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Low Mach number limit for the quantum hydrodynamics system

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Abstract

In this paper, we deal with the low Mach number limit for the system of quantum hydrodynamics, far from the vortex nucleation regime. More precisely, in the framework of a periodic domain and ill-prepared initial data we prove strong convergence of the solutions toward regular solutions of the incompressible Euler system. In particular, we will perform a detailed analysis of the time oscillations and of the relative entropy functional related to the system.

Keywords: Compressible and incompressible Navier–Stokes equation, Quantum fluids, Energy estimates, Relative entropy, Acoustic equation

Mathematics Subject Classification: Primary 35L65; Secondary 35L40, 76R50

Contents

1 Introduction

In this paper, we deal with the following system of quantum hydrodynamics:

$$\partial_s \rho + \operatorname{div} J = 0, \tag{1}$$

$$\partial_s J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) + \nabla p(\rho) = \operatorname{div}(\rho \nabla^2 \log \rho), \tag{2}$$

where ρ and J represent the charge and current density, respectively, and $p(\rho)$ is the hydrodynamic pressure, which is a function depending only of ρ, satisfying the following conditions:

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$$p(\rho) \in C[0,\infty) \cap C^1(0,\infty) \tag{3}$$

and

$$\begin{cases} p(0) = 0, \ p'(\rho) \geq a_1 \rho^{\gamma-1} - b & \text{for all } \rho > 0, \\ p(\rho) \leq a_2 \rho^\gamma + b, & \text{for all } \rho \geq 0, \gamma \geq \frac{n}{2}. \end{cases} \tag{4}$$

The computations we are going to perform later on can be easily adapted to the general pressure law (4); however, for simplicity in this paper we will take $p(\rho)$ of the form

$$p(\rho) = \rho^\gamma / \gamma. \tag{5}$$

By using the Madelung formalism, the quantum hydrodynamics with this pressure corresponds to a nonlinear Schrödinger equation with nonlinear self-interaction potential obeying a power law (e.g., the cubic defocusing NLS).

In the paper, [27] regarding hydrodynamic nucleation of quantized vortex pairs in a polariton quantum fluid, theoretical predictions of different flow regimes are reported, to be depending on the Mach number; in particular, the authors show that the low Mach number regime prevents the onset of the vortex nucleation mechanism.

It is well known that a way to obtain the incompressible system from the compressible one is to perform the so-called *incompressible* or low Mach number limit. In fact, if we denote by ε the Mach number

$$\varepsilon = \text{Mach number} = \frac{\text{typical fluid speed}}{\text{sound speed}},$$

it makes sense to consider the limit $\varepsilon \to 0$. When this situation occurs, we observe that the pressure becomes nearly constant and the fluid cannot generate density variations, so it behaves as an incompressible fluid. In order to study this dynamics on the system (1)–(2), we perform the incompressible scaling given by

$$\rho^\varepsilon(x,t) = \rho\left(y\varepsilon^{-2}, s\varepsilon^{-1}\right), \quad J^\varepsilon = \varepsilon^{-1}J\left(y\varepsilon^{-2}, s\varepsilon^{-1}\right). \tag{6}$$

With the scaling (6), the system (1)–(2) becomes

$$\begin{aligned} &\partial_t \rho^\varepsilon + \text{div} J^\varepsilon = 0, \\ &\partial_t J^\varepsilon + \text{div}\left(\frac{J^\varepsilon \otimes J^\varepsilon}{\rho^\varepsilon}\right) + \varepsilon^{-2}\nabla p(\rho^\varepsilon) = \text{div}(\rho^\varepsilon \nabla^2 \log \rho^\varepsilon). \end{aligned} \tag{7}$$

1.1 Statement of the main result

The goal of this paper will be the study of the limiting behavior of the system (7) as $\varepsilon \to 0$. Before giving a precise description of the limiting behavior of our system (7), we need to define the framework where we are going to set up our problem. In this paper, we will always assume that $t \geq 0$ and $x \in \mathbb{T}^n$, where $\mathbb{T}^n$ is the n-dimensional torus.

1.1.1 Weak solutions

To simplify our notations, from now on we will denote by Ψ^ε the renormalized pressure, namely

$$\Psi^\varepsilon = \sqrt{\frac{(\rho^\varepsilon)^\gamma - 1 - \gamma(\rho^\varepsilon - 1)}{\varepsilon^2 \gamma(\gamma-1)}} \tag{8}$$

and by

$$\delta^\varepsilon = \varepsilon^{-1}(\rho^\varepsilon - 1) \tag{9}$$

the density fluctuation. A natural framework to deal with the system (7) is given by the space of finite initial energy. In fact, the energy associated with the system (7) is given by

$$E(t) = \frac{1}{2} \int_{\mathbb{T}^n} \left(|\Lambda^\varepsilon(t)|^2 + (\Psi^\varepsilon)^2 + |\nabla \sqrt{\rho^\varepsilon}|^2 \right) \, dx, \tag{10}$$

where $\Lambda^\varepsilon = J^\varepsilon / \sqrt{\rho^\varepsilon}$. So it seems now natural to introduce the following definition of *weak solution*.

Definition 1.1 We say that $(\rho^\varepsilon, J^\varepsilon)$ is a weak solution for the system (7) with initial data $(\rho_0^\varepsilon, J_0^\varepsilon)$ if there exists locally integrable functions $\sqrt{\rho^\varepsilon}, \Lambda^\varepsilon$, such that $\sqrt{\rho^\varepsilon} \in L_{\text{loc}}^2((0, T); H_{\text{loc}}^1(\mathbb{T}^n))$, $\Lambda^\varepsilon \in L_{\text{loc}}^2((0, T); L_{\text{loc}}^2(\mathbb{T}^n))$ and by defining $\rho^\varepsilon = (\sqrt{\rho^\varepsilon})^2$, $J^\varepsilon = \sqrt{\rho^\varepsilon}\Lambda^\varepsilon$ the following integral identity hold for any test function $\varphi \in C_0^\infty([0, T] \times \mathbb{T}^n)$, $\varphi(\cdot, T) = 0$

$$\int_0^T \int_{\mathbb{T}^n} (\rho^\varepsilon \partial_t \varphi + J^\varepsilon \cdot \nabla \varphi) \, dx \, dt + \int_{\mathbb{T}^n} \rho_0^\varepsilon \varphi(0) dx = 0 \tag{11}$$

and for any test function $\psi \in C_0^\infty([0, T] \times \mathbb{T}^n; \mathbb{R}^3)$, $\psi(\cdot, T) = 0$

$$\int_0^T \int_{\mathbb{T}^n} \left(J^\varepsilon \partial_t \psi + \Lambda^\varepsilon \otimes \Lambda^\varepsilon : \nabla \psi + \frac{1}{\varepsilon^2} p(\rho^\varepsilon) \, \text{div } \psi \right.$$
$$\left. + 4 \nabla \sqrt{\rho^\varepsilon} \otimes \nabla \sqrt{\rho^\varepsilon} : \nabla \psi - \rho^\varepsilon \nabla \, \text{div } \psi \, dx \, dt \right) + \int_{\mathbb{T}^n} J_0^\varepsilon \psi(0) dx = 0. \tag{12}$$

The existence of irrotational weak solutions, including vacuum states, for finite energy large data which are obtained by an H^1 wave function via a Madelung transformation has been proved by Antonelli and Marcati [1,2], by using dispersive analysis, local smoothing effects and polar factorization methods. Moreover, by means of the convex integration methods, in [9] it has been proved that the system admits on the torus infinitely many global-in-time weak solutions for any sufficiently smooth initial data including the case of a vanishing initial density—the vacuum zones. Existence of smooth solutions away from vacuum was proved before by Li and Marcati [21] for small perturbations of quantum subsonic steady states. Related results concerning the dynamics of quantum hydrodynamics systems can also be found in [15,18,19].

1.1.2 Initial data

Since it is natural to work with weak solutions, which have bounded energy (10), it is quite obvious to require that the initial data satisfy the following condition:

$$E(0) = \frac{1}{2} \int_{\mathbb{T}^n} \left(|\Lambda_0^\varepsilon|^2 + (\Psi_0^\varepsilon)^2 + |\nabla \sqrt{\rho_0^\varepsilon}|^2 \right) dx < +\infty, \tag{13}$$

where

$$\rho^\varepsilon|_{t=0} = \rho_0^\varepsilon, \quad J^\varepsilon|_{t=0} = J_0^\varepsilon, \quad \Lambda_0^\varepsilon = J_0^\varepsilon / \sqrt{\rho_0^\varepsilon}.$$

We perform our analysis by considering sufficiently general initial data, which in a weak sense can be called ill-prepared initial data, since we do not assume $\rho_0^\varepsilon = 1$ and div $\Lambda_0^\varepsilon = 0$, but we simply require that

$$\Lambda_0^\varepsilon \to \tilde{v}_0 \qquad \text{strongly in } L^2(\mathbb{T}^n), \tag{14}$$

$$\Psi_0^\varepsilon \to \delta_0 \qquad \text{strongly in } L^2(\mathbb{T}^n), \tag{15}$$

$$\sqrt{\rho_0^\varepsilon} - 1 \to 0 \qquad \text{strongly in } H^1(\mathbb{T}^n). \tag{16}$$

In particular, (15) implies that

$$\delta_0^\varepsilon \to \tilde{\delta}_0 \qquad \text{strongly in } L^\gamma(\mathbb{T}^n). \tag{17}$$

In order to apply in the sequel the relative entropy method, we need some additional regularity assumptions: $\sigma_0 \in H^s(\mathbb{T}^n)$, $\tilde{v}_0 \in H^s(\mathbb{T}^n)$, $P\tilde{v}_0 \in H^s(\mathbb{T}^n)$, $Q\tilde{v}_0 \in H^{s-1}(\mathbb{T}^n)$, for any $s \geq n/2 + 1$, where P and Q denote the Leray projectors on the divergence-free vector fields and the gradient vector fields, respectively.

1.1.3 The limiting system

As already said, the aim of this paper is to perform the low Mach number limit for the system (7). If we look at the second equation of (7) (linear momentum equation), we can deduce that as $\varepsilon \to 0$, ρ^ε behaves like $\tilde{\rho} + \varepsilon^2$ (where $\tilde{\rho}$ is a constant which by a simple scaling can always be assumed to be as $\tilde{\rho} = 1$). So, at a formal level, we can see that as $\varepsilon \to 0$, the density ρ^ε becomes constant, and Λ^ε converges to a solenoidal vector field v. Hence, we end up with the following incompressible Euler system:

$$\begin{cases} \operatorname{div} v = 0, \\ \partial_t v + \operatorname{div}(v \otimes v) + \nabla\pi = 0, \\ v(x, 0) = P\tilde{v}_0 = v_0. \end{cases} \tag{18}$$

It is worthwhile, at this point, to recall the following classical result on the existence of regular solutions for the incompressible Euler system (18), see Kato [20] and Lions [23].

Proposition 1.2 *Let the initial velocity field satisfy* $v_0 \in H^s(\text{or } H^{s+1})$, $s \geq \frac{n}{2} + 2$ *with* $\operatorname{div} v_0 = 0$. *Then, there exists* $0 < T^* < \infty$, *the maximal existence time, and a unique smooth solution* (v, π) *of the incompressible Euler equation* (18) *on* $[0, T^*)$ *with initial data* v_0, *satisfying for any* $T < T^*$

$$\sup_{0 \leq t < T^*} (\|v\|_{H^s} + \|\partial_t v\|_{H^{s-1}} + \|\nabla\pi\|_{H^s} + \|\partial_t \nabla\pi\|_{H^{s-1}}) \leq M(T).$$

1.1.4 Main result

Now we are ready to state the main result we are going to prove in the paper.

Theorem 1.3 *Assume that* $(\rho^\varepsilon, \Lambda^\varepsilon)$ *is a weak solution of the quantum hydrodynamic system* (7), *in the sense of the Definition* 1.1 *and that initial data verify the conditions of Sect.* 1.1.2. *Let* T^* *and* T^{**} *be as in the Propositions* 1.2, 3.2, *respectively, then as* $\varepsilon \to 0$ *and for all* $T < \min(T^*, T^{**})$, *we have*

i) $\Lambda^\varepsilon \rightharpoonup v$ *weakly in* $L^\infty(0, T; L^2(\mathbb{T}^n))$,
ii) $P\Lambda^\varepsilon \to v$ *strongly in* $L^\infty(0, T; L^2(\mathbb{T}^n))$,

where v *is the unique local in time solution of the Euler system* (18) $(v \in L^\infty_{\text{loc}}(0, T^*, H^s(\mathbb{T}^n))$, $s > \frac{n}{2} + 2)$.

1.2 Plan of the paper

The low Mach number limit for fluid dynamic models has been studied by many authors. See, for example, the paper by Lions and Masmoudi [24], Desjardin et al. [5], Desjardin and Grenier [4] for the case of the compressible Navier–Stokes equations. The mathematical analysis is completely different in the case of well-prepared initial data ($\rho_0^\varepsilon = 1$,

$\mathrm{div}(\Lambda_0^\varepsilon/\sqrt{\rho^\varepsilon}) = 0$) or in the case of "ill-prepared" data. In the latter case, the fluctuation of the fluid density is of the same order of the Mach number and the gradient part of the velocity develops fast time oscillations. In fact, the main issue in treating this kind of limits is the presence of acoustic waves that propagate with high speed of order $1/\varepsilon$ and are supported by the gradient part of the velocity field. The main consequence is the loss of compactness of the velocity field or of the momentum and the impossibility to define the limit of nonlinear quantities such as the convective term. The analysis at this point is different according to the space domain of the problem. We can say that in the case of the incompressible limit the acoustic waves in general are well described by a wave equation with a source term bounded in some suitable space. Then, in the case of an unbounded domain (whole or exterior domain) we can observe that the acoustic waves redistribute their energy in the space and so one can exploit the dispersive properties of these waves to get the local decay of the acoustic energy and to recover compactness in time, see, for example, [4,6,7,12]. In the case of a periodic domain, we do not have a dispersion phenomenon, but the waves interact with each other, so in the spirit of Schochet [28] and [29] one has to introduce an operator that describes the oscillations in time so that they can be included in the energy estimates, see [14,22,25,26].

In this paper, we will study the incompressible limit in a periodic domain for the system of quantum hydrodynamics (7), and, as explained above, the main issue is to control the time oscillations of the density fluctuation and of the momentum J^ε. In the Sect. 2, we start by recovering the standard energy estimates satisfied by the weak solutions of (7). Then, in Sect. 3 we introduce the operator $\mathcal{L}$ which describes the time oscillations and will give a careful analysis of all its properties. In Sect. 4, in order to include the time oscillation in the energy and to study the convergence of our sequences, we introduce the relative entropy functional $H^\varepsilon(t)$. This functional computes the error terms due to the fast time oscillations, namely the difference of our sequence and the limit solutions. In an heuristic way, we can say that the entropy measures the error that we have when we pass from the weak to the strong convergence. We will be able to show that as $\varepsilon \to 0$ the entropy goes to zero and so we get the strong convergence of our sequences. This will lead to the proof of the main result in Sect. 5.

For completeness, we conclude this section by mentioning that in the same framework of the incompressible limits and its related problems and techniques fits also the so-called quasineutral limit in plasma physics or the zero electron mass limit, see, for example, [3,8,10,11,13,16,17,30,31].

2 Energy inequality and its consequences

Taking into account the existence result of Sect. 1.1.1, we know that the weak solutions of the system (7) satisfy the following energy bound:

$$E(t) = \frac{1}{2} \int_{\mathbb{T}^n} \left(|\Lambda^\varepsilon(t)|^2 + (\Psi^\varepsilon)^2 + |\nabla\sqrt{\rho^\varepsilon}|^2 \right) \mathrm{d}x \leq E(0). \tag{19}$$

Hence, by virtue of (13) and by the convexity of the function $z \to z^\gamma - 1 - \gamma(z-1)$ for $z \geq 0$, the following convergence (up to a subsequence) hold:

$$\rho^\varepsilon - 1 \to 0 \quad \text{strongly in } L^\infty([0,T]; L^\gamma(\mathbb{T}^n)), \tag{20}$$

$$\Lambda^\varepsilon \rightharpoonup v \quad \text{weakly in } L^\infty([0,T]; L^2(\mathbb{T}^n)). \tag{21}$$

Moreover, since

$$|\sqrt{z} - 1|^2 \leq M|z - 1|^\gamma, \quad |z - 1| \geq \eta, \quad \gamma \geq 1,$$
$$|\sqrt{z} - 1|^2 \leq M|z - 1|^2, \quad z \geq 0,$$

from (20), we have

$$\sqrt{\rho^\varepsilon} - 1 \to 0 \quad \text{strongly in } L^\infty([0, T]; L^2(\mathbb{T}^n)). \tag{22}$$

By rewriting the continuity equation $(7)_1$ in the following way:

$$\partial_t(\rho^\varepsilon - 1) + \text{div}((\sqrt{\rho^\varepsilon} - 1)\Lambda^\varepsilon) + \text{div}\,\Lambda^\varepsilon = 0,$$

as $\varepsilon \to 0$, we infer that $v(x, t)$ is a divergence-free vector field. Unfortunately, the previous convergences are not enough to pass into the limit in the system (7) since we still do not control the oscillations in time. This will be argument of the next session.

3 Study of the time oscillations

In this section, we try to understand the behavior of the oscillations in time in order to prove that they do not affect the limit system. In order to describe the time oscillations (following [29]), we introduce the group $\mathcal{L}(\tau)$, $\tau \in \mathbb{R}$, defined by $e^{\tau L}$, where L is the operator on the space $\mathcal{D}'_0 \times \mathcal{D}'$, where $\mathcal{D}'_0 = \{\phi \in \mathcal{D}' \mid \int \phi = 0\}$ given by

$$L\begin{pmatrix} \phi \\ v \end{pmatrix} = -\begin{pmatrix} \text{div}\,v \\ \nabla\phi \end{pmatrix}. \tag{23}$$

Notice that $\mathcal{L}$ is an isometry in any H^s space, $s \in \mathbb{R}$. Now, we introduce the following notations that we are going to use later on:

$$U^\varepsilon = \begin{pmatrix} \delta^\varepsilon \\ Q(J^\varepsilon) \end{pmatrix}, \qquad V^\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)U^\varepsilon, \tag{24}$$

$$\bar{U}^\varepsilon = \begin{pmatrix} \Psi^\varepsilon \\ Q(\Lambda^\varepsilon) \end{pmatrix}, \qquad \bar{V}^\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)\bar{U}^\varepsilon, \tag{25}$$

and the following approximation holds:

$$\|U^\varepsilon - \bar{U}^\varepsilon\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\mathbb{T}^n))} \longrightarrow 0 \quad \text{as } \varepsilon \to 0. \tag{26}$$

By using the notations (8) and (9), we rewrite the system (7) as follows:

$$\varepsilon\partial_t\delta^\varepsilon + \text{div}\,Q(J^\varepsilon) = 0, \qquad \varepsilon\partial_t Q(J^\varepsilon) + \nabla\delta^\varepsilon = \varepsilon G^\varepsilon \tag{27}$$

where

$$G^\varepsilon = -Q\left[\text{div}\left(\frac{J^\varepsilon \otimes J^\varepsilon}{\rho^\varepsilon}\right) - \text{div}(\rho^\varepsilon \nabla^2 \log \rho^\varepsilon)\right] - (\gamma - 1)\nabla(\Psi^\varepsilon)^2. \tag{28}$$

By means of (24), the system (27) has also the following form:

$$\partial_t U^\varepsilon = \frac{1}{\varepsilon}LU^\varepsilon + \begin{pmatrix} 0 \\ G^\varepsilon \end{pmatrix}, \tag{29}$$

which is equivalent to

$$\partial_t V^\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)\begin{pmatrix} 0 \\ G^\varepsilon \end{pmatrix}. \tag{30}$$

From the energy bounds (19), we get that G^ε is bounded in $L^2([0,T];H^{-s}(\mathbb{T}^n))$, $s > 0$ uniformly in ε; hence, V^ε is compact in time (the oscillations have been canceled) and since $V^\varepsilon \in L^\infty([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n))$, uniformly in ε, we get as $\varepsilon \to 0$,

$$V^\varepsilon \to \bar{V} \text{ strongly in } L^p\left([0,T];H^{-s'}(\mathbb{T}^n)\right) \quad \text{for all } s' > s \quad \text{and} \quad 1 < p < \infty.$$

At this point, it is important to remark that $\mathcal{L}\left(-\frac{t}{\varepsilon}\right)(0,G^\varepsilon)$ can be considered as an almost periodic function in $\tau = t/\varepsilon$ and computing its mean values yields the definitions of the following bilinear forms (see [26,28]).

Definition 3.1 For all divergence-free vector field $v \in L^2(\mathbb{T}^n)$ and all $V = (\psi, \nabla q) \in L^2(\mathbb{T}^n)$, we define the following linear and bilinear symmetric forms in V:

$$B_1(v,V) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathcal{L}(-s) \begin{pmatrix} 0 \\ \text{div}(v \otimes \mathcal{L}_2(s)V + \mathcal{L}_2(s)V \otimes v) \end{pmatrix} ds, \tag{31}$$

and

$$B_2(V,V) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathcal{L}(-s) \begin{pmatrix} 0 \\ \text{div}(\mathcal{L}_2(s)V \otimes \mathcal{L}(s)V + (\gamma-1)\nabla(\mathcal{L}_1(s)V)^2) \end{pmatrix} ds, \tag{32}$$

Now, in the same spirit as in [26], if we pass into the limit in (30) we get that $\bar{V}$ satisfies the following equation:

$$\partial_t \bar{V} + B_1(v,\bar{V}) + B_2(\bar{V},\bar{V}) = 0, \tag{33}$$

where B_1 and B_2 are as in (31) and (32), respectively. In a way similar to [26], we obtain the following local existence result for the system (33).

Proposition 3.2 *Let us consider the following system:*

$$\begin{cases} \partial_t V^0 + B_1(v,V^0) + B_2(V^0,V^0) = 0, \\ V^0(0) = (\sigma_0, Q\tilde{v}_0), \end{cases} \tag{34}$$

where v is the solution of the incompressible Euler problem (18) and $(\sigma_0, Q\tilde{v}_0)$ satisfy the regularity conditions of Sect. 1.1.2. Then, there exists a maximal existence time $0 < T^{**} < \infty$ and a unique local strong solution V^0 of (34) such that $V^0 \in L^\infty([0,T^{**}];H^{s-1}(\mathbb{T}^n)) \cap L^2([0,T^{**}];H^s(\mathbb{T}^n))$, for any $s \geq n/2 + 1$.

Remark 3.3 It is important to notice that in the case of well-prepared initial data, i.e., $V^0(0) = (\sigma_0, Q\tilde{v}_0) = 0$, the solution of the system (34) is given by $V^0 = 0$. This means that the oscillations with respect to time vanish and so $\Lambda^\varepsilon \to v$ strongly in $L^\infty([0,T], L^2(\mathbb{T}^n))$. But, for the general initial data, since the oscillation part with respect to time t/ε does not vanish, there are oscillations in time of the solution sequence.

Now we report three technical proposition concerning the properties of the linear and bilinear forms B_1 and B_2 that we will use in the sequel. For their proofs, we refer to [26].

Proposition 3.4 *For all v, V, V_1, V_2, we have*

$$\int B_1(v,V)V = 0 \quad \text{and} \quad \int B_1(v,V_1)V_2 + B_1(v,V_2)V_1 = 0, \tag{35}$$

$$\int B_2(V,V)V = 0 \quad \text{and} \quad \int B_2(V_1,V_1)V_2 + 2B_2(V_1,V_2)V_1 = 0. \tag{36}$$

Proposition 3.5 *For all $v \in L^p(0, T; L^2(\mathbb{T}^n))$ and $V \in L^q(0, T; L^2(\mathbb{T}^n))$, as $\varepsilon \to 0$, we have the following weak convergence (p and q are such that the products are well defined)*

$$\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\begin{pmatrix} 0 \\ \operatorname{div}(v \otimes \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V + \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V \otimes v) \end{pmatrix} \xrightarrow[\text{weakly}]{} B_1(v, V), \tag{37}$$

$$\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\begin{pmatrix} 0 \\ \operatorname{div}(\mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V \otimes \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V) + (\gamma - 1)\nabla(\mathcal{L}_1\left(\frac{t}{\varepsilon}\right) V)^2 \end{pmatrix} \xrightarrow[\text{weakly}]{} B_2(V, V). \tag{38}$$

Proposition 3.6 *For any $V_1 \in L^q(0, T; H^s(\mathbb{T}^n))$ and $V_2 \in L^p(0, T; H^{-s}(\mathbb{T}^n))$, with $s \in \mathbb{R}$, $1/p + 1/q = 1$, one has as $\varepsilon \to 0$*

$$\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\begin{pmatrix} 0 \\ \operatorname{div}(\mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V_1 \otimes \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V_2 + \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V_2 \otimes \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V_1) \end{pmatrix}$$

$$+ \mathcal{L}\left(-\frac{t}{\varepsilon}\right)\begin{pmatrix} 0 \\ (\gamma - 1)\nabla(\mathcal{L}_1\left(\frac{t}{\varepsilon}\right) V_1 \mathcal{L}_1\left(\frac{t}{\varepsilon}\right) V_2) \end{pmatrix} \xrightarrow[\text{weakly}]{} B_2(V_1, V_2). \tag{39}$$

It is also possible to extend (39) to the case where we replace V_2 in the left-hand side by a sequence V_2^ε that converges strongly to V_2 in $L^p(0, T; H^{-s}(\mathbb{T}^n))$.

4 Relative entropy

In order to prove the convergence stated in Theorem 1.3, we introduce the following relative entropy functional:

$$\mathcal{H}^\varepsilon(t) = \frac{1}{2}\int_{\mathbb{T}^n}\left\{|\Lambda^\varepsilon - v - \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V^0|^2 + |\Psi^\varepsilon - \mathcal{L}_1\left(\frac{t}{\varepsilon}\right) V^0|^2 + |\nabla\sqrt{\rho^\varepsilon}|^2\right\}\mathrm{d}x. \tag{40}$$

The entropy describes the difference between the solutions of the scaled quantum hydrodynamic system (7) and the limit solution, namely the solution v of the Euler system (18) and the fast time oscillations. The goal of this section is to recover uniform estimates in ε for $\mathcal{H}^\varepsilon(t)$ and to show that the relative entropy vanishes as $\varepsilon \to 0$, yielding the strong convergence of our solutions. First of all, we recall that the solutions of (7) satisfy the following energy bound

$$\frac{1}{2}\int_{\mathbb{T}^n}\left(|\Lambda^\varepsilon(t)|^2 + (\Psi^\varepsilon)^2 + |\nabla\sqrt{\rho^\varepsilon}|^2\right)\mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{T}^n}\left(|\Lambda_0^\varepsilon|^2 + (\Psi_0^\varepsilon)^2 + |\nabla\sqrt{\rho_0^\varepsilon}|^2\right)\mathrm{d}x. \tag{41}$$

Moreover, the solution v of the Euler system (18) satisfies the conservation of energy

$$\frac{1}{2}\int_{\mathbb{T}^n}|v|^2\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^n}|v_0|^2\mathrm{d}x, \tag{42}$$

while the solution V^0 of (34), taking into account (35) and (36), satisfies

$$\frac{1}{2}\int_{\mathbb{T}^n}|V^0|^2\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^n}(|\sigma_0|^2 + |Q\tilde{v}_0|^2)\mathrm{d}x, \tag{43}$$

Now if we use $\mathcal{L}_1\left(\frac{t}{\varepsilon}\right) V^0$ as test function for the weak formulation of the mass conservation equation (7)$_1$, we have

$$\int_{\mathbb{T}^n}\mathcal{L}_1\left(\frac{t}{\varepsilon}\right) V^0 \delta^\varepsilon\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}^n}\mathcal{L}_1\left(\frac{s}{\varepsilon}\right) \partial_s V^0 \delta^\varepsilon\,\mathrm{d}x\,\mathrm{d}s$$

$$+ \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^n}\left(\operatorname{div}(\sqrt{\rho^\varepsilon}\Lambda^\varepsilon)\mathcal{L}_1\left(\frac{s}{\varepsilon}\right) V^0 + \operatorname{div}\left(\mathcal{L}_2\left(\frac{s}{\varepsilon}\right) V^0\right) \delta^\varepsilon\right)\mathrm{d}x\,\mathrm{d}s$$

$$= \int_{\mathbb{T}^n}\sigma_0 \delta^\varepsilon(0)\mathrm{d}x. \tag{44}$$

By using v and then $\mathcal{L}_2\left(\frac{t}{\varepsilon}\right)V^0$ as test functions in the momentum equation $(7)_2$, we have

$$
\int_{\mathbb{T}^n}\sqrt{\rho^\varepsilon}\,\Lambda^\varepsilon v\mathrm{d}x + \int_0^t\int_{\mathbb{T}^n}\sqrt{\rho^\varepsilon}\,\Lambda^\varepsilon(v\cdot\nabla v + \nabla\pi)\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\mathbb{T}^n}\Lambda^\varepsilon\otimes\Lambda^\varepsilon\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}s
$$
$$
+ \int_0^t\int_{\mathbb{T}^n}\nabla\sqrt{\rho^\varepsilon}\otimes\nabla\sqrt{\rho^\varepsilon}:\nabla v\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{T}^n}\sqrt{\rho_0^\varepsilon}\Lambda_0^\varepsilon v_0\,\mathrm{d}x, \tag{45}
$$

$$
\int_{\mathbb{T}^n}\sqrt{\rho^\varepsilon}\,\Lambda^\varepsilon\mathcal{L}_2\left(\frac{t}{\varepsilon}\right)V^0\mathrm{d}x - \int_0^t\int_{\mathbb{T}^n}\Lambda^\varepsilon\otimes\Lambda^\varepsilon\cdot\nabla\mathcal{L}_2\left(\frac{s}{\varepsilon}\right)V^0\,\mathrm{d}x\,\mathrm{d}s
$$
$$
+ \int_0^t\int_{\mathbb{T}^n}4\nabla\sqrt{\rho^\varepsilon}\otimes\nabla\sqrt{\rho^\varepsilon}:\nabla\mathcal{L}_2\left(\frac{s}{\varepsilon}\right)V^0 - \rho^\varepsilon\Delta\operatorname{div}\mathcal{L}_2\left(\frac{s}{\varepsilon}\right)V^0\,\mathrm{d}x\,\mathrm{d}s
$$
$$
- \int_0^t\int_{\mathbb{T}^n}\mathcal{L}_2\left(\frac{s}{\varepsilon}\right)\partial_s V^0\sqrt{\rho^\varepsilon}\Lambda^\varepsilon - \frac{1}{\varepsilon}\sqrt{\rho^\varepsilon}\Lambda^\varepsilon\nabla\mathcal{L}_1\left(\frac{s}{\varepsilon}\right)V^0\,\mathrm{d}x\,\mathrm{d}s
$$
$$
- \int_0^t\int_{\mathbb{T}^n}\left(\frac{1}{\varepsilon}\delta^\varepsilon + (\gamma-1)(\Psi^\varepsilon)^2\right)\operatorname{div}\left(\mathcal{L}_2\left(\frac{s}{\varepsilon}\right)V^0\right)\mathrm{d}x\,\mathrm{d}s
$$
$$
= \int_{\mathbb{T}^n}\sqrt{\rho_0^\varepsilon}\Lambda_0^\varepsilon Q\tilde{v}_0\,\mathrm{d}x. \tag{46}
$$

Now taking into account that

$$
\int_0^t\int_{\mathbb{T}^n}\mathcal{L}\left(\frac{s}{\varepsilon}\right)\partial_s V^0 U^\varepsilon\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_{\mathbb{T}^n}\partial_s V^0 V^\varepsilon\,\mathrm{d}x\,\mathrm{d}s,
$$

we sum up (41), (42), (43) and subtract (44), (45), (46); therefore, we get that following inequality for $H^\varepsilon(t)$

$$
\mathcal{H}^\varepsilon(t) \le I^\varepsilon + A^\varepsilon + B^\varepsilon + C^\varepsilon, \tag{47}
$$

where we set

$$
I^\varepsilon = \mathcal{H}^\varepsilon(0) + \int_{\mathbb{T}^n}v\mathcal{L}_2\left(\frac{t}{\varepsilon}\right)V^0\mathrm{d}x + \int_{\mathbb{T}^n}\left(\sqrt{\rho_0^\varepsilon}-1\right)\Lambda_0^\varepsilon(v_0 + Q\tilde{v}_0)\,\mathrm{d}x \tag{48}
$$

$$
A^\varepsilon = -\int_{\mathbb{T}^n}(\Psi^\varepsilon - \delta^\varepsilon)\mathcal{L}_1\left(\frac{t}{\varepsilon}\right)V^0\mathrm{d}x + \int_{\mathbb{T}^n}(\sqrt{\rho^\varepsilon}-1)\Lambda^\varepsilon(v + \mathcal{L}_2\left(\frac{t}{\varepsilon}\right)V^0)\,\mathrm{d}x, \tag{49}
$$

$$
B^\varepsilon = \int_0^t\int_{\mathbb{T}^n}\left(\sqrt{\rho^\varepsilon}\Lambda^\varepsilon(v\cdot\nabla v + \nabla\pi) - \Lambda^\varepsilon\otimes\Lambda^\varepsilon\cdot\nabla\left(v + \mathcal{L}_2\left(\frac{s}{\varepsilon}\right)V^0\right)\right)\mathrm{d}x\,\mathrm{d}s
$$
$$
+ \int_0^t\int_{\mathbb{T}^n}(\gamma-1)(\Psi^\varepsilon)^2\operatorname{div}\left(\mathcal{L}_2\left(\frac{s}{\varepsilon}\right)V^0\right)\mathrm{d}x\,\mathrm{d}s
$$
$$
+ \int_0^t\int_{\mathbb{T}^n}(B_1(v,V^0)V^\varepsilon + B_1(V^0,V^0)V^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s, \tag{50}
$$

$$
C^\varepsilon = \int_0^t\int_{\mathbb{T}^n}4\nabla\sqrt{\rho^\varepsilon}\otimes\nabla\sqrt{\rho^\varepsilon}:\left(\nabla v + \nabla\mathcal{L}_2\left(\frac{s}{\varepsilon}\right)V^0\right)\mathrm{d}x\,\mathrm{d}s
$$
$$
- \int_0^t\int_{\mathbb{T}^n}\rho^\varepsilon\Delta\operatorname{div}\mathcal{L}_2\left(\frac{s}{\varepsilon}\right)V^0\,\mathrm{d}x\,\mathrm{d}s. \tag{51}
$$

4.1 Uniform estimates for $\mathcal{H}^\varepsilon(t)$

Here we will estimate uniformly in ε the right-hand side of (47). In what follows we will denote by $r^\varepsilon(t)$ any term such that $r^\varepsilon(t) \to 0$, as $\varepsilon \to 0$ and by $M(T)$ a constant that depends only on $T = \min(T^*, T^{**})$. We start with I^ε. By taking into account the assumptions on the initial data of Sect. 1.1.2, the properties v solution of the system (18) and of the operator $\mathcal{L}$ we get

$$
|I^\varepsilon| \le C\mathcal{H}^\varepsilon(0) + \varepsilon E(0)\|v_0 + Q\tilde{v}_0\|_{H^s(\mathbb{T}^n)} \le \mathcal{H}^\varepsilon(0) + r^\varepsilon(t), \tag{52}
$$

where $C > 0$ is a constant. In order to estimate A^ε, we use (26) and the regularity of V^0 and we get

$$|A^\varepsilon| \leq r^\varepsilon(t)M(T) + \left| \int_{\mathbb{T}^n} (\Psi^\varepsilon - \delta^\varepsilon)\mathcal{L}_1 \left(\frac{t}{\varepsilon} \right) V^0 dx \right|$$

$$\leq r^\varepsilon(t)M(T) + M(T)\|U^\varepsilon - \bar{U}^\varepsilon\|_{L^\infty(0,T;L^{2\gamma/\gamma+1})} \leq r^\varepsilon(t)M(T). \qquad (53)$$

In the same spirit, we estimate C^ε and we end up with

$$|C^\varepsilon| \leq M(T) \int_0^t \|\nabla\sqrt{\rho^\varepsilon}\|_{L^2(\mathbb{T}^n)} ds + 2\int_0^t \int_{\mathbb{T}^n} (\sqrt{\rho^\varepsilon} - 1)\nabla\sqrt{\rho^\varepsilon}\nabla\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 dx\, ds$$

$$+ 2\int_0^t \int_{\mathbb{T}^n} \nabla\sqrt{\rho^\varepsilon}\nabla\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 dx\, ds$$

$$\leq M(T) \int_0^t \mathcal{H}^\varepsilon(s) ds. \qquad (54)$$

The term B^ε deserves some more attention, first of all we split it in three parts as follows:

$$|B^\varepsilon| \leq |B_1^\varepsilon| + |B_2^\varepsilon| + |B_3^\varepsilon|, \qquad (55)$$

and then we estimate each one of the three parts. We start with B_1^ε,

$$|B_1^\varepsilon| = \left| \int_0^t \int_{\mathbb{T}^n} \sqrt{\rho^\varepsilon}\Lambda^\varepsilon \nabla\pi\, dx\, ds \right| \leq r^\varepsilon(t) + M(T) \int_0^t \mathcal{H}^\varepsilon(s) ds. \qquad (56)$$

Then, as $\varepsilon \to 0$ we also have:

$$|B_2^\varepsilon| = \left| \int_0^t \int_{\mathbb{T}^n} (B_1(v, V^0)V^\varepsilon + B_2(V^0, V^0)V^\varepsilon) dx\, ds \right|$$

$$\leq \int_0^t \int_{\mathbb{T}^n} \left| (B_1(v, V^0)\bar{V} + B_2(V^0, V^0)\bar{V}) \right| dx\, ds + r^\varepsilon(t), \qquad (57)$$

$$|B_3^\varepsilon| = \left| \int_0^t \int_{\mathbb{T}^n} \sqrt{\rho^\varepsilon}\Lambda^\varepsilon (v \cdot \nabla)v\, dx\, ds - \int_0^t \int_{\mathbb{T}^n} \Lambda^\varepsilon \otimes \Lambda^\varepsilon \nabla \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) dx\, ds \right.$$

$$\left. - \int_0^t \int_{\mathbb{T}^n} (\gamma - 1)(\Psi^\varepsilon)^2 \operatorname{div} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) dx\, ds \right|$$

$$\leq \left| -\int_0^t \int_{\mathbb{T}^n} \left(\Lambda^\varepsilon - v - \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \otimes \left(\Lambda^\varepsilon - v - \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \right.$$

$$\cdot \nabla \left(v + \mathcal{L} \left(\frac{s}{\varepsilon} \right) V^0 \right) dx\, ds$$

$$-(\gamma - 1) \int_0^t \int_{\mathbb{T}^n} \left| \Psi^\varepsilon - \mathcal{L}_1 \left(\frac{s}{\varepsilon} \right) V^0 \right|^2 \operatorname{div} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) dx\, ds$$

$$-\int_0^t \int_{\mathbb{T}^n} \left[\Lambda^\varepsilon \otimes \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) + \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \otimes \Lambda^\varepsilon \right]$$

$$\cdot \nabla \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) dx\, ds$$

$$+\int_0^t \int_{\mathbb{T}^n} \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \otimes \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right)$$

$$\cdot \nabla \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) dx\, ds$$

$$+\int_0^t \int_{\mathbb{T}^n} \left\{ (\gamma - 1) \left| \mathcal{L}_1 \left(\frac{s}{\varepsilon} \right) V^0 \right|^2 \operatorname{div} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \right.$$

$$\left. -(\gamma - 1)\mathcal{L}_1 \left(\frac{s}{\varepsilon} \right) V^0\Psi^\varepsilon \operatorname{div} \left(\mathcal{L}_2 \left(\frac{t}{\varepsilon} \right) V^0 \right) \right\} dx\, ds \Bigg| + r^\varepsilon(t). \qquad (58)$$

Now by using (38), we have

$$
\int_0^t \int_{\mathbb{T}^n} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \otimes \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \cdot \nabla \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \, dx \, ds
$$

$$
\cdot (\gamma - 1) \int_0^t \int_{\mathbb{T}^n} \left| \mathcal{L}_1 \left(\frac{s}{\varepsilon} \right) V^0 \right|^2 \operatorname{div} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \, dx \, ds
$$

$$
= - \int_0^t \int_{\mathbb{T}^n} \left[\operatorname{div} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \otimes \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \right.
$$

$$
\left. + (\gamma - 1) \nabla \left| \mathcal{L}_1 \left(\frac{s}{\varepsilon} \right) V^0 \right|^2 \right] \cdot \left(V^0 + \begin{pmatrix} 0 \\ v \end{pmatrix} \right) dx \, ds
$$

$$
= - \int_0^t \int_{\mathbb{T}^n} \mathcal{L} \left(\frac{s}{\varepsilon} \right) V^0 \begin{pmatrix} 0 \\ \operatorname{div} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \otimes \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) + (\gamma - 1) \nabla \left| \mathcal{L}_1 \left(\frac{s}{\varepsilon} \right) V^0 \right|^2 \end{pmatrix}
$$

$$
\cdot \left(V^0 + \begin{pmatrix} 0 \\ v \end{pmatrix} \right) dx \, ds - \int_0^t \int_{\mathbb{T}^n} B_2(V^0, V^0) \cdot \left(V^0 + \begin{pmatrix} 0 \\ v \end{pmatrix} \right) dx \, ds + r^\varepsilon(t) = r^\varepsilon(t).
$$

$$(59)$$

In the same way if we use (39), we get

$$
- \int_0^t \int_{\mathbb{T}^n} \left[\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \otimes \Lambda^\varepsilon + \Lambda^\varepsilon \otimes \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right] \cdot \nabla \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) dx \, ds
$$

$$
- \int_0^t \int_{\mathbb{T}^n} (\gamma - 1) \mathcal{L}_1 \left(\frac{s}{\varepsilon} \right) V^0 \Psi^\varepsilon \operatorname{div} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) dx \, ds
$$

$$
= \int_0^t \int_{\mathbb{T}^n} \left[\operatorname{div} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \otimes \Lambda^\varepsilon + \Lambda^\varepsilon \otimes \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) \right.
$$

$$
\left. + (\gamma - 1) \nabla \left(\mathcal{L}_1 \left(\frac{s}{\varepsilon} \right) V^0 \Psi^\varepsilon \right) \cdot \left(v + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) dx \, ds \right.
$$

$$
= \int_0^t \int_{\mathbb{T}^n} \mathcal{L} \left(\frac{s}{\varepsilon} \right) \begin{pmatrix} 0 \\ \operatorname{div} \left(\mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \otimes \Lambda^\varepsilon + \Lambda^\varepsilon \otimes \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \right) + (\gamma - 1) \nabla \left(\mathcal{L}_1 \left(\frac{s}{\varepsilon} \right) V^0 \Psi^\varepsilon \right) \end{pmatrix}
$$

$$
\cdot \left(V^0 + \begin{pmatrix} 0 \\ v \end{pmatrix} \right) dx \, ds
$$

$$
= \int_0^t \int_{\mathbb{T}^n} \left(2 B_2(V^0, \bar{V}) + B_1(v, V^0) \right) \cdot \left(V^0 + \begin{pmatrix} 0 \\ v \end{pmatrix} \right) dx \, ds + r^\varepsilon(t)
$$

$$
= \int_0^t \int_{\mathbb{T}^n} 2 B_2(V^0, \bar{V})(v, V^0) \cdot \left(V^0 + \begin{pmatrix} 0 \\ v \end{pmatrix} \right) dx \, ds + r^\varepsilon(t). \qquad (60)
$$

By standard computations, we also get

$$
\int_0^t \int_{\mathbb{T}^n} \operatorname{div}(v \otimes \Lambda^\varepsilon + \Lambda^\varepsilon \otimes v) \cdot \left(v + \mathcal{L} \left(\frac{s}{\varepsilon} \right) V^0 \right) dx \, ds
$$

$$
= \int_0^t \int_{\mathbb{T}^n} B_1(v, \bar{V}) V^0 dx \, ds + r^\varepsilon(t) \qquad (61)
$$

and

$$
\int_0^t \int_{\mathbb{T}^n} \operatorname{div}(v \otimes \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 + \mathcal{L}_2 \left(\frac{s}{\varepsilon} \right) V^0 \otimes v) \cdot \left(v + \mathcal{L} \left(\frac{s}{\varepsilon} \right) V^0 \right) dx \, ds
$$

$$
= \int_0^t \int_{\mathbb{T}^n} B_1(v, V^0) V^0 dx \, ds + r^\varepsilon(t) = r^\varepsilon(t). \qquad (62)
$$

By adding up (56)–(55) and by using the properties (35) and (36), the term (55) assumes the form

$$
\begin{aligned}
|B^\varepsilon| \leq\ & r^\varepsilon(t) + M(T)\int_0^t H^\varepsilon(\tau)\mathrm{d}\tau \\
& + \int_0^t \int_{\mathbb{T}^n} (2B_2(V^0, \bar{V})V^0 + B_2(V^0, V^0)\bar{V})\mathrm{d}x\,\mathrm{d}s \\
& + \int_{\mathbb{T}^n} (B_1(v, \bar{V})V^0 + B_1(v, V^0)\bar{V})\mathrm{d}x\,\mathrm{d}s \\
=\ & r^\varepsilon(t) + M(T)\int_0^t H^\varepsilon(s)\mathrm{d}s.
\end{aligned}
\tag{63}
$$

By considering (53), (54) and (63) together, we can conclude that the relative entropy $\mathcal{H}^\varepsilon(t)$ satisfies the following inequality:

$$
\mathcal{H}^\varepsilon(t) \leq C\mathcal{H}^\varepsilon(0) + M(T)\int_0^t \mathcal{H}^\varepsilon(s)\mathrm{d}s + r^\varepsilon(t),
\tag{64}
$$

from which, since $r^\varepsilon(t) \to 0$ as $\varepsilon \to 0$, and, by using Gronwall's inequality, we get there exists a constant $M > 0$ such that

$$
\mathcal{H}^\varepsilon(t) \leq M \qquad \text{for any } t \in [0, T], \text{ uniformly in } \varepsilon.
\tag{65}
$$

4.2 Convergence of the relative entropy

Because of the bound (65), it makes sense to define the following quantity:

$$
\eta(t) = \limsup_{\varepsilon \to 0} \mathcal{H}^\varepsilon(t).
$$

We get from (64) that

$$
\eta(t) \leq \eta(0) + M(T)\int_0^t \eta(s)\mathrm{d}s.
\tag{66}
$$

Since the initial conditions (14)–(16) entail that $\eta(0) \equiv 0$, from (66) we can conclude that

$$
\eta(t) = \limsup_{\varepsilon \to 0} \mathcal{H}^\varepsilon(t) = 0 \qquad \text{for any } t \in [0, T].
\tag{67}
$$

5 Proof of Theorem 1.3

Because of the previous estimates, we have now the uniform "a priori" bounds needed to prove Theorem 1.3. Therefore, we have that (i) is a consequence of (21), while (ii) follows from (67) and the following estimate

$$
\begin{aligned}
\sup_{0 \leq t \leq T} \|P\Lambda^\varepsilon - v\|_{L^2(\mathbb{T}^n)} &= \sup_{0 \leq t \leq T} \left\| P\left(\Lambda^\varepsilon - v - \mathcal{L}_2\left(\frac{t}{\varepsilon}\right)V^0\right)\right\|_{L^2(\mathbb{T}^n)} \\
&\leq \sup_{0 \leq t \leq T} \left\|\Lambda^\varepsilon - v - \mathcal{L}_2\left(\frac{t}{\varepsilon}\right)V^0\right\|_{L^2(\mathbb{T}^n)} \\
&\leq \sup_{0 \leq t \leq T} \mathcal{H}^\varepsilon(t) \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

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References
1. Antonelli, P., Marcati, P.: On the finite energy weak solutions to a system in quantum fluid dynamics. Commun. Math. Phys. **287**, 657–686 (2009)
2. Antonelli, P., Marcati, P.: The quantum hydrodynamics system in two space dimensions. Arch. Ration. Mech. Anal. **203**, 499–527 (2012)

3. Chen, L., Donatelli, D., Marcati, P.: Incompressible type limit analysis of a hydrodynamic model for charge-carrier transport. SIAM J. Math. Anal. **45**(3), 915–933 (2013)
4. Desjardins, B., Grenier, E.: Low Mach number limit of viscous compressible flows in the whole space. Proc. R. Soc. Lond. A Math. Phys. Eng. Sci. **455**(1986), 2271–2279 (1999)
5. Desjardins, B., Grenier, E., Lions, P.-L., Masmoudi, N.: Incompressible limit for solutions of the isentropic Navier–Stokes equations with Dirichlet boundary conditions. J. Math. Pures Appl. **78**(5), 461–471 (1999)
6. Donatelli, D., Feireisl, E., Novotný, A.: On incompressible limits for the Navier–Stokes system on unbounded domains under slip boundary conditions. Discrete Contin. Dyn. Syst. Ser. B. **13**(4), 783–798 (2010)
7. Donatelli, D., Feireisl, E., Novotný, A.: On the vanishing electron-mass limit in plasma hydrodynamics in unbounded media. J. Nonlinear Sci. **22**(6), 985–1012 (2012)
8. Donatelli, D., Feireisl, E., Novotný, A.: Scale analysis of a hydrodynamic model of plasma, M3AS. Math. Models Methods Appl. Sci. **25**, 371–394 (2015)
9. Donatelli, D., Feireisl, E., Marcati, P.: Well/ill posedness for the Euler–Korteweg–Poisson system and related problems. Commun. Partial Diff. Equ. **40**(7), 1314–1335 (2015)
10. Donatelli, D., Marcati, P.: A quasineutral type limit for the Navier–Stokes–Poisson system with large data. Nonlinearity **21**(1), 135–148 (2008)
11. Donatelli, D., Marcati, P.: Analysis of oscillations and defect measures for the quasineutral limit in plasma physics. Arch. Ration. Mech. Anal. **206**(1), 159–188 (2012)
12. Donatelli, D., Marcati, P.: Low Mach number limit on exterior domains. Acta Math. Sci. **32**(1), 164–176 (2012)
13. Donatelli, D., Marcati, P.: Quasineutral limit, dispersion and oscillations for Korteweg type fluids. SIAM J. Math. Anal. **47**(3), 2265–2282 (2015)
14. Donatelli, D., Trivisa, K.: From the dynamics of gaseous stars to the incompressible Euler equations. J. Differ. Equ. **245**, 1356–1385 (2008)
15. Gasser, I., Markowich, P.A., Schmidt, D., Unterreiter, A.: Macroscopic theory of charged quantum fluids. Mathematical problems in semiconductor physics (Rome, 1993), 42–75, Pitman Res. Notes Math. Ser., **340**, Longman, Harlow (1995)
16. Jiang, S., Wang, S.: The convergence of the Navier–Stokes–Poisson system to the incompressible Euler equations. Commun. Partial Differ. Equ. **31**(4–6), 571–591 (2006)
17. Ju, Q., Li, F., Wang, S.: Convergence of the Navier–Stokes–Poisson system to the incompressible Navier–Stokes equations. J. Math. Phys. **49**(7), 073515 (2008). 8
18. Jüngel, A.: Transport Equations for Semiconductors. Lecture Notes in Physics, vol. 773. Springer, Berlin (2009)
19. Jüngel, A., Li, H., Markowich, P., Wang, S.: Recent progress on quantum hydrodynamic models for semiconductors. In: Hou, T.Y., Tadmor, E. (eds.) Hyperbolic Problems: Theory, Numerics, Applications, pp. 217–226. Springer, Berlin (2003)
20. Kato, T.: Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. J. Funct. Anal. **9**, 296–305 (1972)
21. Li, H., Marcati, P.: Existence and asymptotic behavior of multi-dimensional quantum hydrodynamical model for semiconductors. Commun. Math. Phys. **245**, 215–247 (2004)
22. Li, H., Lin, C.-K.: Zero Debye length asymptotic of the quantum hydrodynamic model for semiconductors. Commun. Math. Phys. **256**, 195–212 (2005)
23. Lions, P.-L.: Mathematical Topics in Fluid Dynamics, Incompressible Models. Clarendon Press, Oxford Science Publications, Oxford (1996)
24. Lions, P.-L., Masmoudi, N.: Incompressible limit for a viscous compressible fluid. J. Math. Pures Appl. **77**(6), 585–627 (1998)
25. Masmoudi, N.: From Vlasv Poisson system to the incompressible Euler system. Commun. Partial Diff. Equ. **26**, 1913–1928 (2001)
26. Masmoudi, N.: Incompressible, inviscid limit of the compressible Navier–Stokes system. Ann. Inst. H. Poincaré Anal. Non Linéaire **18**, 199–224 (2001)
27. Nardin, G., Grosso, G., Léger, Y., Piętka, B., Morier-Genoud, F., Deveaud-Plédran, B.: Hydrodynamic nucleation of quantized vortex pairs in a polariton quantum fluid. Nat. Phys. **7**(8), 635–641 (2011)
28. Schochet, S.: Fast singular limits of hyperbolic PDEs. J. Differ. Equ. **114**, 476–512 (1994)
29. Schochet, S.: The mathematical theory of low Mach number flows, M2AN. Math. Model. Numer. Anal. **39**, 441–458 (2005)
30. Wang, S.: Quasineutral limit of Euler–Poisson system with and without viscosity. Commun. Partial Differ. Equ. **29**(3–4), 419–456 (2004)
31. Wang, S., Jiang, S.: The convergence of the Navier–Stokes–Poisson system to the incompressible Euler equations. Commun. Partial Differ. Equ. **31**, 571–591 (2006)