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A fixed point approach to the solution of singular fractional differential equations with integral boundary conditions

Kalaivani Chandran¹, Kalpana Gopalan^{1*}, Sumaiya Tasneem Zubair¹ and Thabet Abdeljawad^{2,3,4*} 

*Correspondence:

kalpanag@ssn.edu.in;
tabdeljawad@psu.edu.sa

¹Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Kalavakkam, Chennai, 603 110, India

²Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

Full list of author information is available at the end of the article

Abstract

In this article, we first demonstrate a fixed point result under certain contraction in the setting of controlled b -Branciari metric type spaces.

- Thereafter, we specifically consider a following boundary value problem (BVP) for a singular fractional differential equation of order α :

$${}^c D^\alpha v(t) + h(t, v(t)) = 0, \quad 0 < t < 1,$$

$$v''(0) = v'''(0) = 0,$$

$$v'(0) = v(1) = \beta \int_0^1 v(s) ds,$$

where $3 < \alpha < 4$, $0 < \beta < 2$, ${}^c D^\alpha$ is the Caputo fractional derivative and h may be singular at $v = 0$.

- Eventually, we investigate the existence and uniqueness of solutions of the aforementioned boundary value problem of order α via a fixed point problem of an integral operator.

MSC: 34B15; 54H25

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1 Introduction

As regards science and engineering disciplines, fractional differential equations occur in the fields of aerodynamics, chemistry, physics, or polymer rheology electrostatics, such as the mathematical simulation of structures and processes. The derivatives of fractional order [28, 35, 38] are concerned in this sort of equations. Exceedingly, fractional-order differential equations often become tools for various perspectives on control systems, fluid dynamics, and so forth.

The significance of studying fractional-order differential equations arises from the factuality that fractional-order models are more precise than integer-order models; it seems

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to be so that the fractional-order models have more degrees of freedom. Recent findings can be found in [3, 4, 8, 14, 19, 34, 37, 45] on fractional differential equations.

Integral boundary conditions have several applications in the areas such as problems with blood flow, thermo-elasticity, underground water supply, and population dynamics. We refer the reader to certain recent publications [11, 17, 22, 36, 42–44, 46, 48] and the references therein for a comprehensive explanation of the integral boundary conditions.

Many researchers have pointed out the importance of the existence and uniqueness of fractional differential equations of different orders [29, 30, 32, 35, 40]. The existence of positive solutions was considered especially in the context of cone metric spaces [9, 12, 47].

On the other hand, fixed point theory can be used as a definitive modeling method in numerous fields and/or engineering to achieve solutions and/or research findings. In general, it has become one of the most effective features of modern mathematics and in particular of functional analysis. Fixed point theorems are concerned with the existence, uniqueness and characteristics of a specified operator's fixed points. The contraction mapping theorem due to Banach [10] is a very important and valuable finding on fixed point theory. Recent advancement in exploring new generalized metric spaces (and/or related results) has provoked great attention in metric fixed point theory (see [1, 5–7, 13, 16, 18, 20, 23–27, 31, 39]).

2 Preliminaries

In the year 1993, Czerwik [15] initiated the idea of b -metric spaces as a generalization of metric spaces by multiplying a constant b on the right side of the equation of triangle inequality.

Definition 2.1 Let $X \neq \emptyset$ and $b \geq 1$ be a given real number. A function $d_b : X \times X \rightarrow [0, +\infty)$ is a b -metric if and only if for each $r, s, t \in X$ the following conditions are satisfied:

- (1) $d_b(r, s) \geq 0$ for all $r, s \in X$ and $d_b(r, s) = 0$ if and only if $r = s$;
- (2) $d_b(r, s) = d_b(s, r)$ for all $r, s \in X$;
- (3) $d_b(r, s) \leq b[d_b(r, t) + d_b(t, s)]$ for all $r, s, t \in X$.

Then d_b is called a b -metric on X and (X, d_b) is called a b -metric space.

Nabil Mlaiki et al. [33] defined a new type of generalized b -metric spaces, namely controlled metric type spaces, as follows:

Definition 2.2 Let $X \neq \emptyset$ and $\omega : X \times X \rightarrow [1, +\infty)$. A function $d_\omega : X \times X \rightarrow [0, +\infty)$ is called a controlled metric type if:

- (1) $d_\omega(r, s) = 0$ if and only if $r = s$;
- (2) $d_\omega(r, s) = d_\omega(s, r)$;
- (3) $d_\omega(r, s) \leq \omega(r, t)d_\omega(r, t) + \omega(t, s)d_\omega(t, s)$

for all $r, s, t \in X$. The pair (X, d_ω) is called a controlled metric type space.

By concatenating the concepts of controlled metric type spaces [33] and extended-Branciari b -distance spaces [2], the authors in [41] proposed a new sort of metric spaces, namely controlled b -Branciari metric type spaces, that are defined now.

Definition 2.3 Let $X \neq \emptyset$ and $\mathfrak{C} : X \times X \rightarrow [1, +\infty)$. A function $d_{\mathfrak{C}} : X \times X \rightarrow [0, +\infty)$ is called a controlled b -Branciari metric type if it satisfies:

- (1) $d_{\mathfrak{C}}(r, s) = 0$ if and only if $r = s$ for all $r, s \in X$;
 - (2) $d_{\mathfrak{C}}(r, s) = d_{\mathfrak{C}}(s, r)$ for all $r, s \in X$;
 - (3) $d_{\mathfrak{C}}(r, s) \leq \mathfrak{C}(r, t)d_{\mathfrak{C}}(r, t) + \mathfrak{C}(t, w)d_{\mathfrak{C}}(t, w) + \mathfrak{C}(w, s)d_{\mathfrak{C}}(w, s)$,
- for all $r, s \in X$ and for all distinct points $t, w \in X$, each distinct from r and s , respectively. The pair $(X, d_{\mathfrak{C}})$ is named a controlled b -Branciari metric type space.

The major aspect of controlled b -Branciari metric type space is the extension of the rectangular inequality.

Example 2.4 ([41]) Let $X = \{1, 2, 3, 4\}$. Define $d_{\mathfrak{C}} : X \times X \rightarrow [0, +\infty)$ as follows:

$$\begin{aligned}
 d_{\mathfrak{C}}(r, r) &= 0, \forall r \in X, d_{\mathfrak{C}}(1, r) = d_{\mathfrak{C}}(r, 1) = 50, \forall r \in X - \{1\}, \\
 d_{\mathfrak{C}}(2, 3) &= d_{\mathfrak{C}}(3, 2) = d_{\mathfrak{C}}(2, 4) = d_{\mathfrak{C}}(4, 2) = 200, \\
 d_{\mathfrak{C}}(4, 3) &= d_{\mathfrak{C}}(3, 4) = 800.
 \end{aligned}$$

Let $\mathfrak{C} : X \times X \rightarrow [1, +\infty)$ be symmetric and can be defined as follows:

$$\begin{aligned}
 \mathfrak{C}(r, r) &= 1, \forall r \in X, \\
 \mathfrak{C}(1, 2) &= 3, \mathfrak{C}(1, 3) = 4, \mathfrak{C}(1, 4) = \mathfrak{C}(2, 3) = 5, \mathfrak{C}(2, 4) = 6, \mathfrak{C}(3, 4) = 2.
 \end{aligned}$$

Hence $(X, d_{\mathfrak{C}})$ is a controlled b -Branciari metric type space. However, we can see that

- (i) $d_{\mathfrak{C}}(3, 4) = 800 > \mathfrak{C}(3, 4)[d_{\mathfrak{C}}(3, 1) + d_{\mathfrak{C}}(1, 2) + d_{\mathfrak{C}}(2, 4)] = 600$,
- (ii) $d_{\mathfrak{C}}(3, 4) = 800 > \mathfrak{C}(3, 1)d_{\mathfrak{C}}(3, 1) + \mathfrak{C}(1, 4)d_{\mathfrak{C}}(1, 4) = 450$.

Thus $(X, d_{\mathfrak{C}})$ is neither an extended Branciari b -distance space nor a controlled metric type space.

For the reader’s convenience, we present some necessary definitions and lemmas from the theory of fractional calculus.

Definition 2.5 ([21]) For a function $v : [0, \infty) \rightarrow R$, the Caputo derivative of fractional order $\alpha > 0, n - 1 < \alpha < n, n \in \mathbb{N}$ is defined as

$${}^c D^\alpha v(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{v^n(s)}{(t - s)^{\alpha - n + 1}} ds, \quad n = [\alpha] + 1, \tag{2.1}$$

where $[\alpha]$ denotes the integer part of the real number α .

Definition 2.6 ([21]) The Riemann–Liouville fractional integral of order α for a function v is defined as

$$I^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} v(s) ds, \quad \alpha > 0, \tag{2.2}$$

provided that such an integral exists.

Lemma 2.7 [21] Given $y \in C(0, 1) \cup L(0, 1)$, $3 < \alpha < 4$ and $0 < \beta < 2$, the unique solution of

$$\begin{aligned}
 {}^c D^\alpha v(t) + y(t) &= 0, \quad 0 < t < 1, \\
 v''(0) &= v'''(0) = 0, \\
 v'(0) = v(1) &= \beta \int_0^1 v(s) ds,
 \end{aligned}$$

is

$$v(t) = \int_0^1 G(t,s)y(s) ds$$

where

$$G(t,s) = \frac{1}{\alpha(2-\beta)\Gamma(\alpha)} \begin{cases} \{\alpha(2-\beta) + 2\beta t(\alpha-1+s)\}(1-s)^{\alpha-1} \\ -\alpha(2-\beta)(t-s)^{\alpha-1}, & \text{if } 0 \leq s \leq t \leq 1, \\ \{\alpha(2-\beta) + 2\beta t(\alpha-1+s)\}(1-s)^{\alpha-1}, & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \quad (2.3)$$

Lemma 2.8 ([28]) *If $\alpha > 0$ and $\beta > 0$, then*

1. ${}^c D^\alpha t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$, for $\beta > n$.
2. ${}^c D^\alpha t^k = 0$, for $k = 0, 1, 2, \dots, n-1$.

Deeply influenced by the foregoing facts, we specifically discuss the following boundary value problem (BVP) for a singular fractional differential equation of order α :

$$\begin{aligned} & {}^c D^\alpha v(t) + h(t, v(t)) = 0, \quad 0 < t < 1, \\ & v''(0) = v'''(0) = 0, \\ & v'(0) = v(1) = \beta \int_0^1 v(s) ds, \end{aligned} \quad (2.4)$$

where $3 < \alpha < 4$, $0 < \beta < 2$, ${}^c D^\alpha$ is the Caputo fractional derivative given by (2.1) and h may be singular at $v = 0$.

Ying He [21] takes into consideration the problem (2.4) for a continuous function h and $\beta \in (0, \alpha)$. The outcomes in the paper correspond to the positive solutions to this problem. Ying He [21] first developed the accurate estimation of the below BVP of the Green’s function, and unveiled some of its properties;

$$\begin{aligned} & {}^c D^\alpha v(t) + y(t) = 0, \quad 0 < t < 1, \\ & v''(0) = v'''(0) = 0, \\ & v'(0) = v(1) = \beta \int_0^1 v(s) ds. \end{aligned} \quad (2.5)$$

- In Sect. 3, we prove a fixed point theorem in the framework of controlled b -Branciari metric type spaces.
- In Sect. 4, an approximation for the Green’s function relevant to the problem is presented and utilized for the solution of the given problem in the proof of the existence and uniqueness theorem.
- Thereafter the BVP given in Eq. (2.4) is transformed into an integral equation and analyzed as a fixed point problem.
- Criteria for the existence and uniqueness of a fixed point for an integral operator are evaluated via complete controlled b -Branciari metric type spaces. Eventually in Sect. 5, an exemplary example is offered to endorse the theoretical result.

3 Fixed point theorem

Throughout this section, in the sense of controlled b -Branciari metric type spaces, we give a fixed point result under specific contraction condition.

Theorem 3.1 *Let $(X, d_{\mathfrak{C}})$ be a complete controlled b -Branciari metric type space with co-efficient function $\mathfrak{C}(r, s) > 1$ for any $r, s \in X$ and $F : X \rightarrow X$ be a mapping satisfying*

$$d_{\mathfrak{C}}(Fr, Fs) \leq \eta \mathfrak{C}(r, s) d_{\mathfrak{C}}(r, s), \quad \text{for all } r, s \in X, \tag{3.1}$$

where $\eta \in [0, 1)$ is such that, for any $r_0 \in X$, we have

$$\limsup_{n, m \rightarrow +\infty} \mathfrak{C}(r_n, r_m) \mathfrak{C}(r_n, r_{n+1}) < \frac{1}{\eta^2} \tag{3.2}$$

and

$$\limsup_{n, m \rightarrow +\infty} \mathfrak{C}(r_n, r_m) \mathfrak{C}(r_{n+1}, r_{n+2}) < \frac{1}{\eta^2} \tag{3.3}$$

and we assume $\limsup_{n \rightarrow +\infty} \mathfrak{C}(r, r_n)$ and $\limsup_{n \rightarrow +\infty} \mathfrak{C}(r_n, r)$ exist for any $r \in X$. Then F has a fixed point in X . Moreover, suppose that, for any $r, s \in X$, we have

$$\limsup_{n \rightarrow +\infty} \mathfrak{C}(F^n r, F^n s) < \frac{1}{\eta} \tag{3.4}$$

where $F^n r = F^{n-1}(Fr)$. Then the fixed point of F is unique.

Proof Let $r_0 \in X$ and define an iterative sequence $\{r_n\}$ by

$$r_0, \quad Fr_0 = r_1, \quad Fr_1 = r_2 \quad \Rightarrow \quad r_2 = F^2 r_0, \dots, r_{n+1} = F^{n+1} r_0.$$

Consider

$$\begin{aligned} d_{\mathfrak{C}}(r_n, r_{n+1}) &= d_{\mathfrak{C}}(Fr_{n-1}, Fr_n) \\ &\leq \eta \mathfrak{C}(r_{n-1}, r_n) d_{\mathfrak{C}}(r_{n-1}, r_n) \\ &\vdots \\ &\leq \eta^n \prod_{k=1}^n \mathfrak{C}(r_{k-1}, r_k) d_{\mathfrak{C}}(r_0, r_1). \end{aligned} \tag{3.5}$$

Similarly

$$\begin{aligned} d_{\mathfrak{C}}(r_n, r_{n+2}) &= d_{\mathfrak{C}}(Fr_{n-1}, Fr_{n+1}) \\ &\leq \eta \mathfrak{C}(r_{n-1}, r_{n+1}) d_{\mathfrak{C}}(r_{n-1}, r_{n+1}) \\ &\vdots \\ &\leq \eta^n \prod_{k=1}^n \mathfrak{C}(r_{k-1}, r_{k+1}) d_{\mathfrak{C}}(r_0, r_2). \end{aligned} \tag{3.6}$$

Now to show that $\{r_n\}$ is Cauchy, we consider $d_{\mathfrak{C}}(r_n, r_{n+p})$ in two cases.

Case 1: If p is odd, say $2m + 1$, then using (3.5), we obtain

$$\begin{aligned}
 d_{\mathcal{C}}(r_n, r_{n+2m+1}) &\leq \mathcal{C}(r_n, r_{n+1})d_{\mathcal{C}}(r_n, r_{n+1}) + \mathcal{C}(r_{n+1}, r_{n+2})d_{\mathcal{C}}(r_{n+1}, r_{n+2}) \\
 &\quad + \mathcal{C}(r_{n+2}, r_{n+2m+1})d_{\mathcal{C}}(r_{n+2}, r_{n+2m+1}) \\
 &\quad \vdots \\
 &\leq \mathcal{C}(r_n, r_{n+1})d_{\mathcal{C}}(r_n, r_{n+1}) + \mathcal{C}(r_{n+1}, r_{n+2})d_{\mathcal{C}}(r_{n+1}, r_{n+2}) \\
 &\quad + \sum_{i=\frac{n}{2}+1}^{\frac{n+2m-2}{2}} [\mathcal{C}(r_{2i}, r_{2i+1})d_{\mathcal{C}}(r_{2i}, r_{2i+1}) + \mathcal{C}(r_{2i+1}, r_{2i+2})d_{\mathcal{C}}(r_{2i+1}, r_{2i+2})] \\
 &\quad \times \prod_{j=\frac{n}{2}+1}^i \mathcal{C}(r_{2j}, r_{n+2m+1}) + \prod_{i=\frac{n}{2}+1}^{\frac{n+2m}{2}} \mathcal{C}(r_{2i}, r_{n+2m+1})d_{\mathcal{C}}(r_{n+2m}, r_{n+2m+1}) \\
 &\leq \sum_{i=\frac{n}{2}}^{\frac{n+2m}{2}} \mathcal{C}(r_{2i}, r_{2i+1})d_{\mathcal{C}}(r_{2i}, r_{2i+1}) \prod_{j=\frac{n}{2}}^i \mathcal{C}(r_{2j}, r_{n+2m+1}) \\
 &\quad + \sum_{i=\frac{n}{2}}^{\frac{n+2m-2}{2}} \mathcal{C}(r_{2i+1}, r_{2i+2})d_{\mathcal{C}}(r_{2i+1}, r_{2i+2}) \prod_{j=\frac{n}{2}}^i \mathcal{C}(r_{2j}, r_{n+2m+1}) \\
 &\leq \sum_{i=\frac{n}{2}}^{\frac{n+2m}{2}} \mathcal{C}(r_{2i}, r_{2i+1}) \prod_{j=\frac{n}{2}}^i \mathcal{C}(r_{2j}, r_{n+2m+1}) \eta^{2i} \prod_{k=1}^{2i} \mathcal{C}(r_{k-1}, r_k)d_{\mathcal{C}}(r_0, r_1) \\
 &\quad + \sum_{i=\frac{n}{2}}^{\frac{n+2m-2}{2}} \mathcal{C}(r_{2i+1}, r_{2i+2}) \prod_{j=\frac{n}{2}}^i \mathcal{C}(r_{2j}, r_{n+2m+1}) \eta^{2i+1} \prod_{k=1}^{2i+1} \mathcal{C}(r_{k-1}, r_k)d_{\mathcal{C}}(r_0, r_1).
 \end{aligned} \tag{3.7}$$

Let

$$a_i = \eta^{2i} \prod_{j=\frac{n}{2}}^i \mathcal{C}(r_{2j}, r_{n+2m+1}) \prod_{k=1}^{2i+1} \mathcal{C}(r_{k-1}, r_k)d_{\mathcal{C}}(r_0, r_1)$$

and

$$b_i = \eta^{2i+1} \prod_{j=\frac{n}{2}}^i \mathcal{C}(r_{2j}, r_{n+2m+1}) \prod_{k=1}^{2i+2} \mathcal{C}(r_{k-1}, r_k)d_{\mathcal{C}}(r_0, r_1).$$

By utilizing (3.2) and (3.3), we obtain

$$\limsup_{i \rightarrow +\infty} \frac{a_{i+1}}{a_i} = \limsup_{i, m \rightarrow +\infty} \eta^2 \mathcal{C}(r_{2i+2}, r_{n+2m+1}) \mathcal{C}(r_{2i+2}, r_{2i+3}) < 1$$

and

$$\limsup_{i \rightarrow +\infty} \frac{b_{i+1}}{b_i} = \limsup_{i, m \rightarrow +\infty} \eta^2 \mathcal{C}(r_{2i+2}, r_{n+2m+1}) \mathcal{C}(r_{2i+3}, r_{2i+4}) < 1.$$

Thereby we deduce that

$$\sum_{i=\frac{n}{2}}^{+\infty} \eta^{2i} \prod_{j=\frac{n}{2}}^i \mathfrak{C}(r_{2j}, r_{n+2m+1}) \prod_{k=1}^{2i+1} \mathfrak{C}(r_{k-1}, r_k) d_{\mathfrak{C}}(r_0, r_1) < +\infty$$

and

$$\sum_{i=\frac{n}{2}}^{+\infty} \eta^{2i+1} \prod_{j=\frac{n}{2}}^i \mathfrak{C}(r_{2j}, r_{n+2m+1}) \prod_{k=1}^{2i+2} \mathfrak{C}(r_{k-1}, r_k) d_{\mathfrak{C}}(r_0, r_1) < +\infty.$$

Henceforth

$$\left(\sum_{i=\frac{n}{2}}^{\frac{n+2m}{2}} \eta^{2i} \prod_{j=\frac{n}{2}}^{i+1} \mathfrak{C}(r_{2j}, r_{n+2m+1}) \prod_{k=1}^{2i+1} \mathfrak{C}(r_{k-1}, r_k) d_{\mathfrak{C}}(r_0, r_1) \right)$$

and

$$\left(\sum_{i=\frac{n}{2}}^{\frac{n+2m-2}{2}} \eta^{2i+1} \prod_{j=\frac{n}{2}}^i \mathfrak{C}(r_{2j}, r_{n+2m+1}) \prod_{k=1}^{2i+2} \mathfrak{C}(r_{k-1}, r_k) d_{\mathfrak{C}}(r_0, r_1) \right)$$

are Cauchy sequences in \mathbb{R} .

Case 2: If p is even, say $2m$, using (3.5) and (3.6), we obtain

$$\begin{aligned} d_{\mathfrak{C}}(r_n, r_{n+2m}) &\leq \mathfrak{C}(r_n, r_{n+2}) d_{\mathfrak{C}}(r_n, r_{n+2}) + \mathfrak{C}(r_{n+2}, r_{n+3}) d_{\mathfrak{C}}(r_{n+2}, r_{n+3}) \\ &\quad + \mathfrak{C}(r_{n+3}, r_{n+2m}) d_{\mathfrak{C}}(r_{n+3}, r_{n+2m}) \\ &\quad \vdots \\ &\leq \mathfrak{C}(r_n, r_{n+2}) d_{\mathfrak{C}}(r_n, r_{n+2}) + \mathfrak{C}(r_{n+2}, r_{n+3}) d_{\mathfrak{C}}(r_{n+2}, r_{n+3}) \\ &\quad + \sum_{i=\frac{n+3}{2}}^{\frac{n+2m-3}{2}} [\mathfrak{C}(r_{2i}, r_{2i+1}) d_{\mathfrak{C}}(r_{2i}, r_{2i+1}) + \mathfrak{C}(r_{2i+1}, r_{2i+2}) d_{\mathfrak{C}}(r_{2i+1}, r_{2i+2})] \\ &\quad \times \prod_{j=\frac{n+3}{2}}^i \mathfrak{C}(r_{2j}, r_{n+2m}) + \prod_{i=\frac{n+3}{2}}^{\frac{n+2m-1}{2}} \mathfrak{C}(r_{2i}, r_{n+2m}) d_{\mathfrak{C}}(r_{n+2m-1}, r_{n+2m}) \tag{3.8} \\ &\leq \mathfrak{C}(r_n, r_{n+2}) d_{\mathfrak{C}}(r_n, r_{n+2}) \\ &\quad + \sum_{i=\frac{n+3}{2}}^{\frac{n+2m-1}{2}} \mathfrak{C}(r_{2i}, r_{2i+1}) d_{\mathfrak{C}}(r_{2i}, r_{2i+1}) \prod_{j=\frac{n+3}{2}}^i \mathfrak{C}(r_{2j}, r_{n+2m}) \\ &\quad + \sum_{i=\frac{n+1}{2}}^{\frac{n+2m-3}{2}} \mathfrak{C}(r_{2i+1}, r_{2i+2}) d_{\mathfrak{C}}(r_{2i+1}, r_{2i+2}) \prod_{j=\frac{n+1}{2}}^i \mathfrak{C}(r_{2j}, r_{n+2m}) \\ &\leq \mathfrak{C}(r_n, r_{n+2}) \eta^n \prod_{i=1}^n \mathfrak{C}(r_{i-1}, r_{i+1}) d_{\mathfrak{C}}(r_0, r_2) \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{i=\frac{n+3}{2}}^{\frac{n+2m-1}{2}} \mathfrak{E}(r_{2i}, r_{2i+1}) \prod_{j=\frac{n+3}{2}}^i \mathfrak{E}(r_{2j}, r_{n+2m}) \eta^{2i} \prod_{k=1}^{2i} \mathfrak{E}(r_{k-1}, r_k) d_{\mathfrak{E}}(r_0, r_1) \\
 &+ \sum_{i=\frac{n+1}{2}}^{\frac{n+2m-3}{2}} \mathfrak{E}(r_{2i+1}, r_{2i+2}) \prod_{j=\frac{n+1}{2}}^i \mathfrak{E}(r_{2j}, r_{n+2m}) \eta^{2i+1} \prod_{k=1}^{2i+1} \mathfrak{E}(r_{k-1}, r_k) d_{\mathfrak{E}}(r_0, r_1).
 \end{aligned}$$

Let

$$a_i = \eta^{2i} \prod_{j=\frac{n+3}{2}}^i \mathfrak{E}(r_{2j}, r_{n+2m}) \prod_{k=1}^{2i+1} \mathfrak{E}(r_{k-1}, r_k) d_{\mathfrak{E}}(r_0, r_1)$$

and

$$b_i = \eta^{2i+1} \prod_{j=\frac{n+1}{2}}^i \mathfrak{E}(r_{2j}, r_{n+2m}) \prod_{k=1}^{2i+2} \mathfrak{E}(r_{k-1}, r_k) d_{\mathfrak{E}}(r_0, r_1).$$

By using (3.2) and (3.3), we get

$$\limsup_{i \rightarrow +\infty} \frac{a_{i+1}}{a_i} = \limsup_{i, m \rightarrow +\infty} \eta^2 \mathfrak{E}(r_{2i+2}, r_{n+2m}) \mathfrak{E}(r_{2i+2}, r_{2i+3}) < 1$$

and

$$\limsup_{i \rightarrow +\infty} \frac{b_{i+1}}{b_i} = \limsup_{i, m \rightarrow +\infty} \eta^2 \mathfrak{E}(r_{2i+2}, r_{n+2m}) \mathfrak{E}(r_{2i+3}, r_{2i+4}) < 1.$$

Hence, we deduce that

$$\sum_{i=\frac{n+3}{2}}^{+\infty} \eta^{2i} \prod_{j=\frac{n+3}{2}}^i \mathfrak{E}(r_{2j}, r_{n+2m}) \prod_{k=1}^{2i+1} \mathfrak{E}(r_{k-1}, r_k) d_{\mathfrak{E}}(r_0, r_1) < +\infty$$

and

$$\sum_{i=\frac{n+1}{2}}^{+\infty} \eta^{2i+1} \prod_{j=\frac{n+1}{2}}^i \mathfrak{E}(r_{2j}, r_{n+2m}) \prod_{k=1}^{2i+2} \mathfrak{E}(r_{k-1}, r_k) d_{\mathfrak{E}}(r_0, r_1) < +\infty.$$

Thereby

$$\left(\sum_{i=\frac{n+3}{2}}^{\frac{n+2m-1}{2}} \eta^{2i} \prod_{j=\frac{n+3}{2}}^i \mathfrak{E}(r_{2j}, r_{n+2m+1}) \prod_{k=1}^{2i+1} \mathfrak{E}(r_{k-1}, r_k) d_{\mathfrak{E}}(r_0, r_1) \right)$$

and

$$\left(\sum_{i=\frac{n+1}{2}}^{\frac{n+2m-3}{2}} \eta^{2i+1} \prod_{j=\frac{n+1}{2}}^i \mathfrak{E}(r_{2j}, r_{n+2m+1}) \prod_{k=1}^{2i+2} \mathfrak{E}(r_{k-1}, r_k) d_{\mathfrak{E}}(r_0, r_1) \right)$$

are Cauchy sequences in \mathbb{R} . By (3.7) and (3.8), it implies that $\{r_n\}$ is a Cauchy sequence in X when $n \rightarrow +\infty$. Through completeness of $(X, d_{\mathfrak{C}})$ there exists $r \in X$ such that $\lim_{n \rightarrow +\infty} r_n = r$. We are now attempting to demonstrate that r is a fixed point of F . For $n \in \mathbb{N}$, we have

$$d_{\mathfrak{C}}(r_{n+2}, r) \leq \mathfrak{C}(r_{n+2}, r_{n+1})d_{\mathfrak{C}}(r_{n+2}, r_{n+1}) + \mathfrak{C}(r_{n+1}, r_n)d_{\mathfrak{C}}(r_{n+1}, r_n) + \mathfrak{C}(r_n, r)d_{\mathfrak{C}}(r_n, r).$$

Passing $n \rightarrow +\infty$ in the aforementioned inequality, we find

$$\begin{aligned} \lim_{n \rightarrow +\infty} d_{\mathfrak{C}}(r_{n+2}, r) &= 0, \tag{3.9} \\ d_{\mathfrak{C}}(r, Fr) &\leq \mathfrak{C}(r, r_{n+2})d_{\mathfrak{C}}(r, r_{n+2}) + \mathfrak{C}(r_{n+2}, r_{n+1})d_{\mathfrak{C}}(r_{n+2}, r_{n+1}) + \mathfrak{C}(r_{n+1}, Fr)d_{\mathfrak{C}}(r_{n+1}, Fr) \\ &= \mathfrak{C}(r, r_{n+2})d_{\mathfrak{C}}(r, r_{n+2}) + \mathfrak{C}(r_{n+2}, r_{n+1})d_{\mathfrak{C}}(r_{n+2}, r_{n+1}) + \mathfrak{C}(r_{n+1}, Fr)d_{\mathfrak{C}}(Fr_n, Fr) \\ &\leq \mathfrak{C}(r, r_{n+2})d_{\mathfrak{C}}(r, r_{n+2}) + \mathfrak{C}(r_{n+2}, r_{n+1})d_{\mathfrak{C}}(r_{n+2}, r_{n+1}) \\ &\quad + \mathfrak{C}(r_{n+1}, Fr)\eta\mathfrak{C}(r_n, r)d_{\mathfrak{C}}(r_n, r). \end{aligned}$$

By employing Eq. (3.9) and by the hypothesis of the theorem, we get $d_{\mathfrak{C}}(Fr, r) \leq 0$ as $n \rightarrow +\infty$. Therefore $d_{\mathfrak{C}}(r, Fr) = 0$ i.e., $Fr = r$. As a result we see that r is a fixed point of F .

Unicity: Let r, s be the two fixed points of F where $r \neq s$, then $Fr = r$ and $Fs = s$. Consider

$$d_{\mathfrak{C}}(r, s) = d_{\mathfrak{C}}(Fr, Fs) \leq \eta\mathfrak{C}(r, s)d_{\mathfrak{C}}(r, s) = \eta\mathfrak{C}(F^n r, F^n s)d_{\mathfrak{C}}(r, s).$$

Letting $n \rightarrow +\infty$ in the equation above and utilizing (3.4), we obtain $d_{\mathfrak{C}}(r, s) < d_{\mathfrak{C}}(r, s)$, which is a contradiction. Thereby, r is a unique fixed point of F . □

Corollary 3.2 *Let (X, d_r) be a complete rectangular b -metric space with $b \geq 1$. Let $F : X \rightarrow X$ be a mapping. Assume there exists $l \in [0, 1)$ such that*

$$d_r(Fr, Fs) \leq bld_r(r, s)$$

for all $r, s \in X$. Assume that $b^2l < 1$. Then F has a unique fixed point r^* .

Proof The proof follows from Theorem 3.1 by defining $\mathfrak{C} : X \times X \rightarrow [1, +\infty)$ as $\mathfrak{C}(r, s) = b$. □

4 Existence–uniqueness of the solution of the BVP (2.4)

In this section, we confirm the existence and uniqueness of the solution of the nonlinear BVP (2.4) within controlled b -Branciari metric type spaces.

We commence this section by proposing the Green’s function developed in Ying He [21] relevant to the BVP corresponding to the linear fractional equation (2.5). Thereafter we present an inequality fulfilled by the Green’s function to be utilized on the nonlinear BVP (2.4) in our existence–unicity result. It is proved in Ying He [21] that linear problem (2.5) has a unique solution in $C[0, 1]$ given by

$$v(t) = \int_0^1 G(t, s)y(s) ds, \tag{4.1}$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\alpha(2-\beta)\Gamma(\alpha)} \begin{cases} \{\alpha(2-\beta) + 2\beta t(\alpha-1+s)\}(1-s)^{\alpha-1} \\ -\alpha(2-\beta)(t-s)^{\alpha-1}, & \text{if } 0 \leq s \leq t \leq 1, \\ \{\alpha(2-\beta) + 2\beta t(\alpha-1+s)\}(1-s)^{\alpha-1}, & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

where $3 < \alpha < 4$ and $0 < \beta < 2$. For the properties of the Green's function, we refer to Ying He [21].

In the following, we provide an estimation for the L^2 norm of the Green's function of a fractional differential equation with integral boundary conditions specified in (2.4).

Lemma 4.1 *Let $3 < \alpha < 4$ and $0 < \beta < 2$, $\alpha \neq \beta$ hold. Then, for all $t,s \in (0,1)$, the Green's function $G(t,\cdot) \in L_2$ obeys*

$$\int_0^1 |G(t,s)|^2 ds < \frac{1}{(\Gamma(\alpha))^2} \left[\frac{4}{5} + \frac{12\beta}{5|\beta-2|} + \frac{20\beta^2}{21(\beta-2)^2} \right]. \tag{4.2}$$

Proof If $3 < \alpha < 4$ and $0 < \beta < 2$, $\beta \neq \alpha$, the interpretation of the Green's function specifies:

(1) For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} |G(t,s)| &\leq \frac{2\alpha|\beta-2| + 2\beta t(\alpha-1+s)}{\alpha|\beta-2|\Gamma(\alpha)} (1-s)^{\alpha-1} \\ &\leq \frac{1}{\Gamma(\alpha)} \left(2 + \frac{2\beta}{\alpha|\beta-2|}(\alpha-1+s) \right) (1-s)^{\alpha-1}. \end{aligned} \tag{4.3}$$

(2) For $0 \leq t \leq s \leq 1$,

$$\begin{aligned} |G(t,s)| &\leq \frac{\alpha|\beta-2| + 2\beta t(\alpha-1+s)}{\alpha|\beta-2|\Gamma(\alpha)} (1-s)^{\alpha-1} \\ &\leq \frac{1}{\Gamma(\alpha)} \left(1 + \frac{2\beta}{\alpha|\beta-2|}(\alpha-1+s) \right) (1-s)^{\alpha-1}. \end{aligned} \tag{4.4}$$

Therefore, by Eqs. (4.3) and (4.4), for all $t,s \in (0,1)$, we have

$$|G(t,s)|^2 \leq \frac{1}{(\Gamma(\alpha))^2} \left(4 + \frac{8\beta(\alpha-1+s)}{\alpha|\beta-2|} + \frac{4\beta^2(\alpha-1+s)^2}{\alpha^2(\beta-2)^2} \right) (1-s)^{2\alpha-2}, \tag{4.5}$$

which yields

$$\begin{aligned} &\int_0^1 |G(t,s)|^2 ds \\ &\leq \frac{1}{(\Gamma(\alpha))^2} \left[\int_0^1 4(1-s)^{2\alpha-2} ds + \frac{8\beta}{\alpha|\beta-2|} \int_0^1 (\alpha-1+s)(1-s)^{2\alpha-2} ds \right. \\ &\quad \left. + \frac{4\beta^2}{\alpha^2(\beta-2)^2} \int_0^1 (\alpha-1+s)^2(1-s)^{2\alpha-2} ds \right] \\ &= \frac{1}{(\Gamma(\alpha))^2} \left[\frac{4}{2\alpha-1} + \frac{8\beta}{\alpha|\beta-2|} \left(\frac{2\alpha^2-2\alpha+1}{2\alpha(2\alpha-1)} \right) + \frac{4\beta^2}{\alpha(\beta-2)^2} \left(\frac{2\alpha^2-3\alpha+2}{(2\alpha-1)(2\alpha+1)} \right) \right] \\ &< \frac{1}{(\Gamma(\alpha))^2} \left[\frac{4}{5} + \frac{12\beta}{5|\beta-2|} + \frac{20\beta^2}{21(\beta-2)^2} \right]. \quad \square \end{aligned}$$

Let $h(\cdot, v(\cdot)) \in L_2$ for any $v \in C[0, 1]$ and define a mapping $F : C[0, 1] \rightarrow C[0, 1]$ as

$$Fv(t) = \int_0^1 G(t, s)h(s, v(s)) ds, \quad t \in [0, 1], \tag{4.6}$$

where $s \rightarrow G(t, s)$ is continuous from $[0, 1] \rightarrow L_2$. Let $t_n \in [0, 1]$ with $t_n \rightarrow t$.

As $h(\cdot, v(\cdot)), G(t, \cdot) \in L_2$, for any $v \in C[0, 1]$ and $t \in [0, 1]$, the function $G(t, \cdot)h(\cdot, v(\cdot))$ is integrable. Then, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} Fv(t_n) &= \lim_{n \rightarrow \infty} \int_0^1 G(t_n, s)h(s, v(s)) ds \\ &= \int_0^1 \lim_{n \rightarrow \infty} G(t_n, s)h(s, v(s)) ds \\ &= \int_0^1 G(t, s)h(s, v(s)) ds = Fv(t), \end{aligned}$$

which implies $Fv \in C[0, 1]$. Hence the map $F : C[0, 1] \rightarrow C[0, 1]$ holds.

Let $d_{\mathfrak{C}} : C[0, 1] \times C[0, 1] \rightarrow [0, +\infty)$ be defined by

$$d_{\mathfrak{C}}(v, z) = \sup_{t \in [0, 1]} |v(t) - z(t)|^2. \tag{4.7}$$

Then $d_{\mathfrak{C}}$ is a complete controlled b -Branciari metric type on $C[0, 1]$ with a controlled function

$$\mathfrak{C}(v, z) = \begin{cases} 9 + \sup_{t \in [0, 1]} |v(t) - z(t)|^2, & \text{if } v(t) \neq z(t), \\ 1, & \text{if } v(t) = z(t). \end{cases}$$

Lemma 4.2 *Let $v \in C[0, 1]$ and F be given by Eq. (4.6). Then $v(t)$ is a solution of boundary value problem (2.4) if and only if it is the fixed point of F .*

Proof Let $v(t)$ be a solution of the BVP (2.4). Then, by Lemma 2.7, the unique solution of (2.4) can be represented as

$$v(t) = \int_0^1 G(t, s)h(s, v(s)) ds, \quad t \in [0, 1],$$

where $G(t, s)$ is defined in (2.3). Thereby $v(t)$ is a fixed point of F .

On the other hand, let $v(t)$ be a fixed point of F . As $\beta < n - 1$, by Lemma 2.8, we have

$$\begin{aligned} {}^cD^\alpha(v(t)) &= {}^cD^\alpha\left(\int_0^1 G(t, s)h(s, v(s)) ds\right) \\ &= \frac{1}{\alpha(2 - \beta)\Gamma(\alpha)} \left[{}^cD^\alpha\left(\int_0^t [-\alpha(2 - \beta)(t - s)^{\alpha-1} \right. \right. \\ &\quad \left. \left. + [\alpha(2 - \beta) + 2\beta t(\alpha - 1 + s)](1 - s)^{\alpha-1}\right]h(s, v(s)) ds \right. \\ &\quad \left. + \int_t^1 [\alpha(2 - \beta) + 2\beta t(\alpha - 1 + s)](1 - s)^{\alpha-1}h(s, v(s)) ds \right] \end{aligned}$$

$$\begin{aligned}
 &= -h(t, v(t)) + {}^c D^\alpha \left(\frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s, v(s)) ds \right. \\
 &\quad \left. + \frac{2\beta t}{\alpha(2-\beta)\Gamma(\alpha)} (\alpha-1+s)(1-s)^{\alpha-1} h(s, v(s)) ds \right) \\
 &= -h(t, v(t)).
 \end{aligned}$$

Therefore $v(t)$ satisfies the differential equation (2.4). Moreover it is easy to verify that $v''(0) = v'''(0) = 0$ and $v'(0) = v(1) = \beta \int_0^1 v(s) ds$, which implies $v(t)$ is a solution for the BVP (2.4). Since $v(t)$ is a fixed point of F and F is continuous, $v(t)$ is a continuous solution for the given boundary value problem. Hence, every fixed point of F in $C[0, 1]$ solves the BVP (2.4). \square

We propose the following existence–uniqueness theorem for the solution of the problem (2.4).

Theorem 4.3 *Let $3 < \alpha < 4$ and*

$$\eta = \frac{1}{(\Gamma(\alpha))^2} \left[\frac{4}{5} + \frac{12\beta}{5|\beta-2|} + \frac{20\beta^2}{21(\beta-2)^2} \right] < 1 \tag{4.8}$$

hold for any $0 < \beta < 2, \beta \neq \alpha$. Suppose that for the function $h(\cdot, v(\cdot)) \in L_2$ for any $v \in C[0, 1]$ and for any $v, z \in C[0, 1]$, the inequality

$$|h(s, v(s)) - h(s, z(s))|^2 \leq (9 + |v(s) - z(s)|^2) |v(s) - z(s)|^2, \quad s \in [0, 1], \tag{4.9}$$

holds. Therefore the map F specified in Eq. (4.6) has a unique fixed point, and correspondingly, the BVP (2.4) has a unique solution in $C[0, 1]$.

Proof Through utilizing the Cauchy–Schwarz inequality and the definition of the map F described in Eq. (4.6), we get

$$\begin{aligned}
 |Fv(t) - Fz(t)|^2 &\leq \left| \int_0^1 G(t, s) [h(s, v(s)) - h(s, z(s))] ds \right|^2 \\
 &\leq \left(\int_0^1 |G(t, s)|^2 ds \right) \left(\int_0^1 |h(s, v(s)) - h(s, z(s))|^2 ds \right) \\
 &\leq \eta \int_0^1 (9 + |v(s) - z(s)|^2) |v(s) - z(s)|^2 ds.
 \end{aligned} \tag{4.10}$$

Taking the supremum over $[0, 1]$ along with the definition of the metric (4.7) and a controlled function \mathfrak{C} , we obtain

$$d_{\mathfrak{C}}(Fv, Fz) \leq \eta \mathfrak{C}(v, z) d_{\mathfrak{C}}(v, z)$$

and the inequalities (3.2), (3.3) and (3.4) hold for any $r_0 \in X$. Thus, it appears to follow that the map F defined in Eq. (4.6) meets the criteria of Theorem 3.1 and henceforth has a unique fixed point. Hence by Lemma 4.2, the BVP (2.4) has a unique solution in $C[0, 1]$. \square

5 Example

In this section, we add an example to Theorem 3.1.

Example 5.1 Consider the following differential equation of fractional order:

$${}^c D^{\frac{7}{2}} v(t) + h(t, v(t)) = 0, \quad 0 < t < 1, \tag{5.1}$$

where h is defined by

$$h(t, v(t)) = \begin{cases} \frac{1}{v(t)(1-v(t))}, & \text{if } -1 \leq v < 1, \\ \frac{1}{v(t)(1+v(t))}, & \text{otherwise,} \end{cases}$$

which is singular at $v = 0$ with the boundary conditions

$$v''(0) = v'''(0) = 0, \quad v'(0) = v(1) = \frac{1}{2} \int_0^1 v(s) ds.$$

The solution of the BVP provided in the example can be verified to fulfil the following integral equation:

$$v(t) = \int_0^1 G(t, s)h(s, v(s)) ds,$$

where the Green’s function $G(t, s)$ is given by

$$G(t, s) = \frac{4}{21\Gamma(\frac{7}{2})} \begin{cases} \{\frac{21}{4} + t(\frac{5}{2} + s)\}(1-s)^{5/2} - \frac{21}{4}(t-s)^{5/2}, & \text{if } 0 \leq s \leq t \leq 1, \\ \{\frac{21}{4} + t(\frac{5}{2} + s)\}(1-s)^{5/2}, & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{5.2}$$

Note that, in this example, $\alpha = \frac{7}{2}$, $\beta = \frac{1}{2}$ which fulfill the requirement (4.8), and by Lemma 4.1, we have

$$\eta = \int_0^1 |G(t, s)|^2 ds \leq \frac{103,168}{212,625\pi} \approx 0.15 < 1. \tag{5.3}$$

For the Green’s function acquired in Eq. (5.2) and the related map F defined by Eq. (4.6), we use the Cauchy–Schwarz inequality to derive

$$\begin{aligned} |Fv(t) - Fz(t)|^2 &\leq \left| \int_0^1 G(t, s)[h(s, v(s)) - h(s, z(s))] ds \right|^2 \\ &\leq \left(\int_0^1 |G(t, s)|^2 ds \right) \left(\int_0^1 |h(s, v(s)) - h(s, z(s))|^2 ds \right) \\ &= \eta \left(\int_0^1 |h(s, v(s)) - h(s, z(s))|^2 ds \right), \end{aligned} \tag{5.4}$$

where $\eta < 1$ and

$$|h(s, v(s)) - h(s, z(s))| = \begin{cases} \left| \frac{1}{v(s)(1-v(s))} - \frac{1}{z(s)(1-z(s))} \right|, & \text{if } v, z \in [-1, 1), \\ \left| \frac{1}{v(s)(1+v(s))} - \frac{1}{z(s)(1+z(s))} \right|, & \text{if } v, z \in (-\infty, -1) \cup [1, +\infty), \\ \left| \frac{1}{v(s)(1-v(s))} - \frac{1}{z(s)(1+z(s))} \right|, & \text{if } v \in [-1, 1), z \in (-\infty, -1) \cup [1, +\infty), \\ \left| \frac{1}{v(s)(1+v(s))} - \frac{1}{z(s)(1-z(s))} \right|, & \text{if } v \in (-\infty, -1) \cup [1, +\infty), z \in [-1, 1). \end{cases}$$

For $v, z \in [-1, 1)$, we see that

$$\begin{aligned} |Fv(t) - Fz(t)|^2 &\leq \eta \int_0^1 \left| \frac{1}{v(s)(1-v(s))} - \frac{1}{z(s)(1-z(s))} \right|^2 ds \\ &= \eta \int_0^1 \left| \frac{(v(s))^2 - (z(s))^2 - (v(s) - z(s))}{v(s)z(s)(1-v(s))(1-z(s))} \right|^2 ds \\ &= \eta \int_0^1 |v(s) - z(s)|^2 \left(\frac{(v(s) + z(s)) - 1}{v(s)z(s)(1-v(s))(1-z(s))} \right)^2 ds \\ &\leq \eta \int_0^1 |v(s) - z(s)|^2 \left(\frac{(v(s) + z(s)) + 1}{v(s)z(s)} \right)^2 ds \tag{5.5} \\ &= \eta \int_0^1 |v(s) - z(s)|^2 \left(\frac{(v(s) + z(s))^2 + 2(v(s) + z(s)) + 1}{v(s)z(s)} \right)^2 ds \\ &\leq \eta \int_0^1 |v(s) - z(s)|^2 \left(\frac{(v(s) - z(s))^2 + 4v(s)z(s) + 2(v(s) + z(s)) + 1}{v(s)z(s)} \right)^2 ds \\ &\leq \eta \int_0^1 |v(s) - z(s)|^2 (9 + |v(s) - z(s)|^2) ds. \end{aligned}$$

By taking the supremum over $t \in [0, 1]$ and taking into consideration the metric definition given in (4.7), one has

$$d_{\mathcal{C}}(Fv, Fz) \leq \eta \mathcal{C}(v, z) d_{\mathcal{C}}(v, z), \tag{5.6}$$

where

$$\mathcal{C}(v, z) = 9 + \sup_{t \in [0, 1]} |v(t) - z(t)|^2.$$

Similarly, we can prove the other cases. Therefore the map F specified by (4.6) has a unique fixed point and perhaps the BVP provided in the example does have a unique solution in $C[0, 1]$.

6 Conclusion

Following up Sevinik Adigüzel et al. [40], in this study, we dealt with problem (2.4) in the context of controlled b -Branciari metric type spaces, which is a stronger concept than the concept of extended-Branciari b -distance spaces [2] and controlled metric type spaces [33], providing a disparate approach to the existence and uniqueness of the solution. This

method can also be used for different α -derivative intervals. The theorem of existence–uniqueness in this study strengthens the current research as it provides quite requisites not only for positive solutions but also for any continuous solutions to the problem.

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Author details

¹Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Kalavakkam, Chennai, 603 110, India.

²Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia.

³Department of Medical Research, China Medical University, Taichung, Taiwan. ⁴Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan.

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