Heterogeneous Interrelationships among Attributes in Multi-Attribute Decision-Making: An Empirical Analysis

Zhen-Song Chen[1,2], Xuan Zhang[1,*], Rosa M. Rodríguez[3], Xian-Jia Wang[4], Kwai-Sang Chin[5]

[1]*School of Civil Engineering, Wuhan University, Wuhan, China*

[2]*School of Data Science, City University of Hong Kong, Hong Kong, China*

[3]*Department of Computer Science, University of Jaén, Jaén, Spain*

[4]*Economics and Management School, Wuhan University, Wuhan, China*

[5]*Department of Systems Engineering and Engineering Management, City University of Hong Kong, Hong Kong, China*

ARTICLE INFO

ABSTRACT

Tremendous effort has been exerted over the past few decades to construct multi-attribute decision functions with the capacity to model heterogeneous interrelationships among attributes. In this paper, we report an empirical study aiming to test whether or not considering interrelationships among attributes can benefit the representation of real preferences in multi-attribute ranking tasks. The generalized extended Bonferroni mean (GEBM) has recently been advocated as a promising and efficient tool for modeling heterogeneous interrelationships among attributes. We compare the GEBM with one of its most widely adopted competitors, simple additive weighting (SAW), in terms of their fitting quality when applied to preference elicitation. The attribute performances are manifested uniformly with the use of three widely-adopted utility measurements. Subsequently afterwards, the maximum split approach to establish the constraint objective function in regression for both the GEBM and the SAW to test whether or not all constraints resulting from the subject's ranking can be fulfilled. On this bases, the number of fully or partly fitted subjects, consistency for subjects according to the better fitting model, and reliability of attribute weights learned by either the GEBM or the SAW are empirically examined in a bid to demonstrate the quantitative construction of fitting quality measurement. With the established fitting quality measurement, the necessity of taking heterogeneous interrelationships among attributes into account when constructing multi-attribute decision functions to represent real preferences can be analyzed. The main conclusion from the empirical study suggests that the relative performance of the two aggregation paradigms examined here depends on which fitting quality measurements are adopted. Researchers enthusiastic to discover the heterogeneous interrelationships among attributes when constructing multi-attribute decision functions might find the present results relevant when modeling actual preferences, and consequently this work should serve as a useful reference for enterprises and service providers seeking to strategically drive customer purchasing decisions.

1. INTRODUCTION

Typical multi-attribute decision-making (MADM) structures often involve the following three components: (1) evaluation of various alternatives against certain attributes, (2) determination of the attributes weights reflecting the preference of a decision maker, and (3) aggregation of the attributes evaluations taking account of the attributes weights [1]. A considerable number of MADM methodologies and theories have been developed and applied in a wide range of fields, such as bid evaluation, enterprise strategy planning, quality assessment, and product recommendation [2–7]. However, in many decision-making scenarios, decision makers do not always follow the above procedures in a stepwise manner, since they usually have some rough idea in the first place of the appropriate outputs for some prototype inputs [8,9]. The preference aggregation outputs are usually in the form of numerical data or preferences

(e.g., expressed as linear orderings). For example, a decision maker can offer a ranking of cars *A*, *B*, and *C* simply by a rough evaluation of these three alternatives under various attributes. The logical aggregation adopted by decision makers hidden behind the generation of alternative rankings implies a need for aggregation functions, and this is of fundamental importance for the successful implementation of MADM tasks.

Identification of aggregation functions for preference modeling and elicitation of attributes weighting vectors have been hot research topics for the past few decades [10–15]. Recent efforts in this area have been devoted to the incorporation of the human cognitive process by fitting different types of aggregation functions to empirical data collected from real-life scenarios. Reimann *et al.* [16] fitted the *ordered weighted averaging* (OWA) and *simple additive weighting* (SAW) models to empirical data. In general, the OWA model gives a slightly poorer fit to empirical data. The use of empirical studies has brought a novel perspective to the preference modeling field, allowing the examination of a huge amount of research

data on the aggregation of attributes evaluations. Motivated by [16], this paper initiates an additional effort to test another question: "*Can considering heterogeneous interrelationships among attributes in multi-attribute decision functions provide a better representation of real preferences*?" In particular, heterogeneous interrelationships among attributes in the context of MADM reveal a situation in which both dependency and independency implicitly coexist in the attribute structure and need to be simultaneously taken into consideration, which are demonstrated in recent research efforts to be a significant factor affecting the decision-making process [17,18]. In real-life MADM problems, there usually exist complex interrelationships among the attributes under consideration, which are reflected in the corresponding input arguments [8,16,19]. For example, consider a car choice problem where four attributes, namely, price (A_1), physical characteristic (A_2), service life (A_3), and brand trust (A_4), are taken into account. In this case, we may identify A_2 and A_3 as being interrelated, which can be reflected in the inputs of each alternative. Decision makers commonly judge and aggregate these attributes implicitly to form their preferences with regard to alternatives and to produce the corresponding ranking. The question here is whether or not the interrelationships among attributes will be taken into account in the context of ranking alternatives under given attributes. There has been no previous research demonstrating empirically heterogeneous interrelationships among attributes in multi-attribute decision functions in preference modeling.

The solution to this problem requires the use of several prominent aggregation functions to model the interrelationships among attributes in the construction of multi-attribute decision functions. The Choquet integral (CI) proposed in [20] and the power average (PA) operator introduced in [21] are useful tools to model the interdependence or correlation among input arguments. However, both focus on capturing interrelationships among input arguments depending on the fuzzy measure (FM) determination and the support function construction. The appropriate choice of FM and support function is still a matter of debate, and, in particular, the PA operator can only be used for modeling homogeneous interrelationships [19]. The generalized extended Bonferroni mean (GEBM), which originates from the Bonferroni mean (BM), has been advocated in the recent literature as a promising and efficient tool for modeling heterogeneous interrelationships among attributes. To simplify our empirical study, the GEBM and its competing SAW model are selected when modeling such interrelationships in multi-attribute decision functions to facilitate an explicit discussion of their different aggregation paradigms.

The BM has been one of the most powerful aggregation functions in the field of MADM since its introduction by Bonferroni in [22] in 1950. Owing to the inherent structure of the BM, it can model homogeneous correlations among attributes by connecting each argument with other arguments [8]. On the basis of this property, Dutta *et al.* [23] extended the BM to accommodate even more complex correlations among attributes where some arguments are related to only a nonempty subset of the remaining arguments, while others have no relation to the remaining arguments; this is termed the extended BM (EBM). As a structural generalization of the EBM, the GEBM proposed by Chen *et al.* [24] allows heterogeneous interrelationships among aggregation inputs to be effectively captured in a more flexible way in situations where both dependent and independent attributes need to be simultaneously

taken into consideration. Compared with other aggregation functions capable of modeling interrelationships among attributes, one of the main advantages of the GEBM is that it can be identified as a composite function consisting of averaging and conjunctive operators. The aggregation mechanism is reflected by the composite function, which is convenient for decomposition and substitution with restricted forms of aggregation functions. The specific requirements for various applications can be satisfied by defining appropriate composite aggregation functions, which makes the GEBM one of the most useful aggregation functions in the context of decision-making. The GEBM has attracted a great deal of attention with regard to its theoretical development since its introduction in [23], no work has been done on its fitting to empirical data that represent the real preferences of decision makers [1,25]. In addition, it has yet to be applied to test whether considering interrelationships among attributes can provide a better representation of real preferences.

To fill this research gap, this paper conducts a large-scale empirical study to fit the GEBM and the SAW respectively to empirical data and to test whether considering interrelationship among attributes can indeed provide a better representation of real preferences. There are three main components of this procedure: aggregation functions, attributes evaluations and rankings provided by survey respondents, and constraint functions in regression. In our experiment, for comparison with the GEBM, we use SAW as a benchmark preference-approximation model in which interrelationships among attributes are not considered. More specifically, we provide the performance parameters of products as model inputs and ask the subjects for the rankings of alternative products as outputs. The rankings of alternatives include richer information compared with that obtained from the choice and sorting tasks, and are easier to obtain in real-word scenarios in comparison with accurate aggregated scores. To derive feasible weighting parameters for each model, a maximum split approach is used as the constraint function in regression.

The remainder of the paper is structured as follows. A review of the literature on identification of aggregation functions and elicitation of weighting vector is presented in Section 2. Section 3 provides several definitions of the BM, GBM, and GEBM, and the other two main components used in the development of our proposed approach. We describe our experimental design and data collection in Section 4. In Section 5, a comparison of fitting quality for the GEBM and SAW models is given. Section 6 concludes the paper.

2. LITERATURE REVIEW

The developments of this empirical study consist mainly of (a) aggregation functions for attribute interrelationships elicitation, (b) fitting aggregation functions to empirical data, and (c) detailed analysis of empirical results. As such, the literature review is subdivided as follows.

- **Aggregation functions for attribute interrelationship elicitation.**

In term of the interactions among attributes in MADM, the attribute interrelationships can be categorized into two groups: the *homogeneity* and *heterogeneity* [8,26,27]. Homogeneous interrelationship means that given a set of attributes *X*, each attribute in

X are related to all or part of the remaining attitudes with the structural resemblance. Heterogeneous interrelationship among attributes, however, highlights the situation in which a proportion of all attributes are related in a flexible way to different kinds of subsets of all attributes in X and the others are independent from the rest.

Except for the BM and its generalizations, there exist a considerable amount of aggregation functions capable of modeling different type of attribute interrelationships. Among the existing aggregation functions, CI is commonly considered as one of the most powerful tools for aggregating interacting attributes as an adequate substitute to weighted arithmetic mean or OWA operator [28,29]. CI introduces the renowned concept of FM to allow defining weights of attributes not only on the single-attribute level, but also on each subset of attributes in a nonadditive fashion. FM on each combination of attributes is measured by importance index and interaction index of attributes, thus making it possible to model the interactions among attributes [30]. To take into account the interrelationship between the input arguments in an information fusion, Yager [21] introduced the concept of the PA operator, as mentioned before. In its inherent structure, the PA operator when applied to MADM scenario assumes that the weight of an attribute depends on the interrelationship between this attribute and all the remaining attributes, which is accomplished by calculating the support value from the rest to each attribute, following the simple philosophy that the more similar two attribute inputs are, the more they support each other [26]. Heronian mean (HM) is characterized by the ability to capture the relevance between arguments, which has also been regarded as a useful tool to model attribute interrelationships. Compared with the BM, HM assumes that the self-interaction should be incorporated in the aggregation of attributes, and it distinguishes the interrelationship between attributes x_i and x_j from the interrelationship between x_j and x_i [31]. The Maclaurin symmetric mean (MSM) operator proposed by Maclaurin [32] is an effective approach to capturing the interrelationships among the multi-input arguments. The MSM operator is monotonically decreasing with respect to the values of an additionally introduced parameter, which may be interpreted to be a reflection of the risk preferences of the decision makers in practice [33]. A special degeneration of MSM is the Muirhead mean (MM) operator where all permutations of attributes will be considered, and the additionally introduced parameter is replaced by the weighting vector, which can capture the overall interrelationship of multi-input arguments [34].

- **Fitting aggregation functions to empirical data**.

There have been a number of papers related to fitting aggregation functions with empirical data. Beliakov [11] studied methods for fitting weighted arithmetic means, OWA functions, and CIs, and showed that the linear programming formulation is valid for such cases. Weighted quasi-arithmetic means and OWA functions were identified and described, together with their corresponding weighting vectors, in [13,14,35,36]. Kumar and Tripathi [37] fitted two new aggregation functions called compensatory summation units (CSUs) and compensatory product units (CPUs). Reimann et al. [16] fitted OWA and SAW models to empirical data to test how well the OWA operator represents real preferences. A comprehensive overview of methods for capacity identification in Choquet-integral-based multi-attribute utility theory was conducted in [38], using the Kappalab R package. Beliakov [39]

proposed an approach for fitting triangular norms and conorms to empirical data in which membership functions and aggregation operators were constructed from the data, and they applied it to fuzzy systems. Other aggregation functions fitted in fuzzy control were studied in [40–42]. The fitting process is often related to information from inputs and outputs. Various types of inputs and outputs have been introduced to fit aggregation functions. Typically, inputs and outputs are real numbers, often from the range [0, 1], although other choices are possible, such as discrete sets, intervals, and linguistic labels. In [1,16,43], real numbers were used as inputs and ranking preferences as outputs, and the fitting process was conducted after the inputs had been normalized. Beliakov [11] employed real numbers and the desired overall aggregation scores of each alternative as inputs and outputs, respectively. Beliakov [11] and Shepetukha and Olson [44] introduced a fitting method with interval-valued inputs. The constraint function in regression is also an important part of the fitting procedure. Here, we concentrate on linear programming problems. Grabisch et al. [38] provided a comprehensive overview of constraint functions used for capacity identification in the CI. Beliakov [11] used the "least absolute deviation" attributes proposed by Bloomfield and Steiger [45] as a constraint objective function that minimizes the deviation between aggregated score and desired score. A maximum entropy and minimum variance principle for capacities was introduced by [46–48]. Reimann et al. [16] maximized the deviation of aggregation scores between two different alternatives in different orders, while Ahn [10] used the sum of deviations as an objective function. Geng et al. [49] and Khot et al. [50] argued that approximation algorithms have achieved success in empirical applications. Jiang et al. [48] proposed a weight learning method for medical knowledge networks based on a max-margin framework.

By fitting aggregation functions to empirical data, we can often obtain the weighting vector of attributes, which is a core product of the fitting process. In fact, instead of using a fitting process, many studies have obtained the attribute weights by comparisons of attribute importance [51]. In the approach of Shepetukha and Olson [44], attribute weights were learned using the ranking of attribute importance and the relative weights between adjacent attributes. Solymosi and Dombi [52] compared attribute importance by considering threshold values and introduced the centroid of the feasible set of formulated linear inequalities as a weighting vector for attributes. Mikhailov [53] decomposed the fuzzy comparison judgments of attributes into a series of interval judgments by using α-cuts and obtained a sequence of crisp attribute priorities by applying the fuzzy preference programming method. Instead of the weighting vector, fitting processes can be used for multi-attribute utility measurement. Salminen et al. [54] presented a test, based on pairwise preference information, to identify the class of functions (linear, quasi-concave, or neither) to which a decision-maker's (implicit) value function belongs. In the approach of Aggarwal [1], the utility function was learned from a set of preference pairs using a state-of-the-art support vector machine (RankSVM) based on the learning-to-rank method. A summary of research on fitting aggregation functions to empirical data is given in Table 1.

- **Analyses of empirical data**.

Different proposals have adopted distinct approaches to obtain useful insights from analyses of empirical data. Reimann et al. [16]

Table 1 | Research on fitting aggregation functions to empirical data.

Aggregation Functions	Inputs	Outputs	Objective Functions	Ref.
OWA, SAW	Real numbers	Ranking	Maximum split	[16]
Choquet integrals, triangular norms and conorms	Interval values	Interval values	Least absolute deviation	[11]
AM/GM/QM/OWA/GBM /Choquet integrals	Real numbers	Desired overall score/ranking	Least absolute deviation/maximum split	[12]
OWA	Real numbers	Ranking	Maximum likelihood	[1]
Choquet integrals	Real numbers	Desired overall score/ranking	Least squares/maximum split/minimum variance/minimum distance	[38]

OWA, ordered weighted averaging; SAW, simple additive weighting; GBM, generalized Bonferroni mean.

employed measurements of the consistency between ranking and valuation tasks and between two ranking tasks, the Cognitive Style Index, the fitting ability of rankings of subsets, and the distributions of weights. To test fitting ability, Beliakov et al. [12] applied the percentage of comparisons accurately predicted using various aggregation functions, as well as the results of fitting aggregation functions to raw abundance data. Beliakov and James [55] introduced 10-fold cross-validation accuracy to test the adaptability of aggregation functions. Grabisch et al. [38] analyzed the distributions of weights of different constraint objective functions in regression for empirical results.

3. PRELIMINARIES

In this section, we set out to introduce the necessary background on the SAW and GEBM. After giving some important definitions and properties, we present our constraint objective function and linear constraints in regression for SAW and GEBM. More detailed overviews of aggregation functions can be found in [56,57], and, without further specifications, this paper follows roughly the notation used in [8,27].

Definition 1. [56] A function $f : [0,1]^n \rightarrow [0,1]$ is called an aggregation function if it is monotone nondecreasing in each variable and satisfies $f(0,0, \dots, 0) = 0, f(1,1, \dots, 1) = 1$.

Aggregation functions are classified depending on their behavior with respect to inputs [58]. An aggregation function A is conjunctive if the output is bounded by $0 \leq A(x) \leq \text{Min}(x)$, disjunctive if the output is bounded by $1 \geq A(x) \geq \text{Max}(x)$, or averaging the output is bounded by $\text{Min}(x) \leq A(x) \leq \text{Max}(x)$. Otherwise, it is included in the mixed class of aggregation functions.

Definition 2. [56] For a given strictly monotone and continuous function $g : [0,1] \rightarrow [-\infty, +\infty]$, which is called a generating function or generator, and a weighting vector $\mathbf{w}$, the weighted quasi-arithmetic mean is the function

$$M_{\mathbf{w},g}(\mathbf{x}) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right).$$

In particular, if the generating function is $g(x) = x$, it degenerates into the weighted average mean operator $\text{WAM}(x) = w_i x_i$.

The SAW method, also known as the weighted sum method, is the most widely used MADM method [59]. The basic principle of SAW is to obtain a weighted sum of the performance ratings of each alternative under all attributes [60]. The SAW method consists of two

basic steps [59,60]: (1) scale the values of all attributes to make them comparable; (2) aggregate the values of all the attributes for each alternative with weighted average mean operator.

As an effective aggregation function, the BM has been successfully applied in such fields as image processing [61,62], environmental modeling [63], water quality analysis [24], and social network analysis [64]. The BM has such a wide range of applications because of its decomposable structure and distinguishable components. Its original functional form is composed of two averaging operators and a conjunctive operator that can be interpreted as involving a product of each argument with the average of the other arguments [65].

Definition 3. [22] Let $p, q \geq 0$ and $x_i \geq 0, i = 1, \dots, n$. The BM is the function

$$B^{p,q}(\mathbf{x}) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} x_i^p x_j^q\right)^{1/(p+q)}.$$

The BM is an averaging aggregation operator. It can be interpreted as computing the average satisfaction of each input x_i AND (the power mean of the remaining inputs). Given the special case in which $n = 2$ and $p = q$, the BM is equivalent to the geometric mean. If $q = 0$, the BM becomes the power mean.

As an extension of this basic concept, Yager [65] replaced the simple average operator by other types of mean aggregation function such as the OWA function and the discrete CI and introduced arguments with varying importance weights. Beliakov et al. [66] then studied various weighted quasi-arithmetic means and t-norm functions to replace the averaging operators and conjunctive function, respectively, and investigated the properties of this generalized BM (GBM).

Definition 4. [66] Let M denote a three-tuple aggregation function $\langle M_1, M_2, C \rangle$ with $M_1 : [0,1]^n \rightarrow [0,1]$, $M_2 : [0,1]^{n-1} \rightarrow [0,1]$, and $C : [0,1]^2 \rightarrow [0,1]$, with the diagonal of C denoted by $\delta_C(t) = C(t,t)$ and the inverse diagonal δ_C^{-1}. The GBM is given by

$$B_M^G(\mathbf{x}) = \delta_C^{-1}\left(M_1\left(C\left(x_1, M_2\left(\mathbf{x}_{j \neq 1}\right)\right), \dots, C\left(x_n, M_2\left(\mathbf{x}_{j \neq n}\right)\right)\right)\right).$$

Both M_1 and M_2 are averaging operators, while C is a conjunctive operator. The function can be interpreted as the average satisfaction of each input x_i AND (the average of the remaining inputs). In the special case in which M_1 is the arithmetic mean, M_2 is the power

mean with power q, and $C(x, y) = x^p y^q$, the GBM operator degenerates into the conventional BM.

Beliakov et al. [66] found that the GBM has sufficient modeling capabilities for the concepts of partial conjunction, partial disjunction, k-tolerance, and k-intolerance. The GBM has also been found to be capable of modeling any number of mandatory requirements by iteration of the inner mean with Bonferroni-type functions, and this widens its range of application to real-world problems [27].

The GBM refers to a connection of each argument x_i with all other arguments $\mathbf{x}_{j \neq i}$, from which we can see that the aggregation function assumes that each attribute c_i has a connection with all the other attributes $C \setminus \{c_i\}$, where $C = \{c_1, c_2, ..., c_n\}$ denotes a set of attributes. However, in the real world, the attributes may not follow homogeneous interrelationship patterns, and some attributes may depend only on a nonempty subset B_i of the set $C \setminus \{c_i\}$ that constitutes an attribute subset Π_D, while others are mutually independent and constitute a subset Π_{ID}. Obviously, Π_D and Π_{ID} are two disjoint subsets of the attribute set C. To model this scenario, the EBM operator is introduced.

Definition 5. [23] For any $p > 0$ and $q \geq 0$, the EBM operator of dimension n is a mapping denoted by $B_E^{p,q}(\mathbf{x}) : [0,1]^n \rightarrow [0,1]$ such that

$$B_E^{p,q}(\mathbf{x}) = \frac{[[\neg I']]}{n} \left(\frac{1}{[[\neg I']]} \sum_{i \in \neg I'} (x_i)^p \left(\frac{1}{[[I_i]]} \sum_{j \in I_i} (x_j)^q \right) \right)^{\frac{1}{p+q}}$$

$$+ \frac{[[I']]}{n} \left(\frac{1}{[[I']]} \sum_{i \in I'} (x_i)^p \right)^{\frac{1}{p}},$$

where I_i are the indices of the attributes set B_i that is related to c_i, and I' and $\neg I'$ are the indices of the attributes in Π_{ID} and Π_D, respectively. The cardinality of any set K is denoted by $[[K]]$.

As indicated by this formula, the EBM operator is divided into two components: the dependent part (the former) and the independent part (the latter). If $[[\neg I']] = n$, and each input argument is dependent on all other input arguments, the EBM operator reduces to the BM. Similar to the construction of the general form of the BM, the EBM was generalized by Chen et al. [8] as follows.

Definition 6. [8] Let $\mathcal{M}$ denote a quintuple aggregation function $\langle M_1, M_2, M_3, M_4, C \rangle$, with $M_1 : [0,1]^{[[\neg I']]} \rightarrow [0,1]$, $M_2 : [0,1]^{[[\neg I']]-1} \rightarrow [0,1]$, $M_3 : [0,1]^{[[I']]} \rightarrow [0,1]$, $M_4 : [0,1]^2 \rightarrow [0,1]$, and $C : [0,1]^2 \rightarrow [0,1]$, with the diagonal of C denoted by $\delta_C(t) = C(t, t)$ and the inverse diagonal by δ_C^{-1}. The GEBM is given by

$$B_{\mathcal{M}}^{GE}(\mathbf{x}) = M_4 \left(\delta_C^{-1} \left(M_1 \left(C^{[[\neg I']]} \left(x_{i \in \neg I'}, M_2 \left(\mathbf{x}|_{I_i} \right) \right) \right) \right) \right),$$

$$M_3 \left(\mathbf{x}|_{I'} \right) \right),$$

where $\mathbf{x}|_K$ denotes the restriction of $\mathbf{x}$ when regarded as a function from $[n]$ to $[0,1]$ to the subset K. In other words, $\mathbf{x}|_K$ is the $[[K]]$-tuple obtained by considering only the coordinates of $\mathbf{x}$ indexed by the elements of K in increasing order.

In addition to aggregation operators, utility measurement is also an important part of the empirical-data-fitting process. In this paper,

we use linear, concave and convex utility functions for utility measurement, as suggested in Reimann et al. [16]:

linear utility function:

$$u\left(a_{i,k}\right) = \frac{a_{i,k} - \underline{a_k}}{\overline{a_k} - \underline{a_k}},$$

concave utility function:

$$u\left(a_{i,k}\right) = \left(\frac{a_{i,k} - \underline{a_k}}{\overline{a_k} - \underline{a_k}} \right)^{\frac{1}{2}},$$

convex utility function:

$$u\left(a_{i,k}\right) = \left(\frac{a_{i,k} - \underline{a_k}}{\overline{a_k} - \underline{a_k}} \right)^2,$$

where $\overline{a_k}$ and $\underline{a_k}$ denote the best and worst value of any alternative in attribute k.

We apply the maximum split approach to establish the constraint objective function in regression, which can be interpreted as

$$\text{Max } z$$
$$\text{s.t. } f_w(\mathbf{u}_i) - f_w(\mathbf{u}_j) \geq z, \forall i, j : A_i \succ A_j,$$
$$\sum_{k=1}^{n} w_k = 1,$$

where $f_w(\mathbf{u}_i)$ is the aggregated value of the utility values $\mathbf{u}_i = (u_{i,1}, u_{i,2}, ..., u_{i,k})$ with attribute weighting vector $\mathbf{w} = (w_1, w_2, ..., w_k)$ by the aggregation function f.

From the general form of the constructed constraint objective function, we can see that only a positive value of the deviation variable z will indicate that all constraints resulting from the subject's ranking can be fulfilled. A negative value of z indicates that some comparisons in a ranking cannot be made and that the model is not able to fit the ranking. For the SAW model, the objective function can be specified as

$$\text{Max } z$$
$$\text{s.t. } \sum_{k=1}^{n} w_k u\left(a_{i,k}\right) - \sum_{k=1}^{n} w_k u\left(a_{j,k}\right) \geq z, \forall i, j : A_i \succ A_j,$$
$$\sum_{k=1}^{n} w_k = 1.$$

For the GEBM, we use a weighting vector $\mathbf{v} \in [0,1]^{[[\neg I']]}$ and for each $\mathbf{x}|_{I_i}$ we associate an $[[I_i]]$-dimensional vector $\mathbf{v}^i$ with weights $v_j / \sum_{j \in I_i} v_j$ for all j. Two main approaches to fitting a weighting vector to an objective function are introduced in [55]: fitting v_{ij} weights to product pairs and fitting w_i weights with fixed u. The latter approach is considered in this paper. The weights of each $\mathbf{x}|_{I_i}$ are assigned with $1 / [[I_i]]$ equally. By using the SAW model for M_1,

M_2, and M_3, the product of the conjunctive operator, we can obtain the objective function in regression as

Max z

s.t.
$$
\frac{[[\neg I']]}{n}\left(\left(\sum_{k\in\neg I'} w_k u\left(a_{i,k}\right)\left(\frac{1}{[[I_k]]}\sum_{t\in I_k} u\left(a_{i,t}\right)\right)\right)^{\frac{1}{2}}\right.
$$
$$
\left. -\left(\sum_{k\in\neg I'} w_k u\left(a_{j,k}\right)\left(\frac{1}{[[I_k]]}\sum_{t\in I_k} u\left(a_{j,t}\right)\right)\right)^{\frac{1}{2}}\right)
$$
$$
+\frac{[[I']]}{n}\left(\sum_{k\in I'} w_k u\left(a_{i,k}\right) - w_k u\left(a_{j,k}\right)\right) \geq z, \forall i,j : A_i \succ A_j,
$$
$$
\sum_{k=1}^{n} w_k = 1.
$$

4. EXPERIMENTAL DESIGN AND DATA COLLECTION

To compare the fitting quality of GEBM and SAW towards empirical data, we took smart phones as the experimental materials because they are widely used and their configuration parameters are easily understood. Students were taken as subjects because the majority of students possess at least one smart phone. In this experiment, we provided various configuration parameters of several smart phones and asked the subjects to rank these phones according to their preferences. To avoid any influence of brand effect on our empirical results, we did not include any brand information on the experimental materials in the questionnaire, and no photographs or models were shown to the subjects. We employed the following attributes and configuration parameters.

- Battery capacity (A_1): with increasing use of smart phones, battery capacity has become of great importance. A small battery capacity often leads to low power in a smart phone or even to the phone switching itself off, causing disruption to the user's everyday activities. Therefore, mobile phone manufacturers expend much effort in improving battery capacity.

- Thickness (A_2): thickness is another factor influencing the selection of smart phones. A thin phone can be handled more easily and comfortably than a thick one. In addition, thinner phones are lighter and have a more attractive appearance.

- Screen size (A_3): more and more attention is now being paid to screen size when selecting a smart phone. A large screen gives users a better experience when watching movies, communicating, and reading.

- Cruising power (A_4): cruising power indicates how long a smart phone can be used continuously for a single function such as watching a movies, playing a game, or making a call. There are many tests to measure cruising time for various smart phones, and we used these to obtain necessary data for our experiment. The data were measured in an environment where smart phones were used for playing games continuously.

- Camera (A_5): taking photographs is a vital function for smart phones. A higher number of pixels gives clearer photographs.

- ROM (A_6): a larger ROM capacity gives smoother and longer use.

A heterogeneous interrelationship can be established on the basis of these six attributes. In general, cruising power is related to battery capacity, screen size, and phone thickness, and a thinner phone often implies a smaller battery capacity. The interrelationships among the attributes are shown in Figs. 1–5. The configuration parameters of the smart phones in our experiment are given in Table 2. Utility measurements are given in Tables 3–5.

We conducted our experiment using an online questionnaire that asked three questions of each subject: their gender and age and their preference rankings for the seven smart phones. To improve the analysis, we conducted two surveys under the same conditions. The first survey involved 208 subjects, who took an average time of 87.28 s each to complete the questionnaire. After excluding subjects who completed the questionnaire in less than 20 s, we were left with 149 answers from 70 male (46.98%) and 79 female (53.02%) subjects, with an average age of 21.8 years and an average completion time of 105.05 s. In the second survey, 223 subjects were involved and provided 175 available answers. In this survey, 76 participants (43.43%) were male and 99 (56.57%) female, with an average age of 21.5 years and an average completion time of 94.55 s.

5. COMPARISON OF FITTING QUALITY OF AGGREGATION OPERATORS

According to the empirical surveys designed above, the fitting quality of GEBM and SAW can be compared. In this section, the number of fully or partly fitted subjects, consistency for subjects according to the better fitting model, and reliability of attribute weights learned by either the GEBM or the SAW are empirically examined in a bid to demonstrate the quantitative construction of fitting quality measurement.

5.1. Descriptive Statistics

The responses provided by our subjects were quite heterogeneous. As can be seen from Tables 6 and 7, except for smart phones A and G, the distributions of ranks for the other phones are even, with standard deviations below 10.4 in both surveys, which means that the viewpoint of subjects on ranking is highly decentralized. We can also observe that some alternatives (B and D in both surveys) have almost the same chance of being ranked first and last. Thus, the preferences of the subjects cover a wide range of possible rankings and provide a rich sample to which the two models can be fitted.

To test whether considering interrelationships among attributes can provide a better representation of real preferences, we need to compare the fitting quality of the GEBM operator with that of the SAW model. A comparative analysis between the GEBM and SAW models will be conducted with regard to the following aspects.

5.2. Number of Fully Fitted Subjects

As mentioned above, only a positive z can indicate that a model can be fully fitted to the rankings. Thus, the fitting quality of the two
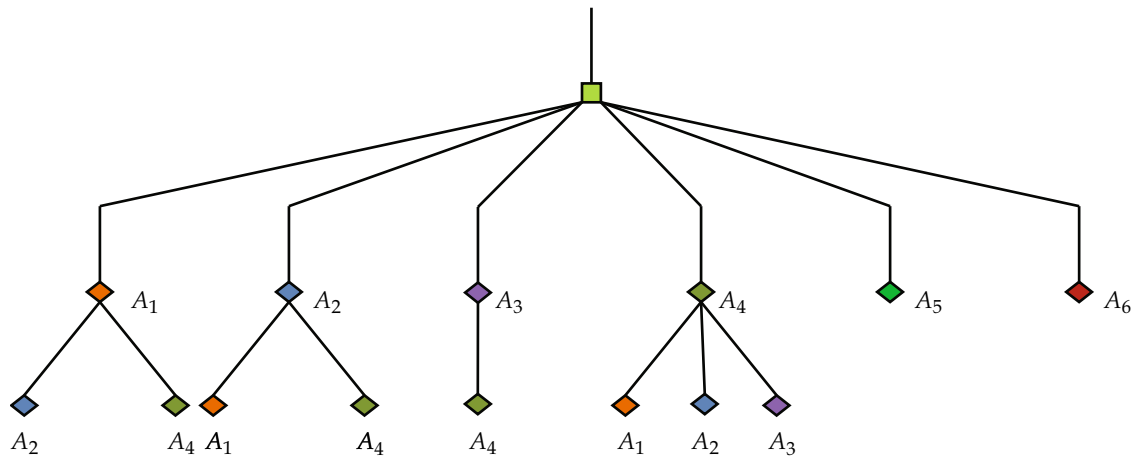
Figure 1 | Interrelationships among selected attributes of smart phones.

Figure 2 | Available questionnaires of survey 1.

Figure 3 | Available male subjects of survey 1.

Table 2 | Configuration parameters of smart phones.

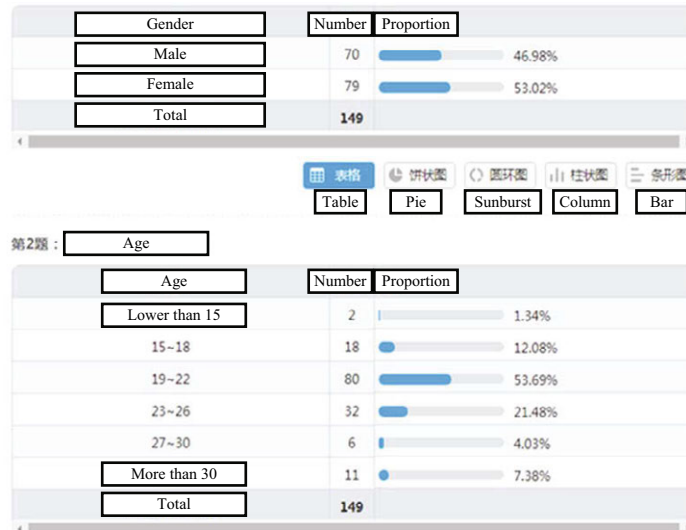| Smart Phone	Battery Capacity (mAh)	Thickness (mm)	Screen Size (inches)	Cruising Power (hours)	Camera (megapixels)	ROM (GB)
A	4000	7.80	6.10	7.56	20	64
B	3300	8.60	6.30	6.15	12	128
C	3500	8.50	6.20	7.18	12	64
D	3060	11.1	5.70	7.00	19	128
E	3300	7.25	6.00	7.00	20	128
F	3520	7.62	6.00	6.00	12	256
G	2716	7.70	5.80	6.80	12	256
$\overline{a_k}$	4000	7.25	6.30	7.56	20	256
$\underline{a_k}$	2716	11.1	5.80	6.00	12	64

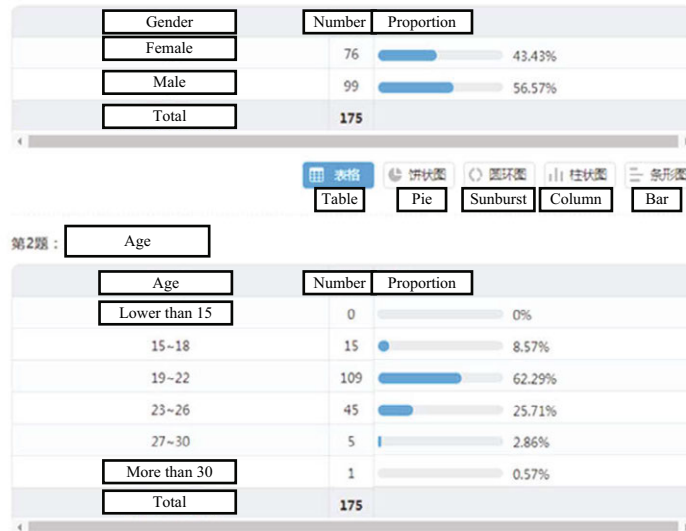**Figure 4** | Genders and ages of available subjects of survey 1.

Figure 5 | Genders and ages of available subjects of survey 2.

Table 3 | Linear utility measurement.

Smart Phone	Battery Capacity	Thickness	Screen Size	Cruising Power	Camera	ROM
A	1.000	0.857	0.667	1.000	1.000	0
B	0.455	0.649	1.000	0.096	0	0.333
C	0.611	0.675	0.833	0.756	0	0
D	0.268	0	0	0.641	0.875	0.333
E	0.455	1.000	0.500	0.641	1.000	0.333
F	0.626	0.904	0.500	0	0	1
G	0	0.883	0.167	0.513	0	1

Table 4 | Concave utility measurement.

Smart Phone	Battery Capacity	Thickness	Screen Size	Cruising Power	Camera	ROM
A	1.000	0.926	0.817	1.000	1.000	0
B	0.675	0.806	1.000	0.310	0	0.577
C	0.782	0.822	0.913	0.869	0	0
D	0.518	0	0	0.801	0.935	0.577
E	0.675	1.000	0.707	0.801	1.000	0.577
F	0.791	0.951	0.707	0	0	1
G	0	0.913	0.409	0.716	0	1

Table 5 | Convex utility measurement.

Smart Phone	Battery Capacity	Thickness	Screen Size	Cruising Power	Camera	ROM
A	1.000	0.766	0.449	1.000	1.000	0
B	0.207	0.421	1.000	0.009	0	0.111
C	0.373	0.456	0.694	0.572	0	0
D	0.072	0	0	0.411	0.766	0.111
E	0.207	1.000	0.250	0.411	1.000	0.111
F	0.392	0.817	0.250	0	0	1
G	0	0.780	0.028	0.263	0	1

Table 6 | Distribution of ranks for all alternatives in survey 1.

Rank	1	2	3	4	5	6	7
A	62	18	12	9	8	21	19
B	12	24	22	21	34	24	12
C	12	16	34	15	24	21	27
D	16	19	22	38	24	14	16
E	17	26	30	34	27	12	3
F	20	29	14	15	19	41	11
G	10	17	15	17	13	16	61

Table 7 | Distribution of ranks for all alternatives in survey 2.

Rank	1	2	3	4	5	6	7
A	85	18	14	10	12	18	18
B	14	28	26	24	40	28	15
C	9	27	39	27	18	29	26
D	13	23	24	46	32	18	19
E	16	26	32	34	39	21	7
F	21	33	27	18	21	40	15
G	17	20	13	16	13	21	75

Table 8 | Proportion of positive z.

	Linear	Concave	Convex
Survey 1			
SAW	0.1342	0.1074	0.0805
GEBM	0.0201	0.0067	0.0268
Survey 2			
SAW	0.0971	0.0743	0.0801
GEBM	0.0457	0.0229	0.0571

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

Table 9 | Mean of z in different models.

	Linear	Concave	Convex
Survey 1			
SAW	−0.0791	−0.0706	−0.0905
GEBM	−0.0834	−0.0836	−0.1046
Survey 2			
SAW	−0.0852	−0.0699	−0.1042
GEBM	−0.0834	−0.0786	−0.1063

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

aggregation operators can be revealed by the proportion of positive z. The calculation results are given in Table 8, and the means of z from the two models are given in Table 9.

From these tables, we observe that the GEBM operator has a lower proportion of positive z values in fitting to empirical data than the SAW model. The GEBM operator fails to fit fully to more subjects. The SAW model is better at fitting numbers. In addition, except for the linear utility measurement in survey 2, the GEBM operator has a smaller mean of z values than the SAW model. However, the fitting results of both aggregation operators are poor owing to their low proportion of positive z values, which can also be seen from Table 9, where the mean z values for both models are negative. The SAW model with linear utility function in survey 2 can be fitted to the largest number of rankings, but this is still less than one-fifth of the total.

5.3. Number of Partly Fitted Subjects

As revealed in Section 5.2, the fits of both aggregation functions to the complete rankings are poor. Therefore, we consider both models with all possible subsets of these alternatives ($n = 2, \dots, 6$) and analyze the quality of the fit to the subsets of rankings for both models.

From Tables 10–12, we can see that the SAW model fits better for all possible subsets of alternatives. In particular, with increasing subset size, the quality of fit of the GEBM operator falls rapidly, as a result

Table 10 | Proportion of positive z for alternative subsets with linear utility measurement.

Subset Size	2	3	4	5	6
Survey 1					
SAW	0.9741	0.9223	0.7785	0.5407	0.2982
GEBM	0.9111	0.7166	0.4754	0.2557	0.1074
Survey 2					
SAW	0.9766	0.9298	0.7856	0.5265	0.2678
GEBM	0.9118	0.7159	0.4740	0.1257	0.0457

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

Table 11 | Proportion of positive z for alternative subsets with concave utility measurement.

Subset Size	2	3	4	5	6
Survey 1					
SAW	0.9741	0.9108	0.7421	0.4775	0.2378
GEBM	0.9112	0.7093	0.4740	0.2432	0.0911
Survey 2					
SAW	0.9767	0.9207	0.7660	0.5088	0.2359
GEBM	0.9118	0.7198	0.4908	0.2574	0.1020

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

Table 12 | Proportion of positive z for alternative subsets with convex utility measurement.

Subset Size	2	3	4	5	6
Survey 1					
SAW	0.9741	0.9089	0.7283	0.4548	0.2205
GEBM	0.9112	0.7031	0.4491	0.2375	0.0959
Survey 2					
SAW	0.9766	0.9161	0.7303	0.4405	0.2082
GEBM	0.9118	0.7022	0.4465	0.2446	0.1102

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

of which it gives a far poorer fit for larger subset sizes. For example, for $n = 5$ and 6 with linear utility measurement in survey 2, the quality of fit of the SAW model is about four times higher than that of the GEBM operator. These results indicate that the GEBM operator fails to fit to a larger number of subjects either fully or partially.

5.4. Consistency for Subjects According to the Better Fitting Model

The subjects can be separated into those whose preferences are better explained by the SAW model and those whose preferences are better represented by the GEBM operator by comparing the z values for the two aggregation operators. The number of subjects for the better fitting model is given in Table 13.

According to rational choice theory, the fitting quality of a preference model is a function of the consistency of the preferences manifested by subjects toward an object [16]. Therefore, to evaluate the quality of fit of aggregation operators, consistency for subjects according to the better fitting model may be another important aspect. An aggregation operator is expected to give a better fit to the preferences of rational and consistent subjects than to those of irrational and inconsistent subjects. We calculate the distance between rankings a and b, and the overall distance as

$$C_{ab} = \frac{1}{n} \sum_{i=1}^{n} \left| r_{a,i} - r_{b,i} \right|,$$

$$C_o = \sum_{a=1}^{N} \sum_{b=1, b \neq a}^{N} CD_{ab},$$

respectively, where $r_{a,i}$ and $r_{b,i}$ are the positions of alternative i in rankings a and b, respectively. Table 14 presents the calculation results.

In both models, the GEBM operator has a higher consistency for subjects than the SAW model. We can therefore conclude that although GEBM can only fit a smaller number of rankings than SAW, the rationality and consistency of these rankings are better.

Table 13 | Number of subjects for better fitting model.

	Linear	Concave	Convex
Survey 1			
SAW	59	115	81
GEBM	90	34	68
Survey 2			
SAW	62	117	81
GEBM	113	58	94

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

Table 14 | Number of subjects for better fitting model.

	Linear	Concave	Convex
Survey 1			
SAW	2.111	2.176	2.228
GEBM	1.901	1.739	1.785
Survey 2			
SAW	2.179	2.151	2.167
GEBM	1.885	1.727	1.822

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

5.5. Comparison of Attribute Weights Based on Individual Views

Attribute weights are the product of the fitting process. The fitting ability of aggregation operators can also be measured by the reliability of attribute weights. In this subsection, we compare attribute weights learned by the SAW and the GEBM based on individual views, i.e., we compare the weighting vectors of subjects. The group weighting vector is used to represent individual weighting vectors in each model and to compare the fitting quality of aggregation operators. We calculate the group weighting vector with the minimum deviation principle:

$$\text{Min} \ \sum_{s=1}^{S} \sum_{k=1}^{K} \left(w_k - w_{\bar{k}} \right)^2$$

$$\text{s.t.} \ \sum_{k=1}^{6} w_k = 1,$$

where S and K are the numbers of subjects and attributes respectively, and $\mathbf{w}_{\bar{k}}$ are the individual weighting vectors. Tables 15 and 16 give the group weighting vectors of both models.

We can see that the group weighting vectors in the same model of both surveys are very similar, which indicates that the empirical data shown in the surveys are reliable. However, there is an obvious difference between the SAW group weighting vectors and the GEBM group weighting vectors. For example, the weight of ROM obtained by the GEBM operator is nearly twice as high as that from the SAW model, which indicates a difference in their fitting quality. Which operator is more reasonable? To compare the two operators effectively, we conducted a questionnaire online involving 50 subjects and learned the importance of these attributes for their preferences. After analysis of the empirical results, we found that the average attribute weighting vector in these subjects was $(0.154, 0.146, 0.151, 0.175, 0.180, 0.194)$. Obviously, the weighting vectors of the GEBM operator is farther from the real weighting vector, because the ROM weight for the GEBM operator is so high. However, the weighting vector of the GEBM operator is more relevant to the real weighting vector than the SAW model.

5.6. Consistency between TOPSIS Ranking and Real Ranking

In this subsection, we compare the consistency between TOPSIS ranking and real ranking to measure the fitting quality of GEBM and SAW. TOPSIS ("technique for order of preference by similarity to ideal solution") ranks the alternatives by calculating the distance of each alternative from the ideal solution and the negative ideal solution (NIS) [59]. Using individual weighting vectors, we can obtain a new ranking with the TOPSIS decision-making approach. In this way, the reliability of attribute weights on individual views can also be represented how the TOPSIS ranking represents the real smart phone ranking, in other words, the consistency between the TOPSIS ranking and the real ranking. The process can be conducted according to the following steps:

Step 1: Normalize the indicator matrix

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{n} x_{ij}^2}}.$$

Table 15 | Group weighting vectors of both models in survey 1.

Survey 1	Linear	Concave	Convex
SAW	(0.038, 0.007, 0.228, 0.250, 0.067, 0.410)	(0.197, 0.013, 0.160, 0.274, 0.002, 0.354	(0.012, 0.009, 0.269, 0.148, 0.122, 0.440
GEBM	(0.031, 0.003, 0.092, 0.012, 0.122, 0.740)	(0.030, 0.005, 0.040, 0.069, 0.035, 0.721	(0.037, 0.006, 0.079, 0.003, 0.219, 0.656

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

Table 16 | Group weighting vectors of both models in survey 2.

Survey 2	Linear	Concave	Convex
SAW	(0.039, 0.009, 0.222, 0.252, 0.063, 0.415	(0.204, 0.019, 0.150, 0.268, 0.002, 0.357	(0.009, 0.012, 0.272, 0.136, 0.131, 0.440
GEBM	(0.030, 0.001, 0.094, 0.016, 0.122, 0.737)	(0.132, 0.005, 0.035, 0.076, 0.030, 0.722	(0.023, 0.002, 0.085, 0.003, 0.240, 0.647

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

Table 17 | Average distance between TOPSIS rankings and real rankings.

	Linear	Concave	Convex
Survey 1			
SAW	2.4698	2.4545	2.4966
GEBM	2.3797	2.3969	2.3950
Survey 2			
SAW	2.5208	2.5649	2.5371
GEBM	2.4473	2.4784	2.4686

SAW, simple additive weighting; GEBM, generalized extended Bonferroni mean.

Step 2: Determine the positive ideal solution (PIS) $A^+ = \left\{ r_1^+, \ldots, r_j^+, \ldots, r_6^+ \right\}$ and the NIS $A^- = \left\{ r_1^-, \ldots, r_j^-, \ldots, r_6^- \right\}$.

Step 3: Calculate the distance and relative closeness of each smart phone from the PIS and NIS:

$$
\begin{aligned}
d^+ &= \sqrt{ \sum_{j=1}^{m} \left(w_k r_{ij} - w_k r_j^+ \right)^2 }, \\
d^- &= \sqrt{ \sum_{j=1}^{m} \left(w_k r_{ij} - w_k r_j^- \right)^2 }, \\
RC_i &= \frac{d^-}{d^+ + d^-}.
\end{aligned}
$$

Step 4: Rank the smart phones according to RC_i.

Step 5: Calculate the distance between the TOPSIS rankings and the real rankings as $C_{ab} = (1/n) \sum_{i=1}^{n} |r_{a,i} - r_{b,i}|$. The average distance for each model is given in Table 17.

The GEBM operator has a smaller average distance between TOPSIS rankings and real rankings for both models, which reflects the same conclusion as in Section 5.5. So as Section 5.5, this section conclude that on individual views, GEBM receives more reliable weighting vectors of subjects than SAW model.

5.7. Comparing Attribute Weights Based on the Attribute View

The weighting vectors obtained by the fitting process provide some important insights about interrelationships among attributes. From the study above, we know that the individual weighting vectors learned by GEBM are more relevant to the actual cognition by the subjects. Meanwhile, we can gain some important information on attributes by comparing weights of various attributes among

individual subjects, which is based on the attribute view. Using this information, we can compare the fitting quality of the aggregation operators in an effective manner.

We know that the weights of all attributes add up to 1. There is a negative correlation between the weights of independent attributes. However, for dependent attributes, the correlation may be positive or negative, depending on the dependence. In this way, from the attribute weights, we can reveal interrelationships among attributes as perceived by the subjects. We shall just analyze attribute weights in linear utility measurement in survey 1. The results are shown as Tables 18 and 19.

Interrelationships among attributes are reflected in these tables. For example, the correlation between battery capacity and cruising power is positive in both tables, which means that they are strongly dependent. We may conclude from this that the GEBM provides a good representation of human cognition. For instance, the correlation between battery capacity and screen size is as high as 0.359. The attribute weights obtained by the SAW model are closer to those given by the human cognitive process.

6. DISCUSSION AND CONCLUSION

We have investigated whether considering interrelationships among attributes can provide a better representation of real preferences by fitting the SAW and GEBM models to empirical data. From an analysis of the results of fitting, we can draw the following conclusions.

1. The GEBM operator does not provide a realistic model of preferences that fits those of most people; i.e., most people do not consider interrelationships among attributes when making decisions. Decision-making is usually a time-consuming task and decision makers are not always rational, so it is understandable that interrelationships among attributes are ignored by decision makers.

2. The GEBM operator shows a higher consistency for subjects than the SAW model. The GEBM models the preferences of more rational and consistent decision makers particularly well; i.e., rational and consistent decision makers consider interrelationships among attributes when making decisions. Considering such interrelationships aids rational decision-making.

Table 18 | Pearson correlation coefficient of attributes weights for SAW model in linear utility measurement in survey 1.

	Battery Capacity	Thickness	Screen Size	Cruising Power	Camera	ROM
Battery capacity	1.000	−0.044	−0.268	0.202	−0.231	−0.286
Thickness	−0.044	1.000	0.107	−0.184	−0.087	−0.064
Screen size	−0.268	0.107	1.000	−0.201	0.086	−0.542
Cruising power	0.202	−0.184	−0.201	1.000	−0.326	−0.518
Camera	−0.231	−0.087	0.086	−0.326	1.000	−0.136
ROM	−0.286	−0.064	−0.542	−0.518	−0.136	1.000

Table 19 | Pearson correlation coefficient of attributes weights for GEBM operator in linear utility measurement in survey 1.

	Battery Capacity	Thickness	Screen Size	Cruising Power	Camera	ROM
Battery capacity	1	−0.060	0.359	0.170	−0.230	−0.397
Thickness	−0.060	1	−0.125	0.021	0.385	−0.336
Screen size	0.359	−0.125	1	−0.195	−0.182	−0.557
Cruising power	0.170	0.021	−0.195	1	−0.159	−0.056
Camera	−0.230	0.385	−0.182	−0.159	1	−0.592
ROM	−0.397	−0.336	−0.557	−0.056	−0.592	1

3. More people make decisions that are in line with those indicated by the SAW model than by the GEBM, as described in Sections 5.2 and 5.3. However, the results for the weighting vectors learned by the GEBM are more relevant to the real weighting vectors than those of the SAW model.

4. The GEBM operator considers interrelationship among attributes. However, the attribute weights learned by the GEBM operator fail to reflect this information, and indeed they do not even obey the interrelationships.

We can therefore conclude that considering interrelationship among attributes when representing real preferences is scenario-dependent. Rational and consistent decision makers would consider interrelationships among attributes when making decisions. However, most people may not be rational and consistent in real life. Thus, although consideration of interrelationships among attributes might represent the preferences of rational and consistent people, this is not necessarily the case for most people. This conclusion can provide useful guidance for manufacturing enterprises. If consumers want to spend more on a product, they will consider the interrelationships among attributes. This means that an attribute will have a strong impact on the consciousness of a decision maker only when independent attributes have a similar impact.

There are also some limitations to the approach adopted in this paper. There are many aggregation operators that consider interrelationships among attributes, such as the CI and the PA operator [20,21,67,68], but we have only studied the fitting quality of the GEBM operator to test whether considering interrelationships among attributes can provide a better representation of real preferences. Furthermore, the GEBM operator is a composite operator, and the selection of averaging operators, conjunctive operator, and related parameters has an important impact on the fitting quality of the GEBM. In addition, further studies are needed to improve on the empirical results presented here.

CONFLICT OF INTEREST

none.

AUTHORS' CONTRIBUTIONS

Z.S.C. and X.Z. conceived and designed the framework; Z.S.C. and X.Z. wrote the paper; Z.S.C. and X.Z. collected and analyzed the data; R.M.R., X.J.W. and K.S.C. finally checked and revised the paper. All authors read and approved the final manuscript.

ACKNOWLEDGMENTS

REFERENCES

[1] M. Aggarwal, On learning of weights through preferences, Inf. Sci. 321 (2015), 90–102.

[2] G.-H. Tzeng, J.-J. Huang, Multiple Attribute Decision Making: Methods and Applications, Chapman and Hall/CRC, New York, 2011.

[3] C.A.B. e Costa, É.C. Corrêa, J.-M. De Corte, J.-C. Vansnick, Facilitating bid evaluation in public call for tenders: a socio-technical approach, Omega. 30 (2002), 227–242.

[4] Y. Wu, Y. Dong, J. Qin, W. Pedrycz, Flexible linguistic expressions and consensus reaching with accurate constraints in group decision-making, IEEE Trans. Cybern. (2019).

[5] L. Martínez, R.M. Rodríguez, F. Herrera, The 2-tuple Linguistic Model: Computing with Words in Decision Making, Springer, Berlin, Heidelberg, 2015.

[6] R.M. Rodríguez, Á. Labella, L. Martínez, An overview on fuzzy modelling of complex linguistic preferences in decision making, Int. J. Comput. Intell. Syst. 9 (2016), 81–94.

[7] S.-H. Xiong, Z.-S. Chen, K.-S. Chin, A novel MAGDM approach with proportional hesitant fuzzy sets, Int. J. Comput. Intell. Syst. 11 (2018), 256–271.

[8] Z.-S. Chen, K.-S. Chin, Y.-L. Li, Y. Yang, On generalized extended Bonferroni means for decision making, IEEE Trans. Fuzzy Syst. 24 (2016), 1525–1543.

[9] Z.-S. Chen, Y. Yang, X.-J. Wang, K.-S. Chin, K.-L. Tsui, Fostering linguistic decision-making under uncertainty: a proportional interval type-2 hesitant fuzzy TOPSIS approach based on Hamacher aggregation operators and andness optimization models, Inf. Sci. 500 (2019), 229–258.

[10] B.S. Ahn, Preference relation approach for obtaining OWA operators weights, Inf. Sci. 47 (2008), 166–178.

[11] G. Beliakov, Aggregation functions from data using linear programming, Fuzzy Sets Syst. 160 (2009), 65–75.

[12] G. Beliakov, S. James, D.G. Nimmo, Using aggregation functions to model human judgements of species diversity, Inf. Sci. 306 (2015), 21–33.

[13] T. Calvo, G. Beliakov, Identification of weights in aggregation operators, in: H. Bustince, F. Herrera, J. Montero (Eds.), Fuzzy Sets and Their Extensions: Representation, Aggregation and Models, Springer, Berlin, Heidelberg, 2008, pp. 145–162.

[14] D. Filev, R.R. Yager, On the issue of obtaining OWA operator weights, Fuzzy Sets Syst. 94 (1998), 157–169.

[15] Z.-S. Chen, K.-S. Chin, L. Martinez, K.-L. Tsui, Customizing semantics for individuals with attitudinal HFLTS possibility distributions, IEEE Trans. Fuzzy Syst. 26 (2018), 3452–3466.

[16] O. Reimann, C. Schumacher, R. Vetschera, How well does the OWA operator represent real preferences? Eur. J. Oper. Res. 258 (2017), 993–1003.

[17] Z.-S. Chen, C. Yu, K.-S. Chin, L. Martínez, An enhanced ordered weighted averaging operators generation algorithm with applications for multicriteria decision making, Appl. Math. Model. 71 (2019), 467–490.

[18] Z.-S. Chen, L. Martínez, K.-S. Chin, K.-L. Tsui, Two-stage aggregation paradigm for HFLTS possibility distributions: a hierarchical clustering perspective, Expert Syst. Appl. 104 (2018), 43–66.

[19] B. Dutta, D. Guha, Partitioned Bonferroni mean based on linguistic 2-tuple for dealing with multi-attribute group decision making, Appl. Soft Comput. 37 (2015), 166–179.

[20] G. Choquet, Theory of capacities, Ann. de l'institut Fourier. 5 (1954), 131–295.

[21] R.R. Yager, The power average operator, IEEE Trans. Syst. Man Cybern. Part A Syst. Hum. 31 (2001), 724–731.

[22] C. Bonferroni, Sulle medie multiple di potenze, Bollettino dell'Unione Matematica Italiana. 5 (1950), 267–270.

[23] B. Dutta, D. Guha, R. Mesiar, A model based on linguistic 2-tuples for dealing with heterogeneous relationship among attributes in multi-expert decision making, IEEE Trans. Fuzzy Syst. 23 (2014), 1817–1831.

[24] R. Sadiq, S. Tesfamariam, Probability density functions based weights for ordered weighted averaging (OWA) operators: an example of water quality indices, Eur. J. Oper. Res. 182 (2007), 1350–1368.

[25] G. Beliakov, S. James, R. Mesiar, A generalization of the Bonferroni mean based on partitions, in 2013 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), IEEE, Hyderabad, 2013, pp. 1–6.

[26] L. Sun, H. Dong, A.X. Liu, Aggregation functions considering criteria interrelationships in fuzzy multi-criteria decision making: state-of-the-art, IEEE Access. 6 (2018), 68104–68136.

[27] Z.-S. Chen, K.-S. Chin, K.L. Tsui, Constructing the geometric Bonferroni mean from the generalized Bonferroni mean with several extensions to linguistic 2-tuples for decision-making, Appl. Soft Comput. 78 (2019), 595–613.

[28] K. Ishii, M. Sugeno, A model of human evaluation process using fuzzy measure, Int. J. Man Mach. Stud. 22 (1985), 19–38.

[29] M. Grabisch, The application of fuzzy integrals in multicriteria decision making, Eur. J. Oper. Res. 89 (1996), 445–456.

[30] C. Tan, X. Chen, Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making, Expert Syst. Appl. 37 (2010), 149–157.

[31] D. Yu, Y. Wu, Interval-valued intuitionistic fuzzy Heronian mean operators and their application in multi-criteria decision making, Afr. J. Bus. Manage. 6 (2012), 4158–4168.

[32] C. Maclaurin, A second letter to martin folkes, esq.; concerning the roots of equations, with the demonstartion of other rules in algebra, phil, Transactions. 36 (1729), 59–96.

[33] G. Wei, M. Lu, Pythagorean fuzzy maclaurin symmetric mean operators in multiple attribute decision making, Int. J. Intell. Syst. 33 (2018), 1043–1070.

[34] J. Qin, X. Liu, 2-tuple linguistic Muirhead mean operators for multiple attribute group decision making and its application to supplier selection, Kybernetes. 45 (2016), 2–29.

[35] J. Renaud, E. Levrat, C. Fonteix, Weights determination of OWA operators by parametric identification, Math. Comput. Simul. 77 (2008), 499–511.

[36] V. Torra, On the learning of weights in some aggregation operators: the weighted mean and OWA operators, Math. Soft Comput. 6 (1999), 249–265.

[37] S. Kumar, B.K. Tripathi, On the learning machine with compensatory aggregation based neurons in quaternionic domain, J. Comput. Design Eng. 6 (2019), 33–48.

[38] M. Grabisch, I. Kojadinovic, P. Meyer, A review of methods for capacity identification in Choquet integral based multi-attribute utility theory: applications of the kappalab r package, Eur. J. Oper. Res. 186 (2008), 766–785.

[39] G. Beliakov, Monotone approximation of aggregation operators using least squares splines, Int. J. Uncertain. Fuzziness Knowl. Based Syst. 10 (2002), 659–676.

[40] S. Angilella, S. Greco, F. Lamantia, B. Matarazzo, Assessing non-additive utility for multicriteria decision aid, Eur. J. Oper. Res. (2004), 734–744.

[41] G. Beliakov, R. Mesiar, L. Valaskova, Fitting generated aggregation operators to empirical data, Int. J. Uncertain. Fuzziness Knowl. Based Syst. 12 (2004), 219–236.

[42] Y. Narukawa, V. Torra, Fuzzy measure and probability distributions: distorted probabilities, IEEE Trans. Fuzzy Syst. 13 (2005), 617–629.

[43] G. Beliakov, M. Gagolewski, S. James, S. Pace, N. Pastorello, E. Thilliez, R. Vasa, Measuring traffic congestion: an approach based on learning weighted inequality, spread and aggregation indices from comparison data, Appl. Soft Comput. 67 (2018), 910–919.

[44] Y. Shepetukha, D. Olson, Comparative analysis of multiattribute techniques based on cardinal and ordinal inputs, Math. Comput. Model. 34 (2001), 229–241.

[45] P. Bloomfield, W.L. Steiger, Least Absolute Deviations: Theory, Applications and Algorithms, Springer, Boston, 1983.

[46] R. Fullér, P. Majlender, On obtaining minimal variability OWA operator weights, Fuzzy Sets Syst. 136 (2003), 203–215.

[47] R. Fullér, P. Majlender, An analytic approach for obtaining maximal entropy OWA operator weights, Fuzzy Sets Syst. 124 (2001), 53–57.

[48] J. Jiang, J. Xie, C. Zhao, J. Su, Y. Guan, Q. Yu, Max-margin weight learning for medical knowledge network, Comput. Methods Prog. Biomed. 156 (2018), 179–190.

[49] B. Geng, X. Zhou, J. Zhu, Y. Hung, S.T. Wong, Comparison of reversible-jump markov-chain-monte-carlo learning approach with other methods for missing enzyme identification, J. Biomed. Inform. 41 (2008), 272–281.

[50] T. Khot, S. Natarajan, K. Kersting, J. Shavlik, Gradient-based boosting for statistical relational learning: the Markov logic network and missing data cases, Mach. Learn. 100 (2015), 75–100.

[51] W. Edwards, How to use multiattribute utility measurement for social decisionmaking, IEEE Trans. Syst Man Cybern. 7 (1977), 326–340.

[52] T. Solymosi, J. Dombi, A method for determining the weights of criteria: the centralized weights, Eur. J. Oper. Res. 26 (1986), 35–41.

[53] L. Mikhailov, Deriving priorities from fuzzy pairwise comparison judgements, Fuzzy Sets Syst. (2003), 365–385.

[54] P. Salminen, P. Korhonen, J. Wallenius, Testing the form of a decision-maker's multiattribute value function based on pairwise preference information, J. Oper. Res. Soc. 40 (1989), 299–302.

[55] G. Beliakov, S. James, Using linear programming for weights identification of generalized Bonferroni means in R, in International Conference on Modeling Decisions for Artificial Intelligence, Springer, Berlin, Heidelberg, 2012, pp. 35–44.

[56] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, vol. 127, Cambridge University Press, Cambridge, England, 2009.

[57] D. Dubois, H. Prade, On the use of aggregation operations in information fusion processes, Fuzzy Sets Syst. 142 (2004), 143–161.

[58] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation functions: Construction methods, conjunctive, disjunctive and mixed classes, Inf. Sci. 181 (2011), 23–43.

[59] C.L. Hwang, K. Yoon, Multiple Attribute Decision Making-Method and Applications, A State-of-the-Art Survey, Springer-Verlag, Berlin, Heidelberg, 1981.

[60] S.-Y. Chou, Y.-H. Chang, C.-Y. Shen, A fuzzy simple additive weighting system under group decision-making for facility location selection with objective/subjective attributes, Eur. J. Oper. Res. 189 (2008), 132–145.

[61] D. Paternain, J. Fernández, H. Bustince, R. Mesiar, G. Beliakov, Construction of image reduction operators using averaging aggregation functions, Fuzzy Sets Syst. 261 (2015), 87–111.

[62] D. Paternain, H. Bustince, J. Fernández, G. Beliakov, R. Mesiar, Some averaging functions in image reduction, in: N. García-Pedrajas, F. Herrera, C. Fyfe, J.M. Benítez, M. Ali (Eds.), International Conference on Industrial, Engineering and Other Applications of Applied Intelligent Systems, Springer, Berlin, Heidelberg, 2010, pp. 399–408.

[63] G. Bordogna, M. Boschetti, A. Brivio, P. Carrara, M. Pagani, D. Stroppiana, Fusion strategies based on the OWA operator in environmental applications, in: R.R. Yager, J. Kacprzyk, G. Beliakov (Eds.), Recent Developments in the Orderd Weighted Averaging Operators: Theory and Practice, Springer, Berlin, Heidelberg, 2011, pp. 189–207.

[64] M. Fedrizzi, M. Fedrizzi, M. Brunelli, OWA-based fuzzy m-ary adjacency relations in social network analysis, Technical Report, University of Trento, Trento, 2009.

[65] R.R. Yager, On generalized Bonferroni mean operators for multi-criteria aggregation, Int. J. Approx. Reason. 50 (2009), 1279–1286.

[66] G. Beliakov, S. James, J. Mordelová, T. Rückschlossová, R.R. Yager, Generalized Bonferroni mean operators in multi-criteria aggregation, Fuzzy Sets Syst. 161 (2010), 2227–2242.

[67] S.-H. Xiong, Z.-S. Chen, J.-P. Chang, K.-S. Chin, On extended power average operators for decision-making: a case study in emergency response plan selection of civil aviation, Comput. Ind. Eng. 130 (2019), 258–271.

[68] Y. Yang, Z.-S. Chen, Y.-H.Chen, K.-S. Chin, Interval-valued Pythagorean fuzzy Frank power aggregation operators based on an isomorphic Frank dual triple, Int. J. Comput. Intell. Syst. 11 (2018), 1091–1110.