The Size of Strength-Maximal Graphs

Hong-Jian Lai
DEPARTMENT OF MATHEMATICS
WEST VIRGINIA UNIVERSITY
MORGANTOWN, WEST VIRGINIA 26506

ABSTRACT

Let G be a graph and let $\kappa'(G)$ be the edge-connectivity of G. The *strength* of G, denoted by $\bar{\kappa}'(G)$, is the maximum value of $\kappa'(H)$, where H runs over all subgraphs of G. A simple graph G is called *k-maximal* if $\bar{\kappa}'(G) \le k$ but for any edge $e \in E(G^c)$, $\bar{\kappa}'(G + e) \ge k + 1$. Let G be a k-maximal graph of order n. In [3], Mader proved $|E(G)| \le (n - k)k + \binom{k}{2}$. In this note, we shall show $(n - 1)k - \binom{k}{2}\lfloor n/(k + 2)\rfloor \le |E(G)|$, and characterize the extremal graphs. We shall also give a characterization of all k-maximal graphs.

I. INTRODUCTION

We shall adopt the notation and terminology of Bondy and Murty [2] except for contractions. A graph G may have multiple edges but loops are not allowed. Let G^c denote the complement of a simple graph G. Let H and G be graphs. By $H \subseteq G$ we mean that H is a subgraph of G and by $H \cong G$ we mean that H is isomorphic to G. If X is a subset of $V(G)$ or of $E(G)$, then $G[X]$ denotes the subgraph of G induced by X. An *edge-cut* of a graph G is an edge subset whose removal increases the number of components of G. A *bond* of G is a minimal edge-cut. An edge-cut of size k is called a k-*edge-cut*. We use $\mathbb{N}$ to denote the set of all positive integers. The *floor* of a real number x, denoted by $\lfloor x\rfloor$, is the greatest integer not larger than x. For any $k \in \mathbb{N}$, we define $\binom{k}{2} = [k(k - 1)]/2$ and so $\binom{1}{2} = 0$. The *join* of two graphs G and H, denoted by $G \vee H$, has

$$V(G \vee H) = V(G) \cup V(H)$$

and

$$E(G \vee H) = E(G) \cup E(H) \cup \{uv\colon u \in V(G) \text{ and } v \in V(H)\}.$$

Journal of Graph Theory, Vol. 14, No. 2, 187–197 (1990)
 CCC 0364-9024/90/020187-11$04.00

We use the following definition for *contractions*. Let G be a graph and let X be a subset of $E(G)$. We use G/X to denote the graph obtained from G by identifying the two ends of each edge in X and then deleting all loops produced.

We define the *strength* of G to be the following number:

$$\overline{\kappa}'(G) = \max_{H \subseteq G} \kappa'(H). \tag{1}$$

The invariant $\overline{\kappa}'(G)$, first introduced by Matula [4], has been studied by Boesch and McHugh [1], by Matula [4,5], by Mitchem [6], and implicitly by Mader [3], among others. In [5], Matula gave a polynomial algorithm to determine $\overline{\kappa}'(G)$.

Fix $k \in \mathbb{N}$. A simple graph G is k-*maximal* if $|V(G)| > k$, $\overline{\kappa}'(G) \leq k$ and if for any edge e of G^c, $\overline{\kappa}'(G + e) > k$.

Define, for $n, k \in \mathbb{N}$, $n > k \geq 1$,

$$f(n,k) = \min\{|E(G)|: G \text{ is simple, of order } n, \text{ and } k\text{-maximal}\},$$

and

$$F(n,k) = \max\{|E(G)|: G \text{ is simple, of order } n, \text{ and } k\text{-maximal}\}.$$

Since K_{k+1} is the only k-maximal graph of order $k + 1$,

$$f(k + 1, k) = F(k + 1, k) = \binom{k+1}{2}. \tag{2}$$

Let $\mathscr{E}\chi(f; n, k)$ ($\mathscr{E}\chi(F; n, k)$, respectively) denote the set of simple k-maximal graphs of order n and with strength at most k such that $|E(G)| = f(n,k)$ ($|E(G)| = F(n,k)$, respectively). In [3], Mader proved

Theorem (Mader [3]). For $n, k \in \mathbb{N}$ with $n > k \geq 1$, we have

(a) $F(n,k) = (n - k)k + \binom{k}{2}$;

(b) $G \in \mathscr{E}\chi(F; n, k)$ if and only if $G = K_{k+1}$ or G has a vertex of v of degree k such that $G - v \in \mathscr{E}\chi(F; n - 1, k)$.

In this note, we shall show

$$f(n,k) = (n - 1)k - \binom{k}{2}\left\lfloor \frac{n}{k + 2} \right\rfloor, \tag{3}$$

and shall characterize $\mathscr{E}\chi(f; n, k)$.

II. EXAMPLES

The following examples of k-maximal graphs will be used in this paper.

Example 1. For $n, k \in \mathbb{N}$ with $n > k + 1 > 2$, we define a graph $H(k, n - k)$ to be the simple graph $K_k \vee (n - k)K_1$. If $H = H(k, n - k)$, we use H^1 to denote the subset of $V(H)$ that corresponds to the $n - k$ K_1's.

Example 2. For $n, k, r \in \mathbb{N}$ with $n > k + 1 > 2$ and with $r \geq 2$, and for $m_1, m_2, \ldots, m_r \in \mathbb{N}$ with $k \geq m_i \geq 2$, $1 \leq i \leq r$, and with $m_1 + m_2 + \cdots + m_r = n - rk$, and for any tree T with $V(T) = \{v_1, v_2, \ldots, v_r\}$, we define $T(k; m_1, m_2, \ldots, m_r)$ to be the simple graph obtained from T by replacing each v_i by a graph $H_i = H(k, m_i)$ and by replacing each edge of T, say $v_s v_t$, $1 \leq s, t \leq r$, by a set $E_{s,t}$ of k edges such that each vertex in $H_s^1 \cup H_t^1$ is incident with at least one edge of $E_{s,t}$. Since this graph is simple, $E_{s,t}$ has at most $m_s m_t$ edges, and so $k \leq m_s m_t$ for all s and t such that $v_s v_t \in E(T)$.

III. MAIN RESULTS

We start with some lemmas.

Lemma 1. Let $n = |V(G)|$. Suppose that G has an edge-cut X such that $G[X]$ is spanning subgraph of G.

(a) If $G[X]$ is spanned by a complete bipartite graph, then

$$|X| \geq n - 1 .$$

(b) If G is simple and $G[X]$ is a complete bipartite graph, then either $\kappa'(G) < |X|$ or G is complete and one component of $G - X$ is a single vertex.

Proof. Suppose that X satisfies the hypothesis. If $G[X]$ has a complete bipartite subgraph, then $G[X]$ contains a spanning tree of G and so (a) follows. We only need to prove (b).

Suppose that G is simple and G is not complete. By (a) of this lemma, $|X| \geq n - 1$ and, since G is simple, equality holds only when $G[X]$ is a star $K_{1,n-1}$. Since G is simple and not complete, then G has a vertex of degree smaller than $n - 1$ and so $\kappa'(G) < n - 1 \leq |X|$. ∎

Theorem 1. If $n = |V(G)| > k + 1$ and G is a k-maximal graph, then $\overline{\kappa}'(G) = \kappa'(G) = k$.

Proof. We argue by contradiction and assume that G is k-maximal but $k > \kappa'(G)$. Let X be an edge-cut of G with $|X| = \kappa'(G) < k$ and let G_1 and G_2 be the two components of $G - X$. Clearly $X \neq \emptyset$.

Since $X \neq \emptyset$ and since $k < n - 1$, it follows from (a) of Lemma 1 that there is an edge $e \in E(G^c)$ such that e is incident with a vertex of each component of $G - X$. Since G is k-maximal, $\overline{\kappa}'(G + e) > k$. Let H be a subgraph of $G + e$ such that $\kappa'(H) = \overline{\kappa}'(G + e) > k$. Since $|X \cup \{e\}| \leq k$, H must be a subgraph of either G_1 or G_2. It follows that $\overline{\kappa}'(G) \geq k + 1$, contrary to the fact that $\overline{\kappa}'(G) \leq k$. ∎

The converse of Theorem 1 is false. Let H_1 and H_2 be two $H(2,2)$'s and let $H_1^! = \{w_1, w_2\}$ and $H_2^! = \{u_1, u_2\}$. Let G be the graph obtained from the union of H_1 and H_2 by joining u_i and w_i by a path of length two, $i = 1, 2$. It is easy to see that $\overline{\kappa}'(G) = 2 = \kappa'(G)$ and that G is not 2-maximal.

Lemma 2. Let G be a simple k-maximal graph of order n, where $n > k + 1 > 2$. Then exactly one of the following holds:

(i) $G \cong H(k, 2)$, or
(ii) For any k-edge-cut X of G, each component of $G - X$ is either K_1 or a k-maximal graph that is not K_{k+1}.

Proof. It is clear that (i) and (ii) of Lemma 2 are mutually exclusive.

Let X be a k-edge-cut, and let G_1 and G_2 denote the two components of $G - X$.

Suppose first that $G_2 \cong K_1$. If G_1 is complete, then since $n > k + 1$, $|V(G_1)| \geq k + 1$. Since $\overline{\kappa}'(G) = \kappa'(G) = k$, and since G_1 is complete, G_1 has order at most $k + 1$. Thus $G_1 \cong K_{k+1}$ and so (i) of the lemma holds. Hence we may assume that G_1 is not complete. Let $e \in E(G_1^c)$. Since G is k-maximal, there is a subgraph $L \subseteq G_1 + e$ such that $\kappa'(L) \geq k + 1$. Since L is simple with $\delta(L) \geq \kappa'(L) \geq k + 1$, $|V(G_1)| \geq |V(L)| \geq k + 2$. Hence G_1 is k-maximal and we are done.

Similarly, the lemma will follow if $G_1 \cong K_1$. Hence we may assume that both $|V(G_1)|$ and $|V(G_2)|$ are greater than one. Thus

$$\min\{|V(G_1)|, |V(G_2)|\} \geq 2. \tag{4}$$

Case 1. Suppose that one of the G_i's is complete, say

$$G_1 \cong K_m, \qquad \text{for some } m \geq 2.$$

We shall derive a contradiction.

Since G is k-maximal, $\kappa'(G) \leq k$ and so $m \leq k + 1$. Thus

$$2 \leq m \leq k + 1. \tag{5}$$

Let e be any edge of $E(G^c)$ such that e has exactly one end in $V(G_1)$ and one end in $V(G_2)$. We claim that such an edge e exists. There are $|V(G_1)|\,|V(G_2)|$ pairs (v_1, v_2) in $V(G_1) \times V(G_2)$, and only k of them cannot be the ends of e, because they are joined by one of the k edges of X. By (4),

$$|V(G_1)|\,|V(G_2)| \geq |V(G_1)| + |V(G_2)| = n > k + 1 = |X| + 1,$$

and so e exists, as claimed.

Since G is k-maximal, $G + e$ has a $(k + 1)$-edge-connected subgraph, say $L = L(e)$. If $e \notin E(L)$, then $L \subseteq G$, contrary to $\overline{\kappa}'(G) = k$. Hence, $e \in E(L)$ and it follows that the edge-cut $X \cup \{e\}$ of $G + e$ must be $E(L)$.

Define $L_i = L[V(G_i) \cap V(L)]$, $i \in \{1, 2\}$, and denote $|V(L_1)| = t$.

Without loss of generality, since (4) gives $|V(G_1)| \geq 2$, we can assume that $t \geq 2$: if all edges of X were incident with a single vertex of $V(G)$, then by (4), e can be chosen incident with a vertex of $V(G_1)$ that is not incident with an edge of X.

Also without loss of generality, we can assume that $t \leq k$, as we now show. If the edges of X are incident with fewer than k vertices of $V(G_1)$, then $t \leq k$. If the edges of X are incident with k distinct vertices of $V(G_1)$ then each of those k vertices is incident with exactly one edge of X. By (4), $|V(G_2)| \geq 2$, and so e may be chosen in $E(G^c)$ such that e and some edge of X are incident with a common vertex on $V(G_1)$. Since $k \geq 2$, we can also assume as before that not all edges of $X \cup \{e\}$ are incident with the same vertex in $V(G_1)$. Hence

$$2 \leq t \leq k. \tag{6}$$

Since $\kappa'(L) \geq k + 1$, all t vertices of $V(G_1)$ have degree at least $k + 1$ in L. This, $L_1 \cong K_t$, and $X \cup \{e\} \subseteq E(L)$ give

$$t(k + 1) \leq \sum_{v \in V(L_1)} d_L(v) = 2\binom{t}{2} + |X \cup \{e\}| = t(t - 1) + (k + 1).$$

Thus,

$$(t - 1)(k + 1) \leq t(t - 1).$$

By $2 \leq t$ of (6), we can divide each side by $(t - 1)$ to get $k + 1 \leq t$, which contradicts (6). This concludes Case 1.

Case 2. G_1 is not complete.

For any edge $e \in E(G_1^c) \subseteq E(G^c)$, $G + e$ has a subgraph L with $\kappa'(L) \geq k + 1$. Since $|X| \leq k$, L is a subgraph of G_1. Note that $|V(G_1)| \geq |V(L)| \geq k + 1$. Hence G_1 is a k-maximal if we can show that $\overline{\kappa}'(G_1) \leq k$. By Theorem 1, $\overline{\kappa}'(G_1) \leq \overline{\kappa}'(G) = k$, and so G_1 is k-maximal. Since G_1 is not complete, G_1 is not isomorphic to K_{k+1}.

Similarly, G_2 is k-maximal and is not isomorphic to K_{k+1}. ∎

By Lemma 2, we can determine the minimal k-maximal graphs.

Corollary 2A. Let $n > k > 1$ and let G be a k-maximal simple graph of order n that is not isomorphic to K_{k+1}. If every proper k-maximal subgraph of G is isomorphic to K_{k+1}, then $G \cong H(k, 2)$.

Proof. By Theorem 1, G has an edge-cut X of size k. Let G_1 and G_2 be the two components of $G - X$. By Lemma 2, either $G \cong H(k,2)$ or one of G_1 and G_2 is a proper k-maximal subgraph of G that is not isomorphic to K_{k+1}. By the hypothesis, $G \cong H(k,2)$. ∎

Corollary 2B. Let $k \in \mathbb{N}$ and let G be a k-maximal graph with $|V(G)| = k + 2$, then $G \cong H(k,2)$.

Proof. The proof is trivial if $k = 1$ and so we assume $k > 1$. By Theorem 1 and Lemma 2, G must have a vertex v of degree k. By (2), $G - v \cong K_{k+1}$. Hence $G \cong H(k,2)$. ∎

Let $k \in \mathbb{N}$ and let H_1 and H_2 be two graphs with disjoint vertex sets and with $\max\{|V(H_1)|, |V(H_2)|\} \geq k$. A *k-edge-join* of H_1 and H_2 is a simple graph obtained from the disjoint union of H_1 and H_2 by adding k new edges $e_1, e_1, \ldots, e_k$ to the union of H_1 and H_2 such that each e_i is incident with a vertex of $V(H_1)$ and a vertex of $V(H_2)$. Denote by $[H_1, H_2]_k$ the set of all k-edge-joins of H_1 and H_2. Clearly, $[H_1, H_2]_k = [H_2, H_1]_k$.

Lemma 3. Let $k \in \mathbb{N}$, let H_1 be a k-maximal graph and let H_2 be either a K_1 or a k-maximal graph. Then all graphs in $[H_1, H_2]_k$ are k-maximal.

Proof. Let G be a graph in $[H_1, H_2]_k$. It is easy to see that

$$\overline{\kappa}'(G) \leq \max\{\overline{\kappa}'(H_1), \overline{\kappa}'(H_2)\} \leq k\,.$$

The lemma becomes trivial if $k = 1$. Thus we assume $k > 1$.

By way of contradiction, let $G \in [H_1, H_2]_k$ be a counterexample of minimum number of vertices, for some H_1 and H_2 satisfying Lemma 3, and let $e \in E(G^c)$ such that $\overline{\kappa}'(G + e) \leq k$. Since H_1 and H_2, if nontrivial, are k-maximal graphs, $e \notin E(H_1^c) \cup E(H_2^c)$. Hence we assume that $e = x_1x_2$ with $x_i \in V(H_i)$, $(i = 1, 2)$.

Let $E' = E(G) - E(H_1) - E(H_2)$. Then $|E'| = k$, by the definition of $[H_1, H_2]_k$.

Let X be an edge-cut of $G + e$ with $|X| \leq k$. Applying Theorem 1 to H_1 and H_2, we conclude that neither H_1 nor H_2 has a k-edge-cut. This, together with the fact that the ends of $e = x_1x_2$ are in $V(H_1)$ and $V(H_2)$, respectively, implies that $X \cap (E' \cup \{e\}) = \emptyset$, and so for some $i \in \{1, 2\}$, $X \subseteq E(H_i)$.

Let H_i' and H_i'' be the components of $H_i - X$. Since H_i is k-maximal, by Lemma 2, each of H_i' and H_i'' is either K_1 or k-maximal. Let V' be the vertices of $V(H_i)$ that are incident with edges in $E' \cup \{e\}$. Since X is an edge-cut of $G + e$ and since $(E' \cup \{e\}) \cap X = \emptyset$, either $V' \subseteq V(H_i')$ or $V' \subseteq V(H_i'')$.

Without loss of generality, we assume that $V' \subseteq V(H_i')$, and so the edges in $X \cup \{e\}$ are all incident with vertices of $V(H_i')$. Let $G' = G - V(H_i'')$. Then G'

is a subgraph of G that is in $[H_i', H_{3-i}]_k$ with $e \in E((G')^c)$. By the minimality of G, all graphs in $[H_i', H_{3-i}]_k$ are k-maximal. Thus $\overline{\kappa}'(G' + e) \geq k + 1$, and so $\overline{\kappa}'(G + e) \geq \overline{\kappa}'(G' + e) \geq k + 1$, a contradiction. ∎

Definition of $\mathcal{M}(k)$. For $n, k \in \mathbb{N}$ with $n > k$, let $\mathcal{M}(k)$ denote the family of graphs containing K_{k+1} as the only graph of order $n + 1$, such that a graph G of order $n \geq k + 2$ is in $\mathcal{M}(k)$ if and only if there exist graphs H_1 and H_2, where H_i is either in $\mathcal{M}(k)$ or K_1, $(i = 1, 2)$, and where at least one of the H_i's is not K_1, such that $G \in [H_1, H_2]_k$.

Corollary 3. Let $n, k \in \mathbb{N}$ with $n > k$. A graph G of order n is k-maximal if and only if $G \in \mathcal{M}(k)$.

Proof. It follows from Lemma 3 and induction on $|V(G)|$. ∎

Definition of $\mathcal{F}(k)$. For $n, k \in \mathbb{N}$ with $n > k + 1$, let $\mathcal{F}(k)$ denote the graph family containing $H(k, 2)$ as the only graph of order $k + 2$, such that a graph G of order $n > k + 2$ is in $\mathcal{F}(k)$ if and only if there exist graphs H_1 and H_2 with order $n' = |V(H_1)|$ and $n'' = |V(H_2)|$, where H_i is either in $\mathcal{F}(k)$ or K_1, $(i = 1, 2)$, and where at least one of the H_i's is not K_1, such that $G \in [H_1, H_2]_k$ and such that

$$\left\lfloor \frac{n}{k+2} \right\rfloor = \left\lfloor \frac{n'}{k+2} \right\rfloor + \left\lfloor \frac{n''}{k+2} \right\rfloor. \tag{7}$$

It is clear that $\mathcal{F}(k)$ is a subfamily of $\mathcal{M}(k)$.

Theorem 2. For $n, k \in \mathbb{N}$ with $n > k + 1 \geq 2$, we have

(a) $f(n, k) = (n - 1)k - \binom{k}{2}\lfloor n/(k + 2)\rfloor$;
(b) $G \in \mathcal{E}\chi(f; n, k)$ if and only if $G \in \mathcal{F}(k)$ and $|V(G)| = n$.

Proof. We shall need the following trivial fact: for any real numbers x and y,

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor. \tag{8}$$

Let $r = \lfloor n/(k + 2)\rfloor$. It is easy to check that

$$\begin{aligned} |E(T(k; m_1, m_2, \ldots, m_r))| &= r\binom{k}{2} + (r - 1)k + (m_1 + m_2 + \cdots + m_r)k \\ &= r\binom{k}{2} + (r - 1)k + (n - rk)k \\ &= (n - 1)k - r\binom{k}{2}. \end{aligned}$$

Hence by the definition of $f(n, k)$,

$$f(n, k) \leq (n - 1)k - r\binom{k}{2}. \tag{9}$$

Let $G \in \mathscr{E}\chi(f; n, k)$. We shall prove (a) of Theorem 2 and $G \in \mathscr{F}(k)$ by induction on n. If $n = k + 2$, then by Corollary 2B, $G = H(k, 2)$ and so $G \in \mathscr{F}(k)$ and $f(k + 2, k) = (k + 1)k - \binom{k}{2}$. Hence Theorem 12 holds for $n = k + 2$.

Now suppose that $n > k + 2$. We assume that if $2 \leq k + 1 < m < n$ and if $G \in \mathscr{E}\chi(f; m, k)$, then $G \in \mathscr{F}(k)$ and

$$f(m, k) = (m - 1)k - \binom{k}{2}\left\lfloor \frac{m}{k + 2} \right\rfloor. \tag{10}$$

By Lemma 2, G has a k-edge-cut X such that each component of $G - X$ is either K_1 or a k-maximal graph of order greater than $k + 1$.

An edge-cut X of G is called a *fan* if one component of $G - X$ is a single vertex.

Claim. If $n > k + 2$ and if $(k + 2)$ is a factor of n, then X is not a fan.

By way of contradiction, suppose that $n = r(k + 2)$, for some $r \in \mathbb{N}$, and that X is a fan.

Let v be the vertex of G that is incident with every edge in X, and let $G' = G - v$.

Since G is k-maximal and since $n > k + 2$, it follows from Lemma 2 that G' is k-maximal. Since $n = r(k + 2)$,

$$\left\lfloor \frac{n - 1}{k + 2} \right\rfloor = r - 1. \tag{11}$$

By (9), (10), and (11),

$$\begin{aligned} (n - 1)k - r\binom{k}{2} &\geq f(n, k) \geq f(n - 1, k) + k \\ &= (n - 2)k - \left\lfloor \frac{n - 1}{k + 2} \right\rfloor \binom{k}{2} + k \\ &= (n - 1)k - \left\lfloor \frac{n - 1}{k + 2} \right\rfloor \binom{k}{2} \\ &= (n - 1)k - (r - 1)\binom{k}{2}, \end{aligned}$$

a contradiction. Hence the claim. ∎

Case 1. X is a fan.

By the claim, $(k+2)$ is not a factor of n. Hence

$$r = \left\lfloor \frac{n}{k+2} \right\rfloor = \left\lfloor \frac{n-1}{k+2} \right\rfloor . \tag{12}$$

Let v denote the vertex of G that is incident with edges in X. Then by (9), (10), and (12),

$$\begin{aligned}(n-1)k - r\binom{k}{2} &\geq f(n,k) \geq f(n-1,k) + k \\ &= (n-2)k - \left\lfloor \frac{n-1}{k+2} \right\rfloor \binom{k}{2} + k \\ &= (n-1)k - r\binom{k}{2} .\end{aligned} \tag{13}$$

Thus equalities must hold in (13) and so

$$f(n,k) = (n-1)k - r\binom{k}{2} . \tag{14}$$

By Lemma 2, $G - v$ is k-maximal. By (13) with equalities, $G - v \in \mathscr{E}\chi(f; n-1, k)$. By induction, $G - v \in \mathscr{F}(k)$. Thus $G \in \mathscr{F}'(k)$ by definition.

Case 2. X is not a fan.

Let the two components of $G - X$ be H_1 and H_2, and their orders be n' and n'', respectively. By (8), (9), and (10).

$$\begin{aligned}(n-1)k - r\binom{k}{2} &\geq f(n,k) \geq f(n',k) + f(n'',k) \\ &= (n'-1)k - \left\lfloor \frac{n'}{k+2} \right\rfloor \binom{k}{2} + k + (n''-1)k \\ &\quad - \left\lfloor \frac{n''}{k+2} \right\rfloor \binom{k}{2} \\ &\geq (n-1)k - r\binom{k}{2} .\end{aligned} \tag{15}$$

Thus equalities must hold in (15) and so (7) follows. By Lemma 2, the H_i's are either k-maximal or K_1 and at least one of them is not K_1. By (15) with equalities and by induction, the H_i's are either K_1 or in $\mathscr{F}(k)$. Hence $G \in \mathscr{F}(k)$ and (a) of Theorem 2 holds.

By Corollary 3, every graph in $\mathcal{F}(k)$ is k-maximal. To complete the proof of (b) of Theorem 2, it suffices to show that

$$\text{if } \quad n = |V(G)| \quad \text{and} \quad G \in \mathcal{F}(k), \qquad \text{then } G \in \mathcal{E}\chi(f; n, k). \tag{16}$$

Let $G \in \mathcal{F}(k)$ be a graph of order n with $n > k + 2$.

By the definition of $\mathcal{F}(k)$ and the assumption that $G \in \mathcal{F}(k)$, there exist graphs H_1 and H_2 with the H_i's being in $\mathcal{F}(k)$ or K_1 and not both H_1 and H_2 being K_1 such that $G \in [H_1, H_2]_k$ and that (7) holds. If one of the H_i's is K_1, then by (7) and by the assumption that $n \geq k + 2$, $(k + 2)$ is not a factor of n. Hence (12) holds. By (12) and by induction, we have

$$\begin{aligned} |E(G)| &= f(n - 1, k) + k \\ &= (n - 2)k - \left\lfloor \frac{n - 1}{k + 2} \right\rfloor \binom{k}{2} + k \\ &= (n - 1)k - r\binom{k}{2}. \end{aligned}$$

Thus (14) holds and so $G \in \mathcal{E}\chi(f; n, k)$.

If both H_1 and H_2 are not K_1, then by (7) and by induction, we have

$$\begin{aligned} |E(G)| &= f(n', k) + f(n'', k) \\ &= (n' - 1)k - \left\lfloor \frac{n'}{k + 2} \right\rfloor \binom{k}{2} + k + (n'' - 1)k - \left\lfloor \frac{n''}{k + 2} \right\rfloor \binom{k}{2} \\ &= (n - 1)k - r\binom{k}{2}, \end{aligned}$$

and so (14) holds also. Thus in any case, $G \in \mathcal{E}\chi(f; n, k)$. ∎

ACKNOWLEDGMENT

The author wishes to thank Paul A. Catlin, the author's Ph.D. supervisor, for his shortening the proof of Lemma 2 and other useful suggestions. He also wishes to thank the referees for their helpful comments.

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