LDPC Coding For The Three-Terminal Erasure Relay Channel

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***Abstract*— A three terminal erasure relay channel is considered. It has been shown that appropriately designed maximum distance separable codes achieve the cut-set upper bound on capacity of the three terminal erasure relay channel. This paper presents low-density parity-check (LDPC) coding alternatives for this channel. Design rules for constructing LDPC codes that perform close to the cut-set upper bound on capacity are provided for the general erasure relay channel and also the degraded erasure relay channel, wherein all the information available at the receiver are also available at the relay.**

I. Introduction

A simple relay network comprising of one sender, one receiver, and one intermediate node, called the relay, that participates in the communication by relaying packets from the sender to the receiver, is considered. Relay channels that have been considered are predominantly noisy channels with interferences between the sender and relay transmissions at the receiver. In this paper, we consider a simplistic relay channel, where there is no interference and the only channel impairment are losses or erasures. Consequently, this network with three nodes will be referred to as the three-terminal erasure relay channel. From the standpoint of higher layers in a communication network, the erasure channel model is appealing, since they view communication as over a packet network where packets arrive either error-free or are erased by the lower link-layer error-detection protocols.

Erasure channels are particularly simple to analyze. In [2], it is shown that simple closed form expressions can be derived for erasure relay channels as opposed to relay channels with errors. [3] presents a cut-set upper bound for the achievable rate for the erasure relay channel without interference and shows that this bound is achievable for the degraded relay channel using maximum-distance-separable (MDS) codes. However, MDS codes of long block lengths are impractical for use in practical system design since their decoding complexity increases as $O(n^3)$ for a block length n. Furthermore, a typically large field size is necessary to construct an MDS code of a large block length n.

Hence, motivated by the remarkable success of low-density parity-check (LDPC) codes in performing close to capacity over several point-to-point communication channels, and in particular over the binary erasure channel [4], we propose, as a first step, to use LDPC codes instead to come close to the capacity of the degraded erasure relay channel with no interference. (The no-interference assumption at the receiver can be realized practically if the transmissions from the sender to the receiver and those from the relay to the receiver occur at different frequencies or at different times following an FDMA/TDMA protocol.) Following the density evolution techniques of [4], [5], we derive the necessary conditions on the degree distributions of LDPC codes to be used in the relay channel to come close to the cut-set upper bound of [3]. Further, LDPC-based coding extensions to the non-degraded case is also proposed.

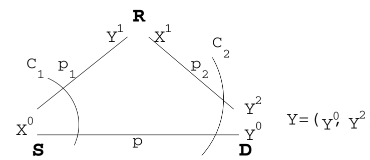

Fig. 1. Erasure Relay Channel

II. Channel model

The communication channel model that is considered in this paper is shown in Figure 1. It comprises of a sender node $\mathbf{S}$, a receiver (or, destination) node $\mathbf{D}$, and an intermediate or relay node $\mathbf{R}$. The relay channel is described by five random variables $\mathbf{X}^0, \mathbf{X}^1, \mathbf{Y}^0, \mathbf{Y}^1$, and $\mathbf{Y}^2$ and a conditional probability density function $p(y^0, y^1, y^2|x^0, x^1)$. This probability density function specifies the probability that when x^0 is sent by $\mathbf{S}$ and x^1 is sent by $\mathbf{R}$, y^0 and y^2 are received at $\mathbf{D}$ and y^1 is received at $\mathbf{R}$. Furthermore, we can view the above relay channel as two separate channels: an erasure-broadcast channel $(\mathbf{X}^0; \mathbf{Y}^0, \mathbf{Y}^1)$ [6] and a point-to-point erasure channel $(\mathbf{X}^1; \mathbf{Y}^2)$. Further, we assume that the erasure loss probabilities on the links from $\mathbf{S}$ to $\mathbf{R}$, from $\mathbf{R}$ to $\mathbf{D}$, and from $\mathbf{S}$ to $\mathbf{D}$ are p_1, p_2, and p, respectively.

III. Cut-set Upper Bound on Achievable Rate

The cut-set upper bound on the achievable rate for the three-terminal relay channel, obtained in [1], is given here.

Theorem 3.1 (Capacity region bound): [1] *The capacity region of the relay channel in Figure 1 is bounded by*

$$R \leq \sup_{p(x^0,x^1)} \min\{I(\mathbf{X}^0;\mathbf{Y}^0,\mathbf{Y}^1) - I(\mathbf{X}^1;\mathbf{Y}^0,\mathbf{Y}^1), I(\mathbf{X}^0;\mathbf{Y}^0) + I(\mathbf{X}^1;\mathbf{Y}^2|\mathbf{X}^0)\}$$

If the relay channel is a degraded channel, i.e., $\mathbf{X}^0 \to \mathbf{Y}^1 \to (\mathbf{Y}^0, \mathbf{Y}^2)$*, is a Markov chain, then the bound becomes*

$$R \leq \sup_{p(x^0,x^1)} \min\{I(\mathbf{X}^0;\mathbf{Y}^1) - I(\mathbf{X}^1;\mathbf{Y}^1), I(\mathbf{X}^0;\mathbf{Y}^0) + I(\mathbf{X}^1;\mathbf{Y}^2|\mathbf{X}^0)\}$$

For the erasure relay channel, the capacity bound simplifies:

Theorem 3.2: [3] *The capacity region over an erasure relay channel is bounded as:*

$$R \leq \max_{\alpha} \min\{(1 - p \cdot p_1), (1 - p) + \alpha(1 - p_2)\}$$

where p_1, p_2, p are the loss probabilities between sender and relay, relay and destination, and sender and destination, respectively, and $0 \leq \alpha \leq 1$ is a coupling parameter that describes the proportion of information sent by the sender that is available at the relay. ($\alpha = 1$ if $R < (1 - p_1)$ and $\alpha = p$ otherwise.)

Under the degraded channel situation, the bound becomes

$$R \leq \max_{\alpha} \min\{(1-p_1), (1-p) + \alpha(1-p_2)\}$$

When $p < p_1$, the relay can be bypassed and the maximum rate achievable is $R \leq (1-p)$. ([3] provides more details.)

A. *Achievability of capacity bound* [3]: *Degraded case*

Consider the relay channel in Figure 1. Suppose $\mathbf{S}$ has k symbols (or, packets) to transmit. Let us suppose that there exists an $[k+m+\ell, k]$ MDS code defined over an appropriate finite field with generator matrix $G = [I_{k\times k} \mid A_{k\times m} \mid B_{k\times \ell}]$. The source $\mathbf{S}$ uses the generator matrix $G_1 = [I_{k\times k} \mid A_{k\times m}]$ to encode. Thus, the rate $R = \frac{k}{k+m}$. (Let $n = k + m$.) Asymptotically, for large n, $\mathbf{R}$ receives around $n(1-p_1)$ symbols. Under the condition that $R \leq 1 - p_1$, an MDS code will, asymptotically with large k and n, ensure perfect communication between $\mathbf{S}$ and $\mathbf{R}$. $\mathbf{R}$ then re-encodes the k information packets of $\mathbf{S}$ by multiplying with generator matrix $G_2 = [B_{k\times \ell}]$. At the destination $\mathbf{D}$, asymptotically, at large $n = k + m$ and ℓ, $n(1-p)$ symbols arrive from $\mathbf{S}$, and $\ell(1-p_2)$ symbols arrive from $\mathbf{R}$. $\mathbf{D}$ decodes the received symbols using the MDS code corresponding to G. As long as at least k symbols are available at $\mathbf{D}$, decoding at $\mathbf{D}$ will result in complete recovery of the k information symbols. Thus, the above coding scheme achieves a rate $R \leq \min\{1 - p_1, \frac{n(1-p)+\ell(1-p_2)}{n}\} = \min\{1-p_1, (1-p) + \alpha(1-p_2)\}$.

IV. Design Rules for LDPC codes

In this section, we use LDPC codes of large block lengths in the relay channel to come close to the cut-set upper bound.

A. *LDPC coding for the binary erasure channel*

Let H be a parity-check matrix of a binary LDPC code $\mathcal{C}$ and G_H the corresponding Tanner graph. Further, let λ, ρ be the variable node and check node degree distributions of G_H [5]. Adopting the notation in [5], we have $\lambda(x) = \sum_{i=2}^{d_v} \lambda_i x^{i-1}$ and $\rho(x) = \sum_{i=2}^{d_c} \rho_i x^{i-1}$, where λ_i (resp., ρ_i) is the fraction of edges in G_H that are incident with degree i variable (resp., check) nodes and d_v and d_c are the maximum degrees of variable (resp., check) nodes in G_H. An LDPC code constructed randomly and having a degree distribution (λ, ρ) has a rate R satisfying

$$R \geq 1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx}.$$

A random LDPC code typically achieves equality in the above.

Suppose an LDPC code of large block length n described by a parity-check matrix H with degree distribution (λ, ρ) is used for transmission across a binary erasure channel (BEC). Then, under the assumption that there are no cycles of length 2ℓ or less in the LDPC constraint graph G_H, we can find the probability of sending an erasure message along an edge from a variable node to a check node in G_H in the ℓ^{th} iteration as a function of the probability of sending an erasure message along an edge from a variable node to a check node in the $(\ell-1)^{th}$ iteration and the erasure probability of the BEC. Let $p^{(0)}$ denote the erasure probability of the BEC and let $p^{(\ell)}$ denote the probability of sending an erasure along an edge from a variable node to a check node in the ℓ^{th} iteration. Assuming that G_H has no cycles of length 2ℓ or less, it is easy to show that the following relation holds [4]

$$p^{(\ell)} = p^{(0)}\lambda(1 - \rho(1 - p^{(\ell-1)}))$$

Defining a function $f(x) = p^{(0)}\lambda(1-\rho(1-x))$, we can obtain the condition for an LDPC code of degree distribution (λ, ρ) to recover all erasures occurring at a BEC erasure probability of $p^{(0)}$ in the asymptotic case where the LDPC code has an arbitrarily large block length and there are no cycles of length 2ℓ or less in the LDPC constraint graph for as large an ℓ as required. Under these assumptions, the condition for recovering all erasures is

$$f(x) < x, \ \forall x \in (0, p^{(0)}]$$

This condition implies that as long as $f(x)$ is strictly less than x for all x in the range $(0, p^{(0)}]$, $p^{(\ell)} \to 0$ as $\ell \to \infty$.

Thus, a rule in designing LDPC codes is to find a degree distribution pair (λ, ρ) that satisfies the following two constraints:

$$\text{capacity of BEC } = 1 - p^{(0)} = \text{rate of LDPC} \geq 1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx}.$$

$$p^{(0)}\lambda(1 - \rho(1-x)) < x, \ \forall x \in (0, p^{(0)}]$$

B. *LDPC coding for the relay channel*

In the following, we adapt the same encoding techniques as with MDS codes in the previous section to obtain design rules for the degree distribution of binary LDPC codes for the relay channel. Let us suppose we have a generator matrix $G = [B_{k\times\ell} | \ A_{k\times m} | \ I_{k\times k}]$. The sender $\mathbf{S}$ encodes the information to be sent using an LDPC code $\mathcal{C}_1$ that is described by a parity check matrix H_1 of size $m \times (k+m)$. Let H_1 have a degree distribution $(\lambda^{(a)}, \rho^{(a)})$. Suppose reducing H_1 to systematic form yields a matrix $H_{1,sys} = [I_{m\times m} | \ A^T_{m\times k}]$, then $G_1 = [A_{k\times m} | \ I_{k\times k}]$ is a generator matrix for the code $\mathcal{C}_1$. Since H_1 is designed randomly, we can assume that it has full rank and therefore there exists a sparse $m \times m$ invertible matrix D such that $H_1 = DH_{1,sys}$. Thus, H_1 can also be written as $H_1 = [D \mid DA^T]$. Let D have a degree distribution $(\lambda^{(1)}, \rho^{(1)})$ and DA^T have a degree distribution $(\lambda^{(2)}, \rho^{(2)})$.

We will now form a new parity check matrix H of size $(m+\ell) \times (k+m+\ell)$ having the following form:

$$H = \left[\begin{array}{c|c|c} X_{\ell\times\ell} & \mathbf{0}_{\ell\times m} & Y_{\ell\times k} \\ \mathbf{0}_{m\times\ell} & D_{m\times m} & (DA^T)_{m\times k} \end{array}\right],$$

where X and Y are sparse matrices and X is invertible.

Suppose reducing H to systematic form yields a matrix H_{sys} of the form

$$H_{sys} = \left[\begin{array}{c|c|c} I_{\ell\times\ell} & \mathbf{0} & B^T_{\ell\times k} \\ \mathbf{0} & I_{m\times m} & A^T_{m\times k} \end{array}\right]$$

Then, H is obtained by left multiplying H_{sys} with $\left[\begin{array}{cc} X & 0 \\ 0 & D \end{array}\right]$. Therefore, both H and H_{sys} are valid parity-check matrices for the generator matrix G. The sub-matrix A in G is obtained by converting H_1 to systematic form, and the sub-matrix B in G is obtained from H as $B = X^{-1}Y$.

Let us suppose that while constructing H, we choose X to be a sparse matrix with degree distribution $(\lambda^{(3)}, \rho^{(3)})$ and the sub-matrix Y to be a sparse matrix with degree distribution $(\lambda^{(4)}, \rho^{(4)})$.

The encoding scheme is the same as before: $\mathbf{S}$ uses the generator matrix $G_1 = [A_{k\times m}|\ I_{k\times k}]$ to encode. The sender's packets arrive at $\mathbf{R}$ via an erasure channel with loss probability p_1 and at $\mathbf{D}$ via an erasure channel with loss probability p.

Suppose the following two conditions hold:

$$\text{rate}R = \frac{k}{k+m} = 1 - \frac{\int_0^1 \rho^{(a)}(x)dx}{\int_0^1 \lambda^{(a)}(x)dx} \leq 1 - p_1 = \text{capacity} \quad (1)$$

$$f(x) = p_1 \lambda^{(a)}(1 - \rho^{(a)}(1-x)) < x,\ \forall x \in (0, p_1] \quad (2)$$

Then, asymptotically, using an LDPC code of large block length $n = k+m$, $\mathbf{R}$ will be able to reconstruct the sender's information perfectly. $\mathbf{R}$ then encodes the information sequence that it has reconstructed by multiplying with generator matrix $G_2 = [B_{k\times \ell}]$. $\mathbf{D}$ receives two sequences of data: one from $\mathbf{R}$ over an erasure channel with loss probability p_2 and the other from $\mathbf{S}$ over an erasure channel with loss probability p. $\mathbf{D}$ uses the sparse parity-check matrix H to iteratively decode the received stream. We will now derive conditions on the degree distributions so that $\mathbf{D}$ can reconstruct the sender's information perfectly and at the same time operate close to capacity.

The analysis of the iterative decoder assuming no-cycles in the LDPC constraint graph for H is as follows: Let L_1 refer to the first ℓ columns of H, L_2 refer to the next m columns, and L_3 refer to the last k columns. Similarly, let R_1 refer to the first ℓ rows of H and R_2 to the remaining m rows. Note that $\mathbf{D}$ receives information from $\mathbf{R}$ for the code symbols corresponding to L_1 and from $\mathbf{S}$ for the code symbols corresponding to L_2 and L_3. Hence, the code symbols corresponding to L_1 arrive at $\mathbf{D}$ with erasure probability p_2 and the code symbols corresponding to L_2 and L_3 arrive at $\mathbf{D}$ with erasure probability p.

Let $p_{L_s}^{(t,i)}$, for $s = 1, 2, 3$, be the probability of sending an erasure message along an edge from a variable node of degree i, belonging to L_s, to a check node in the t^{th} iteration. Similarly, let $q_{R_s}^{(t,i)}$, for $s = 1, 2$, be the probability of sending an erasure message along an edge from a check node of degree i, belonging to R_s, to a variable node in the t^{th} iteration. Further, let $p_{L_s}^{(t)}$, for $s = 1, 2, 3$, be the average probability of sending an erasure message along on edge from a variable node, belonging to L_s, to a check node in the t^{th} iteration. Define $q_{R_s}^{(t)}$, for $s = 1, 2$, analogously. Then, under the cycle-free assumption, we obtain the following conditions on how the probability of erasures, defined above, evolve with the iteration t.

$$p_{L_1}^{(t+1,i)} = p_2 \cdot (q_{R_1}^{(t)})^{i-1}$$

Averaging over all i , we obtain

$$p_{L_1}^{(t+1)} = p_2 \lambda^{(3)}(q_{R_1}^{(t)})$$

Similarly,

$$p_{L_2}^{(t+1)} = p \lambda^{(1)}(q_{R_2}^{(t)})$$

For calculating $p_{L_3}^{(t+1,r)}$, observe that a variable node in L_3 of degree r may be receiving messages along i edges that are connected to check nodes in R_1 and along j edges that are connected to check nodes in R_2, where $i + j = r - 1$. The probability of having such a connection at a degree r variable node in L_2 is $\delta_{i,j}(\lambda^{(4)}, \lambda^{(2)}) = 0.5(\lambda_{i+1}^{(4)}\lambda_j^{(2)} + \lambda_i^{(4)}\lambda_{j+1}^{(2)})$, since the degree r variable node may be passing a message to a check node in R_1 (or to a check node in R_2) after receiving i messages from nodes in R_1 and j messages from nodes in R_2. Thus, averaging over r, we obtain the final recursion for $p_{L_2}^{(t)}$ as

$$p_{L_3}^{(t)} = p \cdot \left(\sum_{r=2}^{d_{v_4}+d_{v_2}} \sum_{i+j=r-1} \delta_{i,j}(\lambda^{(4)}, \lambda^{(2)})(q_{R_1}^{(t-1)})^i (q_{R_2}^{(t-1)})^j \right),$$

where d_{v_4} and d_{v_2} are the maximum variable node degrees in the sub-matrices Y and DA^T, respectively.

Define $\gamma_{i,j}(\rho^{(1)}, \rho^{(2)}) := 0.5(\rho_{i+1}^{(1)}\rho_j^{(2)} + \rho_i^{(1)}\rho_{j+1}^{(2)})$. Then, deriving the recursions for $q_{R_1}^{(t)}$ and $q_{R_2}^{(t)}$ in a similar manner, we obtain

$$q_{R_1}^{(t)} = \sum_{r=2}^{d_{r_3}+d_{r_4}} \sum_{i+j=r-1} \gamma_{i,j}(\rho^{(3)}, \rho^{(4)}) \left(1 - (1 - p_{L_1}^{(t)})^i (1 - p_{L_3}^{(t)})^j\right)$$

$$q_{R_2}^{(t)} = \sum_{r=2}^{d_{r_1}+d_{r_2}} \sum_{i+j=r-1} \gamma_{i,j}(\rho^{(1)}, \rho^{(2)}) \left(1 - (1 - p_{L_2}^{(t)})^i (1 - p_{L_3}^{(t)})^j\right)$$

where $d_{r_1}, d_{r_2}, d_{r_3}, d_{r_4}$ are the maximum row degrees in D, DA^T, X, Y, respectively.

Thus, combining the above recursions, and setting $x = p_{L_1}^{(t)}, y = p_{L_2}^{(t)}, z = p_{L_3}^{(t)}$ and $f(x,y,z) = p_{L_1}^{(t+1)}$, $g(x,y,z) = p_{L_2}^{(t+1)}$, and $h(x,y,z) = p_{L_3}^{(t+1)}$, we can eliminate the $q_{R_s}^{(t)}$'s and obtain a final set of recursions for $p_{L_1}^{(t)}, p_{L_2}^{(t)}, p_{L_3}^{(t)}$. The following conditions must hold for the iterative decoder at $\mathbf{D}$ to recover all the erasures:

$$f(x,y,z) = p_2 \cdot \lambda^{(3)}\left(\sum_{r=2}^{d_{r_3}+d_{r_4}} \sum_{i+j=r-1} \gamma_{i,j}(\rho^{(3)}, \rho^{(4)}) \left(1 - (1-x)^i (1-z)^j\right)\right)$$

$$g(x,y,z) = p \cdot \lambda^{(1)}\left(\sum_{r=2}^{d_{r_1}+d_{r_2}} \sum_{i+j=r-1} \gamma_{i,j}(\rho^{(1)}, \rho^{(2)}) \left(1 - (1-y)^i (1-z)^j\right)\right)$$

$$h(x,y,z) = p \cdot \left(\sum_{r=2}^{d_{v_4}+d_{v_2}} \sum_{i+j=r-1} \delta_{i,j}(\lambda^{(4)}, \lambda^{(2)})(q_{R_1}^{(t)})^i (q_{R_2}^{(t)})^j \right),$$

where $q_{R_1}^{(t)}$ and $q_{R_2}^{(t)}$ are functions of x, y, and z.

$$f(x,y,z) < x, \quad \forall x \in (0, p_2], \forall y, z \in (0, p] \quad (3)$$
$$g(x,y,z) < y, \quad \forall x \in (0, p_2], \forall y, z \in (0, p] \quad (4)$$
$$h(x,y,z) < z, \quad \forall x \in (0, p_2], \forall y, z \in (0, p] \quad (5)$$

The remaining conditions to be satisfied are on the rate R of the LDPC code from $\mathbf{S}$ and the sizes of the sub-matrices D,DA^T, X, Y. For convenience, we choose $\ell = n$. Then,

$$1 - \frac{\int_0^1 \rho^{(1)}(x)dx}{\int_0^1 \lambda^{(1)}(x)dx} = 0,\ 1 - \frac{\int_0^1 \rho^{(3)}(x)dx}{\int_0^1 \lambda^{(3)}(x)dx} = 0 \quad (6)$$

$$1 - \frac{\int_0^1 \rho^{(2)}(x)dx}{\int_0^1 \lambda^{(2)}(x)dx} = \frac{k-m}{k} = 2 - \frac{1}{R} \geq 2 - \max\{\frac{1}{1-p_1}, \frac{1}{1-p+1-p_2}\} \quad (7)$$

$$1 - \frac{\int_0^1 \rho^{(4)}(x)dx}{\int_0^1 \lambda^{(4)}(x)dx} = \frac{k-\ell}{k} = 1 - \frac{1}{R} \geq 1 - \max\{\frac{1}{1-p_1}, \frac{1}{1-p+1-p_2}\} \quad (8)$$

Condition (1) holds as long as (6)-(8) hold. And further, using the analysis for decoding at $\mathbf{D}$, condition (2), for recovering all erasures at $\mathbf{R}$, can be reduced to the following:

$$f_R(x,y) = p_1 \cdot \lambda^{(1)}\left(\sum_{r=2}^{d_{r_1}+d_{r_2}} \sum_{i+j=r-1} \gamma_{i,j}(\rho^{(1)}, \rho^{(2)}) \left(1 - (1-x)^i (1-y)^j\right)\right)$$

$$g_R(x,y) = p_1 \cdot \lambda^{(2)}\left(\sum_{r=2}^{d_{r_1}+d_{r_2}} \sum_{i+j=r-1} \gamma_{i,j}(\rho^{(1)}, \rho^{(2)}) \left(1 - (1-x)^i (1-y)^j\right)\right)$$

$$f_R(x,y) < x, \quad \forall x, y \in (0, p_1] \quad (9)$$
$$g_R(x,y) < y, \quad \forall x, y \in (0, p_1] \quad (10)$$

The conditions (3)-(10) yield the design rules for constructing LDPC codes to achieve or come close to the cut-set upper-bound of capacity for the three-terminal degraded erasure relay channel.

C. Simplification: Decode and Forward

A simpler strategy to adopt for coding in the degraded relay case is the following: $\mathbf{S}$ uses an LDPC code H of rate $\frac{k}{n}$ to transmit to $\mathbf{R}$ and $\mathbf{D}$. As long as the rate $\frac{k}{n} \leq 1 - p_1$, an LDPC code with an appropriate degree distribution (λ, ρ) can be designed to recover all erasures at $\mathbf{R}$. $\mathbf{R}$ decodes the $\mathbf{S}$'s transmission and simply forwards all the n symbols to $\mathbf{D}$. Asymptotically, at large n, $\mathbf{D}$ receives approximately $n(1-p)$ symbols from $\mathbf{S}$ and $n(1-p_2)$ symbols from $\mathbf{R}$. However, approximately only $np(1-p_2)$ symbols from the $\mathbf{R}$'s transmission are new information to $\mathbf{D}$. The total number of unerased symbols at $\mathbf{D}$ is approximately $n(1-p)+np(1-p_2) = n(1-pp_2)$. Thus, $\mathbf{D}$ views the transmission of the sender's codeword as over an erasure channel with effective erasure probability pp_2. $\mathbf{D}$ also uses the LDPC matrix H to decode. This coding scheme achieves a maximum rate $R \leq \min\{1-p_1, 1-pp_2\}$ which is lower than the cut-set bound in Theorem 3.2 for the degraded case. The design rules for the degree-distribution of H are

$$p_1\lambda(1-\rho(1-x)) < x, \ \forall x \in (0, p_1] \text{ to decode at } \mathbf{R} \tag{11}$$

$$pp_2\lambda(1-\rho(1-x)) < x, \ \forall x \in (0, pp_2] \text{ to decode at } \mathbf{D} \tag{12}$$

$$1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx} \leq \min\{1-p_1, 1-pp_2\} \tag{13}$$

Although, the cut-set bound is not achievable using this coding scheme, the simplicity in the LDPC design makes up for the loss in capacity achievable by this method. Furthermore, the design rules say that any (capacity-achieving) LDPC code designed on the binary erasure channel with the desired rate will work well in this scenario.

V. The Non-degraded case

The difference between the degraded case and the non-degraded case is that in the degraded case, the relay has access to all the information from the sender that are available to the destination. The maximum-achievable rate for the erasure relay channel in the non-degraded case is given by

$$R \leq \min\{1-pp_1, (1-p)+\beta(1-p_2)\}$$

for $0 \leq \beta \leq 1$. Here $\beta = 1$ if $R \leq 1-p_1$ and $\beta = p$ otherwise.

The case when $1-p_1 \geq 1-p+1-p_2$ reduces to the degraded case and has already been considered. We will consider some other interesting cases and review the coding method that achieves this upper-bound on the achievable rate using MDS codes as discussed in [7]. Further, we will propose LDPC coding alternatives for these cases.

The general coding scheme is as follows: $\mathbf{S}$ uses an $[n,k]$ MDS code with generator matrix $[I_{k\times k}|\ A_{k\times(n-k)}]$. In the general erasure channel, $\mathbf{R}$ asymptotically, at large n, receives $n(1-p_1)$ symbols sent from $\mathbf{S}$ and $\mathbf{D}$ receives $n(1-p)$ symbols. Of these, asymptotically, $n(1-p_1)(1-p)$ are received by both $\mathbf{R}$ and $\mathbf{D}$, $np(1-p_1)$ symbols are received by $\mathbf{R}$ alone and not by $\mathbf{D}$, and $np_1(1-p)$ symbols are received by $\mathbf{D}$ alone and not by $\mathbf{R}$.

A. Case I: $1-p_1 \leq 1-p_2$

For $R > 1-p_1$, these conditions imply that the maximal achievable rate must satisfy $R \leq \min\{1-pp_1, 1-p+p(1-p_2)\} = 1-pp_1$.

1) MDS coding: Suppose the $[n,k]$ MDS code used by $\mathbf{S}$ has rate $R = \frac{k}{n} > 1-p_1$, then $\mathbf{R}$ receives approximately $k^* = n(1-p_1)$ symbols from $\mathbf{S}$. $\mathbf{R}$ then encodes these k^* symbols using an $[n^*, k^*]$ MDS code of rate $R_2 = \frac{k^*}{n^*} \leq 1-p_2$. This ensures that $\mathbf{D}$ can reconstruct the k^* symbols since the capacity of the $\mathbf{R}$ to $\mathbf{D}$ channel is $1-p_2$. Out of these, approximately pk^* symbols have not been received by $\mathbf{D}$ via the direct link from $\mathbf{S}$. Thus, as long as $n(1-p)+pk^* = n(1-pp_1) \geq k$, the destination $\mathbf{D}$ will be able to reconstruct the k information symbols of $\mathbf{S}$.

2) LDPC coding: $\mathbf{S}$ uses an $[n,k]$ LDPC code $\mathcal{C}$ described by a sparse parity-check matrix H with degree-distribution (λ, ρ). For a large block length n, $\mathbf{R}$, asymptotically receives $k^* = n(1-p_1)$ symbols. The relay uses an $[n^*, k^*]$ LDPC code $\mathcal{C}_2$ of rate $R_2 = \frac{k^*}{n^*} \leq 1-p_2$ and described by a parity-check matrix H_2 with degree-distribution $(\lambda^{(2)}, \rho^{(2)})$. $\mathbf{D}$ first decodes the LDPC code $\mathcal{C}_2$ using iterative decoding. From the previous sections, the design-rules for $(\lambda^{(2)}, \rho^{(2)})$ to ensure perfect recovery of the k^* encoded symbols are

$$f(x) = p_2\lambda^{(2)}(1-\rho^{(2)}(1-x)) < x, \ \forall x \in (0, p_2] \tag{14}$$

$$R_2 = 1 - \frac{\int_0^1 \rho^{(2)}(x)dx}{\int_0^1 \lambda^{(2)}(x)dx} \leq 1-p_2 \tag{15}$$

Once $\mathbf{D}$ has decoded the $k^* = n(1-p_1)$ symbols of the encoded codeword from $\mathbf{R}$, it obtains knowledge of approximately pk^* symbols that were not received via the direct link from $\mathbf{S}$. Thus, $\mathbf{D}$ now has information of $n(1-p) + np(1-p_1) = n(1-pp_1)$ symbols. Hence, the effective erasure probability between $\mathbf{S}$ and $\mathbf{D}$, for decoding the LDPC code $\mathcal{C}$, is pp_1. $\mathbf{D}$ now uses the parity-check matrix H to recover all the erased symbols. Hence, the design rules for (λ, ρ), to ensure perfect recover of all the erased symbols of the codeword sent from $\mathbf{S}$, are

$$f(x) = pp_1\lambda(1-\rho(1-x)) < x, \forall x \in (0, pp_1] \tag{16}$$

$$R = 1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx} \leq 1-pp_1 \tag{17}$$

From the above rules, it is clear that the design of the two LDPC codes used at $\mathbf{S}$ and $\mathbf{R}$ are independent. Hence, LDPC codes with desired rates designed for the BEC will work well in this scenario. Figure 2 shows the performance of two rate 1/2 LDPC codes of a specific irregular degree profile used at $\mathbf{S}$ and at $\mathbf{R}$ for the non-degraded relay channel. The performance for different blocklengths n is shown as a function of the $\mathbf{S}-\mathbf{D}$ erasure probability p. (p_1 and p_2 are fixed at 0.625 and 0.5, respectively. Case I situation arises for $p \geq 0.8$. The irregular LDPC codes that were constructed have poor distances and reveal error floors.)

B. Case II: $1-p_2 \leq 1-p_1$

For $R > 1-p_1$, these conditions imply that the maximal achievable rate must satisfy $R \leq \min\{1-pp_1, 1-p+p(1-p_2)\} = 1-pp_2$.

1) MDS coding: Suppose the $[n,k]$ MDS code used by $\mathbf{S}$ has rate $R = \frac{k}{n} > 1-p_1$, then $\mathbf{R}$ receives approximately $n(1-p_1)$ symbols from $\mathbf{S}$. $\mathbf{R}$ then randomly chooses $k^* = n(1-p_2)$ symbols out of the $n(1-p_1)$ symbols and encodes them using an $[n^*, k^*]$ MDS code of rate $R_2 = \frac{k^*}{n^*} \leq 1-p_2$. This ensures that $\mathbf{D}$ can reconstruct the k^* symbols since the capacity of the $\mathbf{R}$ to $\mathbf{D}$ channel is $1-p_2$. Out of these approximately pk^* symbols have not been received by $\mathbf{D}$ via the direct link from $\mathbf{S}$ to $\mathbf{D}$. Thus, as long as $n(1-p)+pk^* = n(1-pp_2) \geq k$, the destination $\mathbf{D}$ will be able to reconstruct the k information symbols of $\mathbf{S}$.

2) LDPC coding: Again, $\mathbf{S}$ uses an $[n, k]$ linear code $\mathcal{C}$ described by a sparse parity-check matrix H with degree-distribution (λ, ρ). For a large block length n, $\mathbf{R}$ asymptotically receives $k' = n(1 - p_1)$ symbols. $\mathbf{R}$ randomly chooses $k^* = n(1 - p_2)$ of these k' symbols and encodes them using an $[n^*, k^*]$ LDPC code $\mathcal{C}_2$ of rate $R_2 = \frac{k^*}{n^*} \leq 1 - p_2$ and described by a parity-check matrix H_2 with degree-distribution $(\lambda^{(2)}, \rho^{(2)})$. $\mathbf{D}$ first decodes the LDPC code $\mathcal{C}_2$ using iterative decoding. The design-rules for $(\lambda^{(2)}, \rho^{(2)})$ to ensure perfect recovery of the k^* encoded symbols remain the same:

$$f(x) = p_2 \lambda^{(2)}(1 - \rho^{(2)}(1 - x)) < x, \ \forall x \in (0, p_2] \tag{18}$$

$$R_2 = 1 - \frac{\int_0^1 \rho^{(2)}(x)dx}{\int_0^1 \lambda^{(2)}(x)dx} \leq 1 - p_2 \tag{19}$$

Once $\mathbf{D}$ has decoded the $k^* = n(1 - p_2)$ symbols of the encoded codeword from S, it obtains knowledge of approximately pk^* symbols that were not received via the direct link from $\mathbf{S}$. Thus, $\mathbf{D}$ now has information of $n(1 - p) + np(1 - p_2) = n(1 - pp_2)$ symbols. Hence, the effective erasure probability between $\mathbf{S}$ and $\mathbf{D}$, for decoding the LDPC code $\mathcal{C}$, is now pp_2. The decoding rule remains the same as before and the design rules for (λ, ρ) become

$$f(x) = pp_2 \lambda(1 - \rho(1 - x)) < x, \forall x \in (0, pp_2] \tag{20}$$

$$R = 1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx} \leq 1 - pp_2, \tag{21}$$

Here also, the design of the two LDPC codes is independent and therefore, good LDPC codes with desired rates designed on the BEC will work well in this scenario. Figure 3 shows the performance of a rate 1/2 LDPC code of a specific irregular degree profile at $\mathbf{S}$ and a rate 0.375 LDPC code at $\mathbf{R}$ for the non-degraded relay channel. The performance for different blocklengths n is shown as a function of the $\mathbf{S} - \mathbf{D}$ erasure probability p. (p_1 and p_2 are fixed at 0.5 and 0.625, respectively. Case II situation arises for $p \leq 0.8$.)

C. Case III: $1-p+1-p_2 \geq 1-p_1 \geq 1-p+p(1-p_2) = 1-pp_2$

This case has not received adequate attention in [7] and we believe the cut-set bound is not achievable for this case. This is because if a code of rate $R = \min\{1-pp_1, (1-p)+\beta(1-p_2)\}$ is used at $\mathbf{S}$ and $R \leq 1 - p_1$, then we must have $\beta = 1$ by Theorem 3.2. However, to achieve the cut-set bound, we must have $R = \min\{1 - pp_1, 1 - p + 1 - p_2\}$ ($\geq 1 - p_1$ by the hypothesis), which is a contradiction. On the other hand, if a code of rate $R > 1 - p_1$ is used at $\mathbf{S}$, then the coding scheme presented in section III-A will fail since the relay will fail to reconstruct the sender's transmission perfectly.

Hence, the best rate that may be achievable in this case is $R \leq 1 - pp_2$ and the coding scheme to achieve this rate is the same as for the "Decode and Forward" strategy in the degraded situation. Consequently, the design rules for the LDPC code at $\mathbf{S}$ remain the same as in that case. Figure 4 shows the performance of a rate 1/2 LDPC code of a specific irregular degree profile used at $\mathbf{S}$ for both the non-degraded and the degraded relay channel. The relay uses the "Decode and Forward" strategy. The performance for different blocklengths n is shown as a function of the $\mathbf{S} - \mathbf{D}$ erasure probability p. (p_1 and p_2 are fixed at 0.1 and 0.625, respectively. Case III situation in the non-degraded case arises when $p \geq 0.16$. It is interesting to note that at intermediate values of p, the performance of the same code in the degraded channel differs from that in the non-degraded channel.)

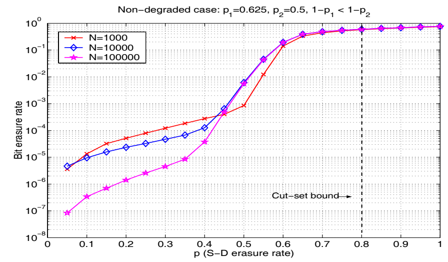

Fig. 2. Performance of irregular rate 1/2 length N LDPC codes at $\mathbf{S}$ and $\mathbf{R}$ as a function of p: $p_1 = 0.625, p_2 = 0.5$, Case I for $p \geq 0.8$

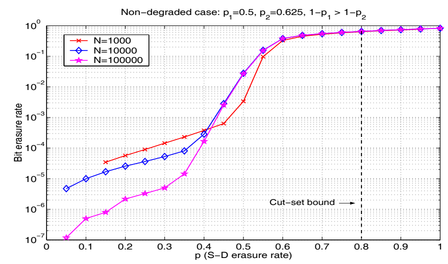

Fig. 3. Performance of rate 1/2 length N LDPC at $\mathbf{S}$, rate 0.375 LDPC at $\mathbf{R}$ as a function of p. $p_1 = 0.5, p_2 = 0.625$, Case II for $p \leq 0.8$

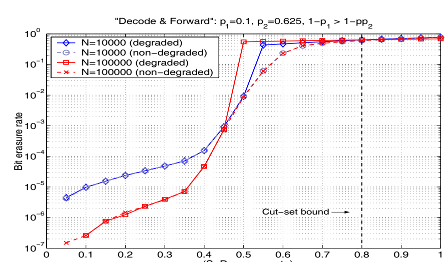

Fig. 4. Performance of rate 1/2 length N LDPC code at $\mathbf{S}$ as a function of p. $p_1 = 0.1, p_2 = 0.625$, Case III for $p \geq 0.16$

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