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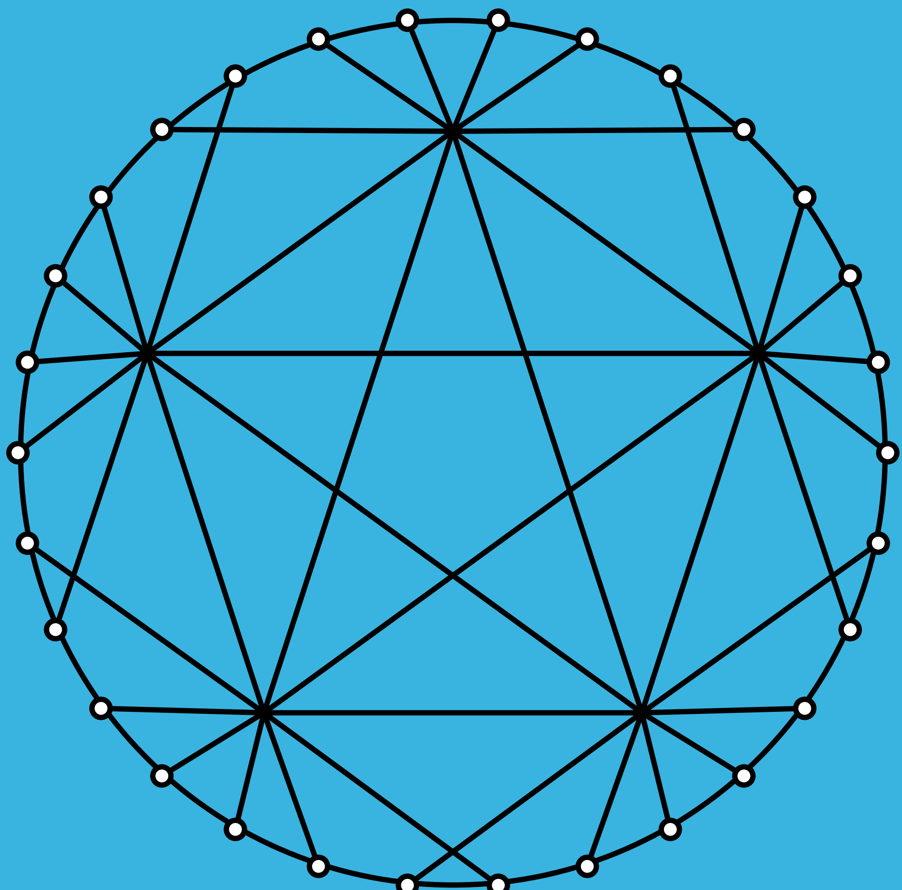

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Enumerating shortest paths and determining edge betweenness centrality in cartesian products of paths and cycles

Alexander Strang[1], Oliver Haynes[2], Rigoberto Flórez[3], and Darren A. Narayan[2,*]

[1] *Case Western Reserve University, 10900 Euclid Ave, Cleveland, OH 44106*
[2] *Rochester Institute of Technology, 1 Lomb Memorial Drive, Rochester, NY 14623*
[3] *The Citadel, 171 Moultrie St, Charleston, SC 29409*
[*] dansma@rit.edu

Abstract: It is well known that the multinomial theorem can be used to enumerate the number of shortest paths in an n-dimensional grid. We explore a property related to shortest paths known as edge betweenness centrality. This property measures the frequency at which an edge appears on a shortest path between two vertices. In this paper we calculate the edge betweenness for all edges in the Cartesian product of paths and cycles. This requires determination of the frequency at which an edge appears in rectangular and torodial prisms.

1 Introduction

In biological, transportation, and social networks, certain vertices and edges play a vital role in the connection of a network. This value can be quantified by *betweenness centrality*, which is the frequency at which a vertex or an edge appears on a shortest path between two distinct vertices.

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In 1977, Freeman defined the *betweenness centrality* of a vertex v as follows.

Definition 1 (Freeman)**.** *The betweenness centrality of a vertex v is denoted $bc(v)$ and is the frequency at which v appears on a shortest path between two other distinct vertices x and y . Let σ_{xy} be the number of shortest paths between distinct vertices x and y, and let $\sigma_{xy}(v)$ be the number of shortest paths between x and y that contain v. Then $bc(v) = \sum_{x,y} \frac{\sigma_{xy}(v)}{\sigma_{xy}}$ (for all distinct vertices v, x, and y).*

In 2002, Girvan and Newman introduced an edge version of betweenness centrality.

Definition 2 (Girvan-Newman)**.** *The betweenness centrality of an edge e is denoted $bc'(e)$ measures the frequency at which e appears on a shortest path between two other distinct vertices x and y. Let σ_{xy} be the number of shortest paths between distinct vertices x and y, and let $\sigma_{xy}(e)$ be the number of shortest paths between x and y that contain e. Then $bc'(e) = \sum_{x,y} \frac{\sigma_{xy}(e)}{\sigma_{xy}}$ (for all distinct vertices x, and y).*

Edges with a high edge betweenness centrality act as bridges between different subgraphs in a graph. Thus severing these edges gives an effective strategy for partitioning the vertex set of a graph into different parts. This idea was used by Girvan and Newman for graph partitioning and detecting communities in social networks. In addition, this concept appears in studies of social networks and neuroscience [3], [4], and [1].

In this paper we use tools from combinatorics to investigate the edge betweenness for Cartesian products of paths and cycles. For a given graph G, we will use $V(G)$ to denote the vertex set of G and $E(G)$ to denote the edge set of G. Given two graphs H and K, with vertex sets $V(H)$ and $V(K)$ the Cartesian product $G = H \times K$ is a graph where $V(G) = \{(u_i, v_j)$ where $u_i \in V(H)$ and $v_j \in V(K)\}$, and $E(G) = \{(u_i, v_j), (u_k, v_l)\}$ if and only if $i = k$ and v_j and v_l are adjacent in K or $j = l$ and u_i and u_k are adjacent in H. For any undefined notation see the textbook by D. B. West [6].

2 Edge betweenness centrality

We begin with an elementary lemma involving cut-edges.

Lemma 3. *Let e be a cut-edge of a graph G where $G - e$ has two vertex disjoint subgraphs H and K. Then $bc'(e) = |V(H)| \cdot |V(K)|$.*

Proof. The proof is straightforward. Every shortest path between any vertex in H and any vertex in K will contain e. It easy to see that that no shortest path entirely within either H or K will use e.

□

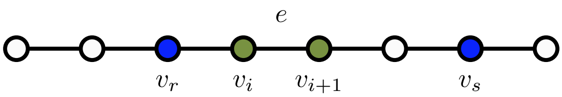


Figure 1: Betweenness edge centrality of a path.

Proposition 4. *Let P_n be a path on vertices $v_1, v_2, ..., v_n$ and edges $v_1v_2, v_2v_3, ..., v_{n-1}v_n$. Then $bc'(v_iv_{i+1}) = 2(i)(n-i)$.*

Proof. We apply Lemma 3 noting that every edge is a cut-edge. We observe that all shortest paths between v_r and v_s will contain v_iv_{i+1} if and only $r \leq i$ and $i+1 \leq s$. Hence to determine $bc'(v_iv_{i+1})$ we count the pairs of vertices (v_r, v_s) where $r \leq i$ and $s \geq n-i$. Finally, doubling to account for both directions gives $bc(v_iv_{i+1}) = (i)(n-i)$. □

We next recall the multinomial theorem from introductory combinatorics.

Lemma 5. *The number of different permutations of n objects, where there are k_i indistinguishable objects of type i, where $\sum_{i=1}^{m} k_i = n$ is*

$$\binom{n}{k_1, k_2, ..., k_n} = \frac{n!}{k_1!k_2! \cdots k_m!}.$$

Lemma 6. *Let $G = P_{k_1} \times P_{k_2}$. If (a_1, a_2) and (b_1, b_2) are vertices where $0 \leq a_1 \leq b_1 \leq k_1$, $0 \leq a_2 \leq b_2 \leq k_2$ then number of shortest paths between (a_1, a_2) and (b_1, b_2) is*

$$\binom{b_1 - a_1 + b_2 - a_2}{b_1 - a_1} = \frac{(b_1 - a_1 + b_2 - a_2)!}{(b_1 - a_1)!(b_2 - a_2)!}.$$

Proof. A shortest path between (a_1, a_2) and (b_1, b_2) can be expressed as a sequence of $(b_1 - a_1) + (b_2 - a_2)$ moves where $b_1 - a_1$ are moves that are in the 'east' direction and $b_2 - a_2$ moves are in the 'north' direction. The total number of shortest paths between (a_1, a_2) and (b_1, b_2) is equivalent to the number of 'words' with $(b_1 - a_1) + (b_2 - a_2)$ letters where $b_1 - a_1$ is the number of Es and $b_2 - a_2$ is the number of Ns (E and N correspond to an east step and a north step, respectively). Then by the definition of the binomial coefficient,

$$\binom{(b_1 - a_1) + (b_2 - a_2)}{b_1 - a_1} = \frac{((b_1 - a_1) + (b_2 - a_2))!}{(b_1 - a_1)!(b_2 - a_2)!}.$$

This completes the proof. $\square$

Lemma 7. *Let $G = P_{k_1} \square P_{k_2}$. Let (a_1, a_2),(b_1, b_2), and (x_1, x_2) be vertices of G where $0 \leq a_1 \leq x_1 \leq b_1 \leq k_1$, $0 \leq a_2 \leq x_2 \leq b_2 \leq k_2$. Then the number of shortest paths between (a_1, a_2) and (b_1, b_2) that contain (x_1, x_2) is*

$$\binom{x_1 - a_1 + x_2 - a_2}{x_1 - a_1} \cdot \binom{b_1 - x_1 + b_2 - x_2}{b_1 - x_1} = \frac{(x_1 - a_1 + x_2 - a_2)!}{(x_1 - a_1)!(x_2 - a_2)!} \cdot \frac{(b_1 - x_1 + b_2 - x_2)!}{(b_1 - x_1)!(b_2 - x_2)!}. \tag{1}$$

Proof. We can use the method from the previous proof to calculate the number of shortest paths from (a_1, a_2) to (x_1, x_2), and then multiplying this number by the number of shortest paths between (x_1, x_2) and (b_1, b_2). $\square$

Let $u_i = (i_1, i_2)$ and $u_j = (j_1, j_2)$ be vertices of $G = P_{k_1} \square P_{k_2}$. We define $R(u_i, u_j) \subseteq G$ to be the rectangle determined by the corner points $u_i = (i_1, i_2)$ and $u_j = (j_1, j_2)$. For example, if $u_i = (i_1, i_2)$ and $u_j = (j_1, j_2)$ satisfy that $i_1 \leq j_1$ and $i_2 \leq j_2$, then $R(u_i, u_j) = \{(k_1, k_2) \,|\, i_r \leq k_r \leq j_r,\ r = 1, 2\}$. Suppose that e is an edge of G with vertices (v_1, v_2) and $(v_1 + 1, v_2)$. We define

$$C_2(u_i, u_j, e) = \frac{(|i_1 - v_1| + |i_2 - v_2|)!}{|i_1 - v_1|!\,|i_2 - v_2|!} \cdot \frac{(|v_1 + 1 - j_1| + |i_2 - v_2|)!}{|v_1 + 1 - j_1|!\,(|i_2 - v_2|)!} \cdot \frac{|i_1 - j_1|!\,|i_2 - j_2|!}{(|i_1 - j_1| + |i_2 - j_2|)!}. \tag{2}$$

and $F_2(u_i, u_j, e) = \begin{cases} C_2(u_i, u_j, e) & \text{if } e \in R(u_i, u_j); \\ 0 & \text{otherwise.} \end{cases}$

The two dimensional case is shown in Figure 2.

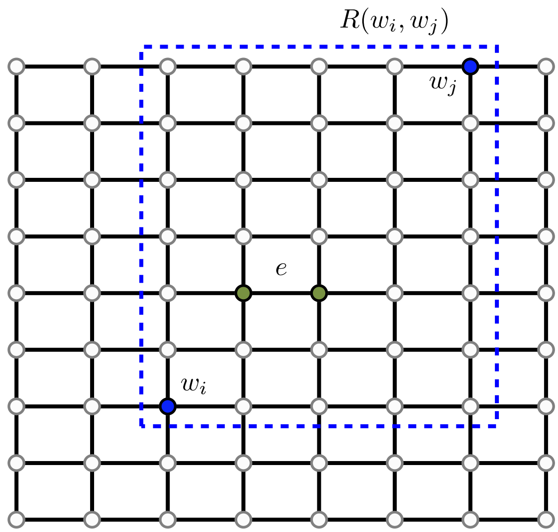


Figure 2: The two dimensional case

Proposition 8. *Let $G = P_{k_1} \square P_{k_2}$. If e is an edge of G with vertices (v_1, v_2) and $(v_1 + 1, v_2)$, then*

$$bc'(e) = \sum_{w_i, w_j \in V(G)} F(w_i, w_j, e).$$

Proof. We suppose that the graph $G = P_{k_1} \square P_{k_2}$ is represented by the grid

$$G = \{(x, y) |\ 0 \leq x \leq k_1 \text{ and } 0 \leq y \leq k_2 \text{ with } x, y \in \mathbb{Z}\}.$$

Let $w_i = (i_1, i_2)$ and $w_j = (j_1, j_2)$ be vertices of G. We prove the proposition for $i_1 \leq j_1$ and $i_2 \leq j_2$ and note the other cases are similar. Let e be an edge with vertices $v = (v_1, v_2)$ and $v' = (v_1 + 1, v_2)$. To find the betweenness edge centrality for e, we count the total number of shortest paths from w_i to w_j passing through the edge e. By Lemma 7, the number of shortest paths from w_i to v is

$$\frac{(|i_1 - v_1| + |i_2 - v_2|)!}{|i_1 - v_1|!\,|i_2 - v_2|!}.$$

By Lemma 7, the number of shortest paths from v' to w_j is

$$\frac{(|v_1 + 1 - j_1| + |i_2 - v_2|)!}{|v_1 + 1 - j_1|!\,(|i_2 - v_2|)!}.$$

The number of shortest paths from w_i to w_j passing through e is

$$\frac{(|i_1 - v_1| + |i_2 - v_2|)!}{|i_1 - v_1|!\,|i_2 - v_2|!} \cdot \frac{(|v_1 + 1 - j_1| + |i_2 - v_2|)!}{|v_1 + 1 - j_1|! \cdot (|i_2 - v_2|)!}.$$

By Lemma 7 the total number of shortest paths between w_i and w_j is $\frac{(|i_1-j_1|+|i_2-j_2|)!}{|i_1-j_1|!|i_2-j_2|!}$.

Hence

$$\begin{aligned} C_2(w_i, w_j, e) &= \frac{(|i_1 - v_1| + |i_2 - v_2|)!}{|i_1 - v_1|!\,|i_2 - v_2|!} \\ &\qquad \cdot \frac{(|v_1 + 1 - j_1| + |i_2 - v_2|)!}{|v_1 + 1 - j_1|! \cdot (|i_2 - v_2|)!} \cdot \frac{|i_1 - j_1|!\,|i_2 - j_2|!}{(|i_1 - j_1| + |i_2 - j_2|)!}. \\ &= \frac{\binom{|i_1-v_1|+|i_2-v_2|}{|i_1-v_1|}\binom{(|v_1+1-j_1|+|i_2-v_2|)}{|v_1+1-j_1|}}{\binom{(|i_1-j_1|+|i_2-j_2|)}{|i_1-j_1|}}. \end{aligned}$$

Then $F_2(w_i, w_j, e) = \begin{cases} C_2(w_i, w_j, e) & \text{if } e \in R(w_i, w_j); \\ 0 & \text{otherwise.} \end{cases}$

Summing over all vertices of G gives

$$bc'(e) = \sum_{w_i, w_j \in V(G)} F(w_i, w_j, e).$$

This completes the proof. $\square$

We can use the same idea to determine the number of shortest paths on a grid with n dimensions.

We next generalize Lemma 6. We note this was previously proven by Handa and Mohanty [5].

Lemma 9 (Handa and Mohanty)**.** *Let $G = P_{k_1} \Box P_{k_2} \Box \cdots \Box P_{k_n}$. Then for all $0 \leq a_i \leq x_i \leq b_i \leq k_i$ the number of shortest paths between*

$(a_1, a_2, ..., a_n)$ *and* $(b_1, b_2, ..., b_n)$ *is*

$$\frac{\left(\sum_{i=1}^{n} |a_i - b_i|\right)!}{\prod_{i=1}^{n} (|a_i - b_i|)!}.$$

Proof. Without loss of generality assume that $a_i \leq b_i$ for all $1 \leq i \leq n$. The total number of edges in a shortest path is $\sum_{i=1}^{n} |a_i - b_i|$. We then count the number of permutations using Lemma 5. $\square$

Let $G = P_{k_1} \square P_{k_2} \square \cdots \square P_{k_n}$ be n dimensional grid. The three dimensional case is given in Figure 3.

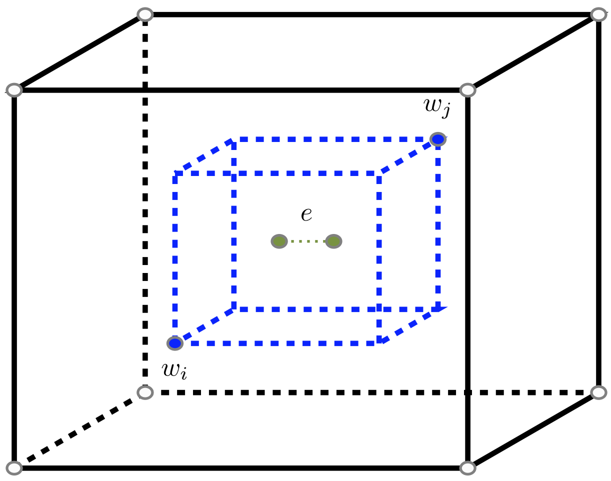


Figure 3: The three dimensional case

Our next theorem gives the edge betweenness centrality of each edge in an n -dimensional grid. Let $u_i = (i_1, i_2, ..., i_n)$ and $u_j = (j_1, j_2, ..., j_n)$ be vertices of G. Let e be an edge of G with vertices $(v_1, v_2, ..., v_n)$ and $(v_1 + 1, v_2, ..., v_n)$. We define $R(u_i, u_j) \subseteq G$ to be the n dimensional rectangular prism determined by the corner points u_i and u_j. For example, if u_i and u_j satisfy that $i_r \leq j_r$ where $r = 1, 2, ..., n$, then rectangular prism is defined by

$$R(u_i,u_j)=\{(k_1,k_2,...,k_n)\,|i_r\le k_r\le j_r,\ r=1,2,...,n\}$$

Then we define

$$C_n(u_i,u_j,e)=\frac{\left(\sum_{t=1}^{n}|i_t-v_t|\right)!}{\prod_{t=1}^{n}(|i_t-v_t|)!}\cdot\frac{\left(|v_1+1-j_1|+\sum_{t=2}^{n}|i_t-v_t|\right)!}{|v_1+1-j_1|!\cdot\prod_{t=2}^{n}(|i_t-v_t|)!}\cdot\frac{\prod_{t=1}^{n}|i_t-j_t|!}{\left(\sum_{t=1}^{n}|i_t-j_t|\right)!}$$

and

$$F_n(u_i,u_j,e)=\begin{cases}C_n(u_i,u_j,e) & \text{if} \qquad e\in R(u_i,u_j);\\ 0 & \text{otherwise.}\end{cases}$$

Now we are in position to prove our main result.

Theorem 10. *Let $G=P_{k_1}\square P_{k_2}\square\cdots\square P_{k_n}$. If e is an edge in G with vertices $(v_1,v_2,...,v_n)$ and $(v_1+1,v_2,...,v_n)$, then*

$$bc'(e)=\sum_{w_i,w_j\in V(G)}F(w_i,w_j,e).$$

Proof. Let $G=P_{k_1}\square P_{k_2}\square\cdots\square P_{k_n}$. To find the betweenness edge centrality for e, first of all we count the total number of shortest paths passing through the edge $e:\{(v_1,v_2,...,v_n),(v_1+1,v_2,...,v_n)\}$. Let $x=(v_1,v_2,...,v_n)$ and $y=(v_1+1,v_2,...,v_n)$ and let $w_i=(i_1,i_2,...,i_n)$ and $w_j=(j_1,j_2,...,j_n)$ be vertices of G. We prove the proposition for $i_r\le j_r$ where $r=1,2,...,n$, and note that the other cases are similar. The number of shortest paths from w_i to x is

$$\frac{\left(\sum_{t=1}^{n}|i_t-v_t|\right)!}{\prod_{t=1}^{n}(|i_t-v_t|)!}.$$

The number of shortest paths from y to w_j is

$$\frac{\left(|v_i+1-j_1|+\sum_{t=2}^{n}|i_t-v_t|\right)!}{|v_i+1-j_1|!\cdot\prod_{t=2}^{n}(|i_t-v_t|)!}.$$

Combining the above two quantities gives that the total number of shortest paths from w_i to w_j is

$$\frac{\left(\sum_{t=1}^{n} |i_t - j_t|\right)!}{\prod_{t=1}^{n} |i_t - j_t|!}.$$

The rest of the proof follows similarly to the proof of Proposition 8. $\square$

2.1 Edge betweenness of the cartesian product of cycles

We can then extend our results by replacing

$$P_{k_1} \Box P_{k_2} \Box \cdots \Box P_{k_n}$$

with

$$C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_n}.$$

The main difference between these two families is that with cycles of cycles, there are 2^n different rectangular prisms that must be considered. To see this note that we have a choice for each dimension. We consider each distinct $R(i, j)$ to be a word, where each letter corresponds to a dimension. Then let T refer to travelling from i_m to j_m without crossing over the modular boundary "looping" and let L refer to travelling from i_m to j_m by crossing the modular boundary. Then for an n dimensional product of cycles, there are as many $R(i, j)$ as there are unique words of length n made up entirely of T and L. Therefore there are 2^n distinct $R(i, j)$.

Example 11. *Let $G = C_8 \Box C_6$. The number of shortest paths between $(2, 2)$ and $(7, 4)$ is $\frac{5!}{3!2!}$.*

The vertices where we seek a shortest path is shown in Figure 4 (a) and the three rectangular prisms are shown in Figure 4 (b), (c), and (d). We consider the shortest paths from vertex $(2, 2)$ to vertex $(7, 4)$. Traversing the rectangle R1 requires 7 steps, traversing rectangle R2 requires 5 steps, and traversing rectangle R3 requires 8 steps. Since R2 has the smallest perimeter, we let $R'((2, 2), (7, 4)) = R2((2, 2), (7, 4))$.

We consider the graph $G = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_n}$. When $n = 1$ we have two prisms and we gain a new prism for each dimension. Hence in the case

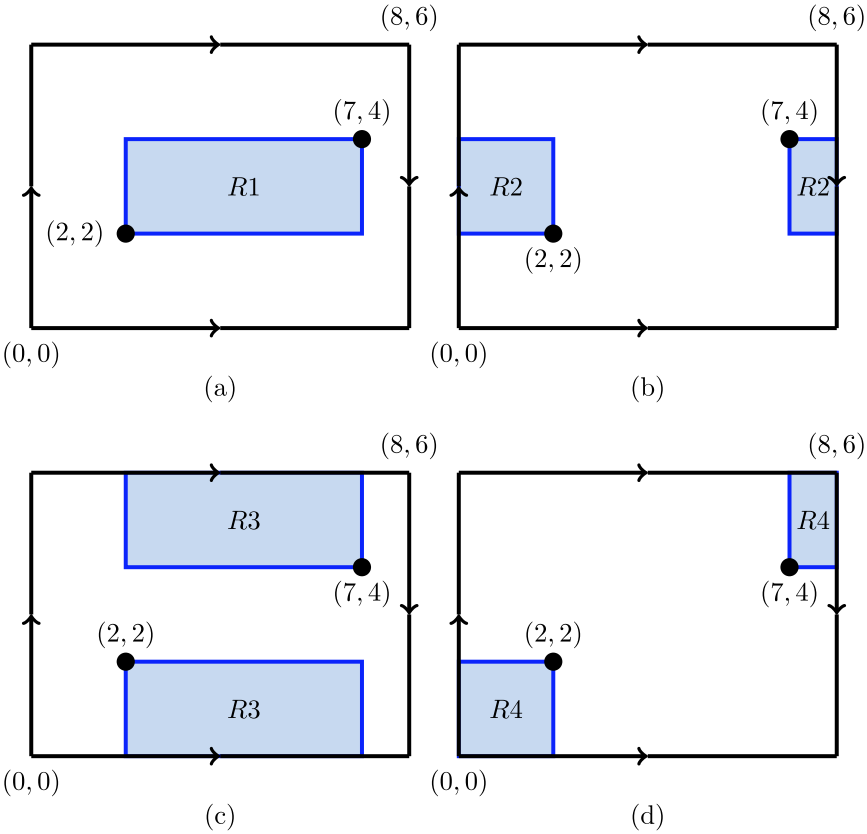


Figure 4: The rectangular prisms for finding shortest paths between $(2, 2)$ and $(7, 4)$ in the torus $C_8 \square C_6$.

with the Cartesian product of n cycles we will have 2^n rectangular prisms to consider, and we seek one of smallest perimeter.

We define the multiplicity M to be the number of dimensions m such that for $i_m < j_m$ the distance $j_m - i_m = \bmod k_m(i_m - j_m)$. Then we have the following result.

Lemma 12. *Let* $G = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_n}$. *Then for all* $0 \leq a_i \leq x_i \leq b_i \leq$

k_i the number of shortest paths between $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ is

$$2^M \frac{\left(\sum_{i=1}^{n} \min\{(a_i - b_i) \bmod k_i, (b_i - a_i) \bmod k_i\}\right)!}{\prod_{i=1}^{n} (\min\{(a_i - b_i) \bmod k_i, (b_i - a_i) \bmod k_i\})!}.$$

Proof. For each i we consider the two types of paths that go in opposite directions. The total number of edges in a shortest path is $\sum_{i=1}^{n} \min\{(a_i - b_i) \bmod k_i, (b_i - a_i) \bmod k_i\}$. We then count the number of permutations using Lemma 5. $\square$

Let $G = C_{k_1} \square C_{k_2} \square \cdots \square C_{k_n}$ be the Cartesian product of n cycles. Our next theorem gives the edge betweenness centrality of each edge in the Cartesian product of n cycles. Let $w_i = (i_1, i_2, ..., i_n)$ and $w_j = (j_1, j_2, ..., j_n)$ be vertices of G and let e be an edge of G with vertices $(v_1, v_2, ..., v_n)$ and $(v_1 + 1, v_2, ..., v_n)$. We define $R(w_i, w_j) \subseteq G$ to be an n dimensional rectangular prism determined by w_i and w_j. For example, if $i_r \leq j_r$ for $r = 1, 2, ..., n$, then $R(w_i, w_j) = \{(k_1, k_2, ..., k_n) \,|\, i_r \leq k_r \leq j_r,\ r = 1, 2, ..., n\}$.

Let $R'(w_i, w_j)$ be a rectangular prism with smallest perimeter. Then we define

$$C_n(w_i, w_j, e) = \frac{\left(\sum_{t=1}^{n} |i_t - v_t|\right)!}{\prod_{t=1}^{n} (|i_t - v_t|)!} \cdot \frac{\left(|v_1 + 1 - j_1| + \sum_{t=2}^{n} |i_t - v_t|\right)!}{|v_1 + 1 - j_1|! \cdot \prod_{t=2}^{n} (|i_t - v_t|)!} \cdot \frac{\prod_{t=1}^{n} |i_t - j_t|!}{\left(\sum_{t=1}^{n} |i_t - j_t|\right)!} \tag{3}$$

and

$$F_n(w_i, w_j, e) = \begin{cases} C_n(w_i, w_j, e) & \text{if } e \in R'(w_i, w_j); \\ 0 & \text{otherwise.} \end{cases}$$

Now we are in position to prove our main result.

Theorem 13. *Let $G = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_n}$. If e is an edge in G with vertices $(v_1, v_2, ..., v_n)$ and $(v_1 + 1, v_2, ..., v_n)$, then*

$$bc'(e) = \sum_{w_i, w_j \in V(G)} F(w_i, w_j, e).$$

Proof. Let $G = C_{k_1} \Box C_{k_2} \Box \cdots \Box C_{k_n}$. To find the betweenness edge centrality for e, we count the total number of shortest paths passing through the edge $e : \{(v_1, v_2, ..., v_n), (v_1 + 1, v_2, ..., v_n)\}$. Let $x = (v_1, v_2, ..., v_n)$ and $y = (v_1 + 1, v_2, ..., v_n)$ and let $w_i = (i_1, i_2, ..., i_n)$ and $w_j = (j_1, j_2, ..., j_n)$ be vertices of G. The number of shortest paths from w_i to x is

$$\frac{\left(\sum_{t=1}^{n} \min\{(i_t - v_t)k_i, (v_t - i_t) \bmod k_i\right)!}{\prod_{t=1}^{n} (\min\{(i_t - v_t)k_i, (v_t - i_t) \bmod k_i)!}.$$

The number of shortest paths from y to w_j is

$$\frac{\begin{array}{l}\Bigg(\min\{(v_1 + 1 - j_1) \bmod k_1, (j_1 - (v_1 + 1)) \bmod k_1\} \\ \qquad\qquad + \sum_{t=2}^{n} \min\{(i_t - v_t) \bmod k_t, (v_t - i_t) \bmod k_t\}\Bigg)!\end{array}}{\begin{array}{l}\min\{(v_1 + 1 - j_1) \bmod k_1, (j_1 - (v_i + 1)) \bmod k_1\}! \\ \qquad\qquad \cdot \prod_{t=2}^{n} \left(\min\{(i_t - v_t) \bmod k_t, (v_t - i_t) \bmod k_t\}\right)!\end{array}}.$$

The total number of shortest paths from w_i to w_j is

$$\frac{\left(\sum_{t=1}^{n} \min\{(i_t - j_t) \bmod k_t, (j_t - i_t) \bmod k_t\}\right)!}{\prod_{t=1}^{n} \min\{(i_t - j_t) \bmod k_t, (j_t - i_t) \bmod k_t\}!}.$$

This completes the proof. $\square$

Recall that the graph $G = P_{k_1} \Box P_{k_2}$ may be represented by the grid

$$G = \{(x, y)|\ 0 \leq x \leq k_1 \text{ and } 0 \leq y \leq k_2 \text{ with } x, y \in \mathbb{Z}\}.$$

We define the graph W_G be the graph that results from G adding "diagonal" edges of the form (x, y)–$(x+1, y+1)$ where $0 \leq x < k_1$ and $0 \leq y < k_2$. We note that in a shortest path the number of these diagonal edges should be maximized.

Lemma 14. *Let $w_1 = (i_1, i_2)$ and $w_2 = (j_1, j_2)$ be vertices in W_G with $i_1 \leq j_1$ and $i_2 \leq j_2$. Let $n = \min\{|i_1 - j_1|, |i_2 - j_2|\}$ and $m = ||i_1 - j_1| - |i_2 - j_2||$. The number of shortest paths from w_1 to w_2 is $\binom{m+n}{n}$.*

Proof. Let P be a shortest path from $w_1 = (i_1, i_2)$ and $w_2 = (j_1, j_2)$. Note that when a path includes diagonal edges of the form $NE := (x, y)$–$(x+1, y+1)$ the path becomes shorter than if it only contains edges of either forms $E := (x, y)$–$(x, y+1)$ or $N := (x, y)$–$(x+1, y)$. So, the maximum number of edges of the form NE that P may have is $n = \min\{|i_1 - j_1|, |i_2 - j_2|\}$. When P reaches the maximum number of edges of the form NE, then P will have only edges of either forms (but not both) E or N. The number of edges of the form E or N that P may have after reaching the maximum number of edges of the form NE is $m = ||i_1 - j_1| - |i_2 - j_2||$. Then the number of shortest paths from w_1 to w_2 is $\binom{m+n}{n}$. □

Lemma 15. *Let $w_1 = (i_1, i_2)$ and $w_2 = (j_1, j_2)$ be vertices in W_G and let e be an edge in W_G with vertices (v_1, v_2) and $(v_1 + 1, v_2)$ where $i_1 \leq v_1 < j_1$ and $i_2 \leq v_2 \leq j_2$. If $n = \min\{|i_1 - v_1|, |i_2 - v_2|\}$, $n' = \min\{|j_1 - v_1 - 1|, |j_2 - v_2|\}$, $m = ||i_1 - v_1| - |i_2 - v_2||$, $m' = ||j_1 - v_1 - 1| - |j_2 - v_2||$, then the number of shortest paths from w_1 to w_2 containing e is $\binom{m+n}{n}\binom{m'+n'}{n'}$.*

Proof. This follows from Lemma 14. □

The proof of the following lemma is similar to the proof of Lemma 7.

Lemma 16. *Let $w_1 = (i_1, i_2)$ and $w_2 = (j_1, j_2)$ be vertices in W_G and let e be an edge in W_G with vertices (v_1, v_2) and $(v_1 + 1, v_2)$ where $i_1 \leq v_1 < j_1$ and $i_2 > v_2 > j_2$. If $n_1 = i_1 - v_1$, $n_2 = i_2 - v_2$, $n_3 = v_1 + 1 - j_1$ and $n_4 = v_2 - j_2$, then the number of shortest paths from w_1 to w_2 containing e is $\binom{n_1+n_2}{n_1}\binom{n_3+n_4}{n_3}$.*

Let $w_i = (i_1, i_2)$ and $w_j = (j_1, j_2)$ be vertices in W_G. Let $e = (v_1, v_2) - (v_1 + 1, v_1)$ be an edge in W_G. If $i_1 \leq v_1 < j_1$ and $i_2 \leq v_2 \leq j_2$, we define $C_d(w_i, w_j, e) = \binom{m+n}{n}\binom{m'+n'}{n'}$ where $n = \min\{|i_1 - v_1|, |i_2 - v_2|\}$, $n' = \min\{|j_1 - v_1 - 1|, |j_2 - v_2|\}$, $m = ||i_1 - v_1| - |i_2 - v_2||$, $m' = ||j_1 - v_1 - 1| - |j_2 - v_2||$. If $i_1 \leq v_1 < j_1$ and $i_2 > v_2 > j_2$, then we define $C_u(w_i, w_j, e) = \binom{m+n}{n}\binom{m'+n'}{n'}$ where $n_1 = i_1 - v_1$, $n_2 = i_2 - v_2$, $n_3 = v_1 + 1 - j_1$ and $n_4 = v_2 - j_2$.

We define $R_W(w_i, w_j) \subseteq W_G$ to be the rectangle determined by the corner points w_i and w_j. Hence we have:

$$F_R(w_i, w_j, e) = \begin{cases} C_d(w_i, w_j, e) & \text{if } e \in R_W(w_i, w_j),\ i_1 < j_1 \text{ and } i_2 \leq j_2; \\ C_u(w_i, w_j, e) & \text{if } e \in R_W(w_i, w_j),\ i_1 < j_1 \text{ and } i_2 > j_2; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 17. *Let $G = P_{k_1} \square P_{k_2}$. If e is an edge in W_G with vertices (v_1, v_2) and $(v_1 + 1, v_2)$, then*

$$bc'(e) = \sum_{w_i, w_j \in V(W_G)} F_R(w_i, w_j, e).$$

Proof. Let $w_i = (i_1, i_2)$ and $w_j = (j_1, j_2)$ be vertices of W_G. For this prove we consider three cases.

Case 1. It easy to see that every shortest path connecting w_i to w_j is in the rectangle $R_W(w_i, w_j)$ (defined by w_i and w_j). Therefore, if $e = (v_1, v_2) - (v_1 + 1, v_1)$ does not belong to $R_W(w_i, w_j)$, then every path from w_i to w_j passing through e is not a shortest path. Thus, $F_R(w_i, w_j, e) = 0$.

Case 2. Suppose that $e \in R_W(w_i, w_j)$ with $i_1 \leq v_1 \leq j_1$ and $i_2 < v_1 + 1 \leq j_2$. From Lemma 15 the total number of shortest paths from w_i to w_j passing through e is $\binom{m+n}{n}\binom{m'+n'}{n'}$, where $n = \min\{|i_1 - v_1|, |i_2 - v_2|\}$, $n' = \min\{|j_1 - v_1 - 1|, |j_2 - v_2|\}$, $m = ||i_1 - v_1| - |i_2 - v_2||$, $m' = ||j_1 - v_1 - 1| - |j_2 - v_2||$.

Case 3. Suppose that $e \in R_W(w_i, w_j)$ with $i_1 \leq v_1 < j_1$ and $i_2 > v_2 > j_2$. From Lemma 16 the total number of shortest paths from w_i to w_j passing e is $\binom{n_1+n_2}{n_1}\binom{n_3+n_4}{n_3}$ where $n_1 = i_1 - v_1$, $n_2 = i_2 - v_2$, $n_3 = v_1 + 1 - j_1$ and $n_4 = v_2 - j_2$.

Summing over all vertices of W_G yields $bc'(e) = \displaystyle\sum_{w_i, w_j \in V(W_G)} F_R(w_i, w_j, e)$.

This completes the proof. $\square$

3 Asymptotics

In this section we investigate the asymptotic behavior of two types of edges: edges that are incident to a corner and edges that are incident to a center vertex. These appear to be the two types of edges with the smallest and largest edge betweenness centrality values. See Figures 5,6, and 7.

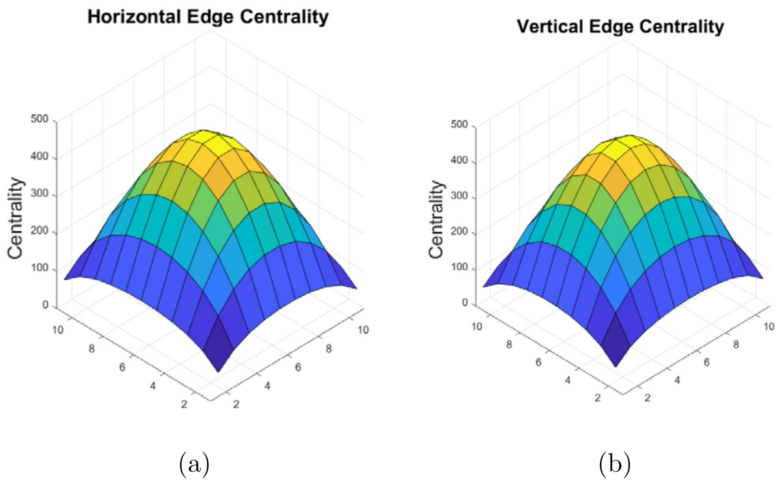
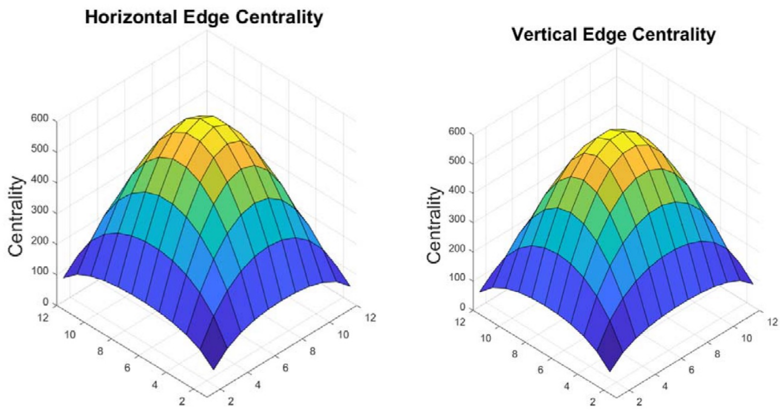


Figure 5: Edge betweenness centrality of edges in a 11×11 grid.

We next explore the asymptotic behavior of corner and central edges in a two dimensional lattice (or grid graph). In the following proposition we extract the two-dimensional case from Theorem 10.



(a) (b)

Figure 6: Edge betweenness centrality of edges in a 12×12 grid.

Proposition 18. *Let $H = P_n \times P_n$. Suppose that the edge e_{cr} with vertices $\{1, 2\}$ is incident to a corner vertex of H and that the edge e_{cn} with vertices $\{a, b\}$ is incident to a center vertex of H, where*

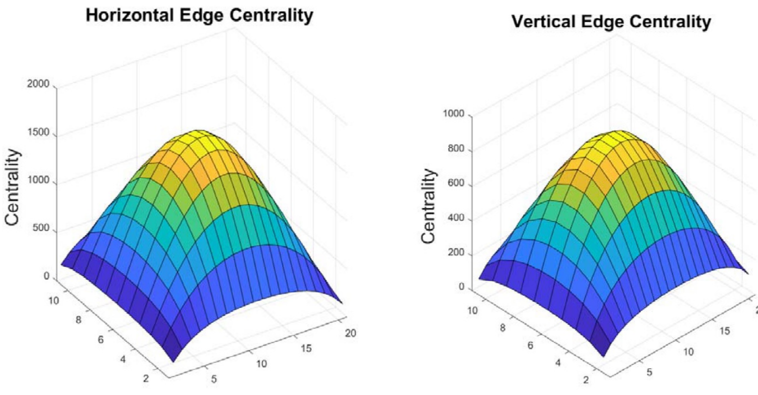
1. $a = (n-1)^2/2$ *and* $b = \left(n^2+1\right)/2$, *if n is even and*
2. $a = \lfloor (n-2)/2 \rfloor (n+1) + 1$ *and* $b = a + n$, *if n is odd, then*

   (a) $$bc'(e_{cr}) = 2\sum_{j=1}^{n-1}\left(\sum_{i=1}^{n-1}\frac{1}{\binom{i+j}{i}}\right) + n^2 - 1.$$

   (b) *If n is odd then,*
   $$bc'(e_{cn}) = 4\sum_{t=0}^{\frac{n-1}{2}-1}\sum_{r=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}}\sum_{i=0}^{\frac{n-1}{2}}\frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+r+t+1}{i+j}} - 4\binom{\lfloor\frac{n+1}{2}\rfloor}{2}.$$

   (c) *If n is even then*
   $$bc'(e_{cn}) = 4\sum_{t=0}^{\frac{n-2}{2}}\sum_{r=0}^{\frac{n-2}{2}}\sum_{j=0}^{\frac{n}{2}}\sum_{i=0}^{\frac{n-2}{2}}\frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+r+t+1}{i+j}} - \frac{n^2}{2}.$$

Next, we analyze the asymptotics of an edge incident to a corner vertex and an edge incident to a center vertex. We first investigate a lower bound of $bc'(e_{cr})$.



(a) (b)

Figure 7: Edge betweenness centrality of edges in a 11×21 grid.

Claim 19. *When i is sufficiently large, $\binom{i+j}{i} \leq e^{i+j}$.*

Proof. $\binom{i+j}{i} \leq \left(\frac{i+j}{i}\right)^i e^i = \left(1+\frac{t}{i}\right)^i e^i$ where j is fixed and $i \to \infty$ we have that $\lim\limits_{i\to\infty} \binom{i+j}{i} \leq \lim\limits_{i\to\infty} \left(1+\frac{j}{i}\right)^i e^i = e^i e^j = e^{i+j}$. $\square$

Now using Claim 19, when $i \to \infty$ and j is fixed, we have that:

$$\binom{i+j}{i} \leq e^{i+j} \Rightarrow \frac{1}{e^{i+j}} < \frac{1}{\binom{i+j}{i}}.$$

Then $\sum\limits_{j=1}^{m}\sum\limits_{j=1}^{k} \frac{1}{e^i e^j} < \sum\limits_{j=1}^{m}\sum\limits_{j=1}^{k} \frac{1}{\binom{i+j}{i}}$

$\Rightarrow \ \frac{1}{e^2}\sum\limits_{j=1}^{m}\left(\frac{1}{e}\right)^{j-1}\sum\limits_{j=1}^{k}\left(\frac{1}{e}\right)^{i-1} < \sum\limits_{j=1}^{m}\sum\limits_{i=1}^{k} \frac{1}{\binom{i+j}{i}}$

As $m \to \infty$ and $k \to \infty$ we have the geometric series $\lim\limits_{m\to\infty}\sum\limits_{j=1}^{m}\left(\frac{1}{e}\right)^{j-1}$ and $\lim\limits_{k\to\infty}\sum\limits_{i=1}^{k}\left(\frac{1}{e}\right)^{i-1}$. Summing these series gives $\frac{1}{1-\frac{1}{e}}$ in each case.

As $m \to \infty$ and $k \to \infty$ we have $\frac{1}{e^2}\left(\frac{1}{1-\frac{1}{e}}\right)\left(\frac{1}{1-\frac{1}{e}}\right) \leq \sum_{j=1}^{m}\sum_{i=1}^{k}\frac{1}{\binom{i+j}{i}}$

$\Rightarrow \left(\frac{1}{e-1}\right)^2 \leq \sum_{j=1}^{m}\sum_{i=1}^{k}\frac{1}{\binom{i+j}{i}}$.

We consider the lower bound of $bc'(e_{cr})$ when $m \to \infty$ and $k \to \infty$. The case of $m = n-1$ and $k = n-1$ is given by

$2\left(\frac{1}{e-1}\right)^2 + n^2 - 1 \leq 2\sum_{j=1}^{m}\sum_{i=1}^{k}\frac{1}{\binom{i+j}{i}} + n^2 - 1 = bc'(e_{cr})$.

Next we consider the lower bound of $bc'(e_{cn})$.

Claim 20. *If $i \to \infty$ and t is fixed, $\left(\frac{i+t}{i}\right) \leq \binom{i+t}{i}$ which implies $e^t \leq \binom{i+t}{i}$ and $e^r \leq \binom{j+r}{j}$.*

It is easy to see that

$$\begin{aligned}
\frac{1}{e^{i+j+1}} &\leq \frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+t+r+1}{i+j}} \\
&\Rightarrow \sum_{t=0}^{\frac{n-2}{2}}\sum_{r=0}^{\frac{n-2}{2}} e^{-(i+j+1)} \leq \sum_{t=0}^{\frac{n-2}{2}}\sum_{r=0}^{\frac{n-2}{2}}\frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+t+r+1}{i+j}} \\
&\Rightarrow \frac{(n/2)^2}{e^{i+j+1}} \leq \sum_{t=0}^{\frac{n-2}{2}}\sum_{r=0}^{\frac{n-2}{2}}\frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+t+r+1}{i+j}} \\
&\Rightarrow \left(\frac{n}{2}\right)^2\sum_{t=0}^{\frac{n-2}{2}}\sum_{r=0}^{\frac{n-2}{2}}\frac{1}{e^i e^j e} \leq \sum_{t=0}^{\frac{n-2}{2}}\sum_{r=0}^{\frac{n-2}{2}}\frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+t+r+1}{i+j}}.
\end{aligned}$$

Let $n = m$. Then it is easy to see that

$$\frac{1}{e}\left(\frac{n}{2}\right)^2\sum_{i=0}^{\frac{m-2}{2}}\frac{1}{e^i}\sum_{j=0}^{\frac{m-2}{2}}\frac{1}{e^j} \leq \sum_{t=0}^{\frac{n-2}{2}}\sum_{r=0}^{\frac{n-2}{2}}\frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+t+r+1}{i+j}}.$$

When $m \to \infty$ we have:

$$\sum_{i=0}^{\frac{m-2}{2}} \frac{1}{e^i} = 1 + \frac{e^{-1}}{1-\frac{1}{e}} = 1 + \frac{e^{-1}e}{e-1} \text{ and } \sum_{j=0}^{\frac{m-2}{2}} \frac{1}{e^i} = 1 + \frac{e^{-1}}{1-\frac{1}{e}} = 1 + \frac{e^{-1}e}{e-1}$$

$$\lim_{m\to\infty} \frac{1}{e}\left(\frac{n}{2}\right)^2 \sum_{i=0}^{\frac{m-2}{2}} \frac{1}{e^i} \sum_{j=0}^{\frac{m-2}{2}} \frac{1}{e^j} \le \lim_{m\to\infty} 4 \sum_{t=0}^{\frac{n-2}{2}} \sum_{r=0}^{\frac{n-2}{2}} \frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+t+r+1}{i+j}}$$

$$\Rightarrow \frac{4}{e}\left(\frac{n}{2}\right)^2 \left(1 + \frac{1}{e-1}\right)^2 - \frac{n^2}{2} \le 4 \sum_{t=0}^{\frac{n-2}{2}} \sum_{r=0}^{\frac{n-2}{2}} \frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+t+r+1}{i+j}} - \frac{n^2}{2} (\text{as } m \to \infty).$$

Next we investigate an upper bound for a corner edge.

Using Claim 20 it is easy to see that:

$$\sum_{j=1}^{k}\sum_{i=1}^{m} \frac{1}{\binom{i+j}{i}} < m \sum_{i=1}^{m} \frac{1}{e^j}.$$

Then fixing m large and letting $k \to \infty$, we have that

$$\begin{aligned} \lim_{k\to\infty} \sum_{j=1}^{k}\sum_{i=1}^{m} \frac{1}{\binom{i+j}{i}} &< m \lim_{k\to\infty} \sum_{j=1}^{k} \frac{1}{e^j} \\ &= \frac{m}{e} \sum_{j=1}^{\infty} \frac{1}{e^{j-1}} = \frac{m}{e}\left(\frac{1}{1-\frac{1}{e}}\right) = \frac{m}{e}\left(\frac{e}{e-1}\right). \end{aligned}$$

Thus for k and m large enough we have that:

$$\sum_{j=1}^{k}\sum_{i=1}^{m} \frac{1}{\binom{i+j}{i}} < m\left(\frac{1}{e-1}\right).$$

Replacing k and m with $n-1$ we have:

$$n^2 - 1 + 2\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} \frac{1}{\binom{i+j}{i}} < n^2 - 1 + (n-1)\left(\frac{1}{e-1}\right).$$

Next we investigate an upper bound for a central edge.

As proved in the upper bound case we have that:

$$e^{t+r+1} < \binom{i+j+t+r+1}{i+j}$$

if i and j are large.

By Claim 19 we have

$$\binom{i+t}{i} \leq e^t e^i = e^{i+t} \text{and} \binom{j+r}{r} \leq e^{j+r}.$$

$$\Rightarrow \frac{\binom{i+j}{i}\binom{j+r}{r}}{\binom{i+j+t+r+1}{i+j}} \leq \frac{e^{i+t}e^{j+r}}{e^{t+r+1}} = e^{i+j-1}.$$

Then

$$\begin{aligned}
\sum_{t=0}^{\frac{n-2}{2}} \sum_{r=0}^{\frac{n-2}{2}} \frac{\binom{i+j}{i}\binom{j+r}{r}}{\binom{i+j+t+r+1}{i+j}} \leq \sum_{t=0}^{\frac{n-2}{2}} \sum_{r=0}^{\frac{n-2}{2}} e^{i+j-1} &= \left(\frac{n}{2}+1\right)\left(\frac{n}{2}+1\right) e^{i+j-1} \\
&\Rightarrow \sum_{i=0}^{\frac{n-2}{2}} \sum_{j=0}^{\frac{n}{2}} \sum_{t=0}^{\frac{n-2}{2}} \sum_{r=0}^{\frac{n-2}{2}} \frac{\binom{i+t}{i}\binom{j+r}{j}}{\binom{i+j+r+t+1}{i+j}} \\
&\leq \sum_{i=0}^{\frac{n-2}{2}} \sum_{j=0}^{\frac{n}{2}} \left(\frac{n}{2}\right)^2 e^{i+j-1} \\
&= \left(\frac{n}{2}\right)^2 \sum_{i=0}^{\frac{n-2}{2}} e^i \sum_{j=0}^{\frac{n}{2}} e^{i-1} \\
&= \left(\frac{n}{2}\right)^2 \frac{\left(1-e^{\frac{n-2}{2}+1}\right)}{1-e} \frac{\left(1-e^{\frac{n}{2}+1}\right)}{1-e} e^{-1}
\end{aligned}$$

and hence

$$\sum_{t=0}^{\frac{n-2}{2}} \sum_{r=0}^{\frac{n-2}{2}} \frac{\binom{i+j}{i}\binom{j+r}{r}}{\binom{i+j+t+r+1}{i+j}} \leq \left(\frac{n}{2}\right)^2 \frac{\left(1-e^{\frac{n-2}{2}+1}\right)\left(1-e^{\frac{n}{2}+1}\right)}{e(1-e)^2}.$$

Then if n is even then we have that:

$$4\left(\sum_{t=0}^{\frac{n-2}{2}}\sum_{r=0}^{\frac{n-2}{2}}\frac{\binom{i+j}{i}\binom{j+r}{r}}{\binom{i+j+t+r+1}{i+j}}\right)-\frac{n^2}{2}<4\left(\frac{n}{2}\right)^2\frac{\left(1-e^{\frac{n-2}{2}+1}\right)\left(1-e^{\frac{n}{2}+1}\right)}{e(1-e)^2}-\frac{n^2}{2}$$
$$=\frac{n^2}{2}\left(2\frac{\left(1-e^{\frac{n}{2}}\right)\left(1-e^{\frac{n}{2}+1}\right)-e(1-e)^2}{e(1-e)^2}\right).$$

We have proved the following upper and lower bounds for corner and central edges.

Theorem 21.

$$2\left(\frac{1}{e-1}\right)^2+n^2-1\leq bc'(e_{cr})<n^2-1+(n-1)\left(\frac{e}{2\,(e-1)}\right).$$

This shows that $bc'(e_{cr})$ is $\Theta(n^2)$.

Theorem 22.

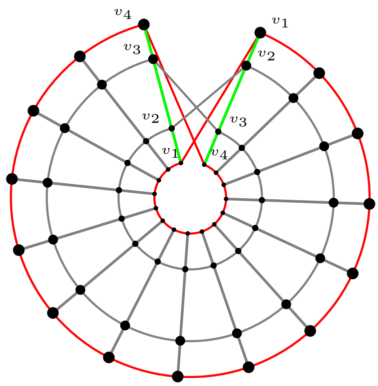
$$\frac{1}{e}\left(\frac{n}{2}\right)^2\left(1+\frac{1}{e-1}\right)^2\leq bc'(e_{cn})\leq 4\left(\frac{n}{2}\right)^2\frac{\left(1-e^{\frac{n}{2}}\right)\left(1-e^{\frac{n}{2}+1}\right)}{e(1-e)^2}-\frac{n^2}{2}.$$

4 Open problems

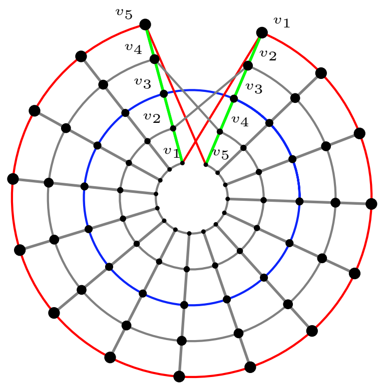
In Section 3 we noticed that in an a two-dimensional lattice central edges seem to have highest edge betweenness centrality and the corner edges seem to have the lowest edge betweenness centrality. We pose the following conjecture.

Conjecture 23. *In an n-dimensional lattice, the central edges have the highest edge betweenness centrality and the corner edges have the lowest edge betweenness centrality.*

In Example 11 and Lemma 12 we investigated the edge betweenness centrality for graphs on a torus. It would be interesting to consider edge betweenness centrality on other surfaces such as Möbius band. Connecting the left and right boundaries identifies vertices close to the top of the left side with vertices close to the bottom of the right side. As a result new challenges arise, including the need different cases where the number of



(a)



(b)

Figure 8:

vertices in the horizontal direction is even or odd. We show two cases in Figures 8 (a) and (b).

We pose the general case as a problem for continued study.

Problem 24. *Determine the edge betweenness centrality for Möbius band graphs.*

A further challenge would be to explore the variants involving the Klein bottle.

Problem 25. *Determine the edge betweenness centrality for Klein bottle graphs.*

Another area for continued research is to refine the asymptotic bounds given in the paper.

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