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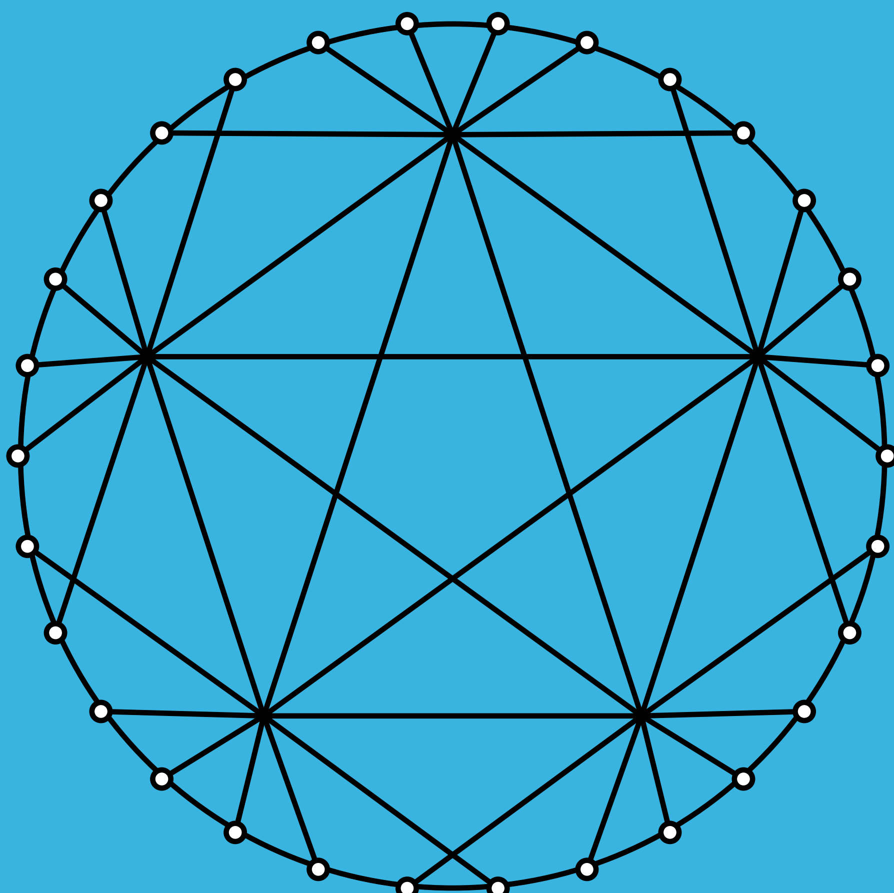

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3-GDDs with block size 4

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Abstract:

We define a 3-GDD$(n, 2, k, \lambda_1, \lambda_2)$ by extending the definitions of a group divisible design and a t-design and give some necessary conditions for its existence. We prove that these necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ except possibly when $n \equiv 1, 3 \pmod 6$, $n \neq 3, 7, 13$ and $\lambda_1 > \lambda_2$. It is known that a partition of all 3-subsets of a 7-set into 5 Steiner triple systems (a large set for 7) does not exist, but we show that the collection of all 3-sets of a 7-set along with a Steiner triple system on the 7-set can be partitioned into 6 Steiner triple systems. Such a partition is then used to prove the existence of all possible 3-GDDs for $n = 7$.

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1 Preliminaries

We begin with some well known definitions and results from Graph Theory and Design Theory.

1.1 1- and 2- factorizations of a complete graph

Definition 1.1. A complete graph K_n is a graph on n vertices where each distinct pair of vertices is connected by an edge.

Definition 1.2. A 1-factor of a graph G is a set of pairwise disjoint edges which partition the vertex set of G.

Definition 1.3. A 1-factorization of a graph G is a partition of the edge set of G into 1-factors.

Definition 1.4. A 2-factor of a graph is a set of edges in which each vertex appears exactly twice.

Definition 1.5. A 2-factorization of the complete graph K_n is a set of 2-factors that partitions the edges of the complete graph.

It is well known that a complete graph K_n on an even number of vertices n has a 1-factorization with $(n-1)$ 1-factors. Also, when n is odd, it is known that there exists a 2-factorization of a complete graph K_n with $\frac{(n-1)}{2}$ 2-factors [4].

1.2 BIBDs and α-RBIBDs

Definition 1.6. A Balanced Incomplete Block Design, BIBD(v, b, r, k, λ), is a collection of b k-subsets (called blocks) of a v-set V, such that each element appears in exactly r blocks, every pair of distinct elements of V occurs in λ blocks and $k < v$. A BIBD(v, b, r, k, λ) is also written as a BIBD(v, k, λ).

Definition 1.7. Suppose (X, A) is a BIBD(v, k, λ). A parallel class in (X, A) is a subset of disjoint blocks from A whose union is X. A partition of A into r parallel classes is called a resolution, and (X, A) is said to be a resolvable BIBD, RBIBD, if A has at least one resolution.

A BIBD is called α-resolvable BIBD if its blocks can be partitioned into classes in which each point occurs α times.

There are well known existence results for BIBDs with block size 3, viz., (1) a BIBD$(v,3,1)$ exists for $v \equiv 1,3 \pmod 6$ and has $\frac{v(v-1)}{6}$ blocks, (2) a BIBD$(v,3,2)$ exists for $v \equiv 0,1 \pmod 3$, (3) a BIBD$(v,3,3)$ exists for all $v \equiv 1 \pmod 2$, and (4) a BIBD$(v,3,6)$ exists for all $v \geq 3$ as well as the following results, e.g. see [4], [5], and [6].

Theorem 1.1. The necessary conditions for the existence of an α-resolvable BIBD$(v,3,\lambda)$ are sufficient except for $v = 6, \alpha = 1$, and $\lambda \equiv 2 \pmod 4$.

Hence,

(i) A 3-resolvable BIBD$(v,3,6)$ exists for all $v \geq 3$, with $(v-1)$ classes.

(ii) A resolvable BIBD$(v,3,1)$ exists for $v \equiv 3 \pmod 6$.

(ii) A resolvable BIBD$(v,3,2)$ exists for $v \equiv 0 \pmod 3$ except for $v = 6$.

1.3 *t*-designs and GDDs

Definition 1.8. A t-(v,k,λ) design, or a t-design is a pair (X,B) where X is a v-set of points and B is a collection of k-subsets (blocks) of X with the property that every t-subset of X is contained in exactly λ blocks. The parameter λ is called the index of the design.

A quadruple $(\lambda;t,k,v)$ is admissible if each $\lambda_s = \frac{\lambda\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$ for $0 \leq s \leq t$ is an integer. An admissible quadruple $(\lambda;t,k,v)$ is denoted by t-(v,k,λ). An admissible t-(v,k,λ) is realizable if a t-(v,k,λ) design exists. Admissible but not realizable parameter quadruples for $t = 3$ and $v \leq 30$ are 3-$(11,5,2)$, 3-$(16,6,2)$, 3-$(22,10,6)$ and 3-$(26,10,3)$ ([3], Page 84).

Definition 1.9. A Steiner Quadruple System (SQS) is an ordered pair (V,B) where V is a finite set of v symbols and B is a collection of 4-subsets of V called blocks (quadruples) with the property that every 3-subset of V is a subset of exactly one quadruple B.

A SQS is just a particular example of a t-design. The following 3-designs with block size 4 exist ([3], pp 82-83):

1. 3-$(n, 4, 1)$ for $n \equiv 2, 4 \pmod 6$,
2. 3-$(n, 4, 2)$ for $n \equiv 2, 4, 5 \pmod 6$,
3. 3-$(n, 4, 3)$ for even $n \geq 4$, and
4. 3-$(2^n + 1, 4, 6t)$ for any $n \geq 2$ and $1 \leq t \leq \frac{2^{n-1}-1}{3}$.

Definition 1.10. A group divisible design GDD$(n, m, k, \lambda_1, \lambda_2)$ is a collection of k-subsets, called blocks, of an nm-set X, where the elements of X are partitioned into m subsets (called groups) of size n each; pairs of distinct elements within the same group are called first associates of each other and appear together in λ_1 blocks while any two elements not in the same group are called second associates and appear together in λ_2 blocks.

1.4 A new concept : 3-GDDs

It is possible to generalize the concepts of GDDs and t-designs in many ways but for GDDs with two groups and block size k, the concepts generalize in a natural and beautiful way:

Definition 1.11. A 3-GDD$(n, 2, k, \lambda_1, \lambda_2)$ is a set X of $2n$ elements partitioned into two parts of size n called groups together with a collection of k-subsets of X called blocks, such that

(i) every 3-subset of each group occur in λ_1 blocks, and

(ii) every 3-subset where two elements are from one group and one element from the other group occurs in λ_2 blocks.

Example 1.1. A 3-GDD$(3, 2, 4, 3, 1)$: Let $X = \{1, 2, 3, a, b, c\}$, $G_1 = \{1, 2, 3\}$ and $G_2 = \{a, b, c\}$. Then $B = \{\{1, 2, 3, a\}, \{1, 2, 3, b\}, \{1, 2, 3, c\}, \{a, b, c, 1\}, \{a, b, c, 2\}, \{a, b, c, 3\}\}$ gives the required blocks of the GDD.

The following Lemmas are very useful.

Lemma 1.2. If a 3-$(2n, 4, \lambda_2)$, (i.e., a 3-GDD$(n, 2, 4, \lambda_2, \lambda_2)$) and a 3-$(n, 4, \lambda_1 - \lambda_2)$ exists, then a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ exists.

Proof. Let G_1 and G_2 be two disjoint sets of cardinality n. The blocks of three designs: (i) a 3-$(n, 4, \lambda_1 - \lambda_2)$ on G_1 (ii) a 3-$(n, 4, \lambda_1 - \lambda_2)$ on G_2, and

(iii) a 3-$(2n,4,\lambda_2)$ on $G_1 \cup G_2$, taken together give a 3-GDD$(n,2,4,\lambda_1,\lambda_2)$ with groups G_1 and G_2.

□

Lemma 1.3. If a 3-$(n,4,2)$ exists, then a 3-GDD$(n,2,4,\lambda_1,\lambda_2)$ for all even λ_1 and even λ_2 exists.

Proof. Let $\lambda_1 = 2t$, and $\lambda_2 = 2s$ for positive integers s and t. Let G_1 and G_2 be two disjoint sets of cardinality n. The blocks of t copies of a 3-$(n,4,2)$ on G_1 as well as on G_2 together with the blocks of s copies of a 3-GDD$(n,2,4,0,2)$ with groups G_1 and G_2, (see Theorem 3.1), give the required 3-GDDs. □

Remark 1.

(i) When $\lambda_1 = \lambda_2$, a 3-GDD$(n,2,k,\lambda_1,\lambda_2)$ is a 3-$(2n,k,\lambda_1)$.

(ii) Every 3-GDD is also a 2-GDD as shown in the next section.

(iii) As a 3-GDD $(n,2,3,\lambda_1,\lambda_2)$ is obtained by a collection of λ_i copies of all subsets of size 3 of G_i, $i = 1,2$ and λ_2 copies of all other 3-subsets of $G_1 \cup G_2$, one can assume that for non-trivial 3-GDDs, $k \geq 4$.

In the next section we obtain some necessary conditions for the existence of a 3-GDD$(n,2,k,\lambda_1,\lambda_2)$. Towards this aim, assuming a 3-GDD exists, we count the number of blocks containing a given element x (called the replication number r for x), the number of blocks, r_1, containing a first associate pair, r_2, the number of blocks containing a second associate pair and the required number of blocks, say b, for the 3-GDD.

2 Necessary conditions

Suppose a 3-GDD$(n,2,k,\lambda_1,\lambda_2)$ exists with groups G_1 and G_2. Without loss of generality, let $x \in G_1$ and let r be the replication number for x. There are $\binom{n-1}{2}$ 3-subsets containing x, where all elements are from the same group G_1. Also there are $(n-1)n$ 3-subsets where x occurs with an element from G_1 and one from G_2 and there are $\binom{n}{2} = \frac{n(n-1)}{2}$ 3-subsets containing x where the other two elements are from G_2. Then as

$$\frac{(k-1)(k-2)r}{2} = \frac{(n-1)(n-2)}{2}\lambda_1 + (n(n-1) + \frac{n(n-1)}{2})\lambda_2,$$

$$r = \frac{(n-1)(n-2)\lambda_1 + 3n(n-1)\lambda_2}{(k-2)(k-1)}.$$

Hence a necessary condition for the existence of 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ is that

$$(n-1)(n-2)\lambda_1 \equiv 0 \pmod{6}. \tag{1}$$

As an application of this condition, a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ for $n \equiv 0 \pmod{3}$ and $\lambda_1 \equiv 1, 2 \pmod{3}$ does not exist.

If a block of size k contains a pair $\{x, y\}$, then the block has $(k-2)$ 3-subsets containing x and y. On the other hand, let r_1 denote the number of times a first associate pair, say $\{x, y\}$, occurs in a 3-GDD$(n, 2, k, \lambda_1, \lambda_2)$. Then as there are $n-2$ 3-subsets of the group containing x, y and a 3^{rd} element from the same group and there are n 3-subsets containing x, y and a 3^{rd} element from a different group, we have

$$\lambda_1(n-2) + \lambda_2(n) = (k-2)r_1.$$

Hence for even k,

$$(\lambda_1 + \lambda_2)n \equiv 0 \pmod{2}. \tag{2}$$

Therefore we obtain a necessary condition:

Lemma 2.1. A necessary condition for the existence of a 3-GDD$(n, 2, k, \lambda_1, \lambda_2)$ for odd n and k even is that λ_1 and λ_2 must be of the same parity.

Now, let r_2 denote the replication number of pairs $\{a, x\}$ where a and x are second associates. As there are no first associate triples containing $\{a, x\}$, there are exactly $2(n-1)$ triples which contain $\{a, x\}$ and each of these triples occurs λ_2 times. Therefore $(k-2)r_2 = 2(n-1)\lambda_2$ and

$$r_2 = \frac{2(n-1)\lambda_2}{k-2} \tag{3}$$

Hence, the expression for r_2, unlike r_1, does not give any divisibility restrictions for $k = 4$.

Now we obtain the number of blocks needed for a 3-GDD$(n, 2, k, \lambda_1, \lambda_2)$ if it exists. There are $2\binom{n}{3}$ 3-subsets which occur in λ_1 blocks and $2n\binom{n}{2}$ 3-subsets where 2 elements are from one group and one element from the other group and each block has $\binom{k}{3}$ 3-subsets, hence we have

$$\binom{k}{3}b = \lambda_1 2\binom{n}{3} + \lambda_2 n^2(n-1)$$

Hence, for $k = 4$,

$$b = \frac{\lambda_1 n(n-1)(n-2)+3\lambda_2 n^2(n-1)}{12}.$$

From the requirement that b is an integer, we have

$$\lambda_1 n(n-1)(n-2) + 3\lambda_2 n^2(n-1) \equiv 0 \pmod{12}. \tag{4}$$

However,[1] Equation 4 does not give any further restrictions on n. Firstly, 6 is a factor of both terms in Equation 4. Secondly, if n is even, 4 is a factor of both terms. Thirdly, if n is odd, then since λ_1 and λ_2 are of the same parity (Lemma 2.1), both terms are even and are congruent to $\lambda_1(n-1)$ modulo 4, hence their sum is $0 \pmod 4$.

Based on the divisibility requirements from the expressions for r (Equation 1), and r_1 (Equation 2), we have following necessary conditions on n for the existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$. The values of λ_1 and λ_2 are given modulo 6:

λ_1/λ_2	0	1	2	3	4	5
0	all n	n even	all n	n even	all n	n even
1	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$
2	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$
3	n even	all n	n even	all n	n even	all n
4	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$
5	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$

Table 1

[1]We thank an unknown mathematician for providing the following nicer argument.

Remark 2. We may have a collection of b blocks satisfying the values of r_1 and r_2 but still does not give a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$. For example, a 3-GDD$(7, 2, 4, 3, 1)$, must have $b = 126$, $r_1 = 11$, and $r_2 = 6$. Now we will construct a collection of 126 blocks, with $r_1 = 11$, and $r_2 = 6$ where each 3-subset of a group occurs $\lambda_1 = 3$ times but still, the collection is not the required 3-GDD: First recall that a large set of STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$ and $v \neq 7$ ([2], Page 65). But we can partition the set of all 3-subsets of G_1, which is a BIBD$(7, 3, 5)$, into five 3-resolvable classes ([1], Page 130). Now we construct blocks of size 4 from i^{th} resolvable class by taking the union of each block of the resolvable class with the i^{th} element of G_2, for $i = 1, 2, 3, 4$ and 5 for some arbitrary ordering of the elements of G_2. We further construct blocks by taking union of the blocks of a BIBD$(7, 3, 1)$ on G_1 with the 6^{th} element of G_2 and union of each block of a BIBD $(7, 3, 1)$ on G_1 with the 7^{th} element of G_2. Similarly, we construct blocks by reversing the roles of G_1 and G_2. The collection of blocks so constructed along with 2 copies of a BIBD$(7, 4, 2)$ obtained by complementing each triple of BIBD$(7, 3, 1)$ on each of the groups G_1 and G_2 have the required values r_1, r_2 and b of a 3-GDD$(7, 2, 4, 3, 1)$. But λ_2 is not 1 as not all of the 3-resolvable classes are BIBDs. Note that each copy of a BIBD$(7, 3, 1)$ and the BIBD$(7, 4, 2)$ obtained by complementing the triples of the BIBD$(7, 3, 1)$ on G_i contains each 3-subset of G_i once for $i = 1, 2$. Hence every first associate triple occurs exactly three times, but still we do not have a 3-GDD as $\lambda_2 \neq 1$.

3 A fundamental construction

Theorem 3.1. A 3-GDD$(n, 2, 4, 0, 1)$ exists for even n and a 3-GDD$(n, 2, 4, 0, 2)$ exists for all positive integers n.

Proof. Let G_1 and G_2 be two sets of the same cardinality n. A K_n on G_i means the vertices of the complete graph K_n are labeled with the elements of G_i, $i = 1$, 2. Let n be even, say $n = 2t$. Then the complete graph K_n on G_1 (respectively K_n on G_2) has a 1-factorization, say $\{E_1, E_2, , \cdots, E_{n-1}\}$ (respectively $\{F_1, F_2, \cdots, F_{n-1}\}$).

For $l = 1, \cdots, n-1$, if $E_l = \{e_1, e_2, \cdots, e_t\}$ and $F_l = \{f_1, f_2, \cdots, f_t\}$, then form blocks $e_i \cup f_j$ of size 4, for $1 \leq i, j \leq t$. It is easy to see that we have a 3-GDD$(n, 2, 4, 0, 1)$ as follows: First no block contains three elements from the same group and hence $\lambda_1 = 0$. Secondly, every pair $\{x, y\}$ of elements of a group is in exactly one 1-factor as an edge, say e. Suppose e is in a 1-factor E_l. Now the blocks which contain pair $\{x,y\}$ (i.e., edge e) are precisely $e \cup f_i$, $i = 1, 2, \cdots, t$. Hence a triple of elements $\{x,y,z\}$ where z is an element from G_2 occurs in exactly one block. By symmetry, a triple containing two elements from G_2 and a third element from G_1, also occurs in exactly one block. Hence $\lambda_2 = 1$.

Similarly, from any 2-factorizations of a K_n on G_1 and a K_n on G_2, we get a 3-GDD$(n, 2, 4, 0, 2)$. □

Remark 3. The above 3-GDD for even n is also a 2-GDD$(n, 2, 4; \frac{n}{2}, n-1)$ with groups G_1 and G_2. Similarly the 3-GDD$(n, 2, 4, 0, 2)$ for odd n is a 2-GDD$(n, 2, 4; n, 2(n-1))$.

Example 3.1. A 3-GDD$(4, 2, 4, 0, 1)$ with $X = \{1, 2, 3, 4, a, b, c, d\}$, $G_1 = \{a, b, c, d\}$, $G_2 = \{1, 2, 3, 4\}$. Blocks are written as columns:

1	1	1	1	1	1	2	2	2	2	3	3
2	2	3	3	4	4	3	3	4	4	4	4
a	c	a	b	a	b	a	b	a	b	a	c
b	d	d	c	c	d	c	d	d	c	b	d

As a consequence of Theorem 3.1 and known 3-designs, we have:

Theorem 3.2. For $n \equiv 1, 3 \pmod 6$, the necessary conditions as described in Table 2 are sufficient for the existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ when $\lambda_1 \le \lambda_2$.

Proof. When $n \equiv 1, 3 \pmod 6$, λ_1 and λ_2 have the same parity, i.e., $\lambda_2 - \lambda_1 \equiv 0 \pmod 2$. Also for $n \equiv 3 \pmod 6$, $\lambda_1 \equiv 0 \pmod 3$. Hence the blocks of a 3-$(2n, 4, \lambda_1)$ on $G_1 \cup G_2$ and $\frac{\lambda_2 - \lambda_1}{2}$ copies of a 3-GDD$(n, 2, 4, 0, 2)$ together give the blocks of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ □

Similarly the following Theorem 3.3, specially when $\lambda_1 \equiv 0 \pmod 3$, is very useful. Recall, a 3-$(n, 4, 3)$ and a 3-GDD$(n, 2, 4, 0, 1)$ exists for all even $n \ge 4$.

Theorem 3.3. A 3-GDD$(n, 2, 4, 3t, \lambda)$ exists for any $t \ge 1$ and $\lambda \ge 1$. In general, if a 3-$(n, 4, \lambda)$ and a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ exist then a

$$3\text{-GDD}(n, 2, 4, \lambda_1 + t\lambda, \lambda_2)$$

exists for all positive integers t.

Corollary 3.4. The necessary conditions are sufficient for the existence of a 3-GDD$(n,2,4,3t,\lambda_2)$ for any even n and hence the necessary conditions are sufficient for the existence of a 3-GDD$(n,2,4,\lambda_1,\lambda_2)$ for $n \equiv 0 \pmod 6$.

We have not proved that the necessary conditions are sufficient for the existence of 3-GDDs with block size 4, $n \equiv 1,3 \pmod 6$ and $\lambda_1 > \lambda_2$, but Theorem 3.3 and the following two results demonstrate how infinite families can be obtained for these cases. The first result, Lemma 3.5 is especially useful for $n \equiv 1 \pmod 6$.

Lemma 3.5. If a 3-$(n,4,4)$ exists, then a 3-GDD$(n,2,4,\lambda_1,\lambda_2)$ exists for (i) $\lambda_1 \equiv 0 \pmod 4$ and even λ_2 and (ii) for $\lambda_1 \equiv 2 \pmod 4$ and even $\lambda_2 \geq 2$.

Proof. A 3-GDD$(n,2,4,4t,2s)$ is obtained by t copies of a 3-$(n,4,4)$ and s copies of a 3-GDD$(n,2,4,0,2)$ for any n for which a 3-$(n,4,4)$ exists. Then we use a 3-GDD$(n,2,4,4t,2(s-1))$ and two copies of a 3-$(2n,4,1)$ to construct all 3-GDD$(n,2,4,4t+2,2s)$ for $s \geq 1$. We note that specifically when $n \equiv 1,2,4,5 \pmod 6$, $2n \equiv 2 \pmod 6$ or $2n \equiv 4 \pmod 6$. Hence a 3-$(2n,4,1)$ exists. □

Note that the set of all 4-subsets of an n-set is a 3-$(n,4,n-3)$. Also, there exists a 3-GDD$(n,2,4,0,2)$ for all n. Hence as an application of Theorem 3.3 we have

Theorem 3.6. A 3-GDD$(n,2,4,\lambda_1 = (n-3)t, \lambda_2 = 2a)$ exists for all positive integers a and t. In particular, a 3-GDD$(6s+1,2,4,\lambda_1 = 6(3s-1)a, \lambda_2 = 6t)$ exists for all positive integers a, s and t. Similarly a 3-GDD$(6s+3,2,4,\lambda_1 = 6sa, \lambda_2 = 6t)$ exists for all positive integers a, s and t.

In the next section, we prove a complete existence result for $n \equiv 2,4,5 \pmod 6$.

4 $n \equiv 2,4,5 \pmod 6$

For $n \equiv 2,4,5 \pmod 6$, a 3-GDD$(n,2,4,0,2)$ and a 3-$(2n,4,1)$ (i.e., a 3-GDD$(n,2,4,1,1)$) exists. Hence a 3-GDD$(n,2,4,\lambda,\mu = \lambda + 2s)$ for any non-negative integers λ and s exists.

4.1 Even λ_1 and λ_2

Let $\lambda_2 = 2t$. Then for any $\lambda_1 = 2s$, we have two cases, viz, $\lambda_1 \leq \lambda_2$ and $\lambda_1 > \lambda_2$.

4.1.1 $\lambda_1 \leq \lambda_2$

The blocks of $\lambda_1 = 2s$ copies of a 3-GDD$(n, 2, 4, 1, 1)$ along with the blocks of $(t - s)$ copies of a 3-GDD$(n, 2, 4, 0, 2)$ give the required 3-GDD.

4.1.2 $\lambda_1 > \lambda_2$

For $n \equiv 2, 4, 5 \pmod 6$, a 3-$(n, 4, 2)$ exists. Hence $(s - t)$ copies of 3-$(n, 4, 2)$ on G_1 and G_2 and $2t$ copies of a 3-GDD$(n, 2, 4, 1, 1)$ give the required 3-GDD$(n, 2, 4, 2s, 2t)$.

4.2 Odd λ_1 and λ_2

The following Lemma 4.1, which is useful for $n \equiv 1 \pmod 6$ as well, completes this case.

Lemma 4.1. A 3-GDD$(n, 2, 4, \lambda'_1, \lambda'_2)$ exists for all even λ'_1, λ'_2 if and only if a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ exists for all odd λ_1 and λ_2.

Proof. We use a 3-GDD$(n, 2, 4, \lambda_1 - 1, \lambda_2 - 1)$, and a 3-$(2n, 4, 1)$. For example, to construct a 3-GDD$(n, 2, 4; 2t, 2s)$, given that a 3-GDD$(n, 2, 4; 2t - 1, 2s - 1)$ exists, we use the blocks of the 3-GDD$(n, 2, 4; 2t - 1, 2s - 1)$ together with the blocks of 3-$(2n, 4, 1)$. □

Remark 4. To apply Lemma 4.1 to prove that the necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, \lambda_1, 1)$ for some $n \equiv 1 \pmod 6$, we need a 3-$(n, 4, 2)$. For example, to make a 3-GDD$(n, 2, 4; 3, 1)$ we need a 3-GDD$(n, 2, 4, 1, 1)$ which exists along with a 3-GDD$(n, 2, 4; 2, 0)$. However, a 3-$(n, 4, 2)$ required for the existence of 3-GDD$(n, 2, 4; 2, 0)$ may not be known or may not exist.

4.3 λ_1 and λ_2 of opposite parity

In this case, n has to be even and for the purpose of this section, $n \equiv 2, 4 \pmod 6$. Therefore a 3-GDD$(n, 2, 4, 0, 1)$, a 3-$(n, 4, 1)$ and a 3-$(2n, 4, 1)$ exist. Hence we use λ_1 copies of a 3-$(n, 4, 1)$ on G_1 and G_2 along with λ_2 copies of a 3-GDD$(n, 2, 4, 0, 1)$ on groups G_1 and G_2 to obtain the following result.

Theorem 4.2. For $n \equiv 2, 4, 5 \pmod 6$, the necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$.

We note that Theorem 4.2 and Corollary 3.4 give the following result:

Theorem 4.3. Necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ for $n \equiv 0, 2, 4, 5 \pmod 6$.

5 Small values of n: $n = 7, 13, 19$

First we recall that if $\lambda_1 = 0$, then we have $r_1 = \frac{(\lambda_1+\lambda_2)n-2\lambda_1}{2} = \frac{\lambda_2 n}{2}$. If n is even, the smallest $\lambda_2 = 1$ and if n is odd λ_2 must be even and the smallest $\lambda_2 = 2$. Hence Theorem 3.1 implies that the necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, 0, \lambda_2)$.

Next, we note that in view of Lemma 1.3 and Lemma 3.5, to obtain complete results on the existence of 3-GDDs for small values of n with $n \equiv 1, 3 \pmod 6$, one needs to construct 3-GDDs with $\lambda_2 = 0$ as well as $\lambda_2 = 1$. Though 3-GDDs with $\lambda_2 = 0$ can be obtained easily as a GDD$(n, 2, k, \lambda_1, 0)$ is nothing but the union of the collection of the blocks of 3-designs on n elements of G_i, $i = 1, 2$ with block size k. Hence necessary and sufficient conditions for the existence of a 3-GDD$(n, 2, k, \lambda_1, 0)$ are the same as the conditions for a 3-(n, k, λ_1), including the case for $k = 4$. Therefore in what follows, we are interested in constructing 3-GDDs with $\lambda_2 = 1$.

5.1 $n = 7$

When λ_1 is odd, the smallest λ_2 is 1. Even though the main problem is to construct a 3-GDD$(7, 2, 4, 3, 1)$, we first construct a 3-GDD$(7, 2, 4, 7, 1)$ to motivate the method of construction for a 3-GDD$(7, 2, 4, 3, 1)$. Let the groups be G_1 and G_2. Observe that a BIBD$(7,3,1)$ obtained by generating difference set $\{1,2,4\}$ on $\mathbb{Z}_7$ and the BIBD$(7,4,2)$ obtained by taking the complement of the blocks of the BIBD$(7,3,1)$ account for all $\binom{7}{3}$ subsets of $\mathbb{Z}_7$ exactly once. Hence if we label the elements of the BIBD$(7,3,1)$ and the BIBD$(7,4,2)$ by elements of G_1, all 3-subsets of G_1 will occur once. Now we construct blocks of size 4 for the required GDD by taking the union of each block of the BIBD$(7,3,1)$ on G_1 with each of the elements of G_2. Notice that in the process every triple with 2 elements from G_1 and one element from G_2 has occurred exactly once. Similarly, we construct more blocks by using the BIBD$(7,3,1)$ and the BIBD$(7,4,2)$ labeled with the elements of G_2 and by taking union of the blocks of the BIBD$(7,3,1)$ on G_2 with the elements from G_1. Note again that every triple with 2 elements from G_2 and one element from G_1 has occurred exactly once. These blocks together with blocks of 7 copies of the BIBD$(7,4,2)$ on G_1 and 7 copies the BIBD$(7,4,2)$ on G_2 give the required 3-GDD$(7,2,4,7,1)$.

To construct a 3-GDD$(7,2,4,3,1)$, the construction and Remark 2 suggest that we should partition 3 copies of a BIBD$(7,3,1)$ along with the 3-sets obtained by the blocks of one copy of the corresponding BIBD$(7,4,2)$ into 7 STSs. All triples of the set $\{1,2,3,4,5,6,7\}$ to be partitioned are given below in a 7 by 7 matrix. The last three columns of the matrix are identical containing triples of the standard Steiner triple system generated by $\{1,2,4\}$.

$$A = \begin{bmatrix} 356 & 357 & 367 & 567 & 124 & 124 & 124 \\ 467 & 461 & 471 & 671 & 235 & 235 & 235 \\ 571 & 572 & 512 & 712 & 346 & 346 & 346 \\ 612 & 613 & 623 & 123 & 457 & 457 & 457 \\ 723 & 724 & 734 & 234 & 561 & 561 & 561 \\ 134 & 135 & 145 & 345 & 672 & 672 & 672 \\ 245 & 246 & 256 & 456 & 713 & 713 & 713 \end{bmatrix}.$$

A partition of the above triples into 7 STS(7)'s is given in the rows below:

$$\begin{array}{c} A_{71}, A_{12}, A_{23}, A_{44}, A_{35}, A_{56}, A_{67} \\ A_{11}, A_{22}, A_{33}, A_{54}, A_{45}, A_{66}, A_{77} \\ A_{21}, A_{32}, A_{43}, A_{64}, A_{55}, A_{76}, A_{17} \\ A_{31}, A_{42}, A_{53}, A_{74}, A_{65}, A_{16}, A_{27} \\ A_{41}, A_{52}, A_{63}, A_{14}, A_{75}, A_{26}, A_{37} \\ A_{51}, A_{62}, A_{73}, A_{24}, A_{15}, A_{36}, A_{47} \\ A_{61}, A_{72}, A_{13}, A_{34}, A_{25}, A_{46}, A_{57} \end{array}$$

Hence, we get 7 triple systems on G_1 if we relabel the elements of $\mathbb{Z}_7$ by the elements of G_1. Now we take the union of each block of i^{th} triple system with i^{th} element of G_2 and then repeat the same by interchanging the roles of G_1 and G_2. These blocks together with the blocks of the remaining 2 copies of a BIBD$(7, 4, 2)$ on each group, give the required 3-GDD$(7, 2, 4, 3, 1)$.

A question "Is it possible to partition the collection of all 3-subsets of the 7-set along with one copy of STS into 6 STSs?" arises naturally. The answer is yes. Below is a partition of the triples given in the first six columns of A above. The triples of each STS are given in the rows:

$$
\begin{array}{lllllll}
\{124 & 135 & 167 & 236 & 257 & 437 & 456\}, \\
\{124 & 136 & 157 & 237 & 256 & 435 & 467\}, \\
\{356 & 457 & 672 & 713 & 461 & 512 & 234\}, \\
\{357 & 346 & 561 & 672 & 471 & 123 & 245\}, \\
\{367 & 235 & 457 & 561 & 712 & 134 & 246\}, \\
\{567 & 235 & 346 & 612 & 724 & 145 & 713\}.
\end{array}
$$

Remark 5. With this partition it was moot to combine two or more copies of STS(7) to the set of all triples of the 7-set and partition into STSs. But the partition given after Matrix A is interesting as it does not have the second "added" STS as is. In fact, all 7 STSs include exactly three sets from the two "added" STSs. We think that this problem of partitioning collection of all subsets of a 7-set along with copies of an STS is interesting in its own right. Hence we discussed it in detail instead of just producing the blocks of a 3-GDD$(7, 2, 4, 3, 1)$ without the background.

As a 3-GDD$(14, 4, 1)$, a 3-$(7, 4, 4)$ and a 3-GDD$(7, 2, 4, 3, 1)$ exist, Theorem 3.3 implies that a 3-GDD$(7, 2, 4, \lambda_1, \lambda_2)$ exists for all odd values of λ_1 and λ_2. Hence, Lemma 4.1 implies that the necessary conditions are sufficient for the existence of a 3-GDD for $n = 7$.

5.2 $n = 13$

A 3-$(13, 3, 2)$ exists, in fact, there exists a partition of all four subsets of a 13-set into five 3-$(13, 3, 2)$'s. ([3], Page 100). Hence Lemma 1.3 implies that the necessary conditions are sufficient for the existence of a 3-GDD$(13, 2, 4, 2t, 2s)$ for all integers $s \geq 0$ and $t \geq 0$. Now Lemma 4.1 and remark after it, imply that the necessary conditions for the existence of a 3-GDD$(13, 2, 4, \lambda_1, \lambda_2)$ are sufficient.

5.3 $n = 19$

A trivial 3-$(19, 4, 16)$ can be partitioned into 4 3-$(19, 4, 4)'s$ ([3], Page 100), so a 3-$(19, 4, 4)$ exists. Hence, we apply Lemma 3.5 and Lemma 4.1 to prove that a 3-GDD$(19, 2, 4, \lambda_1, \lambda_2)$ exists except possibly when $\lambda_2 = 1$, and $\lambda_1 = 3$ or 7.

6 $n = 2^t + 1$, odd t

When t is odd, $2^t + 1 \equiv 3 \pmod 6$ otherwise $2^t + 1 \equiv 5 \pmod 6$. Hence in this section we are only interested in odd t. In this case, as $n \equiv 3 \pmod 6$, from Table 1, $\lambda_1 \equiv 0 \pmod 3$. Also, if $\lambda_1 \equiv 0 \pmod 6$, then λ_2 must be even, and if $\lambda_1 \equiv 3 \pmod 6$, then λ_2 must be odd. if $\lambda_1 \equiv 0 \pmod 6$, then a 3-GDD$(n, 2, 4, 6s, 2t)$ can be obtained by combining s copies of 3-$(n, 4, 6)$, for $n > 3$ on each of the groups and t copies a 3-GDD$(n, 2, 4, 0, 2)$.

If $\lambda_1 \equiv 3 \pmod 6$, then λ_2 is odd. Note that the construction of a 3-GDD(n,2,4,6a+3,6b+1) is enough, because a 3-GDD$(n, 2, 4, 6a+3, 6b+3)$ and a GDD$(n, 2, 4, 6a+3, 6b+5)$ can be obtained using a 3-GDD$(n, 2, 4, 6a+3, 6b+1)$ and a 3-GDD$(n, 2, 4, 0, 2)$. Hence now we construct a 3-GDD$(n, 2, 6a + 3, 6b + 1)$.

A 3-GDD$(n, 2, 4, 3, 5)$ can be constructed for any n, using 3-$(2n, 4, 3)$ and a 3-GDD$(n, 2, 4, 0, 2)$. Hence a 3-GDD$(n, 2, 4, 3, 2t+1)$ exists for all $t \geq 2$. As a consequence using a copies of a 3-$(n, 4, 6)$ one obtains a 3-GDD$(n, 2, 4, 6a+3, 6b+1)$, for all $a \geq 0$ and positive integers $b \geq 1$. Hence the necessary conditions are sufficient for the existence of a 3-GDD$(n = 2^{2s+1}+1, 2, 4, \lambda_1, \lambda_2)$ except possibly a 3-GDD$(n = 2^{2s+1} + 1, 2, 4, 3, 1)$ for $s \geq 1$. We deal with $s = 0$, i.e., $n = 3$, below.

6.1 $n = 3$

A 3-GDD$(3, 2, 4, \lambda_1, 0)$ does not exist as the group size is smaller than the block size. When $n = 3$, to satisfy the condition on λ_1 the whole group has to be a part of λ_1 blocks, forcing $\lambda_2 \geq \lambda_1/3$. Clearly the minimum λ_2 is attained if blocks are formed by $\lambda_1/3$ copies of $G_1 \cup \{a\}$ for all $a \in G_2$ and $\lambda_1/3$ copies of $G_2 \cup \{a\}$ for all $a \in G_1$. Using $\frac{\lambda_2 - \frac{\lambda_1}{3}}{2}$ copies of a

3-GDD$(3, 2, 4, 0, 2)$ and a 3-GDD$(3, 2, 4, \lambda_1, \lambda_1/3)$, we conclude that the necessary conditions are sufficient for the existence of 3-GDD$(3, 2, 4, \lambda_1, \lambda_2)$. Note that the parity conditions imply that $\lambda_2 - \frac{\lambda_1}{3}$ is even.

7 Summary

We define a 3-GDD and prove that the necessary conditions given in the paper are sufficient for the existence of 3-GDDs with block size 4 for all cases except when $n \equiv 1, 3 \pmod 6$, $n \neq 3, 7, 13$ and $\lambda_1 > \lambda_2$. Also we show that the necessary conditions are sufficient for the existence of a 3-GDD$(n = 2^{2s+1} + 1, 2, 4, \lambda_1, \lambda_2)$ except possibly a 3-GDD$(n = 2^{2s+1} + 1, 2, 4, 3, 1)$ where s is a positive integer.

References

[1] H.J.R. Abel, G.Ge and J.Yin, Resolvable and nearly resolvable designs, in *Handbook of Combinatorial Design Theory, Second Edition*, edited by C.J. Colbourn and J.H. Dinitz, (2007), 124-132

[2] C.J. Colbourn, Triple systems, in *Handbook of Combinatorial Design Theory, Second Edition*, edited by C.J. Colbourn and J.H. Dinitz, (2007), 58-79.

[3] G.B. Khosrovshahi and R. Laue, t-Designs with $t \geq 3$, in *Handbook of Combinatorial Design Theory, Second Edition*, edited by C.J. Colbourn and J.H. Dinitz, (2007), 79-101.

[4] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, 1997.

[5] D.R. Stinson, *Combinatorial Designs: Constructions and Analysis*, Springer-Verlag, New York, 2004.

[6] W.D. Wallis, *Introduction to Combinatorial Designs, Second Edition*, CRC Press, 2011.