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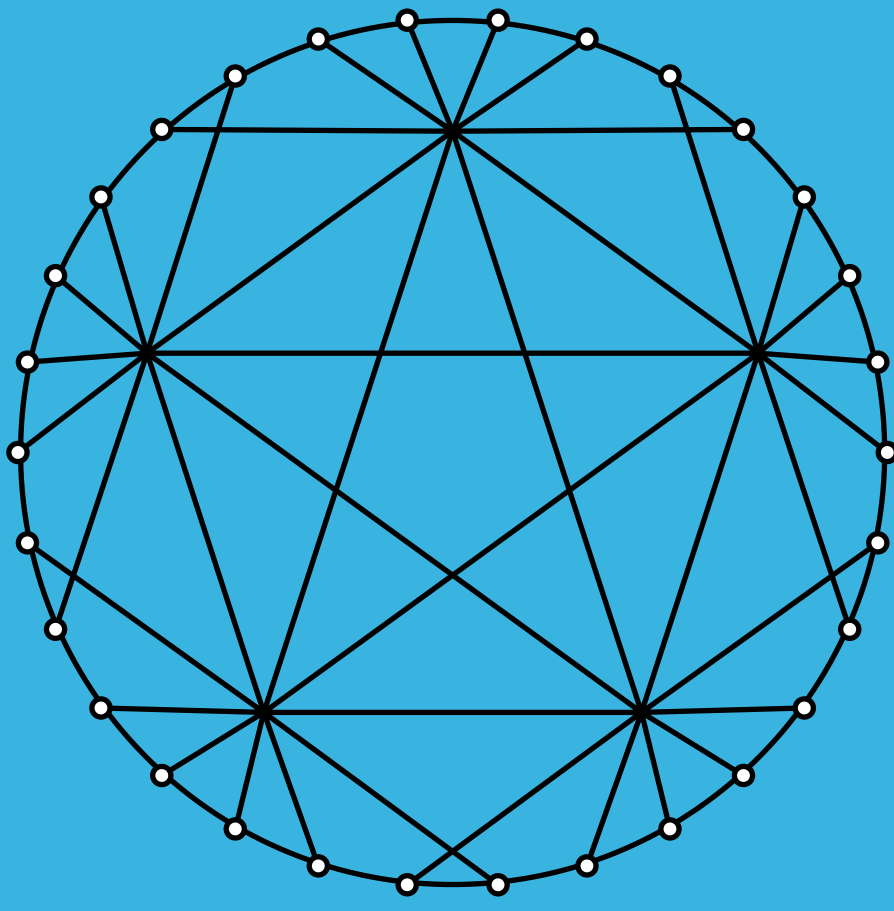

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# Ring dominating functions in graph products

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**Abstract:** An efficient dominating function is a map from the vertex set of a graph to the set $\{0, 1\}$ such that for each vertex in the graph, the sum of the function values in its closed neighborhood is exactly 1. In this paper we introduce a generalization of this idea. Let $R$ be a ring with identity. A ring dominating function is a map from the vertex set of a graph to $R$ such that for each vertex in the graph, the sum of the function values in its closed neighborhood is the identity. We explore the existence of ring dominating functions in the direct, strong and lexicographic products of graphs.

## 1 Introduction

A *dominating set* in a graph $G = (V(G), E(G))$ is a subset $D$ of the vertex set of $G$ such that each vertex in the graph is either in $D$ or is adjacent to an element in $D$. An *efficient dominating set* is a subset $D$ of the vertex set of $G$ such that $|N[v] \cap D| = 1$ for all $v \in V(G)$. If we replace the closed neighborhood $N[v]$ with the open neighborhood $N(v)$ then we say $D$ is an *efficient open dominating set*. Efficient dominating sets (and efficient

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open dominating sets) are also called perfect codes (total perfect codes) in literature and have been widely studied [2], [3], [4], and [6] with numerous applications in efficient resource placement, computer communication and security, and network models. The darkened vertices in Figures 1a and 1b represent efficient and efficient open dominating sets, respectively.

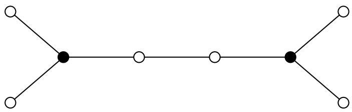

Figure 1a

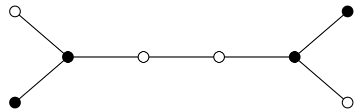

Figure 1b

A dominating set can also be viewed as a map $f : V(G) \to \{0, 1\}$, called a *dominating function*, such that for each vertex $v$ in $G$, the sum of the function values $f(v)$ in its closed neighborhood is at least one. In particular, an *efficient dominating function* is a map $f : V(G) \to \{0, 1\}$ such that for each vertex $v$ in the graph, the sum of the values $f(v)$ in its closed neighborhood is exactly one. Simply using open neighborhoods gives rise to *efficient open dominating functions*. Viewing the dominating sets in Figure 1 as dominating functions we get the labelings in Figure 2. In particular, Figures 2a and 2b illustrate efficient and efficient open dominating functions, respectively.

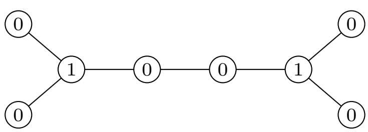

Figure 2a

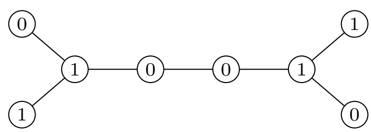

Figure 2b

In the previous definitions we add using regular addition of integers. What happens if in the definition of an efficient dominating function we map into the set $\{0, 1\}$ and add using addition modulo 2? This would give rise to an *odd dominating function* (or odd dominating set) as seen in Figure 3. Recall that $D$ is an odd dominating set of $G$ if and only if $|D \cap N[v]|$ is odd for each $v \in V(G)$. Klaus Sutner proved [9] that every graph has an odd dominating set. We will refer to an odd dominating function as a $\mathbb{Z}_2$ *dominating function.*

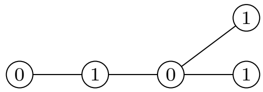


Figure 3

We now introduce a generalization of a dominating function. Given any ring with identity $R$, we define an $R$ *efficient dominating function* to be a map from the vertex set of the graph to the ring such that for each vertex in the graph, the sum of the ring assignments in its closed neighborhood equals the identity. In other words, $f : V(G) \to R$ such that $\sum_{x \in N[v]} f(x) = 1$ for all $v \in V(G)$, where we sum using the addition in $R$. Similarly, we define an $R$ *efficient open dominating function* using open neighborhoods. Figures 4a, 4b and 4c show $\mathbb{Z}_4$ efficient, $\mathbb{R}$ efficient, and $\mathbb{Z}_4$ efficient open dominating functions, respectively.

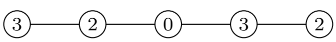


Figure 4a

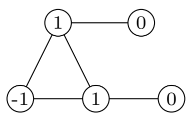


Figure 4b

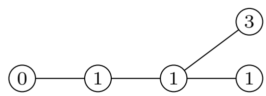


Figure 4c

While every graph has a $\mathbb{Z}_2$ efficient dominating function, it is not the case for rings in general. Determining whether a graph has an $R$ efficient dominating function is equivalent to determining the rings $R$ over which $(A+I)\mathbf{x} = \mathbf{1}$ has a solution, where $A$ is the adjacency matrix of the graph, $I$ is the identity matrix, and $\mathbf{1}$ is the column vector of all ones. For example, there are many graphs which do not admit $\mathbb{R}$ efficient dominating functions. The reader can easily verify that for the graph seen in Figure 5, the system $(A+I)\mathbf{x} = \mathbf{1}$ has no solution.

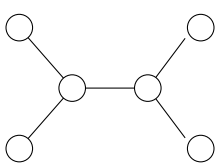

Figure 5

In the remaining sections of this paper, we examine ring dominating functions in the direct, strong and lexicographic products of simple graphs. In particular, we determine the relationship between ring dominating functions in the product and ring dominating functions in the factors of the product.

## 2 The direct product

Given graphs $G$ and $H$, the *direct product* $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G \times H) = \{(g,h)(g',h') \mid gg' \in E(G) \text{ and } hh' \in E(H)\}$. The graphs $G$ and $H$ are called *factors* of the product. Figure 6 shows the direct product $C_4 \times P_3$, where $C_4$ and $P_3$ denote a 4-cycle and path on three vertices.

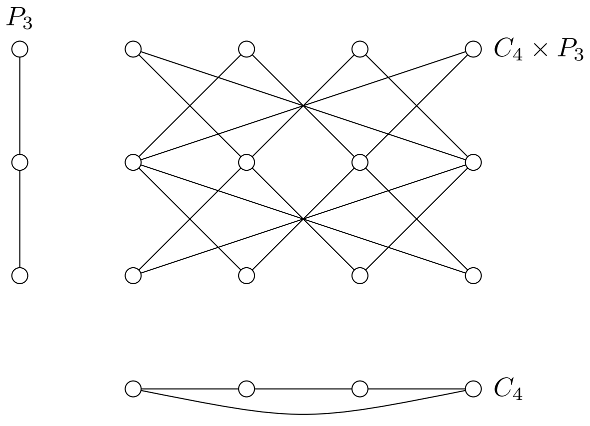


Figure 6

The above definition of the direct product can easily be extended to finitely many graphs. If $G_1, G_2, \ldots, G_n$ are graphs, the *n-fold direct product* is the graph $G_1 \times G_2 \times \cdots \times G_n$ with vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$, and for which the vertices $(g_1, g_2, \ldots, g_n)$ and $(g_1', g_2', \ldots, g_n')$ are adjacent precisely if $g_i g_i' \in E(G_i)$ for every $1 \leq i \leq n$. The direct product is associative and commutative. For a full treatment of this product see [5].

For any $v \in V(G)$, the *fiber* in $G \times H$ above $v$ is the set $G_v = \{(v, h) \mid h \in V(H)\}$. The dark vertices in Figure 7 show the fiber in $C_4 \times P_3$ above the vertex $v$.

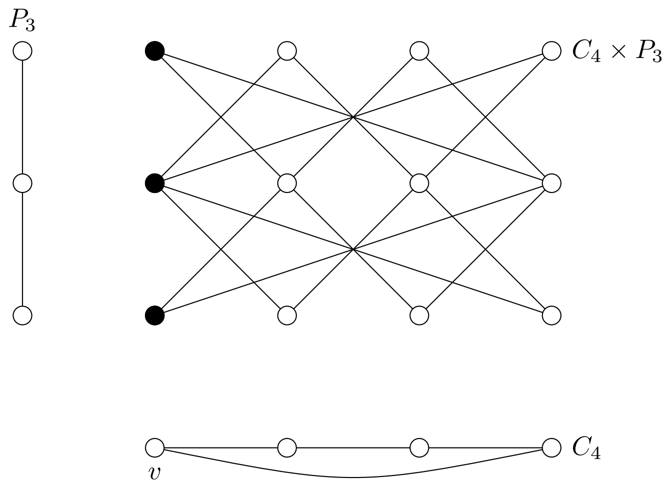


Figure 7

The direct product admits the following property concerning open neighborhoods: for $(g_1, g_2, \ldots, g_n) \in V(G_1 \times G_2 \times \cdots \times G_n)$, $N(g_1, g_2, \ldots, g_n) = N_{G_1}(g_1) \times N_{G_2}(g_2) \times \cdots \times N_{G_n}(g_n)$.

We now examine the relationship between $R$ efficient open dominating functions in the $n$-fold direct product of graphs and $R$ efficient open dominating functions in the factors.

**Theorem 2.1.** *Let $R$ be a ring with identity and $G_1, \ldots, G_n$ be graphs. The direct product $G_1 \times G_2 \times \ldots \times G_n$ has an $R$ efficient open dominating function if and only if each factor has an $R$ efficient open dominating function.*

*Proof.* Suppose $G_1, \ldots, G_n$ have $R$ efficient open dominating functions $f_1, \ldots, f_n$, respectively. We claim that $f : V(G_1 \times \ldots \times G_n) \to R$ defined by $f(g_1, \ldots, g_n) = f_1(g_1) f_2(g_2) \cdots f_n(g_n)$ is an $R$ efficient open dominating function in $G_1 \times \cdots \times G_n$. To see this, let $g = (g_1, \ldots, g_n) \in V(G_1 \times \cdots \times G_n)$.

Then

$$
\begin{aligned}
&\sum_{(g'_1,\ldots,g'_n)\in N(g)} f(g'_1, g'_2, \ldots, g'_n) \\
&\qquad = \sum_{(g'_1,\ldots,g'_n)\in N(g)} f_1(g'_1)\cdots f_n(g'_n) \\
&\qquad = \sum_{g'_1\in N(g_1)} \sum_{g'_2\in N(g_2)} \cdots \sum_{g'_n\in N(g_n)} f(g'_1)f(g'_2)\cdots f(g'_n) \\
&\qquad = \sum_{g'_1\in N(g_1)} \sum_{g'_2\in N(g_2)} \cdots \sum_{g'_{n-1}\in N(g_{n-1})} f(g'_1)f(g'_2)\cdots f(g'_{n-1}) \sum_{g'_n\in N(g_n)} f(g'_n) \\
&\qquad = \sum_{g'_1\in N(g_1)} \sum_{g'_2\in N(g_2)} \cdots \sum_{g'_{n-1}\in N(g_{n-1})} f(g'_1)f(g'_2)\cdots f(g'_{n-1}) \cdot 1 \\
&\qquad \vdots \\
&\qquad = \sum_{g'_1\in N(g_1)} f_1(g'_1) \cdot 1 = 1
\end{aligned}
$$

Hence $f$ is an $R$ efficient open dominating function in $G_1 \times \cdots \times G_n$.

Conversely, suppose $G_1 \times \cdots \times G_n$ has an $R$ efficient open dominating function $f$. Fix any $(g_1, \ldots, g_n) \in V(G_1 \times \cdots \times G_n)$. Then the map $f_i : V(G_i) \to R$ defined by $f_i(x_i) = \sum_{g'_j\in N(g_j), j\neq i} f(g'_1, \ldots, x_i, \ldots, g'_n)$ is an $R$ efficient open dominating function in $G_i$ as follows. Consider a vertex $x_i \in V(G_i)$. Then

$$
\begin{aligned}
\sum_{x'_i\in N(x_i)} f_i(x'_i) &= \sum_{x'_i\in N(x_i)} \sum_{\substack{g'_j \in N(g_j), \\ j\neq i}} f(g'_1, \ldots, x'_i, \ldots, g'_n) \\
&= \sum_{(g'_1,\ldots,x'_i,\ldots,g'_n)\in N(g_1,\ldots,x_i,\ldots,g_n)} f(g'_1, \ldots, x'_i, \ldots, g'_n) = 1
\end{aligned}
$$

Therefore $f_i$ is an $R$ efficient open dominating function in $G_i$. $\square$

Theorem 2.1 is illustrated in Figures 8a and 8b. Figure 8a shows a $\mathbb{Z}_5$ efficient open dominating function in the product obtained from the $\mathbb{Z}_5$

efficient open dominating functions in the factors. Figure 8b shows how to obtain a $\mathbb{Z}_5$ efficient open dominating function in the factor $G$ from the $\mathbb{Z}_5$ efficient open dominating function in the product. In particular, fix a vertex $h \in V(H)$. Then look at $N(h) \times V(G)$ (these vertices are the ones that pass through the dashed lines). The label for any vertex $g \in V(G)$ is simply the sum of the vertices in the fiber above $g$ that lie in $N(h) \times V(G)$. By a similar process we could label the vertices of $H$.

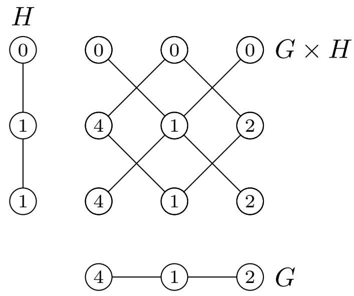


Figure 8a

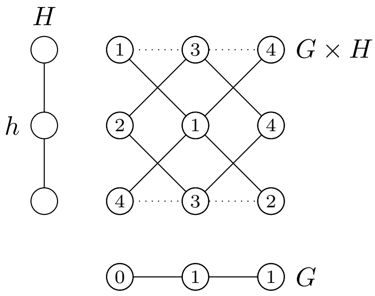


Figure 8b

# 3 The strong product

The *strong product* of $G$ and $H$ is the graph $G \boxtimes H$ whose vertex set is $V(G) \times V(H)$, and for which distinct vertices $(g, h)$ and $(g', h')$ are adjacent precisely if one of the following holds:

1. $g = g'$ and $hh' \in E(H)$
2. $gg' \in E(G)$ and $h = h'$
3. $gg' \in E(G)$ and $hh' \in E(H)$.

This definition extends to an $n$-fold strong product naturally: $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ has vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$ where distinct vertices $(g_1, g_2, \ldots, g_n)$ and $(g'_1, g'_2, \ldots, g'_n)$ are adjacent precisely if $g_i = g'_i$ or $g_i g'_i \in E(G_i)$ for each $1 \leq i \leq n$. The strong product is also associative and commutative. The strong product is sometimes referred to in literature as the *strong direct product*. Figure 9 shows the strong product $P_4 \boxtimes P_3$.

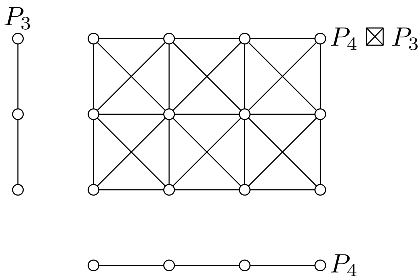


Figure 9

The strong product has the following easily-checked property of closed neighborhoods: for $(g_1, g_2, \ldots, g_n) \in V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n)$,

$$N[(g_1, g_2, \ldots, g_n)] = N_{G_1}[g_1] \times N_{G_2}[g_2] \times \cdots \times N_{G_n}[g_n].$$

We refer to $B((g_1, \ldots, g_n), r)$ as the *ball of radius* $r$ centered at $(g_1, \ldots, g_n)$. This ball contains all vertices in $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ that are within distance $r$ of the vertex $(g_1, g_2, \ldots, g_n)$. By [5], the distance between $(g_1, g_2, \ldots, g_n)$ and $(g_1', g_2', \ldots, g_n')$ in $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ is $\max\{d_{G_i}(g_i, g_i')\}$. Consequently,

$$B((g_1, g_2, \ldots, g_n), r) = B_{G_1}(g_1, r) \times B_{G_2}(g_2, r) \times \cdots \times B_{G_n}(g_n, r).$$

See [5] for more details on the strong product.

We now consider ring dominating functions in the $n$-fold strong product of graphs. We define an $R$ *efficient* $r$*-dominating fuction* to be a map from the vertex set of a graph to $R$ such that for each vertex $v$ in the graph, the sum of the ring assignments in $B(v, r)$ equals the identity. Note that in the case where $r = 1$, we simply refer to an $R$ efficient 1-dominating function as an $R$ efficient dominating function. Figure 10 illustrates a $\mathbb{Z}_5$ efficient 2-dominating function.

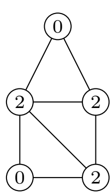


Figure 10

**Theorem 3.1.** *Let $R$ be a ring with identity and let $G_1, G_2, \ldots, G_n$ be graphs. The $n$-fold strong product $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ has an $R$ efficient $r$-dominating function if and only if each factor has an $R$ efficient $r$-dominating function.*

*Proof.* Let $R$ be ring with identity and let $G_1, G_2, \ldots, G_n$ have $R$ efficient $r$-dominating functions $f_1, f_2, \ldots, f_n$, respectively. Then $f : V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n) \to R$ defined by $f(g_1, g_2, \ldots, g_n) = f_1(g_1)f_2(g_2)\ldots f_n(g_n)$ is an $R$ efficient $r$-dominating function in $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$. Let $g = (g_1, g_2, \ldots, g_n) \in V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n)$. Then

$$\begin{aligned}
&\sum_{(g_1',\ldots,g_n') \in B(g,r)} f(g_1', \ldots, g_n') \\
&\qquad = \sum_{(g_1',\ldots,g_n') \in B(g,r)} f_1(g_1')f_2(g_2') \cdots f_n(g_n') \\
&\qquad = \sum_{g_1' \in B(g_1,r)} \sum_{g_2' \in B(g_2,r)} \cdots \sum_{g_n' \in B(g_n,r)} f_1(g_1')f_2(g_2') \cdots f_n(g_n') \\
&\qquad = \sum_{g_1' \in B(g_1,r)} \sum_{g_2' \in B(g_2,r)} \cdots \sum_{g_n' \in B(g_n,r)} f_1(g_1') \cdots f_{n-1}(g_{n-1}') \sum_{g_n' \in B(g_n,r)} f_n(g_n) \\
&\qquad = \sum_{g_1' \in B(g_1,r)} \sum_{g_2' \in B(g_2,r)} \cdots \sum_{g_n' \in B(g_n,r)} f_1(g_1') \cdots f_{n-1}(g_{n-1}') \cdot 1 \\
&\qquad \vdots \\
&\qquad = \sum_{g_1' \in B(g_1,r)} f_1(g_1') \cdot 1 = 1.
\end{aligned}$$

Conversely, suppose $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ has an $R$ efficient $r$-dominating function $f$. Fix $(g_1, g_2, \ldots, g_n) \in V(G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n)$. We claim that for any factor $G_i$, the map $f_i : V(G_i) \to R$ defined by $f_i(x_i) = \sum_{\substack{g_j' \in B(g_j, r), \\ j \neq i}} f(g_1', \ldots, x_i, \ldots, g_n')$ is an $R$ efficient $r$-dominating function. To show this, let $x_i \in V(G_i)$. Then

$$\begin{aligned}
\sum_{x_i' \in B(x_i,r)} f_i(x_i') &= \sum_{x_i' \in B(x_i,r)} \sum_{\substack{g_j' \in B(g_j, r) \\ j \neq i}} f(g_1', \ldots, x_i', \ldots, g_n') \\
&= \sum_{\substack{(g_1', \ldots, x_i', \ldots, g_n') \in \\ B((g_1, \ldots, x_i, \ldots, g_n), r)}} f(g_1', \ldots, x_i, \ldots, g_n') = 1
\end{aligned}$$

Therefore, $f_i$ is a $R$ efficient $r$-dominating function for $G_i$. $\square$

Figure 11 shows a $\mathbb{Z}_5$ efficient 2-dominating function in $G \boxtimes H$ determined from the $\mathbb{Z}_5$ efficient 2-dominating functions in each of the factors.

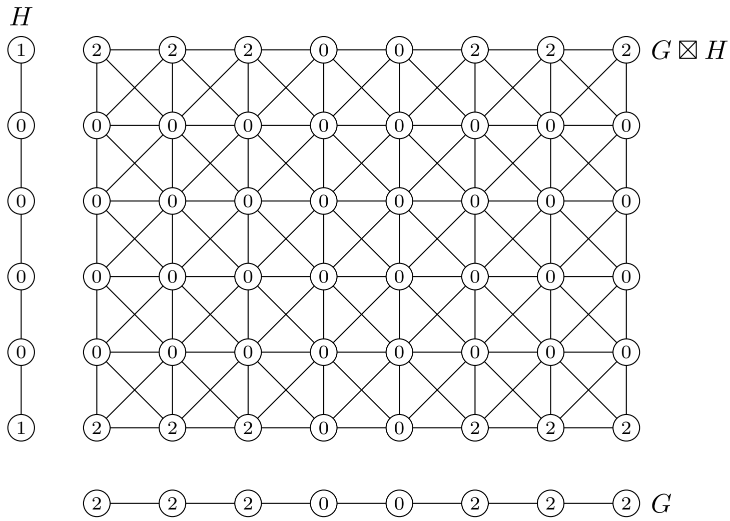


Figure 11

# 4 The lexicographic product

In this section we consider $R$ efficient $r$-dominating functions in the lexicographic product. The *lexicographic product* of graphs $G$ and $H$ is the graph $G \circ H$ with vertex set $V(G) \times V(H)$ and whose edges are the pairs $(g, h)(g', h')$ of distinct vertices where one of the following holds:

1. $gg' \in E(G)$ or
2. $g = g'$ and $hh' \in E(H)$.

The lexicographic product is associative, but in general is not commutative. The lexicographic product is also referred to as *composition* since one can think of replacing each vertex in $G$ with a copy of $H$ and joining each vertex in the fiber $G_v$ to each vertex in the fiber $G_u$ whenever $vu \in E(G)$. Figure 12 shows $P_4 \circ P_3$.

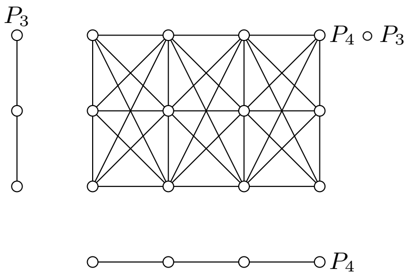


Figure 12

Our final result shows how to obtain $R$ efficient $r$-dominating functions in the product from $R$ efficient $r$-dominating functions in the factor $G$ and vice versa. Notice that the factor $H$ need not have an $R$ efficient $r$-dominating function, but it must have radius at most $r$. We also use the fact that if $\mathrm{rad}(H) \le r$ then $B((g,h),r) = B(g,r) \times V(H)$. This follows immediately from the distance formula for the lexicographic product [5].

**Theorem 4.1.** *Suppose $H$ has radius at most $r$. Then $G$ has an $R$ efficient $r$-dominating function if and only if $G \circ H$ has an $R$ efficient $r$-dominating function.*

*Proof.* Suppose $f$ is an $R$ efficient $r$-dominating function for $G$ and the radius of $H$ is at most $r$. Fix any vertex $c$ in the center of $H$. We claim that $F : V(G \circ H) \to R$ defined by

$$F(g,h) = \begin{cases} f(g) & \text{if } h = c \\ 0 & \text{otherwise} \end{cases}$$

is an $R$ efficient $r$-dominating function for $G \circ H$.

Let $(g,h) \in V(G \circ H)$. Then

$$\sum_{(g',h') \in B((g,h),r)} F(g',h') = \sum_{(g',c) \in B((g,h),r)} f(g') = \sum_{g' \in B(g,r)} f(g') = 1.$$

Hence $F$ is an $R$ efficient $r$-dominating function for $G \circ H$.

Conversely, suppose that $F$ is an $R$ efficient $r$-dominating function in $G \circ H$. We show that $f : V(G) \to R$ defined by $f(g) = \sum_{h' \in V(H)} F(g,h')$ is an $R$

efficient $r$-dominating function in $G$. Let $g \in V(G)$. Then

$$\sum_{g' \in B(g,r)} f(g') = \sum_{g' \in B(g,r)} \sum_{h' \in V(H)} F(g', h').$$

Recall that since $H$ has radius $r$, $\{(g', h') \mid g' \in B(g,r), h' \in V(H)\} = B((g,c), r)$ for any $c$ in the center of $H$. So,

$$\sum_{g' \in B(g,r)} \sum_{h' \in V(H)} F(g', h') = \sum_{(g',h') \in B((g,c),r)} F(g', h') = 1.$$

Hence $f$ is an $R$ efficient $r$-dominating function in $G$. $\square$

This theorem is illustrated in Figures 13 and 14 where $r = 1$ and $R = \mathbb{R}$. Note that in Figure 13, the only central vertex in $H$ has been labeled with $c$.

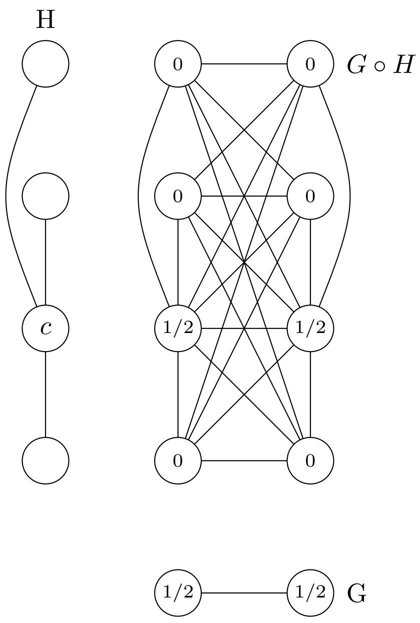


Figure 13

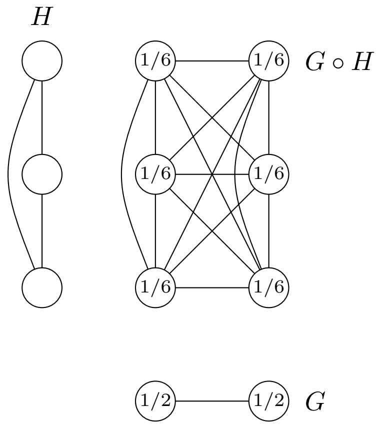


Figure 14

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