{L^2}\,d\tau\\ &\qquad + \int_{\frac{t}{2}}^t\bigl\|\partial_x^{k+m}G(t-\tau) * (1-\Delta)^{-1}\bigl(\Psi(v) - \Psi(w)\bigr)(\tau)\bigr\|_{L^2}\,d\tau\\ &\quad\triangleq J_1 + J_2.\end{aligned} \tag{4.7}$$

By (2.22), we obtain

$$\begin{aligned} J_1 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{k+m}{2}}\big\|\Psi(v)-\Psi(w)\big\|_{L^1}\,d\tau \\ &\quad + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{l}{2}}\big\|\partial_x^{k+m+l-2}\big(\Psi(v)-\Psi(w)\big)\big\|_{L^2}\,d\tau \\ &\triangleq J_{11}+J_{12}. \end{aligned} \tag{4.8}$$

Using (4.3) and (4.5), we arrive at

$$\big\|\Psi(v)-\Psi(w)\big\|_{L^1}\le C\big\|(v,w)\big\|_{\mathcal{E}}^{\theta-1}\|v-w\|_{\mathcal{E}}(1+\tau)^{-\frac{n}{2}(\theta-1)}. \tag{4.9}$$

Noting that $\theta > 1+\frac{2}{n}$ for $n = 1,2$ and $\theta \ge 2$ for $n \ge 3$, using (4.9), we deduce that

$$J_{11}\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\big\|(v,w)\big\|_{\mathcal{E}}^{\theta-1}\|v-w\|_{\mathcal{E}}. \tag{4.10}$$

If $k+m+l-2\le s$, (4.4) gives

$$\big\|\partial_x^{k+m+l-2}\big(\Psi(v)-\Psi(w)\big)(\tau)\big\|_{L^2}\le C\big\|(v,w)\big\|_{\mathcal{E}}^{\theta-1}\|v-w\|_{\mathcal{E}}(1+\tau)^{-\frac{n}{2}(\theta-1)}. \tag{4.11}$$

Take $l = k+2j+2$. By (4.11) and noticing $\theta \ge 2$, we obtain

$$J_{12}\le C(1+t)^{-\frac{k}{2}-j}\big\|(v,w)\big\|_{\mathcal{E}}^{\theta-1}\|v-w\|_{\mathcal{E}}$$

with $0\le m\le s-2k-2j$.

Combining (4.10) and (4.11) yields

$$J_1\le C(1+t)^{-\frac{k}{2}-j}\big\|(v,w)\big\|_{\mathcal{E}}^{\theta-1}\|v-w\|_{\mathcal{E}}$$

with $0\le m\le s-2k-j$. We apply $l = 2$ and $p = 1$ to the term J_2. This yields

$$\begin{aligned} J_2 &\le C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{m}{2}}\big\|\partial_x^{k}\big(\Psi(v)-\Psi(w)\big)(\tau)\big\|_{L^1}\,d\tau \\ &\quad + C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-1}\big\|\partial_x^{k+m}\big(\Psi(v)-\Psi(w)\big)(\tau)\big\|_{L^2}\,d\tau =: J_{21}+J_{22}. \end{aligned} \tag{4.12}$$

It follows from (4.3) and (4.5) that

$$\big\|\partial_x^{k}\big(\Psi(v)-\Psi(w)\big)\big\|_{L^1}\le C\big\|(v,w)\big\|_{\mathcal{E}}^{\theta-1}\|v-w\|_{\mathcal{E}}(1+\tau)^{-\frac{n}{2}(\theta-2)-\frac{n}{4}-\frac{k}{2}-j}, \tag{4.13}$$

where $0\le j\le [\frac{n}{4}]$ and $2k+2j\le s$.

Since $\theta > 1+\frac{2}{n}$ for $n = 1,2$ and $\theta \ge 2$ for $n\ge 3$, using (4.13), we obtain

$$J_{21}\le C(1+t)^{-\frac{k}{2}-j}\big\|(v,w)\big\|_{\mathcal{E}}^{\theta-1}\|v-w\|_{\mathcal{E}}$$

for $0\le j\le [\frac{n}{4}]$ and $2k+2j\le s$.

If $m \leq s - 2k - 2j$, we apply (4.4) to get

$$\left\|\partial_x^{k+m}\big(\Psi(v) - \Psi(w)\big)(\tau)\right\|_{L^2} \leq C\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|_{\mathcal{E}}(1+\tau)^{-d(\theta-1)-\frac{k}{2}-j}, \tag{4.14}$$

which yields

$$J_{22} \leq C(1+t)^{-\frac{k}{2}-j}\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|_{\mathcal{E}},$$

where $0 \leq m \leq s - 2k - 2j$.

Thus

$$J_2 \leq C(1+t)^{-\frac{k}{2}-j}\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|_{\mathcal{E}}. \tag{4.15}$$

We immediately get (4.6).

In the following we prove

$$\left\|\partial_x^k\big(\mathcal{M}[v] - \mathcal{M}[w]\big)(t)\right\|_{H^{s-\varrho(k,n)}} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|_{\mathcal{E}}, \tag{4.16}$$

where $\varrho(k,n) \leq s$.

Assume that k and m are nonnegative integers and $\varrho(k,n) \leq s$. Applying ∂_x^{k+m} to $\mathcal{M}[v] - \mathcal{M}[w]$, then (4.7), (4.8), and (4.10) still hold. Now we estimate J_{12}.

If $k + m + l - 2 \leq s$, we still have (4.11). Taking $l = \varrho(k,n) - k + 2$, then we obtain

$$J_{12} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|_{\mathcal{E}}. \tag{4.17}$$

Combining (4.10) and (4.17) yields

$$J_1 \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|_{\mathcal{E}}. \tag{4.18}$$

Similarly, we have (4.12). Since $\varrho(k,n) \leq s$, in view of (4.3) and (4.5), we get

$$\left\|\partial_x^k\big(\Psi(v) - \Psi(w)\big)\right\|_{L^1} \leq C\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|_{\mathcal{E}}(1+\tau)^{-\frac{n}{2}(\theta-1)-\frac{k}{2}},$$

which yields

$$J_{21} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|.$$

Since $\|\partial_x^{k+m}(\Psi(v) - \Psi(w))\|_{L^2} \leq \|\partial_x^k(\Psi(v) - \Psi(w))\|_{H^m}$, by using (4.4), (4.5) and noticing that $\varrho(k,n) \leq s$, we have

$$\left\|\partial_x^{k+m}\big(\Psi(v) - \Psi(w)\big)\right\|_{L^2} \leq C\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|_{\mathcal{E}}(1+\tau)^{-\frac{n}{2}(\theta-1)-\frac{n}{4}-\frac{k}{2}},$$

which yields

$$J_{22} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v - w\|_{\mathcal{E}}.$$

Thus

$$J_2 \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}} \left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1} \|v-w\|_{\mathcal{E}}. \tag{4.19}$$

Equations (4.18) and (4.19) give (4.16).

It follows from (4.6) and (4.16) that

$$\left\|\mathcal{M}[v]-\mathcal{M}[w]\right\|_{\mathcal{E}} \le C\left\|(v,w)\right\|_{\mathcal{E}}^{\theta-1}\|v-w\|_{\mathcal{E}}. \tag{4.20}$$

Thus for $v, w \in X_R$, we have

$$\left\|\mathcal{M}[v]-\mathcal{M}[w]\right\|_{\mathcal{E}} \le C_1\mathfrak{R}^{\theta-1}\|v-w\|_{\mathcal{E}}.$$

On the other hand, $\mathcal{M}[0](t) = \mathcal{M}_0(t)$ is a solution of the problem (2.1), (1.2). From Theorem 3.1, we know that

$$\left\|\mathcal{M}_0(t)\right\|_{\mathcal{E}} \le C_2E_0,$$

if E_0 is suitably small.

We take $\mathfrak{R} = 2C_2E_0$. If E_0 is suitably small such that $\mathfrak{R} < 1$ and $C_1\mathfrak{R} \le \frac{1}{2}$, then we arrive at

$$\left\|\mathcal{M}[v]-\mathcal{M}[w]\right\|_{X} \le \frac{1}{2}\|v-w\|_{\mathcal{E}}.$$

Therefore, for $v \in \mathcal{E}_{\mathfrak{R}}$, we have

$$\left\|\mathcal{M}[v]\right\|_{\mathcal{E}} \le \left\|\mathcal{M}_0(t)\right\|_{\mathcal{E}} + \frac{1}{2}\mathfrak{R} \le C_2E_0 + \frac{1}{2}\mathfrak{R} = \mathfrak{R}.$$

Thus $u \to \mathcal{M}[u]$ is a contraction mapping on $\mathcal{E}_{\mathfrak{R}}$, so there exists a unique $u \in \mathcal{E}_{\mathfrak{R}}$ satisfying $\mathcal{M}[u] = u$. Therefore, the initial value problem (1.1), (1.2) has a unique solution u satisfying the decay estimates (4.1) and (4.2). We have completed the proof of the theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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