MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. The shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-1}{4}$ and $\frac{x+2}{7}=\frac{y-2}{8}=\frac{z+1}{2}$ is

(1) $\frac{88}{\sqrt{1277}}$ (2) $\frac{78}{\sqrt{1277}}$

(3) $\frac{66}{\sqrt{1277}}$ (4) $\frac{55}{\sqrt{1277}}$

Answer (1)

Sol. $d=\frac{|(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)|}{|\vec{b}_1\times\vec{b}_2|}$

$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 7 & 8 & 2\end{vmatrix}$

$=-26\hat{i}+24\hat{j}-5\hat{k}$, $\vec{a}_2-\vec{a}_1=3\hat{i}+2\hat{k}$

$d=\left|\frac{(3\hat{i}+2\hat{k})\cdot(-26\hat{i}+24\hat{j}-5\hat{k})}{\sqrt{26^2+24^2+5^2}}\right|$

$=\left|\frac{-78-10}{\sqrt{1277}}\right|=\frac{88}{\sqrt{1277}}$

2. In a bag there are 6 white and 4 black balls two balls are drawn at random, then the probability that both ball are white are

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$

(3) $\frac{2}{3}$ (4) $\frac{1}{4}$

Answer (2)

Sol. $P(E)=\frac{{}^6C_2}{{}^{10}C_2}$

$=\frac{15}{45}=\frac{1}{3}$

3. Let $A = \{1, 2, 3\}$ number of non-empty equivalence relations from A to A are

(1) 4 (2) 5

(3) 6 (4) 8

Answer (2)

Sol. The partitions far a set with 3 elements, {1, 2, 3}

{{1},{2},{3}} – Every element is in its own subs

{{1,2},{3}} – Two elements are together, one separate

{{1, 3}, {2}} – Two elements are together, one separate

{{2, 3}, {1}} – Two elements are together, one separate

{{1, 2, 3}} – All elements are together in one subset

∴ Therefore, total possible equivalence relation = 5

4. If $f(x)=16(\sec^{-1}x)^2+(\text{cosec}^{-1}x)^2$. Then the maximum and minimum value of $f(x)$ is

(1) $\frac{1001\pi^2}{33}$ and $\frac{2\pi^2}{9}$ (2) $\frac{1105\pi^2}{68}$ and $\frac{4\pi^2}{17}$

(3) $\frac{1117\pi^2}{59}$ and $\frac{6\pi^2}{19}$ (4) $\frac{1268\pi^2}{27}$ and $\frac{3\pi^2}{16}$

Answer (2)

Sol. $f(x)=(4\sec^{-1}x)^2+(\text{cosec}^{-1}x)^2$

$=(4\sec^{-1}x+\text{cosec}^{-1}x)^2-8\sec^{-1}x\,\text{cosec}^{-1}x$

$=\left(3\sec^{-1}x+\frac{\pi}{2}\right)^2-8\sec^{-1}x\left(\frac{\pi}{2}-\sec^{-1}x\right)$

$= 9(\sec^{-1}x)^2 + \frac{\pi^2}{4} + 3\pi\sec^{-1}x - 4\pi\sec^{-1}x + 8(\sec^{-1}x)^2$

$= 17(\sec^{-1}x)^2 - \pi(\sec^{-1}x) + \frac{\pi^2}{4}$

$= 17\left[(\sec^{-1}x)^2 - \frac{\pi}{17}(\sec^{-1}x) + \frac{\pi^2}{34^2}\right] + \frac{\pi^2}{4} - \frac{17\pi^2}{34^2}$

$= 17\left(\sec^{-1}x - \frac{\pi}{34}\right)^2 + \frac{\pi^2}{4} - \frac{\pi^2}{68}$

$= 17\left(\sec^{-1}x - \frac{\pi}{34}\right)^2 + \frac{4\pi^2}{17}$

Min $= \frac{4\pi^2}{17}$

Max if $\sec^{-1}x = \pi$

$17\left(\pi - \frac{\pi}{34}\right)^2 + \frac{4\pi^2}{17}$

$\frac{1089}{68}\pi^2 + \frac{4\pi^2}{17} = \frac{1105\pi^2}{68}$

5. If $8 = 3 + \frac{1}{4}(3+p) + \frac{1}{4^2}(3+2p) + \ldots \infty$ then the value of p is

(1) $\frac{14}{5}$ (2) $\frac{16}{5}$

(3) $\frac{3}{5}$ (4) $\frac{4}{5}$

Answer (2)

Sol. $8 = \left[3 + \frac{3}{4} + \frac{3}{4^2} + \ldots + \infty\right] + \left[\frac{p}{4} + \frac{2p}{4^2} + \ldots + \infty\right]$

$8 = 3\left[1 + \frac{1}{4} + \frac{1}{4^2} + \ldots + \infty\right] + \left[\frac{p}{4} + \frac{2p}{4^2} + \ldots + \infty\right]$

$8 = 3\left[\frac{1}{1-\frac{1}{4}}\right] + \frac{\frac{p}{4}}{1-\frac{p}{4}}$

$8 = 3\left(\frac{4}{3}\right) + \frac{p}{4-p}$

$4 = \frac{p}{4-p}$

$\Rightarrow 16 - 4p = p$

$\Rightarrow 5p = 16$

$\Rightarrow p = \frac{16}{5}$

6. If $\frac{dx}{dy} + \frac{x}{y^2} = \frac{1}{y^3}$, $x(1) = 1$. Then $x\left(\frac{1}{2}\right)$ equals to

(1) $2 - e$ (2) $3 - e$

(3) $5 - e$ (4) $7 - e$

Answer (2)

Sol. I.F = $e^{\int \frac{1}{y^2}}$

I.F = $e^{-\frac{1}{y}}$

$\therefore \; x \cdot e^{-\frac{1}{y}} = \int e^{-\frac{1}{y}} \left(\frac{1}{y^3}\right) dy$

$x \cdot e^{-\frac{1}{y}} = \int e^{-\frac{1}{y}} \left(\frac{1}{y}\right)\left(\frac{1}{y^2}\right) dy$

Put $\frac{1}{y} = t$

$-\frac{1}{y^2}dy = dt$

$\therefore \; xe^{-t} = -\int e^{-t} \cdot t \, dt$

$xe^{-t} = -\left[te^{-t} - \int\left(\frac{d(t)}{dt} \cdot \int e^{-t} \cdot dt\right)dt\right]$

$xe^{-t} = -\left[-te^{-t} - e^{-t}\right] + c$

$xe^{-t} = te^{-t} + e^{-t} + c$ ...(1)

Given $x(1) = 1$

$e^{-1} = e^{-1} + e^{-1} + c$

$-e^{-1} = c$

$\therefore$ from (1)

$x = t + 1 - (e^{-1} \cdot e^{t})$

Put $y = \frac{1}{2}$

$x = 3 - e$

7. Let $T_r = \frac{(2r-1)(2r+1)(2r+3)(2r+5)}{64}$, then

$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{T_r}$ is equal to

(1) $\frac{22}{45}$ (2) $\frac{32}{35}$

(3) $\frac{27}{45}$ (4) $\frac{32}{45}$

Answer (4)

Sol. $T_r = \frac{(2r-1)(2r+1)(2r+3)(2r+5)}{64}$

$$\Rightarrow \frac{1}{T_r} = \frac{64}{16\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right)\left(r+\frac{5}{2}\right)}$$

$$\Rightarrow \frac{1}{T_r} = \frac{\frac{4}{3}\left[\left(r+\frac{5}{2}\right)-\left(r-\frac{1}{2}\right)\right]}{\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right)\left(r+\frac{5}{2}\right)}$$

$$\Rightarrow \frac{1}{T_r} = \frac{4}{3}\left[\frac{1}{\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right)} - \frac{1}{\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right)\left(r+\frac{5}{2}\right)}\right]$$

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{T_r} = \frac{4}{3}\left[\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}} - \frac{1}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}} + \frac{1}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}} - \frac{1}{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}} \ldots\right]$$

$$= \frac{4}{3}\left[\frac{8}{15}\right] = \frac{32}{45}$$

8. Coefficient of x^{2012} in $(1-x)^{2008}(1+x+x^2)^{2007}$

(1) 0 (2) 1

(3) 2 (4) 3

Answer (1)

Sol. $(1-x)[(1-x)(1+x+x^2)]^{2007}$

$= (1-x)(1-x^3)^{2007}$

$= (1-x^3)^{2007} - x(1-x^3)^{2007}$

[$(1-x^3)^{2007}$ contains 3λ types of exponents while $x(1-x^3)^{2007}$ will have $(3\lambda+1)$ type while 2012 is $(3\lambda+2)$ type] that is not possible $\Rightarrow 0$

Coefficient of x^{2012} in $(1-x^3)^{2007} = 0$

Coefficient of x^{2011} in $(1-x^3)^{2007} = 0$

$\Rightarrow$ Coefficient of x^{2012} in $(1-x)^{2008}(1+x+x^2)^{2007} = 0$

9. If the images of the points $A(1, 3)$, $B(3, 1)$ and $C(2, 4)$ in the line $x + 2y = 4$ are D, E and F respectively, then the centroid of the triangle DEF is

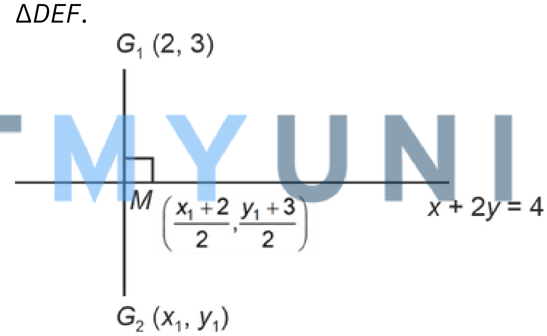
(1) $(3, -1)$ (2) $\left(\frac{3}{5}, -\frac{2}{5}\right)$

(3) $\left(\frac{2}{5}, -\frac{1}{5}\right)$ (4) $\left(\frac{1}{5}, \frac{2}{5}\right)$

Answer (3)

Sol. Centroid of the ΔDEF is the mirror image of the centroid of the ΔABC about the line $x + 2y = 4$.

G_1 = Centroid of $\Delta ABC \equiv (2, 3)$, $G_2 \equiv$ Centroid of ΔDEF.



$$\Rightarrow \frac{y_1-3}{x_1-2} = 2,\quad \frac{x_1+2}{2} + (y_1+3) = 4$$

$$\Rightarrow x_1 = \frac{2}{5},\ y_1 = -\frac{1}{5}$$

$$\Rightarrow G_2 \equiv \left(\frac{2}{5}, -\frac{1}{5}\right)$$

10. If $A = \{1, 2, 3, \ldots, 10\}$.

$B = \left\{\frac{m}{n} : m, n \in A \text{ and } m < n \text{ and gcd of } (m, n) = 1\right\}$.

Then number of elements in set B is

(1) 30 (2) 31

(3) 28 (4) 29

Answer (2)

Sol. $n = 1$ $m \in \varphi$ ...0

$n = 2$ $m = 1 \Rightarrow \frac{m}{n}$ can be $\frac{1}{2}$ ...1

$n = 3$ $m = 1, 2 \Rightarrow \frac{m}{n}$ can be $\frac{1}{3}, \frac{2}{3}$ ...2

$n = 4$ $m = 1, 3 \Rightarrow \frac{m}{n}$ can be $\frac{1}{4}, \frac{3}{4}$ ...2

$n = 5$ $m = 1, 2, 3, 4 \Rightarrow \frac{m}{n} = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ ...4

$n = 6$ $m = 1, 5 \Rightarrow \frac{m}{n} = \frac{1}{6}, \frac{5}{6}$ ...2

$n = 7$ $m = 1, 2, 3, 4, 5, 6 \Rightarrow \frac{m}{n} = \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$ ...6

$n = 8$ $m = 1, 3, 5, 7 \Rightarrow \frac{m}{n} = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ ...4

$n = 9$ $m = 1, 2, 4, 5, 7, 8 \Rightarrow \frac{m}{n} = \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}$ ...6

$n = 10$ $m = 1, 3, 7, 9 \Rightarrow \frac{m}{n} = \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$ ...4

11. How many ways are there to pick 5 letters from English alphabets such that *M* is the middle of the letters (repetition not allowed).

(1) 26C5.5!

(2) 25C4.4!

(3) 26C4.4!

(4) 25C5.5!

Answer (2)

Sol. $A_1 A_2$ __ $\underset{\uparrow \text{fixed}}{M}$ A_3 __ A_4

$^{25}C_4 \times 4!$

12. Let $|Z_i| = 1$ for $i = 1, 2, 3$ satisfying $|\bar{Z}_1 Z_2 + \bar{Z}_2 Z_3 + \bar{Z}_3 Z_1|^2 = a + b\sqrt{2}$, where *a*, *b* an rational numbers such that $\arg(Z_1) = \frac{\pi}{4}$, $\arg(Z_2) = 0$ and $\arg(Z_3) = \frac{-\pi}{4}$, then find (a, b)

(1) (5, 2) (2) (–5, –2)

(3) (5, –2) (4) (–5, 2)

Answer (3)

Sol. $Z_1 = |1| e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}$

$Z_2 = |1| e^{-(0)} = 1 + 0i$

$Z_3 = |1| e^{-\frac{i\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$\bar{Z}_1 Z_2 = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)(1)$

$\bar{Z}_2 Z_3 = 1\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$\bar{Z}_3 Z_1 = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$\bar{Z}_1 Z_2 + \bar{Z}_2 Z_3 + \bar{Z}_3 Z_1 = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{2} - \frac{1}{2}\right) + 2i\left(\frac{1}{2}\right)$

$= \sqrt{2} - \sqrt{2}i + i$

$\Rightarrow |\bar{Z}_1 Z_2 + \bar{Z}_2 Z_3 + \bar{Z}_3 Z_1|^2 = |\sqrt{2} + i(-\sqrt{2} + 1)|^2$

$= \left(\sqrt{(\sqrt{2})^2 + (1 - \sqrt{2})^2}\right)^2$

$= 5 - 2\sqrt{2}$

$(a, b) = (5, -2)$

13. Let a coin is tossed thrice. Let the random variable *x* is tail follows head. Let the mean of *x* is μ and variance is σ2. Find 64 (μ + σ2).

(1) 48 (2) 64

(3) 132 (4) 128

Answer (1)

Sol.

	x_i	P_i
HHH	0	$\frac{1}{8}$
TTT	0	$\frac{1}{8}$
HHT	1	$\frac{1}{8}$
HTH	1	$\frac{1}{8}$
THH	0	$\frac{1}{8}$
TTH	0	$\frac{1}{8}$
THT	1	$\frac{1}{8}$
HTT	1	$\frac{1}{8}$

$\mu = \Sigma P_i x_i = \frac{1}{2}$

$\sigma^2 = \Sigma P_i x_i^2 - \mu^2$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$64\left(\frac{1}{2} + \frac{1}{4}\right) = 64 \times \frac{3}{4} = 48$

14. Let $g(x) = 3f\left(\frac{x}{3}\right) + f(3-x)\ \forall x \in (0,3)$ and $f''(x) > 0\ \forall x \in (0,3)$ then $g(x)$ decreases in interval $(0, \alpha)$, then α is

(1) $\frac{7}{4}$

(2) $\frac{2}{3}$

(3) $\frac{9}{4}$

(4) $\frac{7}{3}$

Answer (3)

Sol. $g(x) = 3f\left(\frac{x}{3}\right) + f(3-x)$

$g'(x) = 3\cdot\frac{1}{3} f'\left(\frac{x}{3}\right) - f'(3-x)$

$= f'\left(\frac{x}{3}\right) - f'(3-x)$

$g''(x) = \frac{f''\left(\frac{x}{3}\right)}{3} + f''(3-x)$

$\Rightarrow g'(x) > 0$

$f'\left(\frac{x}{3}\right) - f'(3-x) > 0$

$f'(x) > 0 \Rightarrow f'(x)$ is increasing

15. Let $\vec{b} = \lambda\hat{i} + 4\hat{k}, \lambda > 0$ and the projection vector of $\vec{b}$ on $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ is $\vec{c}$. If $|\vec{a} + \vec{c}| = 7$, then the area of the parallelogram formed by vector $\vec{b}$ and $\vec{c}$ is (in square units)

(1) 8

(2) 16

(3) 32

(4) 64

Answer (3)

Sol. $\vec{c} = (\vec{b}\cdot\hat{a})\hat{a} = \frac{2\lambda - 4}{6}\vec{a}$

$\because |\vec{a} + \vec{c}| = 7 \Rightarrow \left|\vec{a}\left(1 + \frac{2\lambda - 4}{9}\right)\right| = 7$

$\left|\frac{5 + 2\lambda}{9}\right| \times 3 = 7 \Rightarrow |5 + 2\lambda| = 21$

$\because \lambda > 0 \Rightarrow \lambda = 8$

$\Rightarrow \vec{c} = \frac{4}{3}\vec{a}$ and $\vec{b} = 4(2\hat{i} - \hat{k})$

$\Rightarrow \vec{b} \times \vec{c} = \frac{16}{3}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 2 & 2 & -1 \end{vmatrix} = \frac{16}{3}(-2\hat{i} + 4\hat{j} + 4\hat{k})$

$\Rightarrow |\vec{b} \times \vec{c}| = \frac{32}{3}|-\hat{i} + 2\hat{j} + 2\hat{k}| = 32$

$\Rightarrow$ Area of parallelogram formed by $\vec{b}$ and $\vec{c}$

$\Rightarrow |\vec{b} \times \vec{c}| = 32$

16. Let the parabola $y = x^2 + px - 3$ cuts the coordinate axes at P, Q and R. A circle with centre $(-1, -1)$ passes through P, Q and R, then the area of triangle PQR.

(1) $\frac{5}{2}$ (2) $\frac{3}{2}$

(3) 3 (4) 5

Answer (2)

Sol. Since at $x = 0$, $y = -3$, parabola cuts the coordinate y-axe at $(0, -3)$

$\Rightarrow$ Equation of circle will be

$(x + 1)^2 + (y + 1)^2 = (-1 - 0)^2 + (-1 + 3)^2$

$= 1 + 4 = 5$

$x^2 + 2x + y^2 + 2y = -3 = 0$

Circle cuts x-axis at $y = 0$

$\Rightarrow x^2 + 2x - 3 = 0, (x + 3)(x - 1) = 0$

$(-3, 0), (1, 0)$

Area of Δ

$\Rightarrow \frac{1}{2}\begin{vmatrix}-3 & 0 & 1\\ 1 & 0 & 1\\ 0 & -3 & 1\end{vmatrix} = \frac{1}{2}(3) = \frac{3}{2}$

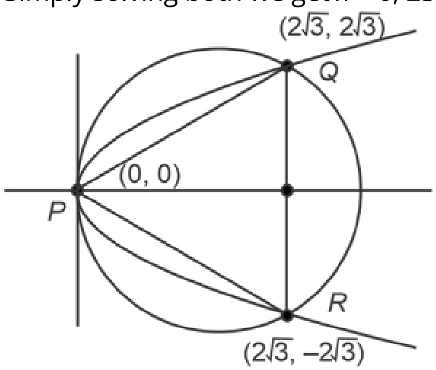
17. If the circle $(x-2\sqrt{3})^2+y^2=12$ and parabola $y^2=2\sqrt{3}x$ intersects at P, Q and R. Then the area of triangle PQR is

(1) 10 sq. units (2) 12 sq. units

(3) 14 sq. units (4) 16 sq. units

Answer (2)

Sol. Simply solving both we get $x = 0, 2\sqrt{3}$



$\Delta PQR = \frac{1}{2}\times(4\sqrt{3})(2\sqrt{3})$

18. A hyperbola with foci $(1, 14)$ and $(1, -12)$ passes through the point $(1, 6)$. The length of the latus rectum of the hyperbola is

(1) $\frac{144}{5}$

(2) 50

(3) $\frac{288}{5}$

(4) 100

Answer (3)

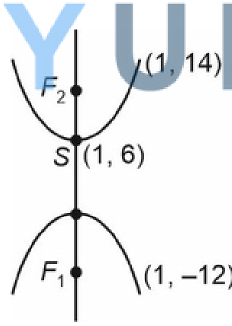
Sol. $|sp - s'p| = 2a$, $ss' = 2ae$

$s(1, 14)$, $s'(1, -12)$, $P(1, 6)$

$\Rightarrow 2a = |8 - 18|$

$\Rightarrow a = 5$; $2ae = 26$

$\Rightarrow ae = 13$



Length of latus rectum $= \frac{2b^2}{a} = \frac{2a^2(e^2-1)}{a}$

$= \frac{2(169 - 25)}{5} = \frac{288}{5}$

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. If A be a 3×3 square matrix such that $\det(A) = -2$. If $\det(3\ \text{adj}(-6\ \text{adj}(3A))) = 2^n \cdot 3^m$, where $m \geq n$, then $4m + 2n$ is equal to

Answer (104)

Sol. Concept: $A.\ \text{adj}(A) = |A|I$, $\det(\lambda A) = \lambda^n \det(A)$

$\Rightarrow \det(A) = |A|^{n-1}$, where n is order

$\Rightarrow \det(3\ \text{adj}(-6\ \text{adj}(3A)))$

$= 3^3 \cdot \det(\text{adj}(-6\ \text{adj}(3A)))$

$= 3^3 \cdot (-6\ \text{adj}(3A))^2$

$= 3^3 \cdot (-6)^6\ |3A|^4$

$= 3^9 \cdot 2^6 \cdot 3^{12} \cdot (-2)^4$

$= 3^{21} \cdot 2^{10}$

$\therefore n = 10, m = 21 \therefore$

$4m + 2n = 104$

22. If $a_1, a_2, a_3 \ldots a_n$ are in geometric progression such that $a_1 a_5 = 28, a_2 + a_4 = 29$, then the value of a_6 is

(1) 635
(2) 784
(3) 872
(4) 898

Answer (2)

Sol. $a_1 a_5 = 28 \Rightarrow a^2 r^4 = 28$

$a_2 + a_4 = 29 \Rightarrow ar + ar^3 = 29$

ar, ar^3 are roots of $k^2 - 29k + 28 = 0$

$\Rightarrow k = 1, k = 28$

$\Rightarrow ar = 1, ar^3 = 28$

$\Rightarrow r^2 = 28, a^2 = \frac{1}{28}$

$a_6 = ar^5 \Rightarrow a_6^2 = a^2 r^{10} = \frac{1}{28} \times (28)^5 = (28)^4$

$\Rightarrow a_6 = (28)^2 = 784$

23.

24.

25.

□ □ □