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Alcalde, José and Romero-Medina, Antonio

University of Alicante, Universidad Carlos III de Madrid

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José Alcalde† Antonio Romero-Medina‡

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†Corresponding Author. Department of Quantitative Methods and Economic Theory, and IUDESP, University of Alicante. jose.alcalde@ua.es

‡Universidad Carlos III de Madrid. aromero@eco.uc3m.es

Abstract

This paper provides an 'escape route' from the efficiency-equity trade-off in the School Choice problem. We achieve our objective by presenting a weak notion of fairness, called τ-fairness, which is always satisfied by some allocation.
Then, we propose the adoption of the Student Optimal Compensating Exchange mechanism, a procedure that assigns a τ-fair allocation to each problem.
We further identify a condition on students' preferences guaranteeing incentive compatibility of this mechanism.

Keywords: School Choice Problem, Fair Matching, Strategy-Proofness.
Journal of Economic Literature Classification Numbers: C78, D63, I28.

1. Introduction

In July 2005, Boston public schools adopted the Student Optimal Stable mechanism as a procedure to distribute the scholar places among the newcomer students. This reform is in the origin of a prolix literature on School Choice. The two main reasons encouraging the Boston School Committee to shift the allocation mechanism were its equitable treatment of students and the strategic properties satisfied by the newly adopted procedure, which are not satisfied by the formerly applied. Unfortunately, the Student Optimal Stable mechanism (SOSM henceforth) fails to select efficient allocations. This lack has motivated an intensive research by several authors to conciliate the three properties, namely efficiency, equity and non-manipulability.

The present paper takes a close look at the these incompatibilities and provides a positive solution to this problem. We identify and characterize a solution that reconciles efficiency and equity in School Choice problems. The allocations meeting such a requirement will be called τ-fair. We then introduce a mechanism selecting one τ-fair allocation. Moreover we identify a class of problems under which this mechanism is strategy-proof: the set of problems satisfying the β-condition.

The analysis of how social conflicts should be solved has generated a vast literature providing a huge number of 'impossibility results' showing the incompatibility of some desirable properties. For instance, de Condorcet (1785) pointed out that majority voting might induce cyclic social preferences; Gibbard (1973) and Satterthwaite (1975) proved that non-trivial mechanisms selecting efficient allocations are manipulable; or Pazner and Schmeidler (1974) provided two examples suggesting that the goals of efficiency and fairness may be mutually incompatible. As a reaction to the negative findings, some authors either concentrate on the analysis of 'reasonable frameworks' where such an incompatibility is escaped or explore alternative solutions avoiding the trade-off. Related to the examples above, Black (1948) introduced the notion of single-peakedness as a condition guaranteeing the transitivity of the majority voting schemes; Moulin (1980) characterizes the set of efficient, non-manipulable voting schemes when agents' preferences fulfill single peakedness; or Pazner and Schmeidler (1978) introduce 'Egalitarian

Equivalence' as an equity criterion which is compatible with efficiency.

Our first concern is related to the trade-off between efficiency and equity. Balinski and Sönmez (1999) introduced an equity criterion for the School Choice framework which might unfulfilled by all the efficient allocations. Equity is very related to the conception of fairness introduced by Foley (1967), and establishes a strong condition to be satisfied by an allocation to be branded as 'unambiguously equitable'. Our approach is closer to the notion of resentment (see Rawls, 1971, pg. 533) in the sense that "those who express resentment must be prepared to show why certain institutions are unjust ..." This argument allows to determine that an allocation is λ-equitable whenever no agent claiming to have suffered an injustice is able to propose an alternative allocation that no one else would declare to be inequitable. Since the set of λ-equitable allocations is larger than that of the equitable ones, we can find efficient allocations fulfilling this weaker equity requirement.

The analysis of τ-fairness, as the confluence of efficiency and λ-equity, yields to identify these allocations as the ones combining the two conditions that have worried some researchers. On the one hand, τ-fair allocations are efficient and, on the other hand, no student prefers her allocation under the SOSM to the school she is assigned to.

The existence of a fairness concept overcoming the incompatibility of efficiency and equity allows to inquiry about the existence of mechanisms selecting τ-fair allocations. When designing such mechanisms we concentrate on fitting two particular restrictions. First, the allocation procedure must be easy-to-be-adopted. Note that there is a large tradition pointing out that societies use to be reluctant to deep reforms.[1] Therefore, the adoption of the new mechanism should not introduce extreme changes on the existing one. We reach this objective by considering a folk modification into the SOSM. Provided that the allocation proposed by the SOSM might fail to be efficient, we might consider it as an 'exchangeable' initial endowment. Students are then allowed to switch their assigned places. As a

[1] An illustrative example of this fact was explained by Al Roth in the 2007 Rosenthal Memorial Lecture, held in Boston University, when explaining why the Top Trading Cycles mechanism (See Abdulkadiroğlu and Sönmez, 2003, Section II.B) was not adopted by the Boston School Committee. A video of this lecture is available at http://www.bu.edu/buniverse/view/?v=grOprKV, accessed on April 2nd, 2014.

consequence of students' interaction, this 'pure exchange market for places' helps to restore efficiency, whereas λ-equity is not lost. These two facts guarantee the τ-fairness of the final assignment. The second limitation we concentrate on is the absence of students' incentives to exhibit a strategic behavior.[2] Our approach to this concern is conditioned by the fact that τ-fairness and strategy-proofness are incompatible in the School Choice context. To reach positive results we proceed by studying how τ-fair mechanisms for School Choice problems behave in restricted domains, from an incentive-compatibility perspective. In particular, we explore the existence of reasonable frameworks where such an incompatibility can be avoided, and we identify a condition, to be named β-Condition, whose fulfillment by the students' sets of preferences yields the existence of strategy-proof, τ-fair mechanisms.

Outline of the Paper

We conclude this section by providing an overview of the remainder of the paper. Section 2 connects our analysis with the existing literature. Section 3 introduces the basic model as well as one of the main contributions of this paper. In particular, we provide a formal definition of τ-fairness and, in Theorem 9, we prove the existence of at least one τ-fair allocation for each School Choice problem. We conclude this section by characterizing the set of τ-fair allocations. Section 4 formally introduces the Student Optimal Compensating Exchange mechanism, which selects a τ-fair allocation for each problem. Therefore, Section 4 can be regarded as a formal proof of Theorem 9. We explore strategic issues related to the application of this mechanism in Section 5. Finally, our main conclusions are provided in Section 6. For exposition simplicity, we relegate some technical proofs to the Appendix.

2. Related Literature

Abdulkadiroğlu and Sönmez (2003) introduced the School Choice model inspired in the far-reaching literature on Matching models. Their analysis is closely related to Balinski and Sönmez (1999) because the former adopts the notion of equity

[2]As Roth points out in the above-mentioned lecture, strategy-proofness is a relevant aspect when designing allocation mechanisms in the School Choice framework.

introduced by the latter. In particular, Balinski and Sönmez (1999) reinterpret the classical notion of stability in Matching models in terms of equity in the School Choice framework. These authors argue that stability embodies the equity of an allocation because it eliminates any justifiable envy and avoids wastefulness.

The School Choice model assumes that each school prioritizes the new students according some given exogenous factors.[3] Schools' priorities are employed to discern whether an allocation is equitable or not, so that any distribution of places in which priorities are not preserved is said inequitable. The close-knit relationship between equity of an allocation and stability of the matching it describes entails a series of well-known facts. Among them, it is relevant to mention the existence of a trade-off between efficiency and equity, as pointed out in Roth (1982).

The general non-existence of fair allocations, i.e. equitable and efficient assignments, has been a central issue in the analysis of mechanisms for School Choice problems. As Abdulkadiroğlu (2013) reports, in the context of School Choice efficiency is a primary objective. This is why some authors concentrate on alternative approaches to reconcile efficiency and equity. For instance, Ergin (2002) and Ehlers and Erdil (2010) study when the SOSM provides fair allocations. These authors identify a condition on the priorities profiles, named acyclicity, which is not only necessary but also sufficient to guarantee that the SOSM allocation is fair for any preferences exhibited by the students. Alternatively, Kesten (2010) designs a mechanism that eliminates the efficiency loss obtained when applying the SOSM. This author reaches his objective by allowing students to forgo priorities that, being relevant to them, are critical to others. This procedure, named 'the Efficiency Adjusted Deferred Acceptance mechanism' (EADAM hereafter), is designed to select efficient allocations, Pareto dominating the SOSM outcome, when the former is not efficient.

[3]Among these factors, the most relevant ones are the distance between the school and the students' residences and the fact that the students have a sibling attending that school or not. In order to break ties, a fair lottery is implemented.

A. **School Choice and Co-operative Game Theory**

There is a close relationship between the literature in School Choice and the analysis of Matching problems. The latter class of models has been extensively explored from a (co-operative) game-theoretical perspective. Within this framework, the absence of (strong) compromise solutions (v.g. core allocations) has been at the origin of the analysis of solutions supported by stability notions involving a weaker compromise from the agents' perspective. In particular, we want to stress that there is a large tradition on exploring different notions of bargaining sets (see, e.g., Aumann and Maschler, 1964; Zhou, 1994) in which agents' ability to object a given allocation is more restrictive than the blocking notion applied when defining the core of a cooperative game (Gillies, 1959), or the equivalent notion of stability in Matching models (Gale and Shapley, 1962). Following this approach of restricting the students' ability to claim inequitable treatments we suggests to adopt λ-equity as a weak notion of equity and then demonstrate that it is compatible with efficiency of an allocation (Theorem 9).

B. **Pareto-Improving the SOSM**

The absence of fair allocations motivated the analysis of mechanisms selecting efficient assignments. Following this line of reasoning, Erdil and Ergin (2008), Pápai (2010) and Azevedo and Leshno (2011) explore natural mechanisms consisting in applying the Gale's Top Trading Cycle (Shapley and Scarf, 1974) to the allocation obtained when applying the SOSM.[4] Let us note that, as argued in the Introduction, it is expected that such mechanisms select τ-fair allocations. It is also remarkable, from a practical applicability perspective, that the New House 4 at MIT applies a similar mechanism to distribute its available dorms among its residents.[5] Nevertheless, these mechanisms have not received recognition in the literature. The main reason is that they are manipulable (see Kesten, 2010,

[4] Since each school might have several places, when applying the Top Trading Cycle each student will be indifferent on exchanging with any of the students having a place at the same school. In order to avoid such indifferences, an exogenous tie-breaking rule is selected. Note that we can describe a family of mechanisms by associating one mechanism to each given tie-breaking rule.

[5] A detailed description of the procedure employed by NH4 can be found on its web page http://scripts.mit.edu/∼nh4/housingrules.php, accessed on April 4, 2014.

Proposition 4), and strategy-proofness is considered by several authors a 'must be satisfied' property by any mechanism for School Choice problems.

Nevertheless, in practice, Kesten's manipulability result might not hold, as agents' preferences over schools exhibit some specificity. In this respect, Abdulkadiroğlu et al. (2011) report: "Families tend to value similar qualities about schools (e.g., safety, academic reputation, etc.), which causes them to have similar ordinal preferences. Indeed, the Boston public schools data exhibit strong correlation in students' preferences over schools." Therefore, these authors suggest that it might be plausible to restrict students' declared preferences to fulfill certain specific properties.

We investigate the conditions avoiding, when satisfied by the admissible sets of student's preferences, Kesten's impossibility result. To be precise, we concentrate in a mechanism named the 'Student Optimal Compensating Exchange' that can be described as the application of the Top Trading Cycle to the SOSM allocation. Our approach parallels that of Alcalde and Barberà (1994) for Matching models. We study conditions on students' preferences guaranteeing that no student benefits from misrepresenting her preferences when the Student Optimal Compensating Exchange mechanism is adopted. To this concern we demonstrate that when the sets preferences satisfy the β-Condition, our mechanism is strategy-proof. Interesting enough, the β-Condition allows situations in which the strong correlation in the students' preferences asserted by Abdulkadiroğlu et al. (2011) is present while SOSM allocation remains inefficient. This condition can be expressed as follows. Let us consider any three schools and two admissible preferences for a given student. Then we require that the two preferences, when restricted to these schools, either locate the same school as the worst one or coincide in which is the median school.

3. Fairness and the School Choice Problem

Let us consider two non-empty, disjoint sets to be termed the students' and the schools' sets. The set of students has n individuals and is denoted $\mathcal{S} = \{s_i\}_{i=1}^{n}$. The set of schools, which is denoted $\mathcal{C} = \{c_j\}_{j=1}^{m}$, has m elements. Each school is endowed a given capacity, which determines the maximum number of places to be distributed among the students. Let $q_j \geq 1$ denote the capacity of school c_j, and let $q = (q_1, \ldots, q_j, \ldots, q_m)$ denote the vector summarizing the capacities of the schools.

Each school prioritizes the students according a linear ordering. Let P_j be ordering describing how school c_j prioritized its potential applicants, and let $P = (P_1, \ldots, P_j, \ldots, P_m)$ be the vector summarizing these priorities, which will be called a priorities profile.

Similarly, each student has linear preferences over the set of schools. Therefore, no student will consider two different schools equivalent. Let $\succ_i$ denote s_i' preferences. For notational convenience, we denote s_i' weak preferences as $\succsim_i$. Thus, $c_j \succsim_i c_h$ will be understood as $c_j \succ_i c_h$ whenever $c_j \neq c_h$. We also extend students' preferences to ensure that all schools are considered acceptable and the 'not having any school place' is the students' worst option. We denote it by stating that, for each student s_i and school c_j, $c_j \ \succ_i \ s_i$. Vector $\succ = (\succ_1, \ldots, \succ_i, \ldots, \succ_n)$ is called a preferences profile.

Therefore, a School Choice problem, also called a Problem, can be described by listing the elements above $[(\mathcal{S}; \succ); (\mathcal{C}; q; P)]$. As the set of students, colleges and quotas will remain fixed in what follows, we will designate each Problem as $(\succ; P)$ or $\mathcal{P}$.

A solution to $\mathcal{P}$ is an application μ that matches students and places observing schools' capacities. Such a correspondence is called a matching. Formally,

Definition 1 A *matching* for $\mathcal{P}$ is a correspondence μ, applying $\mathcal{S} \ \cup \ \mathcal{C}$ to itself such that:

1. For each s_i in $\mathcal{S}$, if $\mu(s_i) \neq s_i$, then $\mu(s_i) \in \mathcal{C}$.
2. For each c_j in $\mathcal{C}$, $\mu(c_j) \subseteq \mathcal{S}$ and $|\mu(c_j)| \leq q_j$.[6]
3. For each s_i in $\mathcal{S}$ and any c_j in $\mathcal{C}$, $\mu(s_i) = c_j$ if and only if $s_i \in \mu(c_j)$.

An efficient assignment denotes the situation in which no re-allocation is weakly preferred by the students, as we describe below.

[6]Throughout this paper, $|T|$ will denote the cardinality of set T.

Definition 2 Given a Problem $(\succ; P)$, we say that matching μ is *efficient* if for any other matching $\mu' \neq \mu$, there is a student, s_i, such that

$$\mu(s_i) \succ_i \mu'(s_i).$$

We now concentrate on two notions of equity. The first one is the classical notion of equity in the context of School Choice whereas the second one is the weak equity we call λ-equity. Definition 6 formalizes the two equity notions, which are illustrated through the next example.

Example 3 There are three students and three schools, having a vacant each. Therefore, $\mathcal{S} = \{1, 2, 3\}$ and $\mathcal{C} = \{a, b, c\}$. Let us assume that the preferences and priorities are

$$\begin{array}{ll} a \succ_1 c \succ_1 b, & 3 \; P_a \; 2 \; P_a \; 1, \\ c \succ_2 a \succ_2 b, & 2 \; P_b \; 1 \; P_b \; 3, \\ c \succ_3 a \succ_3 b, & 1 \; P_c \; 3 \; P_c \; 2. \end{array}$$

Let us consider matching μ described as

$$\mu := \begin{array}{ccc} 1 & 2 & 3 \\ a & b & c \end{array}$$

It is easy to see that student 2 can adduce that μ fails to be equitable because she has priority over student 1 for school a. A way to formalize such a 'complaint' is by proposing matching μ' described as

$$\mu' := \begin{array}{ccc} 1 & 2 & 3 \\ b & a & c \end{array}$$

In such a case we say that student 2 *objects* against matching μ via μ', or equivalently that the pair $(2, \mu')$ constitutes a *fair objection* against μ. Is the presence of such an objection what allows to say that μ is inequitable.

Now, let us consider that student 2's objection is taken into account and matching μ' is implemented. Applying a similar reasoning, we can see that student 1 can object against μ' via μ'' described as

$$\mu'' := \begin{array}{ccc} 1 & 2 & 3 \\ c & a & b \end{array}$$

In such a case we say that student 1 *counter-objects* student 2's objection. The fact that a fair objection is counter-objected allows to disregard it. The essence of λ-equity is the absence of fair objections that cannot be counter-objected.

Definition 4 Let $\mathcal{P} = (\succ; P)$ be a Problem, and let μ be a matching for $\mathcal{P}$. A *fair objection* from student s_i to μ is a pair (s_i, μ') such that

i) $\mu'(s_i) \succ_i \mu(s_i)$, and

ii) $|\mu(c_j)| < q_j$, or $s_i \; P_j \; s_h$ for some $s_h \in \mu(c_j)$, with $c_j = \mu'(s_i)$.

Definition 5 Let (s_i, μ') be a fair objection to matching μ. A *counter-objection* from student s_k to (s_i, μ') is a pair (s_k, μ'') that constitutes a fair objection to matching μ'.
We say that (s_i, μ') is a *justified, fair objection* to μ if it cannot be counter objected.

Definition 6 Let $\mathcal{P} = (\succ; P)$ be a Problem. We say that matching μ is

(a) *equitable* if there is no (s_i, μ') constituting a fair objection against μ; and

(b) λ-*equitable* if there is no (s_i, μ') which constitutes a justified, fair objection against μ.

Note that for a given School Choice problem, any equitable allocation is also λ-equitable. Therefore, the set of λ-equitable matchings also contains all equitable allocations. Thus, as the set of equitable allocations is always non-empty (see, v.g., Gale and Shapley, 1962, Theorem 1), the next statement follows.

Proposition 7 Let $\mathcal{P} = (\succ; P)$ be a Problem. Then, $\mathcal{P}$ has, at least, one λ-equitable matching.

The solution concept that we propose in this section, τ-fairness, combines two central solutions for School Choice problems, namely efficiency and λ-equity.

Definition 8 Let $\mathcal{P} = (\succ; P)$ be a Problem. We say that matching μ is

(a) *fair* if it is efficient and equitable; and

(b) τ*-fair* if it is efficient and λ-equitable.

Let $\mathcal{F}(\mathcal{P})$ denote the set of fair matchings for problem $\mathcal{P}$.

The next question that we address is the existence of τ-fair allocations. Although the sets of equitable and efficient matchings might not intersect, when we focus on λ-equitable allocations rather than equitable ones, such an intersection is always non-empty.

Theorem 9 Let $\mathcal{P} = (\succ; P)$ be a Problem. Then, its set of τ-fair allocations is non-empty.

Since Theorem 9 can be considered a corollary of Theorem 17, its proof is omitted.

A further question of particular relevance is the possibility of characterizing the set of τ-fair allocations. Theorem 11, whose proof is reported in the Appendix A, reports that the only efficient allocations that are λ-equitable are those Pareto dominating the matching proposed by the SOSM.

Previous to introduce the following result, and for the sake of completeness, we need to introduce the Student Optimal mechanism, which is obtained by computing the deferred acceptance algorithm.

Definition 10 Let $(\succ; P)$ be a School Choice problem; we define its *Student Optimal Stable matching*, μ^{SO}, as the outcome of the following algorithm.

Step 1. Each student, s_i, applies to the school that is ranked first after $\succ_i$. Each school, c_j, tentatively accepts up to q_j students, according its priority list. The remaining applications (if any) are rejected.

...

Step k. Each student, s_i, applies to the first school after $\succ_i$ (if any) that has not previously rejected her. If the student has been rejected by all schools, she is unassigned. Each school, c_j, tentatively accepts up to q_j students, according its priority list. Any remaining application is rejected.

The algorithm ends when each student who remains unassigned has been rejected by all schools. Each student is assigned to the school (if any) that accepted her application in the last step.

Theorem 11 Let $\mathcal{P} = (\succ; P)$ be a Problem. μ is τ-fair for $\mathcal{P}$ if and only if

i) μ is efficient, and

ii) for each student, s_i,

$$\mu(s_i) \succsim_i \mu^{SO}(s_i; \mathcal{P}).$$

Note that Theorem 11 has some straightforward implications. The first one, which is the aim of Corollary 12, establishes that the Student Optimal Stable matching is the unique allocation (if any) combining both equity and efficiency. The second consequence is reflected in Corollary 13: it is not worthwhile to distinguish between allocative *fairness* and *τ-fairness* when the former concept is non-empty.

Corollary 12 $\mathcal{F}(\mathcal{P}) \subseteq \left\{\mu^{SO}(\mathcal{P})\right\}$ for each problem $\mathcal{P} = (\succ; P)$.

Corollary 13 Let $\mathcal{P} = (\succ; P)$ be a Problem. If its Student Optimal Stable matching is efficient, then any τ-fair matching is a fair allocation.

4. From Equity to τ-Fairness: The SOCE Mechanism

This section is devoted to the analysis of School Choice mechanisms, i.e. procedures assigning a matching to each School Choice problem. As it is usual in the literature, the properties defined for specific solutions are straightforwardly translated to mechanisms. In particular, a mechanism $\mathcal{M}$ is efficient (resp. equitable, λ-equitable, fair or τ-fair) if for each School Choice problem $\mathcal{P}$, matching $\mathcal{M}(\mathcal{P})$ is efficient (equitable, λ-equitable, fair or τ-fair resp.) with respect to Problem $\mathcal{P}$.

To be precise, we concentrate on introducing the *Student Optimal Compensating Exchange* mechanism, SOCE in short, a mechanism for School Choice problems that always selects a τ-fair allocation.

A simple way to describe the SOCE is by the (sequential) combination of two well-known allocation procedures. The first, the input of which is a School Choice problem, is the SOSM. Once the SOSM is applied, we can consider the Housing Market (see Shapley and Scarf, 1974), in which the students are the agents, and each individual is initially endowed with the place that the SOSM assigned to her. We can then apply Gale's Tops Trading Cycle, introduced by Shapley and Scarf (1974), to achieve a Pareto improvement relative to the initial outcome of the SOSM. Theorem 17 indicates that the outcome of this iterative procedure is always a τ-fair matching.

Therefore, the SOCE can be regarded as a tool to develop a constructive proof of Theorem 9, or alternatively, as a suggestion for improving the system adopted by the Boston School Committee in 2005 (see Abdulkadiroğlu et al., 2006) by guaranteeing allocative efficiency.

Definition 10 described how to compute, for a given Problem $\mathcal{P}$, the Student Optimal Stable matching, $\mu^{SO}(\mathcal{P})$. Therefore, to complete a formal definition of the SOCE, we need to describe how the *Tops Trading Cycle* operates. This is the aim of Definition 15. Appendix B is devoted to illustrate how to compute the SOCE for a specific Problem.

The Housing Market framework, introduced by Shapley and Scarf (1974), involves a set of agents owning one indivisible object each. Agents exhibit preferences over the objects, which will be described through a linear preorder. Following this structure, we assign to each School Choice problem, $\mathcal{P}$, and matching, μ, a Housing problem. In this transition from a School Choice problem and allocation, μ, to a problem reflecting the structure of a Housing Market, there are some specifications that are (or can be considered) natural.

In our *placing market*, student s_i reveals that she wishes to trade with s_h whenever she prefers s_h's place to her own. Let linear preorder $\mathcal{E}_i$ denote student s_i's *preferences for exchange*.[7] The following can be assumed.

(a) Student s_i only wishes to exchange her place with s_h if she benefits from

[7] Let us recall that, as argued in the Introduction, the role of $\mathcal{E}_i$ is just to specify a tie-breaking rule. See footnote 4.

such an exchange,

$$s_h \, \mathcal{E}_i \, s_i \Rightarrow \mu(s_h) \succ_i \mu(s_i).$$

(b) For any two students, s_h and s_k, who have been assigned different schools, any other student, s_i, prefers to exchange with the student who has a place in her preferred school,

$$\mu(s_h) \succ_i \mu(s_k) \Rightarrow s_h \, \mathcal{E}_i \, s_k.$$

Nevertheless, there is no a priori justification to determine how to order two different students who have been assigned a place in the same school. When describing a *compensating market for school places*, we assume that each student prefers to exchange with the individual having the lowest priority for a given school. The rationale of such a hypothesis derives from the manner in which priorities are established in real-life situations. When developing a priority list, schools divide students into categories, primarily based on two factors, namely siblings and residence. Under this method of classifying students, the school prioritizes several students equally at this stage. Lotteries are used to break these ties. Therefore, in most cases, it is expected that the school prioritizes one student relative to another because of some random factor. The Student Optimal Compensating Market's feature of reversing the priority lists is intended to compensate for the random effect introduced by the lottery.[8]

Definition 14 Let $\mathcal{P}$ be a School Choice problem. We define its associated *Student Optimal Compensating Market*, $SOCM(\mathcal{P})$, as the pair $\left(\hat{S}, \mathcal{E}\right)$, where

(1) The set of agents is described as,

$$\hat{S} = \left\{ s_i \in \mathcal{S} : \mu^{SO}(s_i; \mathcal{P}) \in \mathcal{C} \right\}, \text{ and}$$

[8]Alcalde and Romero-Medina (2011) explore a family of *student optimal placing markets*, including the SOCM, each of which yields a τ-fair allocation. In the present paper and for the sake of simplicity, we concentrate on the SOCM.

(2) the *profile of preferences for exchange*, $\mathcal{E} = (\mathcal{E}_i)_{s_i \in \hat{S}}$, is described such that for each $s_i \in \hat{S}$, $\mathcal{E}_i$ is a linear preorder on $\hat{S}$ satisfying

   (a) for each $s_h \in \hat{S}$ such that $\mu^{SO}(s_i; \mathcal{P}) \succsim_i \mu^{SO}(s_h; \mathcal{P})$, $s_i \; \mathcal{E}_i \; s_h$;

   (b) for each $s_h, s_k \in \hat{S}$ such that $\mu^{SO}(s_h; \mathcal{P}) \; \succ_i \; \mu^{SO}(s_k; \mathcal{P})$, $s_h \; \mathcal{E}_i \; s_k$; and

   (c) for any $s_h, s_k \in \hat{S}$ such that $\mu^{SO}(s_h; \mathcal{P}) = \mu^{SO}(s_k; \mathcal{P}) \neq \mu^{SO}(s_i; \mathcal{P})$, $s_h \; \mathcal{E}_i \; s_k$ if and only if $s_k \; P_j \; s_h$.

In a more general context, a *Placing Market* is a pair, $\left(\tilde{S}, \mathcal{E}\right)$, where $\tilde{S}$ is a set of agents and $\mathcal{E}$ denotes the profile of their *preferences for exchange* (i.e., for each s_i in $\tilde{S}$, $\mathcal{E}_i$ is a linear preorder on $\tilde{S}$).

Definition 15 We define the *Tops Trading Cycle* rule as the procedure for assigning to each *Placing Market*, $(S, \mathcal{E})$, the outcome of the following algorithm.

Step 1. Let us consider the digraph whose set of nodes is S, and there is an arc from s_i to s_h if s_h is the maximal for $\mathcal{E}_i$ in S. This digraph has at least one cycle. Let $K(S)$ be the set of students belonging to some cycle. Then, match each student in $K(S)$ to her preferred "mate for exchanging"; i.e., if $s_i \in K(S)$, then $TTC(s_i; (S, \mathcal{E})) = s_h$ whenever s_h is the maximal for $\mathcal{E}_i$ in S.

Let us define $S^2 = S \setminus K(S)$. If S^2 is empty, the algorithm stops. Otherwise, go to Step 2.

...

Step t. Let us consider the digraph whose set of nodes is S^t, and there is an arc connecting s_i to s_h if s_h is the maximal for $\mathcal{E}_i$ in S^t. This digraph has at least one cycle. Let $K(S^t)$ be the set of students belonging to some cycle. Then, match each student in $K(S^t)$ to her preferred "mate for exchanging"; in other words, if $s_i \in K(S^t)$, then $TTC(s_i; (S, \mathcal{E})) = s_h$ whenever s_h is the maximal for $\mathcal{E}_i$ in S^t.

Let us define $S^{t+1} = S^t \setminus K(S^t)$. If S^{t+1} is empty, the algorithm stops. Otherwise, go to Step $(t+1)$.

Because the number of students is finite, and for each t, it holds that $S^{t+1} \subsetneq S^t$, this algorithm stops in a finite number of steps.

We can now provide a formal definition of the *Student Optimal Compensating Exchange* mechanism, which can be straightforwardly introduced by the next sequential procedure.

(1) Given a School Choice problem, say $\mathcal{P} = (\succ; P)$, let us compute its Student Optimal Stable matching, $\mu^{SO}(\mathcal{P})$. Once this is complete, we can distinguish two groups of students:

  i) The unassigned students, namely $\{s_i \in \mathcal{S} \colon \mu^{SO}(s_i; \mathcal{P}) = s_i\}$; and

  ii) The assigned or 'placed' students, namely $\{s_i \in \mathcal{S} \colon \mu^{SO}(s_i; \mathcal{P}) \in \mathcal{C}\}$.

(2) Let us apply the Tops Trading Cycle rule to $SOCM(\mathcal{P})$. Let us observe that in this phase the only students that might participate, and thus improve with respect to $\mu^{SO}(\mathcal{P})$, are the placed ones.

Definition 16 We define the *Student Optimal Compensating Exchange* mechanism as the matching rule assigning each Problem, say $\mathcal{P} = (\succ; P)$, the matching $\mu^{ce}(\mathcal{P})$ described as follows

$$\mu^{ce}(s_i; \mathcal{P}) = \begin{cases} s_i & \text{if } \mu^{SO}(s_i; \mathcal{P}) = s_i \\ \mu^{SO}(TTC(SOCM(\mathcal{P})); \mathcal{P}) & \text{otherwise} \end{cases}.$$

As the next result indicates, for each problem $(\succ; P)$ its SOCE allocation is a τ-fair allocation.

Theorem 17 Let $\mathcal{P} = (\succ; P)$ be a Problem. Its *Student Optimal Compensating Exchange matching,* $\mu^{ce}(\mathcal{P})$, is a τ-fair allocation for such a problem.

A formal proof of this result can be found in Appendix C. We next provide a heuristic explanation justifying why our result holds.

First, let us note that, when considering a given problem, $\mathcal{P}$, for each student s_i and equitable matching μ,

$$\mu^{SO}(s_i; \mathcal{P}) \ \succsim_i \ \mu(s_i);$$

i.e., $\mu^{SO}(\mathcal{P})$ is (constrained) efficient when restricted to the set of equitable allocations.

Second, for each Problem, $\mathcal{P} = (\succ; P)$, $\mu^{ce}(\mathcal{P})$ is an efficient matching that Pareto-dominates $\mu^{SO}(\mathcal{P})$, unless the two matchings coincide.

The two observations above and the fact that $\mu^{SO}(\mathcal{P})$ is equitable are crucial to concluding that $\mu^{ce}(\mathcal{P})$ is τ-fair.

5. Strategic Behavior and the SOCE

A central issue when designing mechanisms for School Choice problems is to guarantee that students are safe when expressing their preferences honestly. This is the essence of strategy-proofness, as we describe in Definition 19.

Let us introduce some formalisms. Given a Problem, $\mathcal{P} = (\succ; P)$, let us assume that some student, say s_i, announces that her preferences are $\succ'_i$ instead of $\succ_i$. Let $\mathcal{P}' = ((\succ'_i, \succ_{-i}); P)$ denote the Problem in which student s_i' preferences have been misrepresented.

We also adopt the following traditional convention. Given a mechanism $\mathcal{M}$, a School Choice problem $\mathcal{P}$, and $x \in \mathcal{S} \cup \mathcal{C}$, we denote $\mathcal{M}(x; \mathcal{P}) = \mu(x)$, where $\mu \equiv \mathcal{M}(\mathcal{P})$.

Definition 18 We say that student s_i *manipulates* School Choice mechanism $\mathcal{M}$ at $\mathcal{P}$ *via* preferences $\succ'_i$ if

$$\mathcal{M}(s_i; \mathcal{P}') \succ_i \mathcal{M}(s_i; \mathcal{P}).$$

A mechanism that cannot be manipulated is strategy-proof.

Definition 19 We say that School Choice mechanism $\mathcal{M}$ is *strategy-proof* if for each Problem, $\mathcal{P} = (\succ; P)$, student s_i and preferences for s_i, say $\succ'_i$,

$$\mathcal{M}(s_i; \mathcal{P}) \succsim_i \mathcal{M}(s_i; \mathcal{P}').$$

The SOCE is a combination of two strategy-proof mechanisms. First, we use the SOSM; then, we apply the TTC. However, the impossibility of finding an efficient and strategy-proof mechanism Pareto-dominating the SOSM (see, e.g., Abdulkadiroğlu et al., 2009 or Kesten, 2010) implies that the SOCE is manipulable.

A way to escape the above negative results is by proposing mild conditions on students' characteristics that prevent them from engaging in strategic behavior. Therefore, our analysis is in line with tradition in Social Choice Theory analyzing the existence of domain restrictions from which strategy-proof mechanisms can be designed. In particular, we concentrate on the possibility of identifying a restriction of students' preferences under which the SOCE is immune to manipulation by the students.

To clarify how manipulability can be prevented, the following examples illustrate the limits of the agents' manipulation.

Example 20 *Benefiting from a loss and the SOCE manipulation*
Let us consider the three-students-three-schools problem, $\mathcal{P}$, where $\mathcal{S} = \{1, 2, 3\}$, $\mathcal{C} = \{a, b, c\}$, and each school quota is $q_j = 1$. The students' preferences and colleges' priorities are

$$\begin{array}{lcr} a \succ_1 b \succ_1 c & \qquad & 3\,P_a\,2\,P_a\,1 \\ b \succ_2 a \succ_2 c & & 1\,P_b\,3\,P_b\,2 \\ b \succ_3 c \succ_3 a & & 2\,P_c\,1\,P_c\,3 \end{array}$$

In such a case the application of the SOCE mechanism yields matching $\mu^{ce}(\mathcal{P})$ described as

$$\mu^{ce}(\mathcal{P}) := \begin{array}{ccc} 1 & 2 & 3 \\ a & b & c \end{array}.$$

Since students 1 and 2 obtain a place at their preferred schools, they cannot manipulate the SOCE at this problem. Nevertheless, when student 3 reports preferences $\succ_3'$, with $b \succ_3' a \succ_3' c$, the application of the SOCE yields matching $\mu^{ce}(\mathcal{P}')$ described as

$$\mu^{ce}(\mathcal{P}') := \begin{array}{ccc} 1 & 2 & 3 \\ a & c & b \end{array}.$$

Note that $\mu^{ce}(3; \mathcal{P}') = b \succ_3 c = \mu^{ce}(3; \mathcal{P})$. This implies that 3 can manipulate the SOCE at $\mathcal{P}$ via $\succ_3'$.

Let us observe that when applying the SOSM to the two examples above we have that $\mu^{SO}(3;\mathcal{P}) = c \succ_3 a = \mu^{SO}(3;\mathcal{P}')$. The intuition behind this example suggests the following possible approach for avoiding misrepresentation. Let us imagine that student s_i modifies her preferences such that, during the SOSM phase, she receives a place at $c_{h'}$, when she was matched to c_h when truthfully reporting her preferences. If all of the places that s_i prefers (when manipulating preferences) to $c_{h'}$ are truthfully worse than c_h, this agent will never benefit from misrepresenting her preferences.

The example above suggests that, to avoid manipulability of the SOCE, no student should be able to exhibit two different preferences having the same best school. Nevertheless, this is not the only option that an student might have to manipulate the SOCE. It is well known that the SOSM mechanism fail to satisfy non-bossiness, i.e. an agent can influence on her rivals' allocation keeping the same place assigned to her. Example 21 points out that this fact can be used by an student to manipulate the SOCE.

Example 21 Let us consider the following instance. $\mathcal{S} := \{1,2,3,4,5,6\}$, $\mathcal{C} := \{a,b,c,d,e,f\}$ with $q_j = 1$ for each j. Students' preferences and schools priorities are[9]

$$\begin{array}{ll}
a \succ_1 b \succ_1 \cdots & 6\ P_a\ 2\ P_a\ 4\ P_a\ 1\ P_a \cdots \\
a \succ_2 c \succ_2 d \succ_2 \cdots & 1\ P_b\ 6\ P_b \cdots \\
d \succ_3 e \succ_3 \cdots & 4\ P_c\ 5\ P_c\ 2\ P_c \cdots \\
a \succ_4 c \succ_4 \cdots & 2\ P_d\ 3\ P_d\ 5\ P_d \cdots \\
d \succ_5 c \succ_5 f \succ_5 \cdots & 3\ P_e \cdots \\
b \succ_6 f \succ_6 a \succ_6 \cdots & 5\ P_f\ 6\ P_f \cdots
\end{array}$$

In such a case, the SOSM yields matching

$$\mu^{SO}(\mathcal{P}) := \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ b & d & e & c & f & a \end{array}.$$

[9]In the following description dots inform that the remaining elements -students or schools- can be placed in any position before the listed ones.

Now, let us consider the alternative preferences for student 2 described as

$$c \succ_2' d \succ_2' a \succ_2' \dots$$

The SOSM associates to the new problem, namely $\mathcal{P}'$, matching

$$\mu^{SO}(\mathcal{P}') := \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ b & d & e & a & c & f \end{matrix}.$$

Let us observe that student 2's misrepresentation modifies the allocations by students 4, 5 and 6 when applying the SOSM. Moreover, if we analyze the consequences of such modifications when SOCE is employed, we have that $\mu^{ce}(2, \mathcal{P}') = c \succ_2 d = \mu^{ce}(2, \mathcal{P})$; i.e. student 2 manipulates the SOCE at $\mathcal{P}$ via $\succ_2'$.

To introduce the β-condition we require some additional notation. Let us consider a set of three schools, namely $\mathcal{C}' = \{a, b, c\}$, and one student, say s_i, with preferences $\succ_i$. We denote by $M(\mathcal{C}', \succ_i)$ the median school on $\mathcal{C}'$ accordingly $\succ_i$; i.e. $b = M(\mathcal{C}', \succ_i)$ if either $a \succ_i b \succ_i c$ or $c \succ_i b \succ_i a$. Similarly, $W(\mathcal{C}', \succ_i)$ denotes the worst school on $\mathcal{C}'$ accordingly $\succ_i$; i.e. $c = W(\mathcal{C}', \succ_i)$ if $a \succ_i c$ and $b \succ_i c$.

Definition 22 The β-Condition
For a given student, say s_i, let Ω_i denote her set of admissible preferences. We say that Ω_i satisfies the β-*Condition* if for each two preferences $\succ_i$ and $\succ_i'$ in Ω_i and any three-school set $\mathcal{C}'$, either $M(\mathcal{C}', \succ_i) = M(\mathcal{C}', \succ_i')$ or $W(\mathcal{C}', \succ_i) = W(\mathcal{C}', \succ_i')$

We can now establish the next result, whose formal proof is gathered to Appendix D.

Theorem 23 Let us assume that, associated with each student s_i, there is a set of preferences Ω_i satisfying the β-Condition. Then, the SOCE rule is strategy-proof when each student is required to declare preferences in Ω_i.

6. Concluding Remarks

This paper provides an alternative approach to circumventing the efficiency-equity dilemma present in the School Choice problem. We propose an alternative interpretation of the legitimacy of an agent's claims against the equity of an allocation. We achieve our objective by employing a weak notion of envy-freeness, which we call λ-equity. This notion requires that any objection to a proposal by the District School Board be disregarded whenever it might be counter-objected. One of the most relevant properties exhibited by this equity criterion is that it is not incompatible with the efficiency requirement. Therefore, it becomes interesting to explore the structure of the set of allocations satisfying both equity (in a weak sense) and efficiency. We call this set of allocations the τ-fair set, and it is characterized as the set of efficient allocations Pareto-improving on the SOSM. As a byproduct, we identify a simple, natural method, we call SOCE, to compute a τ-fair allocation. This rule has a familiar flavor and can be implemented through a minimal reform of certain schooling systems, such as the procedure recently adopted in the Boston area. Loosely, the SOCE can be described as if students were allowed to exchange the places allocated to them in the actual system.

A further question that we address relates to the agents' strategic behavior when the SOCE mechanism is adopted. As Abdulkadiroğlu et al. (2006) claim, strategy-proofness is a crucial property for the survival of an allocation system. Therefore, the (theoretical) manipulability of the SOCE might be regarded as an inconvenience in terms of the practical implementation of this rule. Nevertheless, as we note, in real-life situations, one might expect that no student benefits from strategic behavior. The reason is that, as stated in Theorem 23, the mechanism is manipulable only if agents are allowed to select preferences from a set not fulfilling the β condition. This domain restriction captures, among other possibilities, the case in which students' preferences are highly correlated, as is likely according to Abdulkadiroğlu et al. (2011).

Concerning the real-live adoption of the SOCE, we wish to stress that there are allocation mechanisms, employed in similar frameworks, in which agents are allowed to improve on their initial allocations by exchanging their rights. This evidence is abundant concerning both the civil service and the army. For instance, civil servants are allowed to exchange their placements in Spain,[10] and a resembling exchange can be performed in the US Army under the so-called *Enlisted Assignment Exchanges* (SWAPS).[11] Similar systems, in which agents can exchange goods that they do not own, but for which they *retain some rights*, can be found in some socially accepted systems, such as several *international student exchange programs* or a recent *kidney exchange*[12] program. Additionally, our procedure shares some features with the proposals advanced by the *Ecole Démocratique* to reform the actual system in the French-speaking area of Belgium.[13]

To conclude, let us mention that Theorem 23 is not only valid for the SOCE, but also for any *Student Optimal Placing Market* mechanism, i.e., mechanisms as described in Definition 14, in which condition (2.c) in the description of exchange preferences are not required. To clarify this point, let us mention that our proof of Theorem 23 does not require that such a condition be satisfied. This observation suggests that the question about the maximal domain restriction under which the SOCE is strategy-proof is still an open question that deserves the attention of future research.

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[10]Art. 62 in the Spanish law that governs civil servants or Law 315/1964, B.O.E 15.02.1964. This regulation can be obtained from http://www.ua.es/oia/es/legisla/funcion.htm, accessed on March 3rd, 2015.

[11]The reader is directed to http://usmilitary.about.com/od/armyassign/a/swap.htm, accessed on March 3rd, 2015, for further information on this matter.

[12]Transplant services at the Ronald Reagan UCLA Medical Center provide some relevant information located at http://transplants.ucla.edu/body.cfm?id=112, accessed on March 3rd, 2015.

[13]We would like to acknowledge Estelle Cantillon for highlighting these similarities. The proposals by the Ecole Démocratique can be found in French on its web page http://www.skolo.org/spip.php?article1126&lang=fr, accessed on March 3rd, 2015.

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APPENDIX

A. A Proof of Theorem 11

This appendix introduces a formal proof of Theorem 11, which characterizes the set of τ-fair allocations associated with each specific School Choice problem. Recall that our result establishes that matching μ is τ-fair if it satisfies two properties. The first is Pareto efficiency, which is also required by Definition 8; the second is that each student (weakly) prefers the place that μ assigns to her rather than the place suggested by the SOSM,

$$\mu(s_i) \succsim_i \mu^{SO}(s_i;\mathcal{P}). \tag{1}$$

Proof of Theorem 11

Let $\mathcal{P} = (\succ; P)$ be a Problem, and let μ be a matching. We first show that if μ is τ-fair, then it satisfies condition (1). Note that by Definition 8, μ should be efficient.

To achieve our objective, let us assume by way of contradiction that μ does not satisfy condition (1). There should be a student s_i, such that

$$\mu^{SO}(s_i;\mathcal{P}) \succ_i \mu(s_i). \tag{2}$$

Let us observe that, for each s_i satisfying condition (2) above, the τ-fairness of μ implies that there is $c_j \in \mathcal{C}$ such that $c_j = \mu^{SO}(s_i;\mathcal{P})$. Moreover, such a college satisfies that

(1) $s_h\ P_j\ s_i$, for each s_h in $\mu(c_j)$, and

(2) $\left|\mu^{SO}(c_j;\mathcal{P})\right| = q_j$.

Taking into account that the number of schools is finite, there is an (ordered) set of students $\{s^t\}_{t=1}^{T}$, with $T \leq m$, and schools $\{c^t\}_{t=1}^{T}$ such that, for each t

(a) $\mu(s^t) = c^t$,

(b) $\mu^{SO}(s^t;\mathcal{P}) = c^{t+1}$, with $T+1 = 1$, and

(c) $c^{t+1} \succ_{s^t} c^t$.

Now, let us consider matching μ'' such that

(i) For each $s^t \in \{s^t\}_{t=1}^{T}$, $\mu''(s^t) = c^{t+1}$ (modulo T), and

(ii) For each $s_i \notin \{s^t\}_{t=1}^{T}$, $\mu''(s_i) = \mu(s_i)$.

It is easy to verify that μ'' Pareto-dominates μ, which contradicts our hypothesis.

Now, let us assume that μ is an efficient allocation that satisfies condition (1). We will see that μ is τ-fair.

To reach a contradiction, let us assume that μ is not τ-fair. Then, as μ is efficient, there should be a student-matching pair, (s_i, μ'), that constitutes a justified, fair objection to μ.

Therefore, by Definition 4 and condition (1), we note that

$$\mu'(s_i) \succ_i \mu(s_i) \succsim_i \mu^{SO}(s_i; \mathcal{P}). \tag{3}$$

Note that this relationship implies that $\mu'(s_i) \in \mathcal{C}$. Let c_j denote such a school.

From Martínez et al. (2001), we know that for any matching $\hat{\mu}$, if $\hat{\mu}(s_i) \succ_i \mu^{SO}(s_i; \mathcal{P})$ for some student, s_i, then $\hat{\mu}$ is not equitable. In particular, this implies that matching μ' fails to be equitable. Then, there should be a student, s_h, and a school, c_k, such that

(1) $c_k \succ_h \mu'(s_h)$, and

(2) $s_h \, P_k \, s_l$ for some $s_l \neq s_h$, or $|\mu'(c_k)| < q_k$.

Therefore, if we consider any matching μ'', such that $\mu''(s_h) = c_k$, the pair (s_h, μ'') describes a counter objection from student s_h to (s_i, μ'), which contradicts the hypothesis that μ fails to be τ-fair. ⊡

B. The SOCE: An Example

This appendix provides an example illustrating how to compute the SOCE.

Let us consider the following School Choice problem. $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $\mathcal{C} = \{a, b, c, d\}$; the capacity for each school is 2; and the students' preferences and schools' priorities are

$$\begin{array}{l} b \succ_1 a \succ_1 c \succ_1 d \\ c \succ_2 a \succ_2 d \succ_2 b \\ c \succ_3 b \succ_3 a \succ_3 d \\ d \succ_4 b \succ_4 c \succ_4 a \\ a \succ_5 c \succ_5 d \succ_5 b \\ a \succ_6 b \succ_6 c \succ_6 d \\ a \succ_7 b \succ_7 d \succ_7 c \\ b \succ_8 a \succ_8 c \succ_8 d \end{array}; \qquad \text{and} \qquad \begin{array}{l} 4 \; P_a \; 1 \; P_a \; 2 \; P_a \; 6 \; P_a \; 8 \; P_a \; 7 \; P_a \; 5 \; P_a \; 3 \\ 3 \; P_b \; 2 \; P_b \; 7 \; P_b \; 6 \; P_b \; 4 \; P_b \; 1 \; P_b \; 8 \; P_b \; 5 \\ 7 \; P_c \; 5 \; P_c \; 6 \; P_c \; 8 \; P_c \; 2 \; P_c \; 3 \; P_c \; 1 \; P_c \; 4 \\ 5 \; P_d \; 6 \; P_d \; 3 \; P_d \; 4 \; P_d \; 2 \; P_d \; 8 \; P_d \; 7 \; P_d \; 1 \end{array}$$

The SOSM

As shown in Table 1, the SOSM proposes to allocate a place in school a to students 1 and 2; students 3 and 7 are accepted by school b; students 5 and 6 are placed at school c; and, finally, students 4 and 8 will attend school d. A column in the table is devoted to each school. A row indicates the applications that each school receives at this step. The students framed in a box are those whose applications are refused, whereas the remaining students are tentatively accepted by the school.

The Student Optimal Compensating Market

As we mention in Section 4, a Placing Market is determined by the set of students, $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and a preference profile, $\mathcal{E}$, denoting, for each student, how she orders her "rivals" depending on her initial assignment, μ^{SO}. We interpret the preferences determined by a student as *her inclination to exchange* the place that μ^{SO} assigns to her with the other students. The construction of these preferences is guided by the following two principles.

(a) A student participates in this market if she obtains a positive net profit from the exchange; and

(b) a student, when exchanging, tries to maximize her utility (i.e., she attempts to secure a place at the best school, according to her opinion of the educational institutions).

Step	a	b	c	d
1	$\boxed{5}$, 6, 7	1, 8	2, 3	4
2	6, 7	1, 8	2, $\boxed{3}$, 5	4
3	6, 7	1, 3, $\boxed{8}$	2, 5	4
4	6, $\boxed{7}$, 8	1, 3	2, 5	4
5	6, 8	$\boxed{1}$, 3, 7	2, 5	4
6	1, 6, $\boxed{8}$	3, 7	2, 5	4
7	1, 6	3, 7	$\boxed{2}$, 5, 8	4
8	1, 2, $\boxed{6}$	3, 7	5, 8	4
9	1, 2	3, $\boxed{6}$, 7	5, 8	4
10	1, 2	3, 7	5, 6, $\boxed{8}$	4
11	1, 2	3, 7	5, 6	4, 8
$\boldsymbol{\mu^{SO}} =$	1, 2	3, 7	5, 6	4, 8

Table 1: The SOSM.

In the case of SOCE, a student orders two agents, other than her, who have been assigned the same school by "reversing the school ordering" proposed by the priority lists.

To illustrate how to compute the students' preferences for exchanging, $\mathcal{E}$, we will concentrate on student 6. The process is as follows

(1) Provided that her preferred school is a, and $\mu^{SO}(\text{a}; \mathcal{P}) = \{1, 2\}$, we note that her two "tops for exchanging" are students 1 and 2. Moreover, as $1\ P_a\ 2$, we have that

$$2\ \mathcal{E}_6\ 1.$$

(2) Now, the second school in student 6's preference list is b, the available

places of which have been assigned to students 3 and 7. We note that these students will be placed in the third and fourth positions according to $\mathcal{E}_6$. As $3\ P_b\ 7$, it follows that

$$7\ \mathcal{E}_6\ 3.$$

(3) The third school in student 6's preference list is c, and $\mu^{SO}(\text{c}; \mathcal{P}) = \{5, 6\}$, Principle (a) above indicates that

$$6\ \mathcal{E}_6\ 5.$$

For our purposes, once student 6 has established her own position on $\mathcal{E}_6$, we do not need to continue describing how the remaining students are ordered. Nevertheless, for the sake of completeness, we will also explain how $\mathcal{E}_6$ orders students 4 and 8.

(4) $\mu^{SO}(4; \mathcal{P}) = \mu^{SO}(8; \mathcal{P}) = d$, and $4\ P_d\ 8$. Therefore,

$$8\ \mathcal{E}_6\ 4.$$

Summarizing, the preferences for exchange exhibited by student 6 are

$$2\ \mathcal{E}_6\ 1\ \mathcal{E}_6\ 7\ \mathcal{E}_6\ 3\ \mathcal{E}_6\ 6\ \mathcal{E}_6\ 5\ \mathcal{E}_6\ 8\ \mathcal{E}_6\ 4,$$

or, equivalently,

$$\mathcal{E}_6 := \ 2,\ 1,\ 7,\ 3,\ 6,\ 5,\ 8,\ 4.$$

Applying a similar argument to the remaining students, we can compute $\mathcal{E}$, the description of which is

$$\begin{aligned}
\mathcal{E}_1 &:= \; 7,\ 3,\ 1,\ 2,\ 6,\ 5,\ 8,\ 4 \\
\mathcal{E}_2 &:= \; 6,\ 5,\ 2,\ 1,\ 8,\ 4,\ 7,\ 3 \\
\mathcal{E}_3 &:= \; 6,\ 5,\ 3,\ 7,\ 2,\ 1,\ 8,\ 4 \\
\mathcal{E}_4 &:= \; 4,\ 8,\ 7,\ 3,\ 6,\ 5,\ 2,\ 1 \\
\mathcal{E}_5 &:= \; 2,\ 1,\ 5,\ 6,\ 8,\ 4,\ 7,\ 3 \\
\mathcal{E}_6 &:= \; 2,\ 1,\ 7,\ 3,\ 6,\ 5,\ 8,\ 4 \\
\mathcal{E}_7 &:= \; 2,\ 1,\ 7,\ 3,\ 8,\ 4,\ 6,\ 5 \\
\mathcal{E}_8 &:= \; 7,\ 3,\ 6,\ 5,\ 8,\ 4,\ 2,\ 1
\end{aligned}$$

Therefore, $SOCM\,(\mathcal{P}) = (\mathcal{S}, \mathcal{E})$, where $\mathcal{S} = \{1, \ldots, 8\}$ is the set of students and $\mathcal{E}$ is the preference profile described above.

The Tops Trading Cycle Rule

To continue with our illustrative example, we now compute the outcome of the TTC rule when applied to the Student Optimal Compensating Market that was previously described.

Step 1. At the first step, the set of students is $\mathcal{S}$. To draw the next digraph, we proceed as follows. A node is assigned to each student. We draw an arc connecting s_i to s_h if the latter is the top for $\mathcal{E}_i$.[14]

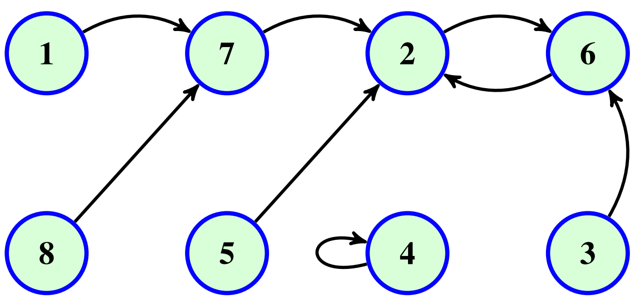

Figure 1: TTC algorithm, first step.

Let us observe that the above digraph contains two cycles; the first includes students 2 and 6, and the second only contains student 4. Therefore, these

[14]For instance, as 7 is the top for $\mathcal{E}_1$, we draw an arc that departs from 1 and is incident to 7.

three students exit the market, and we can describe a "new market" that includes students $S^2 = \{1, 3, 5, 7, 8\}$.

Step 2. At the second step, the set of students is S^2 described above. We proceed in a similar manner to that explained in the previous step and draw its corresponding directed graph.

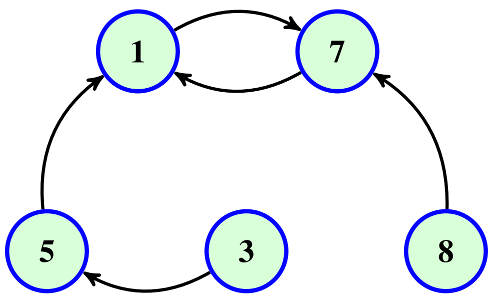

Figure 2: TTC algorithm, second step.

There is clearly a cycle involving students 1 and 7. Therefore, $K(S^2) = \{1, 7\}$, and both students exit the market, allowing us to consider the residual market $S^3 = \{3, 5, 8\}$.

Step 3. In this step, the set of remaining students is S^3. It is easy to see that there one cycle containing student 5. Therefore, $K(S^3) = \{5\}$ and $S^4 = \{3, 8\}$.

Step 4. Now student 3 forms a cycle, and hence $K(S^4) = \{3\}$ and $S^5 = \{8\}$.

Step 5. Since S^5 is a singleton, we note that $K(S^5) = \{8\}$ and $S^6 = \emptyset$. As each student is, at this point, involved in some cycle, the algorithm stops.

Therefore, the outcome of the Tops Trading Cycle, applied to $SOCM(\mathcal{P})$, is

s_i	1	2	3	4	5	6	7	8
$TTC(s_i; (\mathcal{S}, \mathcal{E}))$	7	6	3	4	5	2	1	8

The Compensating Exchange mechanism

To conclude the process, we must move from the initial matching $\mu^{SO}(\mathcal{P})$ to the matching that results from the exchange of places suggested by the Tops Trading Cycle rule. For instance, as $TTC(1) = 7$, then

$$\mu^{ce}(1;\mathcal{P}) = \mu^{SO}(7;\mathcal{P}) = b.$$

Therefore, the outcome of the SOCE for this example is

s_i	1	2	3	4	5	6	7	8
$\mu^{ce}(s_i;\mathcal{P})$	b	c	b	d	c	a	a	d

C. A Proof for Theorem 17

To prove Theorem 17, let us consider a School Allocation problem, $\mathcal{P} = (\succ; P)$, and let $\mu^{SO}(\mathcal{P})$ be its student optimal stable matching.

Let us observe that when describing the Student Optimal Compensating Market, the preferences for exchange held by a student, s_i, such that $\mu^{SO}(s_i;\mathcal{P}) \neq s_i$, verify that for any other student s_h,

$$s_h \ \mathcal{E}_i \ s_i \Rightarrow \mu^{SO}(s_h;\mathcal{P}) \ \succ_i \ \mu^{SO}(s_i;\mathcal{P}).$$

By construction, the Tops Trading Cycle satisfies the requirement that, for each Placing Market, (S,P), and any $s_i \in S$ such that $TTC(s_i;(S,\mathcal{E})) \neq s_i$,

$$TTC(s_i;(S,\mathcal{E})) \ \mathcal{E}_i \ s_i.$$

This implies that for each student, s_i,

$$\mu^{ce}(s_i;\mathcal{P}) \ \succsim_i \ \mu^{SO}(s_i;\mathcal{P}). \tag{4}$$

As $\mu^{SO}(\mathcal{P})$ is equitable, we note the following.

(a) If for some student, s_i, $\mu^{SO}(s_i;\mathcal{P}) = s_i$, then $(\succ, P)$ satisfies that

$$n > \sum_{j=1}^{m} q_j, \text{ and}$$

(b) If there is some school, c_j, that has vacancies at $\mu^{SO}(\mathcal{P})$, i.e.,

$$\left|\mu^{SO}(c_j;\mathcal{P})\right| < q_j,$$

then for each student, s_i,

$$\mu^{SO}(s_i;\mathcal{P}) \succsim_i c_j.$$

To complete our proof, let us assume that $\mu^{ce}(\mathcal{P})$ is not τ-fair. Taking into account equation (4) and Theorem 11, we note that $\mu^{ce}(\mathcal{P})$ should not be efficient. Therefore, there should be a matching, μ, such that for each student, $s_i, \in \mathcal{S}$

$$\mu(s_i) \succsim_i \mu^{ce}(s_i;\mathcal{P}), \text{ and}$$

there should be a student, s_h, such that

$$\mu(s_h) \succ_h \mu^{ce}(s_h;\mathcal{P}). \tag{5}$$

Let S denote the set of students fulfilling Condition (5). By condition (4) and because μ^{SO} is equitable, we note that for each $s_i \in S$, there is another student, s_h, in S such that $\mu(s_i) = \mu^{ce}(s_h;\mathcal{P})$. This finding implies that s_i's preferences for exchange satisfy the requirement that

$$s_h \, \mathcal{E}_i \, TTC(s_i; SOCM(\mathcal{P})). \tag{6}$$

To obtain a contradiction, for each student $s_i \in S$, we let $t(s_i)$ denote the stage at which it is determined $TTC(s_i; SOCM(\mathcal{P}))$. Without loss of generality, let us assume that $s_i \in S$ is such that $t(s_i) \leq t(s_k)$ for each $s_k \in S$. By condition (5), we note that in the digraph for stage $t(s_i)$, there is no arc from s_i to $\mu^{ce}(s_i;\mathcal{P})$.

This finding constitutes a contradiction, which indicates that there is no matching, μ, Pareto-dominating $\mu^{ce}(\mathcal{P})$. $\boxdot$

D. A Proof of Theorem 23

To prove Theorem 23, let us consider a problem $\mathcal{P} \equiv (\succ; P)$, a student $s_i \in \mathcal{S}$ and preferences for s_i, $\succ'_i$. Let $\mathcal{P}' \equiv ((\succ'_i, \succ_{-i}); P)$ be the problem obtained from $\mathcal{P}$ by replacing s_i's preferences.

Let us assume that s_i manipulates the SOCE rule at $\mathcal{P}$ via $\succ'_i$. We will see that any set of preferences for s_i containing $\succ_i$ and $\succ'_i$ fails to satisfy the β-condition.

Note that if $\mu^{SO}(s_i; \mathcal{P}) = s_i \neq \mu^{SO}(s_i; \mathcal{P}')$, we have that $\mu^{SO}(s_i; \mathcal{P}') \succ_i \mu^{SO}(s_i; \mathcal{P})$. This contradicts the non-manipulability of the SOSM (see, v.g., Roth, 1982). Therefore, since s_i manipulates, she is assigned a place in each matchings.

Let us remark the following three facts.

Fact 1. The manipulation hypothesis implies that $\mu^{ce}(s_i; \mathcal{P}') \succ_i \mu^{ce}(s_i; \mathcal{P})$;

Fact 2. by construction, we have that $\mu^{ce}(s_i; \mathcal{P}) \succsim_i \mu^{SO}(s_i; \mathcal{P})$; and

Fact 3. since SOSM is strategy-proof, $\mu^{SO}(s_i; \mathcal{P}) \succsim_i \mu^{SO}(s_i; \mathcal{P}')$.

Now, let us consider the following two cases, which exhaust all the possibilities.

Case 1. $\mu^{SO}(s_i; \mathcal{P}) \succ_i \mu^{SO}(s_i; \mathcal{P}')$. Note that, this implies that $\mu^{SO}(s_i; \mathcal{P}') \succ'_i \mu^{SO}(s_i; \mathcal{P})$. By facts 1 and 2, and transitivity we have that

$$\mu^{ce}(s_i; \mathcal{P}') \succ_i \mu^{SO}(s_i; \mathcal{P}) \succ_i \mu^{SO}(s_i; \mathcal{P}'); \tag{7}$$

facts 1 and 3 imply that $\mu^{ce}(s_i; \mathcal{P}') \neq \mu^{SO}(s_i; \mathcal{P}')$. Therefore, by construction, we have that $\mu^{ce}(s_i; \mathcal{P}') \succ'_i \mu^{SO}(s_i; \mathcal{P}')$. Transitivity yields that

$$\mu^{ce}(s_i; \mathcal{P}') \succ'_i \mu^{SO}(s_i; \mathcal{P}') \succ_i \mu^{SO}(s_i; \mathcal{P}). \tag{8}$$

Note that conditions (7) and (8) indicate that this agent admissible preferences fail to satisfy the β-condition.

Case 2. $\mu^{SO}(s_i;\mathcal{P}) = \mu^{SO}(s_i;\mathcal{P}')$. By fact 1, there should be some school c_j such that

$$c_j \succ_i \mu^{SO}(s_i;\mathcal{P}), \text{ and } \mu^{SO}(s_i;\mathcal{P}) \succ_i' c_j. \tag{9}$$

Note that, otherwise, $\mu^{SO}(\mathcal{P}) = \mu^{SO}(\mathcal{P}')$. Therefore, since the TTC is non-manipulable, $\mu^{ce}(\mathcal{P}) \succsim_i \mu^{ce}(\mathcal{P}')$. Contradicting fact 1.

Therefore, since $\mu^{ce}(s_i;\mathcal{P}') \neq \mu^{SO}(s_i;\mathcal{P})$, condition (9) implies that

$$\mu^{ce}(s_i;\mathcal{P}') \succ_i' \mu^{SO}(s_i;\mathcal{P}) \succ_i' c_j. \tag{10}$$

To conclude, let us observe that conditions (9) and (10) are incompatible with the fulfillment of the β-condition by s_i' admissible preferences.