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IN THIS ISSUE—

Duane W. Bailey, *On symmetry in certain group algebras* 413
Lawrence Peter Belluce and Surender Kumar Jain, *Prime rings with a one-sided ideal satisfying a polynomial identity* 421
L. Carlitz, *A note on certain biorthogonal polynomials* 425
Charles O. Christenson and Richard Paul Osborne, *Pointlike subsets of a manifold* 431
Russell James Egbert, *Products and quotients of probabilistic metric spaces* 437
Moses Glasner, Richard Emanuel Katz and Mitsuru Nakai, *Bisection into small annuli* 457
Karl Edwin Gustafson, *A note on left multiplication of semigroup generators* 463
I. Martin (Irving) Isaacs and Donald Steven Passman, *A characterization of groups in terms of the degrees of their characters. II* 467
Howard Wilson Lambert and Richard Benjamin Sher, *Point-like 0-dimensional decompositions of S^3* 511
Oscar Tivis Nelson, *Subdirect decompositions of lattices of width two* 519
Ralph Tyrrell Rockafellar, *Integrals which are convex functionals* 525
James McLean Sloss, *Reflection laws of systems of second order elliptic differential equations in two independent variables with constant coefficients* 541
Bui An Ton, *Nonlinear elliptic convolution equations of Wiener-Hopf type in a bounded region* 577
Daniel Eliot Wulbert, *Some complemented function spaces in $C(X)$* 589
Zvi Ziegler, *On the characterization of measures of the cone dual to a generalized convexity cone* 603

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ON SYMMETRY IN CERTAIN GROUP ALGEBRAS

DUANE W. BAILEY

A complex Banach algebra A with involution $x \to x^*$ is *symmetric* if Sp $(x^*x) \subset [0, \infty)$ for each $x \in A$. It is shown that (i) if A is symmetric, the algebra of all $n \times n$ matrices with elements from A is symmetric, and (ii) the group algebra of any semi-direct product of a finite group with a locally compact group having a symmetric group algebra is again symmetric.

An involution $x \to x^*$ in A is said to be *hermitian* if Sp $(x) \subset (-\infty, \infty)$ for every self-adjoint $x \in A$. In [1] R. Bonic studied the natural involution in the group algebra of certain discrete groups and raised the question: *Is the group algebra of a semi-direct product of a finite group with a discrete Abelian group necessarily symmetric?* The present work is devoted to proving the more general result that the group algebra of any semi-direct product of a finite group with a locally compact group whose group algebra is symmetric, is again symmetric. The proof in part depends upon showing that the algebra of $n \times n$ matrices with elements from a symmetric Banach algebra has a naturally defined symmetric involution. (We restrict our attention to continuous involutions.)

I am indebted to the referee for pointing out that if G is discrete, our Theorem 2 follows from a result of A. Hulanicki (Corollary 2, page 286 of [4]). Also, while it is easy to show that every symmetric involution is necessarily hermitian and that the notions are equivalent for commutative algebras, the equivalence for noncommutative algebras was an open question until quite recently. Mr. S. Shirali has announced a positive solution to this question which will be contained in his Doctoral Dissertation at Harvard University.

1. Algebras of matrices.

Let A be a Banach algebra with a continuous involution $x \to x^*$. A linear functional f on A is *positive* if $f(x^*x) \geqq 0$ for all $x \in A$. If A contains an identity e, such a functional satisfies $f(y^*x) = f(x^*y)$ for all $x, y \in A$, and if A is symmetric, then

$$\mathrm{Sp}(x) \subset \{f(x) \mid f \text{ a positive functional, } f(e) = 1\} \tag{1.1}$$

whenever $x \in A$ and $x^*x = xx^*$. (For a proof of these and other facts about symmetric Banach algebras, see [5].) In the following, $\nu(x)$ denotes the spectral radius of x.

LEMMA 1. *Let A be a Banach algebra with identity and continuous involution, and let f be a positive linear functional on A. Then*

(i) $|f(x^*hx)| \leqq f(x^*x)\nu(h)$ *whenever* $x, h \in A$ *and* $h^* = h$.

(ii) $\left|f\left(\sum_{i=1}^{n} y_i^* x_i\right)\right|^2 \leqq f\left(\sum_{i=1}^{n} y_i^* y_i\right) f\left(\sum_{i=1}^{n} x_i^* x_i\right)$ *whenever* $x_i, y_i \in A$.

(iii) $f\left(\left(\sum_{i=1}^{n} y_i^* x_i\right)^* \left(\sum_{i=1}^{n} y_i^* x_i\right)\right) \leqq f\left(\sum_{i=1}^{n} x_i^* x_i\right) \nu\left(\sum_{i=1}^{n} y_i^* y_i\right)$ *whenever* $x_i, y_i \in A$.

Proof. For (*i*), see [5, Th. 4.5.2]. Part (ii) is a generalized Cauchy inequality and is easy to prove using the properties of f mentioned above. If the left side of (iii) is 0, there is nothing to prove. Otherwise, we use (i) and (ii) to write

$$\begin{aligned}
&\left(f\left(\left(\sum_{i=1}^{n} y_i^* x_i\right)^* \left(\sum_{j=1}^{n} y_j^* x_j\right)\right)\right)^2 \\
&\quad = \left(f\left(\sum_{i=1}^{n} x_i^* \left(y_i \sum_{j=1}^{n} y_j^* x_j\right)\right)\right)^2 \\
&\quad \leqq f\left(\sum_{i=1}^{n} x_i^* x_i\right) f\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} y_j^* x_j\right) y_i^* y_i \left(\sum_{j=1}^{n} y_j^* y_j\right)\right) \\
&\quad \leqq f\left(\sum_{i=1}^{n} x_i^* x_i\right) f\left(\left(\sum_{j=1}^{n} y_j^* x_j\right)^* \left(\sum_{j=1}^{n} y_j^* x_j\right)\right) \nu\left(\sum_{i=1}^{n} y_i^* y_i\right) .
\end{aligned}$$

We obtain (iii) by cancelling a common factor from both sides.

The set A_n of all $n \times n$ matrices with elements from A can be made into an algebra by defining the operations exactly as for matrices of scalars. Furthermore, if $X \in A_n$, $X = [x_{ij}]$, the mapping $X^* = [y_{ij}]$, where $y_{ij} = x_{ji}^*$, is easily seen to be an involution in A_n. (We use the same symbol for the involution in the two algebras since confusion seems unlikely.) Finally,

$$\| X \| = \max_{i=1,\cdots,n} \sum_{j=1}^{n} \| x_{ij} \| , \qquad X \in A_n ,$$

is a Banach algebra norm for A_n.

THEOREM 1. *If A is symmetric then A_n is symmetric for any positive integer n.*

We note that it is sufficient to prove the theorem for the case in which A has an identity e. For otherwise, let A_e denote the algebra obtained by adjoining an identity to A. It is known [2 or 5] that A_e is symmetric if and only if A is symmetric. So, to show that A_n is symmetric we simply observe that $(A_n)_e$ is *-isomorphic to a closed *-subalgebra of $(A_e)_n$. The isomorphism here is

$$[x_{ij}] + \lambda E \leftrightarrow [x_{ij} + \lambda\delta_{ij}e] .$$

Any closed *-subalgebra of a symmetric Banach algebra is again symmetric, so it is enough to know that $(A_e)_n$ is symmetric.

LEMMA 2. *The theorem is true for* $n = 2$.

Proof. Let $X \in A_2$, $X = [x_{ij}]$. Then $X^*X = [y_{ij}]$ where

$$y_{ij} = x_{1i}^* x_{1j} + x_{2i}^* x_{2j} , \quad \text{and} \quad y_{ij} = y_{ji}^* , \qquad i,j = 1,2 .$$

To prove that A_2 is symmetric, it is enough to show that $-1 \notin \mathrm{Sp}(X^*X)$. That is, if E is the identity matrix in A_n, $E = [\delta_{ij}\, e]$, then $E + X^*X$ possesses an inverse. We will exhibit this inverse.

It is first necessary to establish the invertibility of two elements of A. As in [5], if $x \in A$ satisfies $\mathrm{Sp}(x) \subset [0, \infty)$ we write $x \geqq 0$. The symmetry of A implies [5, Lemma 4.7.10]

$$y_{11} = x_{11}^* x_{11} + x_{21}^* x_{21} \geqq 0 .$$

Thus $e + y_{11}$ has an inverse, say d_1. Next we consider $y_{22} - y_{21}d_1y_{12}$. If f is a positive linear functional on A, $f(e) = 1$, then

$$\begin{aligned} f(y_{21}d_1y_{12}) &\leqq f(y_{21}y_{12})\nu(d_1) \\ &\leqq f(y_{22})\nu(y_{11})\nu(d_1) \\ &\leqq f(y_{22}) \end{aligned}$$

from Lemma 1 (iii) and known properties of ν. It then follows that $f(y_{22} - y_{21}d_1y_{12}) \leqq 0$ and, as a consequence of (1.1),

$$y_{22} - y_{21}d_1y_{12} \geqq 0 .$$

We now know that $e + y_{22} - y_{21}d_1y_{12}$ has an inverse, say d_2. It is then an easy matter to verify that the matrix

$$\begin{bmatrix} d_1 + d_1y_{12}d_2y_{21}d_1 & -d_1y_{12}d_2 \\ -d_2y_{21}d_1 & d_2 \end{bmatrix}$$

is an inverse for $E + X^*X$. Hence A_2 is symmetric.

LEMMA 3. *The theorem holds for* $n = 2^k$, *where* k *is any positive integer.*

Proof. The proof is by induction, the case $k = 1$ being covered by Lemma 2. If we assume the result for $k = m$, then it follows for $k = m + 1$ from the fact that A_{2m+1} is *-isomorphic to $(A_{2m})_2$ by partitioning. In fact, every matrix in A_{2m+1} corresponds to a 2×2 matrix of matrices from A_{2m}, and this correspondence is easily proved to be a *-isomorphism.

Proof of Theorem 1. If n is a positive integer, choose k a positive integer so large that $m = 2^k > n$. Then A_m is symmetric, by Lemma 3, and the closed *-subalgebra of A_m consisting of all matrices with 0 in the last $(m - n)$ rows and columns is obviously *-isomorphic to A_n. It follows that A_n is itself symmetric, and the proof is complete.

2. Group algebras and semi-direct products. If F is a locally compact group, let I_F denote a left invariant Haar integral on F and let Δ_F be the corresponding modular function. Thus $J_F(x) = I_F(x \cdot 1/\Delta_F)$ is a right invariant Haar integral on F. The *group algebra* of F is the Banach space $L^1(F)$ of all complex-valued functions on F which are absolutely integrable with respect to the corresponding left Haar measure, μ_F. This algebra has an involution defined by $x^*(f) = x(f^{-1})\Delta_F(f^{-1}), f \in F$. (Here again we use *, in different positions, to denote both convolution and the involution.)

Let F and G be locally compact groups, and let $f \to \phi_f$ be a homomorphism of F into the group of automorphisms of G such that $(f, g) \to \phi_f(g)$ is a continuous mapping of $F \times G$ into G. In particular, each ϕ_f is continuous (and hence a homeomorphism). Let $S = F \times G$ and define a multiplication in S by

$$(f_1, g_1)(f_2, g_2) = (f_1f_2, g_1\phi_{f_1}(g_2)), \qquad (f_i, g_i) \in S,\ i = 1,2 .$$

Then S becomes a locally compact group which we denote by $F \times_\phi G$. We note in passing that the inverse of (f, g) is $(f^{-1}, \phi_{f^{-1}}(g^{-1}))$.

We now observe that the automorphisms ϕ_f induce a group of bounded linear transformations Φ_f of $L^1(G)$ defined by

$$\Phi_f(x) = x \circ \phi_{f^{-1}} \quad \text{for} \quad f \in F,\ x \in L^1(G) ,$$

and the mapping $f \to \Phi_f$ is a homomorphism of F onto this group. To see that the range of Φ_f is contained in $L^1(G)$, it is sufficient to note that each ϕ_f maps the measurable subsets of G onto measurable subsets, and that for some $\delta(f) > 0$

$$\mu_G(\phi_f(E)) = \delta(f)\mu_G(E) \tag{2.1}$$

is satisfied by every measurable set $E \subset G$. Because ϕ_f is a homeomorphism, it maps Borel sets of G onto Borel sets, and because it is also an automorphism, the measure

$$\mu_G^f(B) = \mu_G(\phi_f(B)) , \qquad B \text{ a Borel set} ,$$

is left-invariant. This measure clearly satisfies conditions (iv)-(vii) of [3, p. 194] and consequently, by the uniqueness of left Haar measure, (2.1) is satisfied for some $\delta(f) > 0$ and all Borel sets. Furthermore, the outer measure

$$\mu^*(E) = \inf \{\mu_G(A) \mid A \text{ is open, } E \subset A\}$$

induced by μ_G also satisfies

$$\mu^*(\phi_f(E)) = \delta(f)\mu^*(E)$$

for every subset $E \subset G$. It is then easy to verify (using [3, Th. 11.32] for example) that (2.1) holds for every measurable set E. In particular, if G is compact, any topological automorphism of G is measure preserving.

Clearly the mapping δ is a homomorphism of F into the multiplicative group of positive real numbers and

$$I_G(\Phi_f(x)) = I_G(x \circ \phi_{f-1}) = \delta(f) I_G(x) , \qquad x \in L^1(G) .$$

In these terms, the modular function for S can be expressed as

$$\Delta_S(f, g) = \delta(f^{-1}) \Delta_F(f) \Delta_G(g) .$$

The principal concern of this paper is the case in which F is finite. In this case the functions Δ_F and δ are obviously identically 1.

THEOREM 2. *Let F be a finite group, and let G be a locally compact group whose group algebra is symmetric. Then any semi-direct product $S = F \times_\phi G$ has a symmetric group algebra.*

Proof. Let $x \in L^1(S)$, $x = x(f, g)$. For each $f \in F$ the function $x_f(g) = x(f, g)$ is, by Fubini's theorem, in $L^1(G)$. Conversely, if $y_f \in L^1(G)$ for each $f \in F$ and y is defined by $y(f, g) = y_f(g)$, then $y \in L^1(S)$. In this manner $L^1(S)$ is identified with the space of all $L^1(G)$-valued functions defined on F. Now,

$$x^*(f, g) = x(f^{-1}, \phi_{f-1}(g^{-1})) \Delta_G(\phi_{f-1}(g^{-1})) = \Phi_f((x_{f-1})^*)(g)$$

and

$$\begin{aligned} x^*x(f, g) &= I_S(x^*[r, s] x[(r, s)^{-1}(f, g)]) \\ &= I_F(I_G(\Phi_r((x_{r-1})^*)(s) \Phi_r(x_{r-1f}) s^{-1} g))) \\ &= \sum_{f \in F} \Phi_r((x_{r-1})^*) * \Phi_r(x_{r-1f})(g) . \end{aligned}$$

To see that $L^1(S)$ is symmetric we must show that $(-x^*x)$ is both right and left quasi-regular. For example, we must exhibit functions $y_f \in L^1(G)$ such that y as defined above satisfies $y + x^*xy - x^*x = 0$. We compute x^*xy.

$$\begin{aligned} x^*xy(f, g) &= I_S(x^*x[p, q]y[(p, q)^{-1}(f, g)]) \\ &= I_F\Big(I_G\Big(\sum_{f\in F} \Phi_r((x_{r-1})^*) * \Phi_r(x_{r-1p})(q)\Phi_p(y_{p-1f}(q^{-1}g))\Big)\Big) \\ &= \sum_{f\in F}\sum_{p\in F} \Phi_r((x_{r-1})^*) * \Phi_r(x_{r-1p}) * \Phi_p(y_{p-1f})(g) \ . \end{aligned}$$

Let the group F be written $F = \{f_1 = e, f_2, \cdots, f_n\}$. Then the equations which must be satisfied are

$$\begin{aligned} y_{f_i} &+ \sum_{j=1}^{n}\sum_{k=1}^{n} \Phi_{r_j}((x_{r_j^{-1}})^*) * \Phi_{r_j}(x_{r_j^{-1}p_k}) * \Phi_{p_k}(y_{p_k^{-1}f_i}) \\ &- \sum_{j=1}^{n} \Phi_{r_j}((x_{r_j^{-1}})^*) * \Phi_{r_j}(x_{r_j^{-1}f_i}) = 0 \ . \qquad i = 1, 2, \cdots, n \ . \end{aligned}$$

These are equivalent to

$$\begin{aligned} y_{f_i} &+ \sum_{j=1}^{n}\sum_{m=1}^{n} \Phi_{r_j}((x_{r_j^{-1}})^*) * \Phi_{r_j}(x_{r_j^{-1}f_iq_m^{-1}}) * \Phi_{f_iq_m^{-1}}(y_{q_m}) \\ &- \sum_{j=1}^{n} \Phi_{r_j}((x_{r_j^{-1}})^*) * \Phi_{r_j}(x_{r_j^{-1}f_i}) = 0 \ . \qquad i = 1, 2, \cdots, n \ . \end{aligned}$$

Transforming both sides by $\Phi_{f_i^{-1}}$ we obtain the equations

$$\begin{aligned} \Phi_{f_i^{-1}}(y_{f_i}) &+ \sum_{j=1}^{n}\sum_{m=1}^{n} \Phi_{f_i^{-1}r_j}((x_{r_j^{-1}})^*) * \Phi_{f_i^{-1}r_j}(x_{r_j^{-1}f_iq_m^{-1}}) * \Phi_{q_m^{-1}}(y_{q_m}) \\ &- \sum_{j=1}^{n} \Phi_{f_i^{-1}r_j}((x_{r_j^{-1}})^*) * \Phi_{f_i^{-1}r_j}(x_{r_j^{-1}f_i}) = 0 \ , \qquad i = 1, 2, \cdots, n \ . \end{aligned}$$

Finally,

$$\begin{aligned} \Phi_{f_i^{-1}}(y_{f_i}) &+ \sum_{k=1}^{n}\sum_{m=1}^{n} \Phi_{s_k}((x_{s_k^{-1}f_i^{-1}})^*) * \Phi_{s_k}(x_{s_k^{-1}q_m}) * \Phi_{q_m}(y_{q_m^{-1}}) \\ &- \sum_{k=1}^{n} \Phi_{s_k}((x_{s_k^{-1}f_i^{-1}})^*) * \Phi_{s_k}(x_{s_k^{-1}}) = 0 \ , \qquad i = 1, 2, \cdots, n \ . \end{aligned} \tag{2.2}$$

It is evidently enough to determine the functions $\Phi_{f_i^{-1}}(y_{f_i})$, for from them the y_{f_i} can be obtained on transforming by Φ_{f_i}. Consider the matrix $A = [a_{ij}]$ of elements from $L^1(G)$ defined by

$$a_{ij} = \Phi_{s_i}(x_{s_i^{-1}f_j^{-1}}) \ , \qquad i, j = 1, 2, \cdots, n \ .$$

Since $L^1(G)$ is symmetric we know, by Theorem 1, that $-A^*A$ has a quasi-inverse, say $C = [c_{ij}]$ with $c_{ij} \in L^1(G)$. It follows from $C + A^*AC - A^*A = 0$ that

$$c_{i1} + \sum_{k=1}^{n}\sum_{m=1}^{n} a_{ki}^* a_{ki} c_{m1} - \sum_{k=1}^{n} a_{ki}^* a_{k1} = 0 \ , \qquad i = 1, 2, \cdots, n \ .$$

Thus $\Phi_{q_m}(q_m^{-1}) = c_{m1}$, $m = 1, 2, \cdots, n$ is a solution of the equations (2.2). A left quasi-inverse for $(-x^*x)$ can be computed in a similar manner. Hence $L^1(S)$ is symmetric.

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PRIME RINGS WITH A ONE-SIDED IDEAL SATISFYING A POLYNOMIAL IDENTITY

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It is known that the existence of a nonzero commutative one-sided ideal in a prime ring implies that the whole ring is commutative. Since rings satisfying a polynomial identity are natural generalizations of commutative rings the question arises as to what extent the above mentioned result can be extended to include these generalizations. That is, if R is a prime ring and I a nonzero one-sided ideal which satisfies a polynomial identity does R satisfy a polynomial identity?

This paper initiates an investigation of this problem. A counter example, given later, will show that the answer to the above question may be negative, even when R is a simple primitive ring with nonzero socle. The main theorem of this paper is Theorem 3 which states:

Let R be a prime ring having a nonzero right ideal which satisfies a polynomial identity. Then, a necessary and sufficient condition that R satisfy a polynomial identity is that R have zero right singular ideal and $\hat{R}$, the right quotient ring of R, have at most finitely many orthogonal idempotents.

2. In the following given a ring R, $R^{\Delta}({}^{\Delta}R)$ denotes the right (left) *singular ideal* of R. Thus $R^{\Delta} = \{x \mid x \in R, x^r \in L^{\Delta}(R)\}$ where $L^{\Delta}(R)$ denotes the set of right ideals of R that meet, in a nonzero fashion, all right ideals of R. Similarly for ${}^{\Delta}R$ and ${}^{\Delta}L(R)$.

If Q is a ring such that R is a subring of Q and $qR \cap R \neq 0$ for each $q \in Q$ then Q is called a right quotient ring for R. Moreover if $Q = \{ab^{-1} \mid a, b \in R, b \text{ regular}\}$ then Q is called a classical right quotient ring. Following [2] we say that a ring R is right quotient simple if and only if it has a classical right quotient ring Q with $Q \cong D_n$, D_n a ring of $n \times n$ matrices over a division ring D.

From [4] we know that if R is a prime ring with $R^{\Delta} = 0$ then R has a unique maximal right quotient ring $\hat{R}$ where $\hat{R}$ is a prime regular ring. Moreover, letting $L(R)$ denote the lattice of right ideals of R, there is a mapping s: $A \to A^s$ of $L(R)$ which is a closure operation satisfying $0^s = 0$, $(A \cap B)^s = A^s \cap B^s$ and $(x^{-1}A)^s = x^{-1}A^s$. The set $L^s(R)$ of closed ideals of R can be made into a lattice in a natural way and it is shown in [4] that $L^s(R) \cong L^s(\hat{R})$ under the mapping $A \to A \cap R$, $A \in L^s(\hat{R})$. We shall have occasion to use the following realization of $\hat{R}$. Let $E = \cup_{A \in L^{\Delta}(R)} \operatorname{Hom}_R(A, R)$. On E

define the relation, $\alpha \equiv \beta$ if for some $A \in L^{\Delta}(R)$, $A \subseteqq \text{Dom}\,\alpha \cap \text{Dom}\,\beta$ and $\alpha(x) = \beta(x)$ for each $x \in A$. It is shown in [5] that $\equiv$ is an equivalence relation and that $E/\equiv$ is a ring and in fact is $\hat{R}$.

The above remarks apply similarly to a prime ring R for which ${}^{\Delta}R = 0$.

3. In this section occur the basic results of this paper. We will have occasion to use the result of Posner [8] stating that if R is a prime ring with polynomial identity then $\hat{R}$ is a classical two-sided quotient ring having the same multilinear identities as R. That part of Posners argument that shows if R has a polynomal identity then so does $\hat{R}$ is a very complicated argument and we take this opportunity to present a simple alternative argument.

LEMMA 1. *Let R be a prime ring with polynomial identity. Then $\hat{R}$ has a polynomial identity.*

Proof. From Posner [8] we know that R has left and right quotient conditions and hence R is right quotient simple, with $\hat{R} \cong D_n$. By a theorem of Faith and Utumi [2] R contains an integral domain K with right quotient ring $\hat{K} \cong \hat{D}$. Since K satisfies a polynomial identity we have by Amitsur [1] that $\hat{K}$ also has a polynomial identity. Thus D, and hence D_n, is finite dimensional over its center; thus D_n, so $\hat{R}$, has a standard identity.

LEMMA 2. *Let R be a prime ring with $R^{\Delta} = 0$, let $A \in L^{\Delta}(R)$ and let $\alpha \in \text{Hom}_R(R, R)$, R considered as a right R-module. If $\alpha(A) = 0$ then $\alpha = 0$.*

Proof. Let $x \in R$; then we have that $x^{-1}A \in L^{\Delta}(R)$. If $r \in x^{-1}A$ then $xr \in A$ and thus $\alpha(xr) = 0$. Since α is a right R-endomorphism, $\alpha(xr) = \alpha(x)\cdot r$: It follows that $\alpha(x)\cdot x^{-1}A = 0$, hence $x^{-1}A \subseteqq \alpha(x)^r$. Thus $\alpha(x)^r \in L^{\Delta}(R)$ and so $\alpha(x) \in R^{\Delta}$. Hence $\alpha(x) = 0$.

The following lemma is trivial in the case R contains a central element. Without a central element the proof is more involved.

LEMMA 3. *Let R be a prime ring with a polynomial identity. Then $\text{Hom}_R(R, R)$ has a polynomial identity, if $R^{\Delta} = 0$.*

Proof. From Lemma 1 we know that $\hat{R}$ has a polynomial identity. Consider $\hat{R}$ realized as $\bigcup_{A \in L(R)} \text{Hom}_R(A, R)/\equiv$. For $\alpha \in \text{Hom}_R(R, R)$ let $\bar{\alpha}$ denote the equivalence class in $\hat{R}$ determined by α. The mapping $\alpha \rightarrow \bar{\alpha}$ is a homomorphism of $\text{Hom}_R(R, R)$ into $\hat{R}$. If $\bar{\alpha} = \bar{\beta}$ then for some $A \in L^{\Delta}(R)$ $\alpha(x) = \beta(x)$, $x \in A$. Thus $(\alpha - \beta)(A) = 0$. By Lemma 2 we see that $\alpha = \beta$. Thus $\alpha \to \bar{\alpha}$ is an injection onto a subring of $\hat{R}$ and so $\mathrm{Hom}_R(R, R)$ has a polynomial identity.

The following theorem provides a sufficient condition on the right ideal I having a polynomial identity to ensure the whole ring has a polynomial identity.

THEOREM 1. *Let R be a prime ring having a right ideal $I \neq 0$, I satisfying a polynomial identity and $I_l = 0$. Then R satisfies a polynomial identity.*

Proof. By assumption I_l, the left annihilator of I, is 0. Hence I is a prime ring itself. Considering I as a left I-module we have by the obvious dual of Lemma 3 that $\mathrm{Hom}_I(I, I)$, (the left I-endomorphisms), has a polynomial identity. For $x \in R$ the mapping $x \to r_x$, right multiplication by x, is an anti-isomorphism of R into $\mathrm{Hom}_I(I, I)$. Thus R itself satisfies a polynomial identity.

THEOREM 2. *Let R be a right quotient simple ring, $I \neq 0$ a right ideal of R satisfying a polynomial identity. Then R satisfies a polynomial identity.*

Proof. From Goldie [3] we have that I contains a uniform right ideal, thus we may assume I is uniform. Since $R^{\Delta} = 0$ it follows that $\{x \mid x \in I,\ x^r \in L^{\Delta}(R)\} = 0$, hence from [6] we have that $K = \mathrm{Hom}_R(I, I)$ is an integral domain. Moreover it is known ([3]) that $\hat{K} \cong D$, D a division ring, where $\hat{R} \cong D_n$. To complete the proof it suffices to show that D has a polynomial identity; the latter will hold provided K has a polynomial identity. To this end consider the homomorphism $a \to l_a$, left multiplication by a, of I into K. Let J denote the image of this map. $J = 0$ implies $I^2 = 0$ which is impossible; hence J is a nonzero subring of K satisfying a polynomial identity. Let $\alpha \in K$ and let $l_a \in J$. Let $x \in I$. Then $\alpha l_a(x) = \alpha(ax) = \alpha(a)\cdot x = l_{\alpha(a)}(x)$. Thus $\alpha l_a = l_{\alpha(a)} \in J$. Hence J is a left ideal of K. Since K is an integral domain we have by an obvious dual to Theorem 1 that K has a polynomial identity.

We now obtain, easily, the following.

THEOREM 3. *Let R be a prime ring having a nonzero right ideal which satisfies a polynomial identity. Then, a necessary and sufficient condition that $\hat{R}$ satisfy a polynomial identity is that $R^{\Delta} = 0$ and $\hat{R}$ have at most a finite number of orthogonal idempotents.*

Proof. Necessity is clear. Conversely, then, since $\hat{R}$ is regular with at most finitely many orthogonal idempotents it follows from [7] that $\hat{R}$ has the descending chain condition (d.c.c.) on right ideals. $\hat{R}$ is prime, thus $\hat{R} \cong D_n$ for some division ring D. Since $L^s(R) \cong L^s(\hat{R})$ we see that $L^s(R)$ has d.c.c. Thus from [4] we see that $\hat{R}$ is a classical right quotient ring, hence Theorem 2 applies.

The following example (communicated orally to S. K. Jain by A. S. Amitsur) shows that the extension of an identity from a right ideal to the entire ring is not always possible. Let F be a field and let F_∞ be the ring of all infinite matrices of finite rank. Let $a = (A_{ij})$ be a matrix such that $a_{11} \neq 0$ and $a_{ij} = 0$ for $i, j \neq 1$. Let $I = aF_\infty$. Then I satisfies the identity $(xy - yx)^2 = 0$ but F_∞ satisfies no identity at all.

4. REMARKS. In the case that R is primitive with a right ideal $I \neq 0$ having a polynomial identity then it is sufficient to assume that R has at most a finite number of orthogonal idempotents to ensure that R also have a polynomial identity.

There are other conditions one may impose upon R and I besides those given here, e.g. if R has at most finitely many orthogonal idempotents and I is a maximal right ideal or if $R^d = 0$ and $I \in L^d(R)$.

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A NOTE ON CERTAIN BIORTHOGONAL POLYNOMIALS

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Konhauser has introduced two polynomial sets $\{Y_n^c(x; k)\}$, $\{Z_n^c(x; k)\}$ that are biorthogonal with respect to the weight function $e^{-x}x^c$ over the interval $(0, \infty)$. An explicit expression was obtained for $Z_n^c(x; k)$ but not for $Y_n^c(x; k)$. An explicit polynomial expression for $Y_n^c(x; k)$ is given in the present paper.

1. Konhauser [2] has discussed two sets of polynomials $Y_n^c(x; k)$, $Z_n^c(x; k)$, $n = 0, 1, \cdots, k = 1, 2, 3, \cdots, c > -1$; $Y_n^c(x; k)$ is a polynomial in x while $Z_n^c(x; k)$ is a polynomial in x^k. Moreover

$$\int_0^\infty e^{-x}x^c Y_n^c(x; k)x^{ki}dx = \begin{cases} 0 & (0 \leqq i < n) \\ \neq 0 & (i = n) \end{cases} \tag{1}$$

and

$$\int_0^\infty e^{-x}x^c Z_n^c(x; k)x^{i}dx = \begin{cases} 0 & (0 \leqq i < n) \\ \neq 0 & (i = n)\ . \end{cases} \tag{2}$$

For $k = 1$, conditions (1) and (2) reduce to the orthogonality conditions satisfied by the Laguerre polynomials $L_n^c(x)$.

It follows from (1) and (2) that

$$\int_0^\infty e^{-x}x^c Y_i^c(x; k)Z_j^c(x; k)dx = \begin{cases} 0 & (i \neq j) \\ \neq 0 & (i = j)\ . \end{cases} \tag{3}$$

The polynomial sets $\{Y_n^c(x; k)\}, \{Z_n^c(x; k)\}$ are accordingly said to be biorthogonal with respect to the weight function $e^{-x}x^c$ over the interval $(0, \infty)$.

Konhauser showed that

$$Z_n^c(x; k) = \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + c + 1)} \tag{4}$$

As for $Y_n^c(x; k)$, he showed that

$$\begin{aligned} Y_n^c(x; k) &= \frac{k}{2i} \int_C \frac{e^{-xt}(t + 1)^{c+kn}}{[(t + 1)^k - 1]^{n+1}} dt \\ &= \frac{k}{n!} \frac{\partial^n}{\partial t^n} \left\{ \frac{e^{-xt}(t + 1)^{c+kn}t^{n+1}}{[(t + 1)^{k+1} - 1]^{n+1}} \right\}_{t=0} . \end{aligned} \tag{5}$$

In the integral in (5), C may be taken as a small circle about the origin in the t-plane.

In the present note we give a generating function and an explicit polynomial expression for the polynomial $Y_n^c(x; k)$. Moreover we show that $Y_n^c(x; k)$ can be identified with a polynomial studied recently by S. K. Chatterjea [1].

2. We apply the Lagrange expansion in the form [4, p. 125]

$$\frac{f(t)}{1 - w\phi'(t)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \left\{\frac{d^n}{dt^n} [f(t)(\phi(t))^n]\right\}_{t=0}, \tag{6}$$

where

$$w = \frac{t}{\phi(t)}, \qquad \phi(t) = a_0 + a_1 t + \cdots \qquad (a_0 \neq 0) .$$

Take

$$f(t) = \frac{e^{-xt}(t+1)^c t}{(t+1)^k - 1}, \qquad \phi(t) = \frac{(t+1)^k t}{(t+1)^k - 1} .$$

Then we have

$$1 - w\phi'(t) = \frac{kt}{(t+1)(t+1)^k - 1},$$

so that

$$\frac{f(t)}{1 - w\phi'(t)} = e^{-xt}(t+1)^{c+1} .$$

Thus, by (5) and (6), we have

$$e^{-xt}(t+1)^{c+1} = \sum_{n=0}^{\infty} Y_n^c(x; k)\left(\frac{t}{\phi(t)}\right)^n . \tag{7}$$

If we put

$$w = \frac{t}{\phi(t)} = \frac{(t+1)^k - 1}{(t+1)^k} = 1 - (t+1)^{-k},$$

then

$$t = (1 - w)^{-1/k} - 1$$

and (7) becomes

$$(1 - w)^{-(c+1)/k} \exp\{-x[(1 - w)^{-1/k} - 1]\} = \sum_{n=0}^{\infty} Y_n^c(x; k) w^n . \tag{8}$$

In the next place, we have

$$
\begin{aligned}
&(1-w)^{-(c+1)/k} \exp\{-x[(1-w)^{-1/k}-1]\} \\
&\quad = (1-w)^{-(c+1)/k} \sum_{r=0}^{\infty} \frac{x^r}{r!} [(1-w)^{-1/k}-1]^r \\
&\quad = \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (1-w)^{-(s+c+1)/k} \\
&\quad = \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \sum_{n=0}^{\infty} \frac{((s+c+1)/k)_n}{n!} w^n \\
&\quad = \sum_{n=0}^{\infty} \frac{w^n}{n!} \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{s+c+1}{k}\right)_n ,
\end{aligned}
$$

where

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1 .$$

It therefore follows from (8) that

$$Y_n^c(x;k) = \frac{1}{n!} \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{s+c+1}{k}\right)_n . \tag{9}$$

3. Chatterjea [1] has defined the polynomial

$$T_{k\ n}^{(\alpha)}(x) - \frac{1}{n!} x^{-\alpha} e^{x^k} D^n (e^{\alpha+n} e^{-x^k}) \tag{10}$$

with $k = 1, 2, 3, \cdots$. The case $\alpha = 0$ had been discussed by Palas [3]. Chatterjea showed that (10) implies

$$T_{k,n}^{(\alpha)}(x) = \sum_{r=0}^{\infty} \frac{x^{kr}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{\alpha+n+ks}{n} . \tag{11}$$

He also obtained operational formulas and a generating function for $T_{k,n}^{(\alpha)}(x)$. The assumption that k is a positive integer is not used in deriving (11).

If we replace k by k^{-1} and α by $k^{-1}\alpha$, (10) becomes

$$T_{k^{-1},k}^{(-1\alpha)}(x) = \sum_{r=0}^{n} \frac{x^{kr}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{k^{-1}(\alpha+s)+n}{n} .$$

On the other hand, since

$$\frac{1}{n!} \left(\frac{s+c+1}{k}\right)_n = \binom{k^{-1}(s+c+1)+n-1}{n} ,$$

(9) gives

$$Y_n^{c+k-1}(x^k;k) = \sum_{r=0}^{n} \frac{x^{kr}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{k^{-1}(s+c)+n}{n} .$$

It follows at once that

$$Y_n^{c+k-1}(x^k;k) = T_{k^{-1},n}^{(k^{-1}c)}(x) , \tag{12}$$

or, if we prefer,

$$Y_n^{k\alpha+k-1}(x^k; k) = T_{k-1,n}^{(\alpha-1)}(x) . \tag{13}$$

4. It may be of interest to point out that a formula equivalent to (9) can be obtained without the use of the Lagrange expansion. In the integral representation (5), put

$$t = (1+u)^{1/k} - 1 .$$

Then (5) becomes

$$Y_n^c(x; k) = \frac{1}{2\pi i}\int_C \frac{\exp\{-x[(1-u)^{1/k}-1]\}(1+u)^{k^{-1}(c+1)+n-1}}{u^{n+1}}du ,$$

where C denotes a small circle about the origin in the u-plane. The numerator of the integral is equal to

$$\begin{aligned}&\sum_{r=0}^{\infty}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}(1+u)^{k^{-1}(c+s+1)+n-1}\\ &\quad=\sum_{m=0}^{\infty}u^m\sum_{r=0}^{m}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^r\binom{r}{s}\binom{k^{-1}(c+s+1)+n-1}{m} .\end{aligned}$$

Taking $m = n$, we therefore get

$$Y_n^c(x; k) = \sum_{r=0}^{n}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^r\binom{r}{s}\binom{k^{-1}(c+s+1)+n-1}{n} . \tag{14}$$

Since

$$\binom{c+n-1}{n} = \frac{(c)_n}{n!} ,$$

it is evident that (14) is identical with (9).

5. Making use of the explicit formulas (4) and (9), we can give a rather brief proof of (3). Indeed we have

$$\begin{aligned}J_{n,m} &= \int_0^\infty e^{-x}x^c Z_n^c(x; k)Y_m^c(x; k)dx\\ &= \frac{\Gamma(kn+c+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{\Gamma(kj+c+1)}\\ &\quad\cdot\frac{1}{m!}\sum_{r=0}^{m}\frac{1}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{s+c+1}{k}\right)_m\cdot\int_0^\infty e^{-x}x^{kj+c+r}dx\\ &= \frac{\Gamma(kn+c+1)}{n!\,m!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\\ &\quad\cdot\sum_{r=0}^{m}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{s+c+1}{k}\right)_m\binom{kj+c+r}{r} .\end{aligned}$$

If $f(x)$ is a polynomial of degree m, it is familiar that

$$f(x) = \sum_{r=0}^{m} \binom{x}{r} \Delta^r f(0) ,$$

where

$$\Delta^r f(0) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f(s) .$$

In particular, for

$$f(x) = \left(\frac{x+c+1}{k}\right)_m ,$$

we have

$$\begin{aligned} \left(\frac{x+c+1}{k}\right)_m &= \sum_{r=0}^{m} \binom{x}{r} \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \left(\frac{s+c+1}{k}\right)_m \\ &= \sum_{r=0}^{m} \binom{+x+r-1}{r} \sum_{s-r}^{n} (-1)^{s} \binom{r}{s} \left(\frac{s+c+1}{k}\right)_m . \end{aligned}$$

For $x = -kj - c - 1$ this reduces to

$$(-j)_m = \sum_{r=0}^{m} \binom{kj+c+r}{r} \sum_{s=0}^{r} (-1)^{s} \binom{r}{s} \left(\frac{s+c+1}{k}\right)_m .$$

Thus

$$\begin{aligned} J_{n,m} &= \frac{\Gamma(kn+c+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(-j)_m}{m!} \\ &= (-1)^m \frac{\Gamma(kn+c+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j}{m} . \end{aligned}$$

Since

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j}{m} = \binom{n}{m} \sum_{j=m}^{n} (-1)^j \binom{n-m}{j-m} = (-1)^m \binom{n}{m} (1-1)^{n-m}$$

it is evident that

$$J_{n,m} = \frac{\Gamma(kn+c+1)}{n!} \delta_{nm} \tag{15}$$

in agreement with (3). In particular

$$J_{n,n} = \frac{\Gamma(kn+c+1)}{n!}$$

as proved in [2].

A little more generally, we have

$$\begin{aligned}J'_{n,m} &= \int_0^\infty e^{-x}x_c Z_n^c(x;k)Y_n^{c'}(x;k)dx \\ &= \frac{\Gamma(kn+c+1)}{n!\,m!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(-j-\frac{c-c'}{k}\right)_m \\ &= (-1)^m\frac{\Gamma(kn+c+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+a}{m},\end{aligned}$$

where $a = (c-c')/k$. It follows that

$$J'_{n,m} = \begin{cases} 0 & (n > m)\,, \\ (-1)^{n+m}\dfrac{\Gamma(kn+c+1)}{n!}\dbinom{a}{m-n} & (n \leqq m)\,. \end{cases} \tag{16}$$

Clearly (16) includes (15).

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POINTLIKE SUBSETS OF A MANIFOLD

C. O. CHRISTENSON AND R. P. OSBORNE

Morton Brown introduced the concept of a cellular subset of S^n. As a consequence of the generalized Schoenflies Theorem it is easy to show that a subset of S^n is pointlike if and only if it is cellular. In this paper the obvious generalization of the definitions of pointlike and cellular sets are made and thier relationship in a manifold is considered. It is easy to show that a cellular subset of a manifold is pointlike. While it is not true that a pointlike subset of a manifold is cellular, it is shown that a pointlike subset of a compact n-manifold lies in a contractible n-manifold with $(n-1)$-sphere boundary. As a consequence of this it is shown that K is a pointlike subset of a compact n-manifold $(n \neq 4)$ if and only if K is cellular. The case $n = 4$ is still unsolved.

DEFINITIONS. An *n-manifold* is a connected separable locally Euclidean metric space. A connected separable metric space in which every point has a neighborhood whose closure is an n-cell is an *n-manifold with boundary*. Note that a manifold is a manifold with boundary boundary but not conversely. A compact connected subset K of an n-manifold M is *pointlike* if $M \sim K$ is homeomorphic with $M \sim \{p\}$ where $p \in M$. A subset K of an n-manifold M is *cellular* if there is a sequence of n-cells $C_1, C_2, \cdots$ such that $C_{i+1} \subset \operatorname{Int} C_i$ and $K = \cap C_i$. An $(n-1)$-sphere S^{n-1} that separates an n-manifold M into components A and B is *collared on the side containing* A if there is an embedding $h: S^{n-1}X[0, 1] \rightarrow \bar{A}$ such that $h(x, 0) = x$. An $(n-1)$-sphere S^{n-1} in an n-manifold M is *bicollared* if there is an embedding $h: S^{n-1}X[0, 1] \rightarrow M$ such that $h(x, 1/2) = x$. A *pseudo-sphere* is a compact manifold that is a homotopy sphere. A compact contractible n-manifold with boundary is called a pseudo-cell. The Poincare Conjecture—known to be true for $n \neq 3, 4$ [7]—says that a pseudo-sphere is a sphere.

PRELIMINARY THEOREMS. The following theorem follows from the corresponding theorem for E^n which is proved by the same methods as used in [4].

THEOREM 1. *A cellular subset of a manifold is pointlike.*

One might think that a pointlike subset of a manifold is cellular. That this is not the case is shown by the following example.

EXAMPLE 1. Let M be E^3 minus the integers on the positive x-axis, and minus 1-spheres of radius 1/4 centered at the negative integers on the x-axis. The 1-sphere of radius 1/4 and center at 0 is pointlike but not cellular. A similar construction using linked 1-spheres gives an example of a pointlike subset of a manifold containing a loop that is homotopically nontrivial in the manifold. A cellular subset of a manifold is not necessarily contractible, for example the crumpled cube bounded by the Alexander Horned sphere is not simply connected even though it is cellular.

LEMMA 2. *Let K be a pointlike subset of a compact manifold M with boundary. Let $h': M \sim K \to M \sim \{p\}$ be a homeomorphism. Then h' can be extended to a continuous map $h: M \to M$ such that $h^{-1}(p) = K$.*

Proof. Define h by

$$h(x) = \begin{cases} h'(x) & \text{for } x \in M \sim K , \\ p & \text{for } x \in K . \end{cases}$$

Let U be an open neighborhood of p. Then $\sim U$ is compact; hence, $h^{-1}(\sim U)$ is compact so $M \sim h^{-1}(\sim U)$ is open. Clearly this set contains K. Thus h is continuous.

LEMMA 3. *If K is a pointlike subset of a compact n-manifold M with boundary and K lies in an open n-cell, then K is cellular.*

Proof. We shall show that if U is a neighborhood of K then there is an n-cell C such that $K \subset \text{Int } C \subset U$. Using this a simple inductive argument completes the proof. Let $h: M \to M$ be the continuous map given by the previous lemma. Then $h(U)$ is a neighborhood of p. Let C' be an n-cell with bicollared boundary in $h(U)$ containing p in its interior. Then $h^{-1}(C') = C$ is a cell by the Generalized Schoenflies theorem.

By obvious modifications of the proof in [8], the Jordan-Brouwer Theorem can be shown to hold in a pseudo-n-sphere. Let K be the closure of one of the complementary domains of S^{n-1}. If an n-cell is sewn to K the result is another pseudo-sphere. Applications of the Van Kampen Theorem, the Mayer-Vietoris Sequence and the Hurewicz Isomorphism show that K is $(n-2)$-connected. Theorem 6.6.5 and Theorem 6.2.20 of [8] show that K is contractible.

LEMMA 4 (*Pseudo Schoenflies Lemma*). *A bicollared $(n-1)$-sphere S^{n-1} in a pseudo-sphere M^n is the common boundary of two psudo-cells.*

MAIN RESULT.

THEOREM 5. *If K is a pointlike subset of a compact manifold M^n and $\dot{K}$ is an $(n-1)$-sphere collared on the side containing K, then K is a pseudo-cell.*

Proof. Assume $n \geqq 3$. Denote by L the set $((M^n \sim K) \cup$ collar of $\dot{K})$. Then L and K are closed and their union is M^n while their intersection is simply connected. By the Van Kampen Theorem $\pi_1(M^n) = \pi_1(L) * \pi_1(K)$, where $*$ denotes the free product. Borsuk [2] has shown that every compact manifold is dominated by a polyhedron, that is there is a finite polyhedron P and continuous maps $f: P \rightarrow M^n$ and $g: M^n \rightarrow P$ such that $f \circ g$ is a homotopic to 1_{M^n}. It follows that $\pi_1(M^n)$ is a finitely presented group. Since K is pointlike, $\pi_1(M^n \sim K) = \pi_1(L) = \pi_1(M^n \sim \{p\}) = \pi_1(M^n)$. We have $\pi_1(M^n) = \pi_1(K) * \pi_1(L) = \pi_1(K) * \pi_1(M^n)$. By Grusko's theorem [6], $\pi_1(K)$ is trivial.

To show that $\pi_q(K)$ is trivial for $q \leqq n$ we show that $H_q(K)$ is trivial for $q \leqq n-2$, then we use duality to get $H_g(K) = 0$ for $q \leqq n$. Since K and L form an excisive couple we may apply the Mayer-Vietoris Sequence to get

$$H_q(K \cap L) \rightarrow H_q(K) \oplus H_q(L) \rightarrow H_q(K \cup L) \rightarrow H_{q-1}(K \cap L) , \qquad 1 \leqq q \leqq n-2 .$$

Since $K \cap L$ is an n-annulus this sequence becomes

$$0 \rightarrow H_q(K) \oplus H_q(L) \rightarrow H_q(K \cup L) \rightarrow 0 ,$$

which implies that $H_q(K) \oplus H_p(L) \approx H_q(K \cup L)$. Since K is pointlike, $H_q(K \cup L) \approx H_q(L)$. Since there is a dominating polyhedron for M^n, $H_q(M^n)$ is a finitely generated group. It follows that $H_q(K)$ is trivial. By the Hurewicz Isomorphism Theorem, $\pi_q(K) = 0$ for $1 \leqq q \leqq n-2$. Let S be the compact manifold obtained by sewing a cell to the boundary of K. Then by duality, S is a homotopy sphere. By Lemma 4, K is contractible.

If $n = 2$ then K can be shown to shown to be a 2-cell by the classification theorem for compact 2-manifolds with contours for boundary.

COROLLARY 6. *Let K be a pointlike subset of a compact manifold M, then K lies in a pseudo-cell with sphere boundary.*

Proof: Let $h: M \rightarrow M$ be the continuous map given by Lemma 2. Let C' be a cell containing p and having a bicollared boundary. Then C' is pointlike so $h^{-1}(C') = C$ is a pointlike subset of M with bicollared sphere boundary. The previous theorem shows that C is a pseudo-cell.

COROLLARY 7. *In a compact manifold in which every pseudo-cell with sphere boundary is a cell, a pointlike subset is cellular.*

LEMMA 8. If K is a pointlike subset of a compact manifold M, then there are infinitely many disjoint homeomorphic copies of K in M.

Proof. Let $p \in M \sim K$ and let $h: M \sim K \rightarrow M \sim \{p\}$ be a homeomorphism. Let $h^{-1}(K) = K_1 \subset M \sim K$. Let g_1 be a homeomorphism of M onto itself such that $g_1(p) = p_1 \notin K \cup K_1$ and $g_1 = 1$ on K. Let $h_1 = g_1 \circ h$. Then $h_1^{-1}(K_1) = K_2$ is homeomorphic with k and

$$h_1^{-1}(K_1) \cap (K_1 \cap K) = \varnothing .$$

Continuing in this fashion we get $K, K_1, K_2, \cdots$.

The complement of two disjoint pointlike subsets of a manifold M need not be homeomorphic with the complement of two points in M; for example two linked 1-spheres in the 3-manifold of Example 1.

THEOREM 9. *A pointlike subset of a compact n-manifold* ($n \neq 4$) *is cellular.*

Proof. By Corollary 6, the pointlike set lies in a pseudo-cell P with sphere boundary. Sew a cell to P along their boundaries to get a homotopy sphere S^n. Since the Poincare Conjecture has been proved [7] for $n \geqq 5$, S^n must be a sphere. The generalized Schoenflies Theorem [3] shows that P is a cell. An application of Lemma 3 completes the proof when $n \geqq 5$. If K is a pointlike subset of a compact manifold M, then there are countably many disjoint homeomorphic copies of K in M. Thus if K is a pointlike subset of M that is not cellular, then M must contain countably many disjoint pseudo-cells that are not cells. If $n = 3$, M is triangulable so an application of Bing's Side Approximation Theorem [1] allows us to assume that each pseudo-cell has a polyhedral sphere boundary. Kneser [5] has shown that such a decomposition can contain only finitely many such sets that are not cells.

We note that we have a generalization of the Generalized Schoenflies theorem: If S^{n-1} is a bicollared $(n-1)$-sphere that separates a compact n-manifold M and one of the components of $M - S^{n-1}$ is pointlike, then that component is a pseudo-cell.

One should observe that the proof the Theorem 5 shows: If K is a pointlike subset of an n-manifold M, $\pi_m(M)$ is finitely generated for $1 \leqq m \leqq n$, and $\dot{K}$ is an $(n-1)$-sphere collared on the side containing K, then K is a pseudo-cell.

Using arguments like those used in the proof of Theorem 5, one can show that a compact n-manifold ($n \neq 4$) can be written as the connected sum of at most finitely many nontrivial summands.

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Question. If we drill countably many disjoint cells out of S^4 and sew in pseudo-cells, is the resulting space ever a manifold?

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PRODUCTS AND QUOTIENTS OF PROBABILISTIC METRIC SPACES

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In this paper some results concerning the products and quotients of probabilistic metric spaces are presented.

Probabilistic metric spaces were first introduced by K. Menger in 1942 and reconsidered by him in the early 1950's [3, 4, 5]. Since 1958, B. Schweizer and A. Sklar have been studying these spaces, and have developed their theory in depth [9, 10, 11, 12, 13]. These spaces have also been considered by several other authors [e. g., 2, 14, 15, 16]. An extensive, detailed up-to-date presentation may be found in [7].

In the sequel, we shall adopt the usual terminology, notation and conventions of the theory of probabilistic metric spaces, with but one exception: In all previous work, the distribution functions which determine the distances between points were required to have supremum one. Our investigations have led us to drop this requirement and the results which we present here show that doing so is natural. It is easy but tedious to check that the restriction to distribution functions with supremum one is not required in any of the previously established results which will be needed in the sequel.

In concluding this introduction we remark that products of probabilistic metric spaces have previously been considered by V. Istratescu and I. Vaduva [2]. However, their definition of Cartesian product employs associative functions which are stronger than Min, the strongest possible triangular norm. Because of this, and in view of the discussion given in [10], their results appear somewhat restrictive. Also, a number of the results concerning finite products, which are presented in § 1 and which were announced in [1], have recently been obtained independently by A. Xavier [17].

1. Product spaces.

DEFINITION 1. Let $(S_1, \mathfrak{F}_1)$ and $(S_2, \mathfrak{F}_2)$ be *PM* spaces and let T be a left-continuous t-norm. The *T-product* $(S_1, \mathfrak{F}_1) \times (S_2, \mathfrak{F}_2)$ of $(S_1, \mathfrak{F}_1)$ and $(S_2, \mathfrak{F}_2)$ is the space $(S_1 \times S_2, T(\mathfrak{F}_1, \mathfrak{F}_2))$, where $S_1 \times S_2$ is the Cartesian product of the sets S_1 and S_2 and $T(\mathfrak{F}_1, \mathfrak{F}_2)$ is the mapping from $(S_1 \times S_2) \times (S_1 \times S_2)$ into the set of distribution functions Δ given by

$$T(\mathfrak{F}_1, \mathfrak{F}_2)(p, q) = T(\mathfrak{F}_1(p_1, q_1), \mathfrak{F}_2(p_2, q_2)) ,$$

for any $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $S_1 \times S_2$.

We shall often denote $S_1 \times S_2$ by S *and* $T(\mathfrak{F}_1, \mathfrak{F}_2)$ by $\mathfrak{F}_T$, and when there can be no doubt, omit the reference to T and write $\mathfrak{F}_T(p, q) = F_{pq}$.

As immediate consequences of Definition 1 we have:

THEOREM 1. *The T-product $(S, \mathfrak{F}_T)$ of two PM spaces $(S_1, \mathfrak{F}_1)$ and $(S_2, \mathfrak{F}_2)$ is a PM space.*

THEOREM 2. *If $(S_1, \mathfrak{F}_1, T)$ and $(S_2, \mathfrak{F}_2, T)$ are Menger spaces under the same left-continuous t-norm T, then their T-product is a Menger space under T.*

COROLLARY 1. *If $(S_1, \mathfrak{F}_1, T_1)$ and $(S_2, \mathfrak{F}_2, T_2)$ are Menger spaces and if there exists a left-continuous t-norm T which is weaker than T_1 and T_2, then their T-product is a Menger space under T.*

We now determine conditions under which the product of equilateral, simple, or α-simple *PM* spaces is again a *PM* space of the same type. We begin with,

THEOREM 3. *If $(S_1, \mathfrak{F}_1)$ and $(S_2, \mathfrak{F}_2)$ are equilateral spaces generated by the same distribution function G, then their Min product $(S_1 \times S_2, \mathfrak{F}_{Min})$ is an equilateral space generated by G.*

Proof. Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be distinct points in $S_1 \times S_2$ and consider

$$F_{pq}(x) = \mathrm{Min}\,(F_{p_1q_1}(x), F_{p_2q_2}(x)) .$$

In all three cases, (1) $p_1 \neq q_1, p_2 \neq q_2$; (2) $p_1 = q_1, p_2 \neq q_2$; (3) $p_1 \neq q_1$, $p_2 = q_2$, we have $F_{pq}(x) = G(x)$ from which the result follows.

It should be noted that the choice of Min in the above theorem is necessary, since we must have

$$T(H(x), G(x)) = T(G(x), G(x)) = G(x) ,$$

where H is the distribution function defined by

$$H(x) = \begin{cases} 0, & x \leqq 0 \\ 1, & x > 0 . \end{cases}$$

In general, this is true only for $T = \mathrm{Min}$. Similarly, it is necessary that $(S_1, \mathfrak{F}_1)$ and $(S_2, \mathfrak{F}_2)$ be generated by the same distribution function.

THEOREM 4. *If* $(S_1, \mathfrak{F}_1)$ *and* $(S_2, \mathfrak{F}_2)$ *are simple spaces generated by the metric spaces* (S_1, d_1) *and* (S_2, d_2), *respectively, and the same distribution function* G, *then their* Min-*product* $(S_1 \times S_2, \mathfrak{F}_{\mathrm{Min}})$ *is a simple space generated by the metric space* $(S_1 \times S_2, \mathrm{Max}\,(d_1, d_2))$ and G.

Proof. Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ belong to $S_1 \times S_2$. It follows from Theorem 1 that $F_{pq} = H$ if and only if $p = q$. Thus we have only to show that whenever $p \neq q$ $F_{pq}(x) = G(x/d(p, q))$, where $d(p, q) = \mathrm{Max}\,(d_1(p_1, q_1), d_2(p_2, q_2))$. There are again three cases to consider:

(1) If $p_1 \neq q_1$ and $p_2 \neq q_2$, then

$$\begin{aligned} F_{pq}(x) &= \mathrm{Min}\,\{G(x/d_1(p_1, q_1)), G(x/d_2(p_2, q_2))\} \\ &= G(x/\mathrm{Max}\,(d_1(p_1, q_1), d_2(p_2, q_2))) = G(x/d(p, q))\,. \end{aligned}$$

(2) If $p_1 = q_1$ and $p_2 \neq q_2$, then $d_1(p_1, q_1) = 0$ and

$$\begin{aligned} F_{pq}(x) &= \mathrm{Min}\,(H(x), G(x/d_2(p_2, q_2)) = G(x/d_2(p_2, q_2)) \\ &= G(x/\mathrm{Max}\,(0, d_2(p_2, q_2))) = G(x/d(p, q))\,. \end{aligned}$$

(3) If $p_1 \neq q_1$ and $p_2 = q_2$, we proceed as in (2) above.

DEFINITION 2. A distance distribution function G is *strict* if it is continuous and strictly increasing on $[0, \infty)$ and with $\mathrm{Sup}_x\, G(x) = 1$.

The restriction of G to $[0, \infty)$ has an inverse which we will denote by G^* and refer to as the inverse of G.

THEOREM 5. *Let* $(S_1, \mathfrak{F}_1)$ *and* $(S_2, \mathfrak{F}_2)$ *be* α-*simple spaces,* $\alpha \geqq 1$, *generated by the metric spaces* (S_1, d_1) *and* (S_2, d_2), *respectively, and the same strict distribution function* G. *Let* T *be the strict t-norm whose additive generator is* $(G^*)^{-m/\alpha}$, *where* $m \geqq 1$ [12]. *Then the* T-*product* $(S_1 \times S_2, \mathfrak{F}_T)$ *is an* α-*simple space generated by the metric space* $(S_1 \times S_2, (d_1^m + d_2^m)^{1/m})$ *and* G.

Proof. Let $d = (d_1^m + d_2^m)^{1/m}$ and let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be distinct points of $S_1 \times S_2$. We have to show that

$$F_{pq}(x) = G(x/d^\alpha(p, q))\,.$$

We again split cases:

(1) If $p_1 \neq q_1$ and $p_2 \neq q_2$, then

$$\begin{aligned} F_{pq}(x) &= T(G(x/d_1^\alpha(p_1, q_1)), G(x/d_2^\alpha(p_2, q_2))) \\ &= f^*\{fG(x/d_1^\alpha(p_1, q_1)) + fG(x/d_2^\alpha(p_2, q_2))\}\,, \end{aligned}$$

where $f = (G^*)^{-m/\alpha}$ and $f^* = G(j^{-\alpha/m})$ and j denotes the identity function. It follows that $fG = j^{-m/\alpha}$, whence

$$\begin{aligned} F_{pq}(x) &= f^*\{x^{-m/\alpha}(d_1^m(p_1, q_1) + d_2^m(p_2, q_2))\} \\ &= G\{x(d_1^m(p_1, q_1) + d_2^m(p_2, q_2))^{-\alpha/m}\} = G(x/d^\alpha(p, q)) . \end{aligned}$$

(2) If $p_1 = q_1$ and $p_2 \neq q_2$, then for $x > 0$

$$F_{pq}(x) = T(H(x), G(x/d_2^\alpha(p_2, q_2))) = G(x/d^\alpha(p, q)) .$$

(3) If $p_1 \neq q_1$ and $p_2 = q_2$, we proceed as in (2).

As a result of Theorem 2 in [12] it follows that for $\alpha > 1$ the α-simple spaces above are all Menger spaces under the t-norm T' whose additive generator is $(G^*)^{1/(1-\alpha)}$. Moreover as B. Schweizer has observed, if we want to have $T = T'$, then α and m must satisfy the equation $1/(1 - \alpha) = -m/\alpha$, from which it follows that

$$1/\alpha + 1/m = 1 .$$

We now turn to the question of topologies on the T-product spaces and state as our final result of this section.

THEOREM 6. *Let $(S_1, \mathfrak{F}_1 T)$ and $(S_2, \mathfrak{F}_2 T)$ be Menger spaces under the same left-continuous t-norm. Let $\mathfrak{B}'$ denote the $\varepsilon - \lambda$ neighborhood system in $(S_1 \times S_2, \mathfrak{F}_T, T)$ and let $\mathfrak{B}$ denote the neighborhood system in $(S_1 \times S_2, \mathfrak{F}_T, T)$ consisting of the Cartesian products $N_{p_1} \times N_{p_2}$, where N_{p_1} and N_{p_2} are $\varepsilon - \lambda$ neighborhoods in the respective component spaces $(S_1, \mathfrak{F}_1, T)$ and $(S_2, \mathfrak{F}_2, T)$. Then $\mathfrak{B}$ and $\mathfrak{B}'$ induce equivalent topologies on $(S_1 \times S_2, \mathfrak{F}_T, T)$.*

Proof. We first note that since T is assumed to be left-continuous, the neighborhood systems $\mathfrak{B}$ and $\mathfrak{B}'$ are in fact bases for their respective topologies [10]. Consequently, it suffices to show that for each B in $\mathfrak{B}$ there exists a B' in $\mathfrak{B}'$ such that $B' \subseteqq B$, and conversely. Let $A_1 \times A_2$ be an element of $\mathfrak{B}$. Then there exist neighborhoods $N_{p_1}(\varepsilon_1, \lambda_1)$ and $N_{p_2}(\varepsilon_2, \lambda_2)$ contained in A_1 and A_2, respectively. Let

$$\varepsilon = \mathrm{Min}\,(\varepsilon_1, \varepsilon_2), \lambda = \mathrm{Min}\,(\lambda_1, \lambda_2)$$

and $p = (p_1, p_2)$. We will show that $N_p(\varepsilon, \lambda) \subseteqq A_1 \times A_2$. To this end, let $q = (q_1, q_2)$ belong to $N_p(\varepsilon, \lambda)$. Then we have

$$\begin{aligned} F_{p_1q_1}(\varepsilon_1) &= T(F_{p_1q_1}(\varepsilon_1), 1) \geqq T(F_{p_1q_1}(\varepsilon_1), F_{p_2q_2}(\varepsilon_2)) \\ &\geqq T(F_{p_1q_1}(\varepsilon), F_{p_2q_2}(\varepsilon)) = F_{pq}(\varepsilon) > 1 - \lambda \geqq 1 - \lambda_1 . \end{aligned}$$

Similarly, $F_{p_2q_2}(\varepsilon_2) > 1 - \lambda_2$. Thus $q_1 \in N_{p_1}(\varepsilon_1, \lambda_1)$ and $q_2 \varepsilon N_{p_2}(\varepsilon_2, \lambda_2)$, from which the result follows.

Conversely, suppose that $N_p(\varepsilon, \lambda)$ is an element of $\mathfrak{B}'$. Since T is left-continuous, $\text{Sup}_{x<1} T(x, x) = 1$, so that there exists an η such that

$$T(1 - \eta, 1 - \eta) > 1 - \lambda .$$

Let $q = (q_1, q_2)$ belong to $N_{p_1}(\varepsilon, \eta) \times N_{p_2}(\varepsilon, \eta)$. Then

$$F_{pq}(\varepsilon) = T(F_{p_1q_1}(\varepsilon), F_{p_2q_2}(\varepsilon)) \geqq T(1 - \eta, 1 - \eta) > 1 - \lambda$$

so that $q \in N_p(\varepsilon, \lambda)$ and $N_{p_1}(\varepsilon, \eta) \times N_{p_2}(\varepsilon, \eta) \subseteqq N_p(\varepsilon, \lambda)$. This completes the proof.

Note that the proof of the first half of Theorem 6, i.e., of the fact that for any B in $\mathfrak{B}$ there exists a B' in $\mathfrak{B}'$ such that $B' \subseteqq B$, is independent of any hypothesis on the t-norm T, while the proof of the second half requires only that $\text{Sup}_{x<1} T(x, x) = 1$.

We conclude this section by remarking that all the above results may be extended in an obvious way to include products of any finite number of *PM* spaces.

2. Diameter of and distance between sets. Throughout this section $(S, \mathscr{F}, T)$ will denote a Menger space with a continuous t-norm.

DEFINITION 3. Let A be a nonempty subset of S. The function D_A, defined by

$$D_A(x) = \text{Sup}_{t<x} \left[\text{Inf}_{p,q \in A} F_{pq}(t) \right] ,$$

will be called the *probabilistic diameter* of A.

We now establish the properties of the probabilistic diameter. Proofs requiring only routine calculations will be omitted.

THEOREM 7. *The function D_A is a distribution function.*

DEFINITION 4. A nonempty subset A of S is *bounded* if $\text{Sup}_x D_A(x) = 1$, *semi-bounded* if $0 < \text{Sup}\, D_A(x) < 1$, and *unbounded* if $D_A = 0$.

THEOREM 8. *If A is a nonempty subset of S, then $D_A = H$ if and only if A consists of a single point.*

THEOREM 9. *If A and B are nonempty subsets of S and $A \subseteqq B$, then $D_A \geqq D_B$.*

THEOREM 10. *If A and B are two nonempty subsets of S such that $A \cap B = \varnothing$, then*

(2.1) $$D_{A\cup B}(x+y) \geqq T(D_A(x), D_B(y)) .$$

Proof. Let x and y be given. To establish (2.1) we first show that

(2.2) $$\operatorname*{Inf}_{p,q\in A\cup B} F_{pq}(x+y) \geqq T\left(\operatorname*{Inf}_{p,q\in A} F_{pq}(x), \operatorname*{Inf}_{p,q\in B} F_{pq}(y)\right) .$$

There are two distinct cases to consider:

Case (1).

(2.3) $$\operatorname*{Inf}_{p,q\in A\cup B} F_{pq}(x+y) = \operatorname*{Inf}_{\substack{p\in A\\ q\in B}} F_{pq}(x+y) .$$

Now for any triple of points p, q and r in S, we have

$$F_{pq}(x+y) \geqq T(F_{pr}(x), F_{rq}(y)).$$

Taking the infinum of both sides of this inequality as p ranges over A, q ranges over B and r ranges over $A \cap B$, and using (2.3) we have,

$$\operatorname*{Inf}_{p,q\in A\cup B} F_{pq}\left(x+y\right) \geqq \operatorname*{Inf}_{\substack{p\in A\\ q\in B\\ r\in A\cap B}} T(F_{pr}(x), F_{rq}(y)\Big) .$$

However, since T is continuous and nondecreasing we obtain

$$\operatorname*{Inf}_{p,q\in A\cup B} F_{pq}(x+y) \geqq T\left(\operatorname*{Inf}_{p,r\in A} F_{pr}(x), \operatorname*{Inf}_{r,q\in B} F_{rq}(y)\right) .$$

Case (2).

$$\operatorname*{Inf}_{p,q\in A\cup B} F_{pq}(x+y) < \operatorname*{Inf}_{\substack{p\in A\\ q\in B}} F_{pq}(x+y) .$$

In this case one of the equalities,

$$\operatorname*{Inf}_{p,q\in A\cup B} F_{pq}(x+y) = \operatorname*{Inf}_{p,q\in A} F_{pq}(x+y)$$

or

$$\operatorname*{Inf}_{p,q\in A\cup B} F_{pq}(x+y) = \operatorname*{Inf}_{p,q\in B} F_{pq}(x+y)$$

must hold. If the first equality holds, we have

$$\begin{aligned}\operatorname*{Inf}_{p,q\in A\cup B} F_{pq}(x+y) &\geqq T\left(\operatorname*{Inf}_{p,q\in A} F_{pq}(x), H(y)\right)\\ &\geqq T\left(\operatorname*{Inf}_{p,q\in A} F_{pq}(x), \operatorname*{Inf}_{p,q\in B} F_{pq}(y)\right) .\end{aligned}$$

The same argument works for the second equality. This establishes (2.2).

Finally, using the fact that the rectangle

$$\{(s, t) : 0 \leqq s \leqq x, 0 \leqq t \leqq y\}$$

is contained in the triangle $\{(s, t) : s, t \geqq 0, s + t < x + y\}$, the inequality (2.2) and the continuity of T we have

$$\begin{aligned} D_{A \cup B}(x + y) &= \operatorname*{Sup}_{s+t<x+y}\Big[\operatorname*{Inf}_{p,q \in A \cup B} F_{pq}(s + t)\Big] \\ &\geqq \operatorname*{Sup}_{\substack{s<x \\ t<y}}\Big[\operatorname*{Inf}_{p,q \in A \cup B} F_{pq}(s + t)\Big] \\ &\geqq T\Big(\operatorname*{Sup}_{s<x}\Big[\operatorname*{Inf}_{p,q \in A} F_{pq}(s)\Big], \operatorname*{Sup}_{t<y}\Big[\operatorname*{Inf}_{p,q \in B} F_{pq}(t)\Big]\Big) \\ &= T(D_A(x), D_B(y)) . \end{aligned}$$

THEOREM 11. *If A is a nonempty subset of S, then $D_A = D_{\bar{A}}$, where $\bar{A}$ denotes the closure of A in the $\varepsilon - \lambda$ topology on S* [10].

Proof. Since $A \subseteqq \bar{A}$, it follows from Theorem 7 that $D_A \geqq D_{\bar{A}}$.

Let $\eta > 0$ be given. In view of the uniform continuity of $\mathfrak{F}$ with respect to the Lévy metric L on Δ [8] there exists an $\varepsilon > 0$ and a $\lambda > 0$ such that for any four points p_1, p_2, p_3 and p_4 in S,

$$L(F_{p_1 p_2}, F_{p_3 p_4}) < \eta$$

whenever $F_{p_1 p_3}(\varepsilon) > 1 - \lambda$ and $F_{p_2 p_4}(\varepsilon) > 1 - \lambda$.

Next, with each point $\bar{p}$ in $\bar{A}$ associate a point $p(\bar{p})$ in $\bar{A}$ such that $F_{p(\bar{p})\bar{p}}(\varepsilon) > 1 - \lambda$. Then, in view of the above for any pair of points $\bar{p}$ and $\bar{q}$ in A,

$$L(F_{p(\bar{p})q(\bar{q})}, F_{\bar{p}\bar{q}}) < \eta .$$

In particular, for all t we have,

$$F_{p(\bar{p})q(\bar{q})}(t - \eta) - \eta \leqq F_{\bar{p}\bar{q}}(t) .$$

Let $A_\eta = \{p(\bar{p}) : \bar{p} \in \bar{A})$. Then since $A_\eta \subseteqq A$,

$$\begin{aligned} \operatorname*{Inf}_{\bar{p},\bar{q} \in \bar{A}} F_{\bar{p}\bar{q}}(t) &\geqq \operatorname*{Inf}_{\bar{p},\bar{q} \in \bar{A}} {}_{p(\bar{p})q(\bar{q})}(t - \eta) - \eta \\ &= \operatorname*{Inf}_{p,q \in A_\eta} F_{pq}(t - \eta) - \eta \geqq \operatorname*{Inf}_{p,q \in A} F_{pq}(t - \eta) - \eta . \end{aligned}$$

Now, taking the supremum for $t < x$ of the above inequality yields

$$D_{\bar{A}}(x) = \operatorname*{Sup}_{t<x}\Big[\operatorname*{Inf}_{\bar{p},\bar{q}\in\bar{A}} F_{\bar{p}\bar{q}}(t)\Big] \geqq \operatorname*{Sup}_{t>x}\Big[\operatorname*{Inf}_{p,q\in A} F_{pq}(t-\eta)\Big] - \eta$$
$$= \operatorname*{Sup}_{t<x-\eta}\Big[\operatorname*{Inf}_{p,q\in A} F_{pq}(t)\Big] - \eta = D_A(x-\eta) - \eta\ .$$

Since the above inequality is valid for all η and since D_A is left-continuous it follows that

$$D_{\bar{A}}(x) \geqq D_A(x)\ .$$

Whence $D_{\bar{A}}(x) = D_A(x)$ and the proof is complete.

DEFINITION 5. Let A and B be nonempty subsets of S. The *probabilistic distance between A and B* is the function F_{AB} defined by

$$F_{AB}(x) = \operatorname*{Sup}_{t<x} T\left(\operatorname*{Inf}_{p\in A}\Big[\operatorname*{Sup}_{q\in B} F_{pq}(t)\Big], \operatorname*{Inf}_{q\in B}\Big[\operatorname*{Sup}_{p\in A} F_{pq}(t)\Big]\right)\ . \tag{2.4}$$

In establishing the properties of F_{AB} we again omit the routine proofs.

THEOREM 12. *F_{AB} is a distribution function.*

THEOREM 13. *If A and B are nonempty subsets of S, then $F_{AB} = F_{BA}$.*

THEOREM 14. *If A is a nonempty subset of S, then $F_{AA} = H$.*

THEOREM 15. *If A and B are nonempty subsets of S, then $F_{AB} = F_{\bar{A}\bar{B}}$.*

Proof. It is sufficient to show that $F_{AB} = F_{A\bar{B}}$ since this result together with Theorem 13 yields

$$F_{AB} = F_{A\bar{B}} = F_{\bar{B}A} = F_{\bar{B}\bar{A}} = F_{\bar{A}\bar{B}}\ .$$

With this in mind we first show that $F_{A\bar{B}} \leqq F_{AB}$. Since $B \subseteqq \bar{B}$ for all t,

$$\operatorname*{Inf}_{q\in B}\Big[\operatorname*{Sup}_{p\in A} F_{pq}(t)\Big] \geqq \operatorname*{Inf}_{\bar{q}\in\bar{B}}\Big[\operatorname*{Sup}_{p\in A} F_{p\bar{q}}(t)\Big]\ . \tag{2.5}$$

Let $\eta > 0$ be given. The argument given in the proof of Theorem 11, establishes that for each point $\bar{q}\in\bar{B}$, there exists a point $q(\bar{q})$ in B such that for all t,

$$F_{p\bar{q}}(t-\eta) - \eta \leqq F_{pq(\bar{q})}(t)\ .$$

Let $B_\eta = \{q(\bar{q})\colon \bar{q}\in\bar{B}\}$. Since $B_\eta \subseteqq B$ we have,

$$\operatorname*{Sup}_{\bar{q}\in\bar{B}} F_{p\bar{q}}(t-\eta)-\eta \leqq \operatorname*{Sup}_{\bar{q}\in\bar{B}} F_{pq(\bar{q})}(t) = \operatorname*{Sup}_{q\in B_\eta} F_{pq}(t) \leqq \operatorname*{Sup}_{q\in B} F_{pq}(t) .$$

Consequently,

$$\operatorname*{Inf}_{p\in A}\Big[\operatorname*{Sup}_{\bar{q}\in\bar{B}} F_{p\bar{q}}(t-\eta)\Big]-\eta \leqq \operatorname*{Inf}_{p\in A}\Big[\operatorname*{Sup}_{q\in B} F_{pq}(t)\Big] .$$

Moreover, taking the supremum on $t<x$ of the above inequality, yields for any η,

$$f(x) \overset{df}{=} \operatorname*{Sup}_{t<x}\Big(\operatorname*{Inf}_{p\in A}\Big[\operatorname*{Sup}_{q\in B} F_{pq}(t)\Big]\Big) \geqq \operatorname*{Sup}_{t<x}\Big(\operatorname*{Inf}_{p\in A}\Big[\operatorname*{Sup}_{\bar{q}\in\bar{B}} F_{p\bar{q}}(t-\eta)\Big]\Big)-\eta$$
$$= \operatorname*{Sup}_{t>x-\eta}\Big(\operatorname*{Inf}_{q\in A}\Big[\operatorname*{Sup}_{\bar{q}\in\bar{B}} F_{p\bar{q}}(t)\Big]\Big)-\eta \overset{df}{=} g(x-\eta)-\eta .$$

Now since both f and g are left-continuous and η is arbitrary, it follows that $f(x) \geqq g(x)$. This together with (2.5), and the continuity of T yields

$$F_{AB}(x) = T\Big\{\operatorname*{Sup}_{t<x}\Big(\operatorname*{Inf}_{p\in A}\Big[\operatorname*{Sup}_{q\in B} F_{pq}(t)\Big]\Big), \operatorname*{Sup}_{t<x}\Big(\operatorname*{Inf}_{q\in B}\Big[\operatorname*{Sup}_{p\in A} F_{pq}(y)\Big]\Big)\Big\}$$
$$\geqq T\Big\{\operatorname*{Sup}_{t<x}\Big(\operatorname*{Inf}_{p\in A}\Big[\operatorname*{Sup}_{\bar{q}\in\bar{B}} F_{p\bar{q}}(t)\Big]\Big), \operatorname*{Sup}_{t<x}\Big(\operatorname*{Inf}_{\bar{q}\in\bar{B}}\Big[\operatorname*{Sup}_{p\in A} F_{p\bar{q}}(t)\Big]\Big)\Big\}$$
$$= \operatorname*{Sup}_{t<x} T\Big(\operatorname*{Inf}_{p\in A}\Big[\operatorname*{Sup}_{\bar{q}\in\bar{B}} F_{p\bar{q}}(t)\Big], \operatorname*{Inf}_{\bar{q}\in\bar{B}}\Big[\operatorname*{Sup}_{p\in A} F_{p\bar{q}}(t)\Big]\Big) = F_{A\bar{B}}(x) .$$

A similar argument shows that $F_{A\bar{B}} \geqq F_{AB}$. Combining these inequalities yields the desired result.

THEOREM 16. *If A and B are nonempty subsets of S, then $F_{AB} = H$ if and only if $\bar{A} = \bar{B}$.*

Proof. Suppose $F_{AB} = H$ and let $\varepsilon > 0$ be given. Then

$$1 = F_{AB}(\varepsilon) = T\Big\{\operatorname*{Sup}_{t<\varepsilon}\Big(\operatorname*{Inf}_{p\in A}\Big[\operatorname*{Sup}_{q\in B} F_{pq}(t)\Big]\Big), \operatorname*{Sup}_{t<\varepsilon}\Big(\operatorname*{Inf}_{q\in B}\Big[\operatorname*{Sup}_{p\in A} F_{pq}(t)\Big]\Big)\Big\}$$
$$= \operatorname*{Sup}_{t<\varepsilon}\Big(\operatorname*{Inf}_{q\in B}\Big[\operatorname*{Sup}_{p\in A} F_{pq}(t)\Big]\Big) = \operatorname*{Inf}_{q\in B}\Big[\operatorname*{Sup}_{p\in A} F_{pq}(\varepsilon)\Big] .$$

So that for any $q \in B$ and every $\lambda > 0$ there exists a point p in A for which $F_{pq}(\varepsilon) > 1-\lambda$. Consequently, q is an accumulation point of A and we have $B \subseteqq \bar{A}$. A similar argument shows that $A \subseteqq \bar{B}$.

Conversely, suppose $\bar{A} = \bar{B}$. Then in view of Theorems 14 and 15, $F_{AB} = F_{\bar{A}\bar{B}} = F_{\bar{A}\bar{A}} = H$.

THEOREM 17. *If A, B and C are nonempty subsets of S, then* *for any x and y*

$$F_{AB}(x + y) \geqq T(F_{AC}(x), F_{BC}(y)) .$$

Proof. Let u and v be given. Then for any triple of points p, q and r in S we have

$$F_{pq}(u + v) \geqq T(F_{pr}(u), F_{qr}(v)) .$$

Making use of the continuity and monotonicity of T we have the following inequality:

$$\operatorname*{Sup}_{q \in B} F_{pq}(u + v) \geqq T\left(\operatorname*{Sup}_{r \in C} F_{pr}(u), \operatorname*{Inf}_{r \in C} \left[\operatorname*{Sup}_{q \in B} F_{qr}(v)\right]\right) .$$

Consequently,

$$\operatorname*{Inf}_{p \in A}\left[\operatorname*{Sup}_{q \in B} F_{pq}(u + v)\right] \geqq T\left(\operatorname*{Inf}_{p \in A}\left[\operatorname*{Sup}_{r \in C} F_{pr}(u)\right], \operatorname*{Inf}_{r \in C}\left[\operatorname*{Sup}_{q \in B} F_{qr}(v)\right]\right) .$$

Similarly,

$$\operatorname*{Inf}_{q \in B}\left[\operatorname*{Sup}_{p \in A} F_{pq}(u + v)\right] \geqq T\left(\operatorname*{Inf}_{r \in C}\left[\operatorname*{Sup}_{p \in A} F_{pr}(u)\right], \operatorname*{Inf}_{q \in B}\left[\operatorname*{Sup}_{r \in C} F_{qr}(v)\right]\right) .$$

Therefore, since T is associative, we have

$$\begin{aligned} &T\left(\operatorname*{Inf}_{p \in A}\left[\operatorname*{Sup}_{q \in B} F_{pq}(u + v)\right], \operatorname*{Inf}_{q \in B}\left[\operatorname*{Sup}_{p \in A} F_{pq}(u + v)\right]\right) \\ &\quad \geqq T\left\{T\left(\operatorname*{Inf}_{p \in A}\left[\operatorname*{Sup}_{r \in C} F_{pr}(u)\right], \operatorname*{Inf}_{r \in C}\left[\operatorname*{Sup}_{p \in A} F_{pr}(u)\right]\right),\right. \\ &\qquad \left. T\left(\operatorname*{Inf}_{q \in B}\left[\operatorname*{Sup}_{r \in C} F_{qr}(v)\right], \operatorname*{Inf}_{r \in C}\left[\operatorname*{Sup}_{q \in B} F_{qr}(v)\right]\right)\right\} . \end{aligned}$$

Now arguing as in the last step of the proof of Theorem 10, we have

$$\begin{aligned} F_{AB}(x + y) &= \operatorname*{Sup}_{u+v<x+y} T\left(\operatorname*{Inf}_{p \in A}\left[\operatorname*{Sup}_{q \in B} F_{pq}(u + v)\right],\right. \\ &\qquad\qquad \left.\operatorname*{Inf}_{q \in B}\left[\operatorname*{Sup}_{p \in A} F_{pq}(u + v)\right]\right) \\ &\geqq \operatorname*{Sup}_{\substack{u<x \\ v<y}} T\left(\operatorname*{Inf}_{p \in A}\left[\operatorname*{Sup}_{q \in B} F_{pq}(u + v)\right], \operatorname*{Inf}_{q \in B}\left[\operatorname*{Sup}_{p \in A} F_{pq}(u + v)\right]\right) \\ &= T\left\{\operatorname*{Sup}_{u<x} T\left(\operatorname*{Inf}_{p \in A}\left[\operatorname*{Sup}_{r \in C} F_{pr}(u)\right], \operatorname*{Inf}_{r \in C}\left[\operatorname*{Sup}_{p \in A} F_{pr}(u)\right]\right),\right. \\ &\qquad \left.\operatorname*{Sup}_{v<y} T\left(\operatorname*{Inf}_{q \in B}\left[\operatorname*{Sup}_{r \in C} F_{qr}(v)\right], \operatorname*{Inf}_{r \in C}\left[\operatorname*{Sup}_{q \in B} F_{qr}(v)\right]\right)\right\} \\ &= T(F_{AC}(x), F_{BC}(y)). \end{aligned}$$

Let $(S, \mathfrak{F}, T)$ be a Menger space under a continuous t-norm, T, and let $\mathfrak{S}$ be a nonempty collection of nonempty subsets of S. Then the function $\mathfrak{F}_{\mathfrak{S}}$ defined for any A and B in $\mathfrak{S}$ by $\mathfrak{F}_{\mathfrak{S}}(A, B) = F_{AB}$, where F_{AB} is given by (2.4), is a mapping from $\mathfrak{S} \times \mathfrak{S}$ into Δ. Furthermore, as a direct consequence of Theorems 12–17 we have,

THEOREM 18. *If each set in $\mathfrak{S}$ is closed, then $(\mathfrak{S}, \mathfrak{F}_{\mathfrak{S}}, T)$ is a Menger space.*

3. Quotient spaces. Let $(S, \mathfrak{F})$ be a *PM* space. In [4] K. Menger introduced three types of distinguishability for pairs of points p, q in S depending upon the behavior of the distance distribution function F_{pq} near zero. These notions may be summarized in the following:

DEFINITION 5. Let $(S, \mathfrak{F})$ be a *PM* space, let p and q be points in S and let $t_{pq} = \operatorname{Inf}\{x : F_{pq}(x) > 0\}$. Then the *distance between p and q is:*

(A) *certainly positive* if $t_{pq} > 0$;

(B) *barely positive* if $t_{pq} = 0$ and $F_{pq}(0^+) = 0$;

(C) *perhaps zero* if $F_{pq}(0^+) > 0$.

In Menger's paper a somewhat different terminology was used. Namely, he said that p and q are: (A) certainly distinguishable if the distance between them is certainly positive; (B) barely distinguishable if the distance between them is barely positive; (C) perhaps indistinguishable if the distance between them is perhaps zero. The reasons for the slight change in the terminology introduced here will become apparent latter (see Definition 6, ff.).

The above mentioned types of distinguishability were recently reconsidered by B. Schweizer [6] who defined two relations C and D on S as follows:

(c) pCq if and only if the distance between p and q is perhaps zero, i.e., if and only if (C) holds.

(d) pDq if and only if the distance between p and q is not certainly positive, i.e., if and only if either (B) or (C) holds.

Concerning these relations, he obtained the following results:

THEOREM 19. *If $(S, \mathfrak{F}, T)$ is a Menger space and T a t-norm such that $T(a, b) > 0$ whenever $a > 0$ and $b > 0$, then the relation C is an equivalence relation.*

THEOREM 20. *Under the hypotheses of Theorem 19, (S, t) is always a pseudo metric space. Moreover, (S, t) is a metric space if and only if the distance between every pair of distinct points of S is certainly positive.*

THEOREM 21. *If the hypotheses of Theorem* 19 *are satisfied, then the relation D on S is an equivalence relation.*

THEOREM 22. *If $(S, \mathfrak{F}, T)$ is a Menger space such that*

$$\operatorname*{Sup}_{a<1} T(a, a) = 1$$

and $T(a, b) > 0$ whenever $a > 0$ and $b > 0$, then the equivalence classes in S determined by the equivalence relation D are closed subsets of S in the $\varepsilon - \lambda$ topology.

In view of the fact that we no longer require that all the distance distribution functions have supremum one, various types of behavior at infinity are possible and can be distinguished. Indeed, the entire preceding discussion concerning behavior at zero can be dualized.

DEFINITION 6. Let $(S, \mathfrak{F})$ be a PM space, let p and q be points in S, let $s_{pq} = \operatorname{Sup}\{x : F_{pq}(x) < 1\}$ and let $F_{pq}(\infty) = \lim_{x\to\infty} F_{pq}(x)$. Then the *distance between p and q is*:

(A′) *perhaps infinite* if $F_{pq}(\infty) < 1$;

(B′) *barely finite* if $s_{pq} = \infty$ and $F_{pq}(\infty) = 1$;

(C′) *certainly finite* if $s_{pq} < \infty$.

We define two relations C' and D' on S which are dual to C and D, respectively, as follows:

(c′) $pC'q$ if and only if the distance between p and q is certainly finite, i.e., if and only if (C′) holds.

(d′) $pD'q$ if and only if the distance between p and q is not perhaps infinite, i.e., if and only if $F_{pq}(\infty) = 1$, or equivalently if and only if (B′) or (C′) hold.

THEOREM 23. *If $(S, \mathfrak{F}, T)$ is a Menger space, then C' is an equivalence relation on S.*

Proof. The fact that C' is reflexive and symmetric is an immediate consequence of the definition of C'. To show that C' is transitive suppose $pC'q$ and $qC'r$, so that $s_{pq} < \infty$ and $s_{qr} < \infty$. Then for any $\varepsilon > 0$,

$$\begin{aligned} F_{pr}(s_{pq} + s_{qr} + \varepsilon) &\geqq T(F_{pq}(s_{pq} + \varepsilon/2), F_{qr}(s_{qr} + \varepsilon/2)) \\ &= T(1, 1) = 1 \,. \end{aligned}$$

Consequently, $s_{pr} \leqq s_{pq} + s_{qr} < \infty$ and $pC'r$.

THEOREM 24. *If $(S, \mathfrak{F}, T)$ is a Menger space in which the distance between every pair of points is certainly finite, then (S, s) is a metric space.*

Proof. In view of the proof of the previous theorem, we need only show that $s_{pq} = 0$ implies $p = q$. To this end let $s_{pq} = 0$, then $\operatorname{Sup}\{x : F_{pq}(x) < 1\} = 0$. Whence, $F_{pq}(0^+) = 1$ and consequently $F_{pq} = H$ so that $p = q$.

THEOREM 25. *If $(S, \mathfrak{F}, T)$ is a Menger space under a continuous t-norm T, then the relation D' on S is an equivalence relation.*

Proof. From $F_{pp}(\infty) = H(\infty) = 1$ and $F_{pq} = F_{qp}$ it follows that D' is reflexive and symmetric. To show that D' is transitive suppose $pD'q$ and $qD'r$. Then for any x,

$$F_{pr}(x) \geqq T(F_{pq}(x/2), F_{qr}(x/2)) .$$

Since T is continuous the above inequality yields

$$F_{pr}(\infty) \geqq T(F_{pq}(\infty), F_{qr}(\infty)) = T(1, 1) = 1$$

and thus $pD'r$.

THEOREM 26. *Let $(S, \mathfrak{F}, T)$ be a Menger space under a continuous t-norm T. Then the equivalence classes in S determined by the equivalence relation D' are closed subsets of S in the $\varepsilon - \lambda$ topology.*

Proof. We first note that since T is continuous on the unit square it is uniformly continuous. Now let $p \in S$ and let $D'(p)$ be the equivalence class determined by p. To show that $D'(p)$ is closed we show that $S - D'(p)$, the complement of $D'(p)$, is open. Let r be any point in $S - D'(p)$. Then there is a $\lambda > 0$ such that $F_{pr}(\infty) = 1 - \lambda$. Since T is uniformly continuous and since $T(a, 1) = a$, there exists an $\varepsilon > 0$ such that $T(a, 1 - \varepsilon) > a - \lambda/2$ for all a in $[0, 1]$. Let $q \in N_r(\varepsilon, \varepsilon)$. Then for any $x > \varepsilon$ we have

$$\begin{aligned} F_{pr}(2x) &\geqq T(F_{pq}(x), F_{qr}(x)) \geqq T(F_{pq}(x), 1 - \varepsilon) \\ &> F_{pq}(x) - \lambda/2 . \end{aligned}$$

Taking the limit as $x \longrightarrow \infty$ yields

$$1 - \lambda = F_{pr}(\infty) \geqq F_{pq}(\infty) - \lambda/2 ,$$

whence $F_{pq}(\infty) \leqq 1 - \lambda/2$. Thus $q \notin D'(p)$ and it follows that

$$N_r(\varepsilon, \varepsilon) \subseteqq S - D'(p) ,$$

hence $S - D'(p)$ is open.

THEOREM 27. *If $(S, \mathfrak{F}, T)$ is a Menger space such that T is continuous and $T(a, b) > 0$ whenever $a > 0$ and $b > 0$, then the equiva-*

lence classes in S determined by p and the equivalence relation C are closed in the $\varepsilon - \lambda$ topology.

Proof. Let $p \in S$ and let $C(p)$ be the equivalence class determined by p. We show that $S - C(p)$ is open. Let $r \in S - C(p)$. Then $F_{pr}(0^+) = 0$ so that F_{pr} is continuous at 0. Hence for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $F_{pr}(\delta) > \varepsilon/2$ and a $\lambda > 0$ such that for all $a \in [0, 1]$ $T(a, 1 - \lambda) > a - \varepsilon/2$. Let $q \in N_r(\delta/2, \lambda)$, then

$$\begin{aligned}\varepsilon/2 > F_{pr}(\delta) &\geqq T(F_{pq}(\delta/2), F_{qr}(\delta/2)) \\ &\geqq T(F_{pq}(\delta/2), 1 - \lambda) > F_{pq}(\delta/2) - \varepsilon/2.\end{aligned}$$

Hence for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $F_{pq}(\delta/2) < \varepsilon$. Consequently, $F_{pq}(0^+) = 0$. Thus $q \in S - C(p)$, whence $N_r(\delta/2, \lambda) \subseteqq S - C(p)$ and $S - C(p)$ is open.

THEOREM 28. *Let $(S, \mathfrak{F}, T)$ be a Menger space under a continuous t-norm T. Let $p \in S$ and let $C'(p)$ be the equivalence class in S determined by p and the equivalence relation C'. Suppose further that there exists a number M such that for any u and v in $C'(p)$ we have $F_{uv}(x) = 1$ whenever $x \geqq M$. Then $C'(p)$ is closed in the $\varepsilon - \lambda$ topology.*

Proof. Suppose q belongs to $\overline{C'(p)}$, the closure of $C'(p)$, but not to $C'(p)$. Then $F_{pq}(x) < 1$ for all finite x, so that for any $t > 0$ there is an $\varepsilon > 0$ such that $F_{pq}(t + M) > 1 - \varepsilon$; and since $q \in \overline{C'(p)}$, there exists a $u \in C'(p)$ such that $F_{qu}(t) > 1 - \varepsilon/2$. Whence,

$$\begin{aligned}1 - \varepsilon > F_{pq}(t + M) &\geqq T(F_{pu}(M), F_{qu}(t)) \\ &= T(1, F_{qu}(t)) = F_{qu}(t) > 1 - \varepsilon/2 ,\end{aligned}$$

which is a contradition. Thus $C'(p) = \overline{C'(p)}$.

The next four theorems show that, under suitable conditions, each of the equivalence relations, C, C', D, D', can be "divided out".

THEOREM 29. *Let $(S, \mathfrak{F}, T)$ be a Menger space under a t-norm T which is continuous and such that $T(a, b) > 0$ whenever $a > 0$ and $b > 0$. For each $p \in S$, let $D(p)$ be the equivalence class in S determined by p and the equivalence relation D and let S/D be the collection of all such equivalence classes. Then $(S/D, \mathfrak{F}_{S/D}, T)$ is a Menger space in which the distance between distinct elememts is certainly positive.*

Proof. The fact that $(S/D, \mathfrak{F}_{S/D}, T)$ is a Menger space follows directly from Theorems 18 and 22.

Let $D(p)$ and $D(q)$ be distinct equivalence classes, and suppose that

$$t_{D(p)D(q)} = 0 \, . \tag{3.2}$$

Since $p \in D(p)$ and $q \in D(q)$, there is an $x_0 > 0$ such that $F_{pq}(x_0) = 0$. In view of (3.2), we thus have

$$\begin{aligned} 0 &< F_{D(p)D(q)}(x_0) \\ &\leqq T\Big(\operatorname*{Inf}_{u \in D(p)} \Big[\operatorname*{Sup}_{v \in D(q)} F_{uv}(x_0)\Big], \operatorname*{Inf}_{v \in D(q)} \Big[\operatorname*{Sup}_{u \in D(p)} F_{uv}(x_0)\Big]\Big) \, . \end{aligned}$$

Hence,

$$0 < \operatorname*{Inf}_{u \in D(p)} \Big[\operatorname*{Sup}_{v \in D(q)} F_{uv}(x_0)\Big] \, ,$$

whence for each $u \in D(p)$

$$\operatorname*{Sup}_{v \in D(q)} F_{uv}(x_0) > 0 \, .$$

Consequently there exists a $q_0 \in D(q)$ such that $F_{pq_0}(x_0) > 0$. Thus since F_{pq_0} is left-continuous, there is an ε, $0 < \varepsilon < x_0$, such that $F_{pq_0}(x_0 - \varepsilon) > 0$. Hence

$$0 = F_{pq_0}(x_0) \geqq T(F_{pq_0}(x_0 - \varepsilon), F_{qq_0}(\varepsilon)) > 0 \, ,$$

since both $F_{pq_0}(x_0 - \varepsilon)$ and $F_{qq_0}(\varepsilon)$ are positive. However, this is a contradiction and hence $t_{D(p)D(q)} > 0$.

THEOREM 30. *Let $(S, \mathfrak{F}, T)$ be a Menger space under a continuous t-norm T. For each $p \in S$ let $D'(p)$ be the equivalence class in S determined by p and the equivalence relation D', and let S/D' be the collection of all such equivalence classes. Then $(S/D', \mathfrak{F}_{S/D'}, T)$ is a Menger space in which the distance between distinct elements is perhaps infinite.*

Proof. In view of Theorems 18 and 26 $(S/D', \mathfrak{F}_{S/D'}, T)$ is a Menger space.
Let $D'(p)$ and $D'(q)$ be distinct equivalence classes and suppose that $F_{D'(p)D'(q)}(\infty) = 1$. Since $p \in D'(p)$ and $q \in D'(q)$, there is an $\varepsilon > 0$ such that $F_{pq}(\infty) < 1 - \varepsilon$. Since T is continuous

$$\begin{aligned} 1 &= F_{D'(p)D'(q)}(\infty) \\ &= \lim_{x \to \infty} \operatorname*{Sup}_{t<x} T\Big(\operatorname*{Inf}_{u \in D'(p)} \Big[\operatorname*{Sup}_{v \in D'(q)} F_{uv}(t)\Big], \operatorname*{Inf}_{v \in D'(q)} \Big[\operatorname*{Sup}_{u \in D'(p)} F_{uv}(t)\Big]\Big) \\ &= \operatorname*{Sup}_{t} T\Big(\operatorname*{Inf}_{u \in D'(p)} \Big[\operatorname*{Sup}_{v \in D'(q)} F_{uv}(t)\Big], \operatorname*{Inf}_{v \in D'(q)} \Big[\operatorname*{Sup}_{u \in D'(p)} F_{uv}(t)\Big]\Big) \\ &= T\Big\{\operatorname*{Sup}_{t}\Big(\operatorname*{Inf}_{u \in D'(p)} \Big[\operatorname*{Sup}_{v \in D'(q)} F_{uv}(t)\Big]\Big), \operatorname*{Sup}_{t}\Big(\operatorname*{Inf}_{v \in D'(q)} \Big[\operatorname*{Sup}_{u \in D'(p)} F_{uv}(t)\Big]\Big)\Big\} \, . \end{aligned}$$

But $T(a, b) = 1$ if and only if $a = b = 1$. Consequently,

$$\operatorname{Sup}_t \left(\operatorname{Inf}_{u \in D'(p)} \left[\operatorname{Sup}_{v \in D'(q)} F_{uv}(t) \right] \right) = 1 .$$

Thus, there exists an x_0 such that

$$\operatorname{Inf}_{u \in D'(p)} \left[\operatorname{Sup}_{v \in D'(q)} F_{uv}(x_0) \right] > 1 - \varepsilon/2 .$$

Hence,

$$\operatorname{Sup}_{v \in D'(q)} F_{pv}(x_0) > 1 - \varepsilon/2 .$$

Since F_{pv} is nondecreasing

$$\operatorname{Sup}_{v \in D'(q)} F_{pv}(\infty) \geqq \operatorname{Sup}_{v \in D'(q)} F_{pv}(x_0) > 1 - \varepsilon/2 .$$

Consequently, there exists a $q_\varepsilon \in D'(q)$ such that

$$F_{pq_\varepsilon}(\infty) < \operatorname{Sup}_{v \in D'(q)} F_{pv}(\infty) - \varepsilon/4 > 1 - 3\varepsilon/4 .$$

and we have

$$1 - \varepsilon > F_{pq}(\infty) \geqq T(F_{pq_\varepsilon}(\infty), F_{qq_\varepsilon}(\infty)) = F_{pq_\varepsilon}(\infty) > 1 - 3\varepsilon/4 .$$

which is a contradiction. Hence $F_{D'(p)D'(q)}(\infty) < 1$ and the distance between distinct equivalence classes is perpaps infinite.

THEOREM 31. *Let $(S, \mathfrak{F}, T)$ be a Menger space under a t-norm T which is continuous and such that $T(a, b) > 0$ whenever $a > 0$ and $b > 0$. For each $p \in S$, let $C(p)$ be the equivalence class in S determined by p and the equivalence relation C, and let S/C be the collection of all such equivalence classes. Then $(S/C, \mathfrak{F}_{S/C}, T)$ is a Menger space. Moreover, if each $C(p)$ in S/C is such that $\operatorname{Inf}_{u,v \in C(p)} F_{uv}(0^+) > 0$, then the distance between distinct elements is not perhaps zero.*

Proof. The first part of this theorem is a direct consequence of Theorems 18 and 27.

To establish the second part, let $C(p)$ and $C(q)$ be distinct equivalence classes, and suppose that $F_{C(p)C(q)}(0^+) > 0$. Since $p \in C(p)$ and $q \in C(q)$, we note first that

$$F_{pq}(0^+) = 0 . \tag{3.3}$$

Next we have

$$
\begin{aligned}
0 &< F_{C(p)C(q)}(0^+) \\
&= \lim_{h\to 0+}\operatorname*{Sup}_{t<h} T\Big(\operatorname*{Inf}_{u\in C(p)}\Big[\operatorname*{Sup}_{v\in C(q)} F_{uv}(t)\Big], \operatorname*{Inf}_{v\in C(q)}\Big[\operatorname*{Sup}_{u\in C(p)} F_{uv}(t)\Big]\Big) \\
&\leqq \lim_{h\to 0^+} T\Big(\operatorname*{Inf}_{u\in C(p)}\Big[\operatorname*{Sup}_{v\in C(q)} F_{uv}(h)\Big], \operatorname*{Inf}_{v\in C(q)}\Big[\operatorname*{Sup}_{u\in C(p)} F_{uv}(h)\Big]\Big) \\
&= T\Big(\lim_{h\to 0^+}\operatorname*{Inf}_{u\in C(p)}\Big[\operatorname*{Sup}_{v\in C(q)} F_{uv}(h)\Big], \lim_{h\to 0^+}\operatorname*{Inf}_{v\in C(q)}\Big[\operatorname*{Sup}_{u\in C(p)} F_{uv}(h)\Big]\Big) ,
\end{aligned}
$$

whence

$$\lim_{h\to 0^+}\Big(\operatorname*{Inf}_{u\in C(p)}\Big[\operatorname*{Sup}_{v\in C(q)} F_{uv}(h)\Big]\Big) = \lambda > 0 .$$

Thus, in particular,

$$\lim_{h\to 0^+}\Big(\operatorname*{Sup}_{v\in C(q)} F_{pv}(h)\Big) \geqq \lambda > \lambda/2 > 0 .$$

Since $\mathrm{Sup}_{v\in C(q)}F_{pv}$ is increasing, for any $h > 0$ we have,

$$\operatorname*{Sup}_{v\in C(q)} F_{pv}(h) > \lambda/2 . \tag{3.4}$$

From (3.4) it follows that for each $h > 0$ there exists a $q_h \in C(q)$ such that

$$F_{pq_h}(h) > \lambda/2 . \tag{3.5}$$

Now let $\mathrm{Inf}_{u,v\in C(q)}F_{uv}(0^+) = \eta$. By hypothesis, $\eta > 0$, whence

$$T(\lambda/2, \eta) > 0 .$$

Moreover, since $q_h \in C(q)$

$$F_{qq_h}(h) \geqq \eta , \tag{3.6}$$

for all $h > 0$. Next, in view of (3.3), there exists an $h_0 > 0$ such that

$$F_{pq}(2h_0) < T(\lambda/2, \eta) . \tag{3.7}$$

Combining the inequalities (3.5), (3.6) and (3.7) we have

$$T(\lambda/2, \eta) > F_{pq}(2h_0) \geqq T({}_{pq_{h_0}}(h_0), F_{qq_{h_0}}(h_0)) \geqq T(\lambda/2, \eta) ,$$

which is a contradiction. Hence $F_{C(p)C(q)}(0^+) = 0$ and the proof is complete.

THEOREM 32. *Let $(S, \mathfrak{F}, T)$ be a Menger space under a continuous t-norm T. For each $p \in S$ let $C'(p)$ be the equivalence class in S determined by p and the equivalence relation C', and let S/C' be the collection of all such equivalence classes. If each $C'(p)$ in S/C' is* *such that for some* M_p, $s_{uv} < M_p$ *for all* u *and* v *in* $C'(p)$, *then* $(S/C', \mathfrak{F}_{S/C'}, T)$ *is a Menger space in which the distance between distinct elements is not certainly finite.*

Proof. In view of Theorems 18 and 28 $(S/C', \mathfrak{F}_{S/C'}, T)$ is a Menger space.

Let $C'(p)$ and $C'(q)$ be distinct equivalence classes and suppose that

$$s_{C'(p)C'(q)} < \infty \ . \tag{3.8}$$

Since $p \in C'(p)$ and $q \in C'(q)$ for each $\lambda > 0$ there is an $\varepsilon > 0$ such that

$$F_{pq}(s_{C'(p)C'(q)} + M_q + \lambda) < 1 - \varepsilon \ , \tag{3.9}$$

where $s_{uv} < M_q$ for all u and v in $C'(q)$. In view of (3.8),

$$\begin{aligned} 1 &= F_{C'(p)C'(q)}(s_{C'(p)C'(q)} + \lambda/2) \\ &= \operatorname*{Sup}_{t < s_{C'(p)C'(q)} + \lambda/2} T\Big(\operatorname*{Inf}_{u \in C'(p)} \Big[\operatorname*{Sup}_{v \in C'(q)} F_{uv}(t) \Big], \\ &\qquad\qquad \operatorname*{Inf}_{v \in C'(q)} \Big[\operatorname*{Sup}_{u \in C'(p)} F_{uv}(t) \Big] \Big) \\ &= T\Big(\operatorname*{Inf}_{u \in C'(p)} \Big[\operatorname*{Sup}_{v \in C'(q)} F_{uv}(s_{C'(p)C'(q)} + \lambda/2) \Big], \\ &\qquad \operatorname*{Inf}_{v \in C'(q)} \Big[\operatorname*{Sup}_{u \in C'(p)} F_{uv}(s_{C'(p)C'(q)} + \lambda/2) \Big] \Big) \ . \end{aligned}$$

Since $T(a, b) = 1$ if and only if $a = b = 1$, it follows that

$$\operatorname*{Inf}_{u \in C'(p)} \Big[\operatorname*{Sup}_{v \in C'(q)} F_{uv}(s_{C'(p)C'(q)} + \lambda/2) \Big] = 1 \ ,$$

whence, in particular,

$$\operatorname*{Sup}_{v \in C'(q)} F_{pv}(s_{C'(p)C'(q)} + \lambda/2) = 1 \ .$$

Thus, there exists a $q_\varepsilon \in C'(q)$ such that

$$F_{pq_\varepsilon}(s_{C'(p)C'(q)} + \lambda/2) > 1 - \varepsilon/2 \ . \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$\begin{aligned} 1 - \varepsilon &> F_{pq}(s_{C'(p)C'(q)} + M_q + \lambda) \\ &\geqq T(F_{pq_\varepsilon}(s_{C'(p)C'(q)} + \lambda/2), F_{qq_\varepsilon}(M_q + \lambda/2)) \\ &= T(F_{pq_\varepsilon}(s_{C'(p)C'(q)} + \lambda/2), 1) \\ &= F_{pq_\varepsilon}(s_{C'(p)C'(q)} + \lambda/2) > 1 - \varepsilon/2 \ . \end{aligned}$$

This is a contradiction, whence $s_{C'(p)C'(q)} = \infty$ and the proof is complete.

In conclusion we note that under the hypotheses of Theorem 31 the equivalence classes in S/C are either bounded or semi-bounded and under the hypotheses of Theorem 32 the equivalence classes in S/C' are bounded.

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BISECTION INTO SMALL ANNULI

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In a Riemannian manifold the modulus of a relatively compact set with border consisting of two sets of components is introduced to measure its magnitude from the viewpoint of harmonic functions. The existence of a subdivision into two sets each having modulus arbitrarily close to one is established.

1. Let M be a Riemannian manifold, i.e. a connected orientable C^∞ n-manifold that carries a metric tensor g_{ij}. Consider a bordered compact region $E \subset M$ whose border is the union of two nonempty disjoint sets α and β of components. We shall call the configuration (E, α, β) an *annulus*.

Let h be the harmonic function on E with continuous boundary values 0 on α and $\log \mu > 0$ on β such that

$$\int_\alpha * dh = 2\pi \ . \tag{1}$$

The number $\mu > 1$ is called the *modulus* of the annulus (E, α, β) and we set

$$\mu = \operatorname{mod}(E, \alpha, \beta) \ .$$

Let w be the *harmonic measure* of β with respect to E, i.e. the harmonic function on E with continuous boundary values 0 on α and 1 on β. By using Green's formula we obtain

$$\log \mu = \frac{2\pi}{D_E(w)} \ , \tag{2}$$

where $D_E(w)$ denotes the Dirichlet integral $\int_E dw \wedge * dw$ of w over E.

An illustration of these concepts is obtained by taking the annulus $E = \{x \mid r \leqq |x| \leqq R\}$ in n-dimensional ($n \geqq 3$) Euclidean space. The harmonic measure of $|x| = R$ with respect to E is

$$w = \frac{|x|^{2-n} - r^{2-n}}{R^{2-n} - r^{2-n}}$$

and the modulus of $(E, |x| = r, |x| = R)$ is given by

$$\log \mu = \pi^{1-(n/2)}(2-n)\Gamma\left(\frac{n}{2}\right)(R^{2-n} - r^{2-n}) \ .$$

Note that $\mu > 1$, in a sense, measures the relative thickness of E and that $\mu \to 1$ as $R - r \to 0$.

Our result gains interest if we generalize the notion of annulus slightly. Let (E_j, α_j, β_j) $(j = 1, \cdots, m)$ be annuli such that $E_i \cap E_j = \varnothing$ for $i \neq j$. Set $E = \bigcup_{j=1}^m E_j$, $\alpha = \bigcup_{j=1}^m \alpha_j$, $\beta = \bigcup_{j=1}^m \beta_j$. Then we shall also call the configuration (E, α, β) an annulus. The modulus $\mu = \mathrm{mod}\,(E, \alpha, \beta)$ and the harmonic measure of E with respect to β are defined exactly as for a connected annulus. Moreover, formula (2) is valid and consequently we have

$$\frac{1}{\log \mu} = \sum_{j=1}^{m} \frac{1}{\log \mu_j}, \tag{3}$$

where $\mu_j = \mathrm{mod}\,(E_j, \alpha_j, \beta_j)$.

2. Let M be a noncompact Riemannian manifold throughout this number. A function which is positive and harmonic on M except for a fundamental singularity is called a *Green's function* if it majorizes no nonconstant positive harmonic functions on M. If a Green's functions exists, then M is called *hyperbolic*; otherwise it is called *parabolic*.

An increasing sequence (Ω_n) of bordered compact regions is called an *exhaustion* of M if $\cup\, \Omega_n = M$. Note that the configuration $(\Omega_{n+1} - \bar{\Omega}_n, \partial\Omega_n, \partial\Omega_{n+1})$ is an annulus and denote its modulus by μ_n.

The parabolicity of a noncompact Riemannian manifold M is characterized by the following

MODULAR CRITERION. *There exists an exhaustion (Ω_n) of M with $\prod \mu_n = \infty$ if and only if M is parabolic.*

In the 2-dimensional case this criterion has been established by Sario [5] and Noshiro [4] and their work can easily be generalized to arbitrary Riemannian manifolds (cf. Smith [7], Glasner [2]).

One naturally asks whether a convergent modular product has any bearing on the hyperbolicity of a manifold. The main result of this paper is that any annulus can be separated into two annuli each having modulus less than $1 + \varepsilon$. This clearly answers the question in the negative and also settles Problem 3 in Sario [6].

3. Suppose the annulus (E, α, β) has components (E_j, α_j, β_j) $(j = 1, \cdots, m)$. Let γ_j be a hypersurface in E_j such that $E_j - \gamma_j = E_j' \cup E_j''$, $E_j' \cap E_j'' = \varnothing$, and $(E_j', \alpha_j, \gamma_j)$ and $(E_j'', \gamma_j, \beta_j)$ are annuli. Set $\gamma = \bigcup_{j=1}^m \gamma_j$. We shall call γ a *bisecting surface* of (E, α, β). Also set $E' = \bigcup_{j=1}^m E_j'$ and $E'' = \bigcup_{j=1}^m E_j''$. We are now able to state the

THEOREM. *Given an annulus (E, α, β) and $\varepsilon > 0$ there exists a bisecting surface γ of (E, α, β) such that*

$$(4) \qquad \operatorname{mod}(E', \alpha, \gamma) < 1 + \varepsilon, \ \operatorname{mod}(E'', \gamma, \beta) < 1 + \varepsilon .$$

This was established by Sario [5] for doubly connected plane regions using Koebe's distortion theorem. All proofs for the 2-dimensional case known to the authors use either a distortion theorem, in essence, or an estimate (cf. Akaza-Kuroda [1]) obtained by means of Möbius transformations (Nakai-Sario [3]) which cannot be generalized to higher dimensions. Therefore, one is led to estimate the Dirichlet integral of the harmonic measure directly and the proof presented here seems to even give a more elementary proof for the 2-dimensional case.

4. Denote by $C(a, b) = C_{x_0}(a, b)$ the Euclidean cylinder

$$(5) \qquad \sum_{j=1}^{n-1} (x^i - x_0^i)^2 < a^2, \ x_0^n < x^n < x_0^n + b ,$$

where $a, b > 0$ and $x_0 = (x_0^1, \cdots, x_0^n)$ is a fixed point. Let $\mathfrak{F}(a, b)$ be the class of C^1 functions f on $C(a, b)$ with continuous boundary values 0 on $\overline{C(a, b)} \cap \{x^n = x_0^n\}$ and 1 on $\overline{C(a, b)} \cap \{x^n = x_0^n + b\}$. Also denote by D^e the Dirichlet integral with respect to the Euclidean metric. We set s equal to the surface area of $\sum_{i=1}^{n-1} (x^i)^2 = 1$, $x^n = 0$ and state the

LEMMA. *For every* $f \in \mathfrak{F}(a, b)$,

$$(6) \qquad D^e_{C(a,b)}(f) \geqq \frac{sa^{n-1}}{b}$$

and equality holds for $f_0(x) = b^{-1}(x^n - x_0^n)$.

Clearly (6) is valid with equality for f_0. To prove (6) for an arbitrary f we may assume $f \in C^1$ in a neighborhood of $\overline{C(a, b)}$. By Green's formula we have

$$D^e_{C(a,b)}(f - f_0, f_0) = \int_{\partial C(a,b)} (f - f_0) \frac{\partial f_0}{\partial n} ds = 0 ,$$

since $f - f_0 = 0$ on the upper and lower boundary of the cylinder and $(\partial f_0 / \partial n) = 0$ on the side of the cylinder. Consequently Schwarz's inequality yields

$$D^e_{C(a,b)}(f) \cdot D^e_{C(a,b)}(f_0) \geqq (D^e_{C(a,b)}(f, f_0))^2 = (D^e_{C(a,b)}(f_0))^2 ,$$

which completes the proof.

5. We are ready to prove the main result. Take a point $x_0 \in \alpha$ and a point $y_0 \in \beta$. Let $x^1, \cdots, x^n$ be a local coordinate system at

$x_0 = (x_0^1, \cdots, x_0^n)$ valid in a neighborhood U of x_0 such that $U \cap \alpha$ is given by $x^n = x_0^n$ and x^n increases as x moves from α to E. Similarly, let $y^1, \cdots, y^n$ be a local coordinate system at $y_0 = (y_0^1, \cdots, y_0^n)$ valid in a neighborhood V of y_0 such that $V \cap \beta$ is given by $y^n = y_0^n$ and y^n increases as y moves from β to E. Choose a constant $c > 0$ so small that

$$\sqrt{g} \mid U \cup V > \sqrt{c} \tag{7}$$

and also

$$(g^{ij} \mid U \cup V)\xi_i \xi_j \geqq \sqrt{c} \sum_{i=1}^{n} (\xi_i)^2 \tag{8}$$

for every vector $(\xi_1, \cdots, \xi_n)$. Now choose $a > 0$ sufficiently small to insure that $\sum_{i=1}^{n-1}(x^i - x_0^i) < a^2$ with $x^n = x_0^n$ and $\sum_{i=1}^{n-1}(y^i - y_0^i)^2 < a^2$ with $y^n = y_0^n$ are contained in $U \cap \alpha$ and $V \cap \beta$, respectively. Finally choose $b > 0$ so that

$$0 < b < \frac{csa^{n-1} \log (1 + \varepsilon)}{2\pi}\,, \tag{9}$$

$$\overline{C_{x_0}(a, b)} - \{x^n = x_0^n\} \subset E, \quad \overline{C_{y_0}(a, b)} - \{y^n = y_0^n\} \subset E$$

and

$$\overline{C_{x_0}(a, b)} \cap \overline{C_{y_0}(a, b)} = \varnothing \,.$$

Now take a bisecting surface γ of (E, α, β) subject to the requirements

$$\gamma \cap (C_{x_0}(a, b) \cup C_{y_0}(a, b)) = \varnothing$$

and

$$\gamma \supset [\overline{C_{x_0}(a, b)} \cap \{x^n = x_0^n + b\}] \cup \overline{C_{y_0}(a, b)} \cap \{y^n = y_0^n + b\}] \,.$$

Let w' (resp. w'') be the harmonic measure of γ (resp. β) with respect to E' (resp. E''). Since $E' \supset C_{x_0}(a, b)$, by using (7) and (8) we obtain

$$D_{E'}(w') > D_{C_{x_0}(a,b)}(w') \geqq cD^e_{C_{x_0}(a,b)}(w') \,. \tag{10}$$

Hence by using (6) and (9) we have

$$\frac{2\pi}{D_{E'}(w')} < \log (1 + \varepsilon)$$

and in view of (2) we conclude that

$$\operatorname{mod} (E', \alpha, \gamma) < 1 + \varepsilon \,.$$

A similar consideration for E'' establishes (4).

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A NOTE ON LEFT MULTIPLICATION OF SEMIGROUP GENERATORS

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It is shown in this note that if A is the infinitesimal generator of a strongly continuous semigroup of contraction operators in any Banach space X, then so is BA for a broad class of bounded operators B; the only requirement on B is that it transforms "in the right direction".

In the recent paper [1] the following interesting result was obtained.

THEOREM 1 (*Dorroh*). *Let X be the Banach space of bounded functions on a set S under the supremum norm, let A be the infinitesimal generator of a contraction semigroup in X, and let B be the operator given by multiplication by p, $pX \subseteqq X$, where p is a positive function defined on S, bounded above, and bounded below above zero. Then BA is also the infinitesimal generator of a contraction semigroup in X.*

This leads naturally to the general question of preservation of the generator property under left multiplication; the purpose of this note is to present Theorem 2 below, which shows that for any Banach space, a large class of operators B are acceptable. In the following, the word "generator" will always mean generator of contraction semigroup.

In this note we will consider only left multiplication by everywhere defined bounded operators B. It is easily seen (e.g., [2, Corollary 3]) that A generates a contraction semigroup if and only if cA does, $c > 0$. Also by [4, Th. 2.1], if A is bounded, BA is a generator if and only if BA is dissipative; in this case clearly right multiplication also yields a generator. See [4, 5] for dissipativeness; we use dissipativeness in the sense [4], and recall that if BA is a generator, then BA is dissipative in all semi-inner products on X.

THEOREM 2. *Let X be any Banach space, A the infinitesimal generator of a contraction semigroup in X, and B a bounded operator in X such that $\|\varepsilon B - I\| < 1$ for some $\varepsilon > 0$. Then BA generates a contraction semigroup in X if and only if BA is dissipative, (i.e., $\operatorname{Re}[BAx, x] \leqq 0$, all $x \in D(A)$, $[u, v]$ a semi-inner product (see [4])).*

Proof. We note that $R(B) = X$ when $\|\varepsilon B - I\| < 1$ for some $\varepsilon > 0$; to show that BA is a generator it suffices to show that εBA is a generator for some positive ε. From the relation $\|\varepsilon B - I\| < 1 \leqq \|(I - \varepsilon BA)^{-1}\|^{-1}$ we have by [2, Lemma 1] that:

$$\beta(I - \varepsilon BA) = \beta((I - \varepsilon BA) + (\varepsilon B - I)) \equiv \beta(\varepsilon B(I - A)) = \beta(\varepsilon B) = 0 ,$$

where $\beta(T) = \dim X/\mathrm{Cl}\,(R(T))$ is the deficiency index of an operator T. A closed implies εBA closed (and therefore $I - \varepsilon BA$ closed), since $\varepsilon BA = A + (\varepsilon B - I)A$ and $\|\varepsilon B - I\| < 1$; BA dissipative implies that $I - \varepsilon BA$ possesses a continuous inverse, so that we therefore have $R(I - \varepsilon BA)$ closed, and thus BA the generator of a contraction semigroup. This result also follows quickly from [2, Theorem 2].

In the above we made use of basic index theory as may be found in [3] and the well-known characterizations of generators as may be found in [3, 4, 5], for example. The index theory notation here is a convenience only; the arguement can be presented without it.

COROLLARY 3. *Theorem* 1 *stated above.*

Proof. As shown in [1], pA is dissipative with respect to the semi-inner product used there, and clearly $0 < m \leqq p(s) \leqq M$ implies that $|\varepsilon p - 1| < 1 - \varepsilon m$ for small enough ε.

COROLLARY 4. *Let B be of the form $cI + C$, $\|C\| < c$, CA dissipative. Then BA is a generator if A is.*

Proof. Clearly $c^{-1}B$ satisfies the conditions of Theorem 2; note $\|\varepsilon B - I\| < 1$ for some $\varepsilon > 0$ if and only if B is of the form $cI + C$, $\|C\| < c$.

Remarks. The condition BA dissipative in Theorem 2, necessary for BA to be a generator, requires (in general) that B be in a "positive" rather than a dissipative direction. For example, if A, B, and BA are self-adjoint operators on a Hilbert space, then A is a generator if and only if A is negative, and then BA is a generator if B is positive.

The condition $\|\varepsilon B - I\| < 1$ in Theorem 2 is easily seen to be equivalent to the condition: B strongly accretive, i.e., $\exists m = m(B)$ such that $\mathrm{Re}\,[Bx, x] \geqq m > 0$ for $\|x\| = 1$, where $[u, v]$ is the semi-inner product being used (see [4]). It is a sharp condition since equality $\|\varepsilon B - I\| = 1$ cannot be permitted in general, as seen from the example $B = 0$, A unbounded, for then BA is not closed.

The effect of Theorem 2 is that, after the application of index theory therein, one sees that the essential question concerning when BA is a generator is the question of when BA is dissipative. Three situations which can then occur are: (i) as in [1], for special operators B, one can find a semi-inner product for which BA is dissipative; (ii) A commutes with B (see [3]), for which one can easily obtain results such as A self-adjoint, dissipative, and B accretive imply BA dissipative; (iii) general (noncommuting) A and B. For case (iii) one can obtain the following interesting result (proof given in forthcoming paper by the author, Math. Zeitschrift). Let $-A$ and B be strongly accretive operators on a Banach space. If

$$\min_{\varepsilon} \| \varepsilon B - I \| \leqq m(-A) \cdot \| A \|^{-1} ,$$

then BA is dissipative. In particular, let A and B be self-adjoint operator: then $(\| B \| - m(B)) \cdot (\| B \| + m(B))^{-1} \leqq m(-A) \cdot \| A \|^{-1}$ is sufficient. Moreover these conditions can be sharpened by introducing the concept of the cosine of an operator. For certain operators the condition for BA to be dissipative can then be written as $\sin B \leqq \cos A$.

The author appreciates useful expository suggestions from the referee. Extensions of these results to unbounded right and left multiplication will appear in a forthcoming paper by the author and G. Lumer.

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A CHARACTERIZATION OF GROUPS IN TERMS OF THE DEGREES OF THEIR CHARACTERS II

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In this paper we continue our study of the relationship between the structure of a finite group G and the set of degrees of its irreducible complex characters. The following hypotheses on the degrees are considered: (A) G has r.x. e for some prime p, i.e. all the degrees divide p^e, (B) the degrees are linearly ordered by divisibility and all except 1 are divisible by exactly the same set of primes, (C) G has a.c. m, i.e., all the degrees except 1 are equal to some fixed m, (D) all the degrees except 1 are prime (not necessarily the same prime) and (E) all the degrees except 1 are divisible by $p^e > p$ but none is divisible by p^{e+1}. In each of these situations, group theoretic information is deduced from the character theoretic hypothesis and in several cases complete characterizations are obtained.

In situation (A), the greater complexity which can occur when $e \geqq p$ is explored and a conjecture concerning p-groups with $e < p$ is studied and certain cases of it are proved. Detailed statements are made about groups G satisfying (B) for which the common set of prime divisors of the degrees does not consist of a single prime for which G has a nonabelian $\mathfrak{S}_p$ subgroup. These results are applied to situation (C), groups with a. c. m, and such groups are completely characterized when m is not a prime power corresponding to a nonabelian Sylow subgroup. If $m = p^e$ and an $\mathfrak{S}_p$ of G is nonabelian then it is shown that G must be nilpotent unless $e = 1$ (in which case G has r.x. 1 and has been completely characterized in [2]). This reduces the study of groups with a.c. m to p-groups and it is shown that a p-group G with a. c. p^e must have an abelian normal subgroup of index p^e unless G has class 2 or 3. Further information is obtained about these "special" class 2 and 3 groups. It is also shown that if $e > 1$ then G must have class $\leqq p$.

Groups satisfying hypothesis (D) are completely characterized and it is shown that in this case there are at most two degrees different from 1. Finally it is shown that if G satisfies hypothesis (E) and has a nonabelian $\mathfrak{S}_p$ subgroup then G is nilpotent and has a.c. p^e. In all the situations considered in this paper, the group in question is shown to be solvable.

We use here the notation and terminology of [2].

1. Groups with r.x. $(p-1)$.

In [4] we classified all groups with

r.x.2. As it turned out, in that study the prime $p = 2$ played a special role. It now appears that in the general classification of groups with r.x. e those primes p with $p \leqq e$ will again play a special role. In the other direction, this means that groups G with r.x. e and $p > e$ are somehow better behaved than the others. In this section we will attempt to justify this last statement.

Let G have r. x. e but not r.x. $(e - 1)$. Then we say that $e = e(G)$ is the character exponent of G. If G has a normal subgroup N of index p with $e(N) = e(G) - 1$, then in terms of the characterization problem, G is trivial. We say that such groups are imprimitive. Otherwise G is primitive. We note that since all groups with r.x. e are M-groups this terminology causes no confusion.

The following result handles the nonnilpotent case. It shows moreover that the nonnilpotent exceptional group of [4](Theorem A (ii)) belongs in some sense to a series of such groups.

THEOREM 1.1. *Let $e(G) = e$ and let $\mathfrak{S}_{p'}(G)$ be noncentral. If $p \neq 2$ and p is not a Mersenne prime or if $p > e$, then G is imprimitive. If $p = e$, then G is imprimitive unless $G/\mathfrak{Z}(G) \cong G_0$ where $G_0 = (V \times_\sigma C) \sim C, |C| = p$, V is elementary abelian and either $p = 2, |V| = 3$ or $p = 2^\alpha - 1$ is a Mersenne prime and $|V| = 2^\alpha$.*

Conversely if $e(G) = e$ and $G/\mathfrak{Z}(G) \cong G_0$ given above, then $p = e$ and G is primitive.

The lemma below is well known.

LEMMA 1.2. *Let π be a set of primes and let arbitrary group G have a normal abelian $\mathfrak{S}_\pi$ subgroup A. Then $A = Z \times B$ where Z and B are characteristic in G and $Z = \mathfrak{Z}(G) \cap A$.*

Proof. Clearly A is characteristic in G and G/A acts on A. Let θ be the endomorphism of A which is given by $\theta(a) = \prod_{x \in G/A} a^x$. Clearly $K = \ker \theta$ and $I = \text{image}\, \theta$ are characteristic subgroups of G and $|K| \cdot |I| = |A|$. If $Z = \mathfrak{Z}(G) \cap A$, then we see easily than $Z \supseteq I$ and $Z \cap K = \langle 1 \rangle$. The latter uses the fact that A is an $\mathfrak{S}_\pi$ subgroup of G. Hence $Z = I$ and $A = Z \times K$.

Proof of Theorem 1.1. Let $H = \mathfrak{S}_{p'}(G)$ and $P = G/\mathfrak{C}(H)$. Let $\hat{H}$ be the group of linear characters of H and let $G_1 = \hat{H} \times_\sigma P$ where P acts faithfully in the natural manner on $\hat{H}$. If there exists $\lambda \in \hat{H}$ with $\mathfrak{C}_P(\lambda) = \langle 1 \rangle$ then choose N with $N \supseteq \mathfrak{C}(H)$ and $[G : N] = p$. By [5](§ 3, in particular the proofs of Theorems 3.1 and 3.2), $e(N) = e(G) - 1$. Now this occurs by Corollary 2.4 (i) of [5] if $p \neq 2$ and p is not a Mersenne prime or if $p > e(G)$. It also occurs for $p = e(G)$

unless G_1 has G_0 as a homomorphic image. This follows by Theorem 2.1 of [5] and a slight modification of Lemma 1.2 of [5] since we do not have to look at subgroups here. We consider this last possibility. Assume G is primitive.

Now P has as a homomorphic imape P_0, a Sylow p-subgroup of G_0. Since $[P_0 : P_0'] = p^2$ we see that P_0 has a nonabelian group of order p^3 as a homomorphic image. Thus there exists $K \triangle G$ with $K \supseteqq \mathfrak{C}(H)$ and G/K nonabelian of order p^3. By [2](p. 885, equation * with $m = 1$) G has a normal subgroup N with $K < N < G, [G : N] = p^2$ and $e(N) \leqq e(G) - 1$. Since $p = e > e(N)$ we conclude by [5] that $[N : \mathfrak{C}(H)] \leqq p^{e(N)}$. Hence $[G : \mathfrak{C}(H)] \leqq p^2 \cdot p^{e(N)} \leqq p^{p+1}$. Since $|P_0| = p^{p+1}$, it follows that $[G : \mathfrak{C}(H)] = p^{p+1}$ and $P \cong P_0$.

Let W be the kernel of the homomorphism $G_1 \rightarrow G_0$. By the above $W \subseteqq \hat{H}$. We show that W is central in G_1. Let $w \in W$ and suppose that $\mathfrak{C}_{G_1}(w) < G_1$. It is easy to see in G_0 that there exists $\bar{a}, \bar{b} \in \hat{H}/W$ such that $|\mathfrak{C}_{P_0}(\bar{a})| = |\mathfrak{C}_{P_0}(\bar{b})| = p$ and $P_0 = \langle \mathfrak{C}_{P_0}(\bar{a}), \mathfrak{C}_{P_0}(\bar{b}) \rangle$. Thus since G_1 acts on $\hat{H}/W$ and $\mathfrak{C}_{G_1}(w) < G_1$ we cannot have both $\mathfrak{C}_{G_1}(w) \supseteqq \mathfrak{C}_{G_1}(\bar{a})$ and $\mathfrak{C}_{G_1}(\bar{b})$. Say $\mathfrak{C}_{G_1}(w) \not\supseteqq \mathfrak{C}_{G_1}(\bar{a})$. Since $|\mathfrak{C}_{P_0}(\bar{a})| = p$ we have $\mathfrak{C}_{G_1}(w) \cap \mathfrak{C}_{G_1}(\bar{a}) = \hat{H}$. Now p-group $\mathfrak{C}_{G_1}(\bar{a})/\hat{H}$ permutes the elements of the coset $aW = \bar{a}$ and $|aW|$ is prime to p. Hence we can choose an element $a \in aW$ which is centralized by $\mathfrak{C}_{G_1}(\bar{a})$. Consider $v = aw \in \hat{H}$. If $x \in \mathfrak{C}_{G_1}(v)$ then $x \in \mathfrak{C}_{G_1}(\bar{v}) = \mathfrak{C}_{G_1}(\bar{a})$. Thus x centralizes $\bar{a}$ and hence $x \in \mathfrak{C}_{G_1}(\bar{a}) \cap \mathfrak{C}_{G_1}(w) = \hat{H}$. Therefore $\mathfrak{C}_{G_1}(v) = \hat{H}$ and this is a contradiction since G is primitive. Thus W is central in G_1 and since $G_1/W \cong G_0, W = \mathfrak{Z}(G_1)$. By Lemma 1.2, $\hat{H} = W \times R$ where $R \triangle G_1$ and $RP \cong G_0$.

Since $\hat{H} = W \times R$ we have $H = Q \times Z$. All linear characters of Z are fixed by P and hence Z is central in G. Also $\hat{Q}P \cong G_0$ and from the nature of G_0 we see easily that $QP \cong G_0$ Moreover $\mathfrak{C}_G(H) = Q \times Z \times S$ where $S = \mathfrak{S}_p(\mathfrak{C}(H))$. We show now that S is central in G.

Choose $\lambda, \mu \in \hat{Q}$ with $|T_P(\lambda)| = |T_P(\mu)| = p$ and $P = \langle T_P(\lambda), T_P(\mu) \rangle$. Let φ be an irreducible character of S. View λ, μ and φ as characters of $\mathfrak{C}(H)$. Let χ be a constituent of $(\lambda\varphi)^*$ so that $\chi \mid \mathfrak{C}(H) = a \sum_1^t (\lambda\varphi)_i$. Clearly $T(\lambda\varphi) = T(\lambda) \cap T(\varphi)$ so $t \geqq p^p$ and $p^p \geqq \deg \chi = at \deg \varphi \geqq p^p \deg \varphi$. Thus $\deg \varphi = 1$ and $t = p^p$. This shows that S is abelian and that $T(\varphi) \supseteqq T(\lambda)$. Similary $T(\varphi) \supseteqq T(\mu)$ and hence $T(\varphi) = G$. Therefore S is central in G and $\mathfrak{Z}(G) = Z \times S$. Hence $G/\mathfrak{Z}(G) \cong QP \cong G_0$ and the result follows.

We show conversely that all the exceptional groups discussed have $e(G) = p$ and are primitive. Let $A = \mathfrak{C}(H)$. Since $G/\mathfrak{Z}(G) \cong G_0$ we see that $A = H\mathfrak{Z}(G)$ is abelian since H is abelian. Also $[G : A] = p^{p+1}$. Let χ be an irreducible character of G and $\chi \mid A = a \sum_1^t \lambda_i$. Then $\deg \chi = at$ and $a^2 t \leqq [G : A] = p^{p+1}$ by Lemma 1.2 of [2]. Thus $\deg \chi \leqq p^{p+1}$ and if $\deg \chi = p^{p+1}$ then $a = 1$ and $t = p^{p+1}$. The latter implies that for $\lambda = \lambda_1$ we have $T(\lambda) = A$. We show that this is not the case. Let $\lambda \mid \mathfrak{Z}(G) = \mu$. Then λ is a constituent of $\tilde{\mu}$ (induction to A) and G/A permutes the linear constituents of $\tilde{\mu}$ since $\mathfrak{Z}(G)$ is central. Now G/A is a p-group and $\deg \tilde{\mu}$ is prime to p so there exists a constituent η of $\tilde{\mu}$ which is fixed by G/A. Since $\lambda \mid \mathfrak{Z}(G) = \eta \mid \mathfrak{Z}(G)$ it follows that $\lambda = \eta\xi$ where ξ is a character with $\xi \mid \mathfrak{Z}(G) = 1$. By properties of G_0, $T(\xi) > A$ and since $T(\eta) = G$, it follows that $T(\eta\xi) > A$. Thus $e(G) \leqq p$. Since $e(G_0) = p$ we have $e(G) = p$.

Suppose G is imprimitive. Let $N \triangle G$ with $[G:N] = p$ and $e(N) = p - 1$. Let χ be a character of G of degree p^p. Since $e(N) = p - 1$ we have $\chi = \varphi^*$ for some irreducible character φ of N. This shows that $N \supseteqq \mathfrak{Z}(G)$. Clearly $N \supseteqq \mathfrak{S}_{p'}(G) = H$ and therefore $N \supseteqq H\mathfrak{Z}(G) = A$. Since $p > e(N)$ and $\mathfrak{C}_N(H)$ is abelian, it follows from § 3 of [5] that $[N:A] = p^{e(N)}$. Hence $[G:A] = p^{e(N)+1} = p^p$, a contradiction. This completes the proof.

We now study p-groups with r.x. e and $p > e$. Here our results are not conclusive.

Let p-group G have $e(G) = e$. We set $\Omega(G)$, the character kernel of G equal to $\Omega(G) = \cap \ker \theta$ where θ runs over all irreducible characters of G of G of degree p^e. If $\Omega(G) = \langle 1 \rangle$, we say G is character regular. In [6](Corollary 2 with $n = p^{e-1}$) we showed that $|\Omega(G)| \leqq \frac{1}{2}(2p^{e-1})!$ We conjecture that if $p > e(G)$ than G is character regular. Reasons for studying this property can be seen in the following result.

Proposition 1.3. Let G be a p-group with $e(G) = e$.

(i) Let $N \triangle G$ with $e(N) = e$. If N is character regular then $\mathfrak{Z}(N) \subseteqq \mathfrak{Z}(G)$.

(ii) Suppose G is primitive and every maximal subgroup is character regular. If $\zeta \in G - \mathfrak{Z}(G)$, then $[G:\mathfrak{C}(\zeta)] \geqq p^2$. Thus if J is a central subgroup of G of order p, then $\mathfrak{Z}(G/J) = \mathfrak{Z}(G)/J$.

Proof. In (i) suppose $\mathfrak{Z}(N) \not\subseteqq \mathfrak{Z}(G)$. Then we can choose $x \in (G, \mathfrak{Z}(N))$ with $x \neq 1$. Since $N \triangle G, x \in N$. Now N is character regular so there exists irreducible character φ of N with $\deg \varphi = p^e$ and $x \notin \ker \varphi$. Let χ be and irreducible constituent of φ^*. Since $\deg \chi \leqq p^e$ we have clearly $\chi \mid N = \varphi$. Thus $\mathfrak{Z}(N)$ is central in the representation associated with χ and $(G, \mathfrak{Z}(N)) \subseteqq \ker \chi \cap N = \ker \varphi$, a contradiction.

We consider (ii). Since $\zeta \notin \mathfrak{Z}(G)$ we have $[G:\mathfrak{C}(\zeta)] \geqq p$. If $[G:\mathfrak{C}(\zeta)] = p$, let $N = \mathfrak{C}(\zeta)$. Then $N \triangle G$, N is character regular and $e(N) = e(G)$ since G is primitive. By (i), $\mathfrak{Z}(N) \subseteqq \mathfrak{Z}(G)$ and hence $\zeta \in \mathfrak{Z}(G)$, a contradiction. Thus $[G:\mathfrak{C}(\zeta)] \geqq p^2$. Clearly $\mathfrak{Z}(G/J) \supseteqq \mathfrak{Z}(G)/J$. Let $\zeta \in G$ be the inverse image of an element of $\mathfrak{Z}(G/J)$. Then $(G, \zeta) \subseteqq J$

and $|J| = p$ so $[G : \mathfrak{C}(\zeta)] \leqq p$. By the above $\zeta \in \mathfrak{Z}(G)$ and the result follows.

We say p-group G has property (*) if $e(G) = e$ and given any $p - e$ nonidentity elements of G there exists an irreducible character χ of G of degree p^e which does not contain any of these elements in its kernel. Note that if $p > e(G)$ and G has property (*), than G is character regular. In [5] we conjectured that every p-group satisfies (*). If this is so the following shows that $p - e$ is best possible.

PROPOSITION 1.4. *Given p and e. If $p \leqq e$, there exists a p-group G with $e(G) = e$ and $\Omega(G) > \langle 1 \rangle$. If $p > e$, then there exists a p-group G with $e(G) = e$ having $p - e + 1$ nonidentity elements with the property that every irreducible character of G of degree p^e contains at least one of these elements in its kernel. Moreover in both cases we can take G to have class 2.*

Proof. Let G be generated by $x_1, \cdots, x_e, y_1, \cdots, y_e, u, v$ all of order p, such that u and v are central, $(x_i, y_i) = uv^i$ for $i = 1, \cdots, e$, and all other commutators are trivial. Set $J = \langle v \rangle$. Clearly G/J is a faithful irreducible linear group of degree p^e. Since $[G : \mathfrak{Z}(G)] \leqq p^{2e}$ we see that $e(G) = e$.

Let $p \leqq e$. We show that $v \in \Omega(G)$. Let χ be an irreducible character of G with $v \notin \ker \chi$. Then for some $i = 1, \cdots, p$ we have $uv^i \in \ker \chi$. Since $p \leqq e$ we see that x_i, y_i exist and that $\bar{x}_i$ and $\bar{y}_i$ are central in $G/\langle uv^i \rangle$. Hence $G/\langle uv^i \rangle$ has r.x. $(e - 1)$ and $\deg \chi \leqq p^{e-1}$. Thus $v \in \Omega(G)$.

Now let $p > e$. Consider the $p - e + 1$ elements $v, uv^{e+1}, \cdots, uv^p$. Let χ be an irreducible character of G containing none of these elements in its kernel. Then for some $i = 1, \cdots, e$ we have $uv^i \in \ker \chi$. As above for such $i \leqq e$, $G/\langle uv^i \rangle$ has r.x. $(e - 1)$ and hence the result follows.

We show now that at least in certain cases (*) holds. For possible later applications we use the following general setup.

Let $\mathscr{S}$ be a class of p-groups closed under taking subgroups and quotient groups. Let G be a member of $\mathscr{S}$ of minimal order which does not satisfy (*) if such exists. We consider properties of this minimal counterexample.

Let $e(G) = e$ and let $x_1, \cdots, x_r$ be $r = p - e$ nonidentity elements of G such that each irreducible character of G of degree p^e contains at least one of the x_i in its kernel. We of course have $r > 0$ and thus $p > e$. Clearly $e > 0$ by Proposition 4.6 of [2]. Hence $r \leqq p - 1$.

We can assume that the x_i are central and have order p as follows. If x is one of the x_i's, then we can find elements $y_1, \cdots, y_k$ such that $h = (x, y_1, \cdots, y_k)$ is a nonidentity central element. If $h \notin \ker \chi$ for some character χ then clearly $x \notin \ker \chi$. Also we can take a suitable power of h to have order p.

We show now that all the x_i are contained in $\Phi(G)$, the Frattini subgroup of G. If not say $x_1 \notin N$ for some maximal subgroup N of G. Since x_1 is central of order p we have $G = N \times \langle x_1 \rangle$ and clearly $e(N) = e(G) = e$. Let ζ be an element of order p in $\mathfrak{Z}(N)$. Then $\langle \zeta, x_1 \rangle$ is central of type (p, p) and has $p + 1$ subgroups of order p. Since $r \leqq p - 1$, we can find one such subgroup J with $x_1, \cdots, x_r, \zeta \notin J$. Then $G = N \times J$ and $\bar{x}_1, \cdots, \bar{x}_r$ are nonidentity elements of $G/J \cong N$, a group with $e(G/J) = e$. Since G is a minimal counterexample, we can find a character φ of G/J of degree p^e with $\bar{x}_i \notin \ker \varphi$. Viewing φ as a character of G yields a contradiction.

Let N be a maximal subgroup of G. Clearly $e(N) \geqq e - 1$. If $e(N) = e$, then since $x_1, \cdots, x_r \in N$ there exists an irreducible character φ of N of degree p^e with $x_i \notin \ker \varphi$ for all i. If X is a constituent of φ^* then since $\deg X \leqq p^e$ we have $\deg X = p^e$ and $X \mid N = \varphi$. Thus $x_i \notin \ker X$ for all i, a contradiction. Therefore $e(N) = e - 1$.

If χ is a character of G of degree p^e, then $\chi = \varphi^*$ for some irreducible character φ of N since $e(N) = e - 1$. Thus $T(\varphi) = N$ and hence $\mathfrak{Z}(G) \subseteqq N$. Therefore $\mathfrak{Z}(G) \subseteqq \mathfrak{Z}(N)$. We show that $\mathfrak{Z}(G) = \mathfrak{Z}(N)$. If not, choose $x_{r+1} \in (G, \mathfrak{Z}(N))$ with $x_{r+1} \neq 1$. Since $e(N) = e - 1$ and $r + 1 = p - (e - 1)$, we can choose an irreducible character θ of N of degree p^{e-1} with $x_i \notin \ker \theta$ for all i. Let χ be a constituent of θ^*. If $\deg \chi = p^e$, then $x_1, \cdots, x_r \notin \ker X$ yields a contradiction. On the other hand, if $\deg \chi = p^{e-1}$, then $\chi \mid N = \theta$ and so $\mathfrak{Z}(N)$ is central in the representation associated with χ. Hence $(G, \mathfrak{Z}(N)) \subseteqq \ker \chi \cap N = \ker \theta$ and this contradicts the fact that $x_{r+1} \notin \ker \theta$. Thus $\mathfrak{Z}(N) = \mathfrak{Z}(G)$.

We show now that $\mathfrak{Z}(G)$ has two generators and is not cyclic. Let G have as a homomorphic image $\bar{G} = G/K$, a faithful irreducible linear group of degree p^e. Suppose $\mathfrak{Z}(G) \cap K$ has a subgroup of type (p,p). Then we can find a central subgroup J of order p with $x_i \notin J$ for all i and $J \subseteqq K$. Then $e(G/J) = e$ and we clearly have a contradiction. Thus $\mathfrak{Z}(G) \cap K$ is cyclic. Since $\mathfrak{Z}(\bar{G})$ is cyclic we see that $\mathfrak{Z}(G)$ has two generators. Let $\zeta \in \mathfrak{Z}_2(G) - \mathfrak{Z}(G)$ with $\zeta^p \in \mathfrak{Z}(G)$. Then the map $g \rightarrow (g, \zeta)$ is a homomorphism of G into the elements of order p in $\mathfrak{Z}(G)$. The kernel is $\mathfrak{C}(\zeta)$ and by the above $[G : \mathfrak{C}(\zeta)] \geqq p^2$. Hence $[G : \mathfrak{C}(\zeta)] = p^2$ and (G, ζ) is abelian of type (p, p). Thus $\mathfrak{Z}(G)$ is not cyclic.

THEOREM 1.5. *If G has class at most 2, then G satisfies* (*).

Proof. Let $\mathscr{S}$ be the family of all p-groups of class at most 2 and let G be a minimal counterexample. Then all of the above applies. Let J be a central subgroup of G with $x_i \notin J$ for all i. Let χ be an irreducible character of G/J viewed as one of G and with $x_1 J \notin \ker \chi$. Let K be the kernel of χ. If U is the subgroup of $\mathfrak{Z}(G)$ of type (p, p) then clearly $J = K \cap U$ and thus $K \cap \mathfrak{Z}(G)$ is cyclic. Let $\bar{G} = G/K$. We show that $\mathfrak{Z}(G/K) = \mathfrak{Z}(G)/K$. Let B be the complete inverse image of $\mathfrak{Z}(\bar{G})$ in G. Clearly $B \supseteq \mathfrak{Z}(G)$. If $B > \mathfrak{Z}(G)$, choose $\zeta \in B - \mathfrak{Z}(G)$ with $\zeta^p \in \mathfrak{Z}(G)$. Since $\bar{\zeta} \in \mathfrak{Z}(\bar{G})$ we have $(G, \zeta) \subseteq K \cap U = J$. Hence $[G : \mathfrak{C}(\zeta)] = p$, a contradiction. Since $x_i \notin \ker \chi$, it follows that $\deg \chi \leqq p^{e-1}$ and so $[\bar{G} : \mathfrak{Z}(\bar{G})] \leqq p^{2e-2}$ by Lemma 2.3 of [2]. Hence $[G : \mathfrak{Z}(G)] \leqq p^{2e-2}$ and G has r.x. $(e-1)$, a contradiction. Thus the theorem is proved.

We now return to our discussion of the general minimal counterexample. Again let $\zeta \in \mathfrak{Z}_2(G) - \mathfrak{Z}(G)$ with $\zeta^p \in \mathfrak{Z}(G)$. Thus if $K = \mathfrak{C}(\zeta)$, then we have $[G : K] = p^2$ and in fact $G/K \cong (G, \zeta)$ is abelian of type (p, p). Let N be any subgroup of G with $G > N > K$. Since $\mathfrak{Z}(K) \not\supseteq \mathfrak{Z}(N)$ and K is character regular we see by Proposition 1.3 (i) that $e(K) < e(N)$. But $e(K) \geqq e(N) - 1$ so $e(K) = e(N) - 1 = e - 2$. In particular $e \geqq 2$.

We show now that $[\mathfrak{Z}(K) : \mathfrak{Z}(G)] = p$ so that $\mathfrak{Z}(\mathfrak{C}(\zeta)) = \langle \mathfrak{Z}(G), \zeta \rangle$. Let θ be an irreducible character of K of degree p^{e-2} (note that $e(K) = e - 2$) with $x_1, \cdots, x_r$ not in its kernel and let $J \subseteq \ker \theta$ where J is central in G of order p. Clearly $J = (T, \zeta)$ for some subgroup T with $G > T > K$. Consider $\bar{G} = G/J$. Since $\bar{x}_i \neq 1$ in $\bar{G}$ we see that $e(\bar{G}) \leqq e - 1$. But $e(\bar{K}) = e - 2$, where of course $\bar{K} = K/J$. Also $\bar{\zeta} \in \mathfrak{Z}(\bar{T}) - \mathfrak{Z}(\bar{G})$. Hence $e(\bar{G}) > e(\bar{T}) \geqq e(\bar{K})$. This yields $e(\bar{G}) = e - 1$ and $e(\bar{T}) = e(\bar{K}) = e - 2$. By Proposition 1.3 (i) we have $\overline{\mathfrak{Z}(K)} \subseteq \mathfrak{Z}(\bar{K}) \subseteq \mathfrak{Z}(\bar{T})$ and thus $(T, \mathfrak{Z}(K)) = J$. Now $T = \langle K, a \rangle$ and the map $b \to (b, a)$ is a homomorphism of $\mathfrak{Z}(K)$ onto J with kernel $\mathfrak{C}(a) \cap \mathfrak{Z}(K) = \mathfrak{Z}(T)$. Hence $[\mathfrak{Z}(K) : \mathfrak{Z}(T)] = p$. But $\mathfrak{Z}(T) = \mathfrak{Z}(G)$ so $[\mathfrak{Z}(K) : \mathfrak{Z}(G)] = p$.

If $e = 2$, then K is abelian and so $\mathfrak{Z}(K) = K$. Hence $[G : \mathfrak{Z}(G)] = p^3$, a contradiction and hence $e \geqq 3$. If we let $\mathscr{S}$ be the set of p-groups with r.x.2, then the above yields:

PROPOSITION 1.6. If G is a p-group with r. x. 2, then G has property (*).

We now discuss an application of the above. Let $\mathscr{T}$ denote a family of character regular p-groups closed under taking subgroups and quotient groups.

PROPOSITION 1.7. Let $G \in \mathscr{T}$ with $e(G) = e$. Let χ be an irreducible character of G of degree p^e and let Z_χ denote the set of elements of G central in the representation associated with χ. Then Z_χ is abelian.

Proof. If Z_χ is central the result is clear. So assume $Z_\chi \not\subseteq \mathfrak{Z}(G)$ and hence $Z_\chi > \mathfrak{Z}(G)$. Choose $\zeta \in Z_\chi - \mathfrak{Z}(G)$ with $\zeta \in \mathfrak{Z}_2(G)$ and $\zeta^p \in \mathfrak{Z}(G)$. Then (ζ, G) is central, elementary abelian and $(\zeta, G) \subseteq \ker \chi$. Clearly $(\zeta, G) \neq \langle 1 \rangle$. If $|(\zeta, G)| \geqq p^2$, choose J_1 and J_2 subgroups of (ζ, G) of order p with $J_1 \cap J_2 = \langle 1 \rangle$. Since $J_i \subseteq \ker \chi$, we have $e(G/J_i) = e$ and hence by induction Z_χ/J_i is abelian. Thus $Z'_\chi \subseteq J_1 \cap J_2 = \langle 1 \rangle$ and Z_χ is abelian.

Thus we can assume that $(\zeta, G) = p$ and hence if $H = \mathfrak{C}(\zeta)$, then $[G : H] = p$. Since H is character regular and $\mathfrak{Z}(H) \not\subseteq \mathfrak{Z}(G)$, Proposition 1.3 (i) yields $e(H) = e - 1$. Thus $\chi \mid H = \sum_1^p \varphi_i$ and χ vanishes off H. This latter fact implies that $Z_\chi \subseteq H$. Now if $\varphi = \varphi_1$, then $\deg \varphi = p^{e-1}$ and $e(H) = e - 1$. Thus in H, Z_φ is abelian. Since clearly $Z_\chi \subseteq Z_\varphi$, the result follows.

COROLLARY 1.8. *Let G have class 2. If $p > e(G) = e$, then G has a normal abelian subgroup of index p^{2e}.*

Proof. Let $\mathscr{T}$ be the family of p-groups of class $\leqq 2$ with $p > e(G)$. By Theorem 1.5 all members of $\mathscr{T}$ are character regular. Let χ be an irreducible character of G of degree p^e. Then by the above Z_χ is a normal abelian subgroup of G. Since G has class 2, $[G : Z_\chi] = p^{2e}$ and the result follows.

2. π-Character groups. In this section we study groups whose irreducible characters have degrees which are powers of a fixed integer m. In fact we consider the more general class of groups defined below. Here $\pi(n)$ denotes the set of prime factors of integer n.

DEFINITION 2.1. Let π be a set of primes. We say group G is a π-character group if the following hold.

(i) The distinct degrees of the irreducible characters of G are $d_0, d_1, \cdots, d_k$ with $k \geqq 1$.

(ii) For all $i \geqq 1$, $d_{i-1} \mid d_i$ and $\pi(d_i) = \pi$.

(iii) If $\pi = \{p\}$, then $\mathfrak{S}_p(G)$ is abelian.

Condition (iii) above is included for convenience in order to avoid overlap with our previous study of r. x. e groups. If H is a homomorphic image of G, then the degrees of the irreducible characters of H forms subset of those of G. Hence if G is a π-character group, then either H is a π-character group or H is abelian. The main result here is as follows.

THEOREM 2.2. *Let G be a π-character group. Suppose the distinct degrees of its irreducible characters are $d_0, d_1, \cdots, d_k$ with $d_{i-1} \mid d_i$. Then G has the following structure.*

(i) *G has a normal abelian $\mathfrak{S}_{\pi'}$ subgroup $H \neq \langle 1 \rangle$ with $G/H \cong \mathfrak{S}_\pi(G)$ abelian.*

(ii) *$A = \mathfrak{C}(H)$ is a normal abelian subgroup of G of index d_k.*

(iii) *There exists a subset $\{a_0, a_1, \cdots, a_r\}$ of $\{0, 1, \cdots, k\}$ with $0 = a_0 < a_1 < \cdots < a_r = k$ such that G/A is abelian of type $(d_{a_1}/d_{a_0}, d_{a_2}/d_{a_1}, \cdots, d_{a_r}/d_{a_{r-1}})$ and $(d_{a_{i+1}}/d_{a_i}) \mid (d_{a_i}/d_{a_{i-1}})$ for all i.*

COROLLARY 2.3. *Suppose the degrees of the irreducible characters of G are all powers of a fixed integer m, with m^s the largest such degree. Let $\pi = \pi(m)$ and assume that $|\pi| > 1$. Then G has a normal abelian subgroup A with G/A abelian of order m^s and type $(m^{s_0}, m^{s_1}, \cdots, m^{s_r})$ for suitable integers s_i. Moreover $\mathfrak{S}_\pi(G)$ is abelian.*

The corollary is of course an immediate consequence of the theorem. The proof of the latter will be in two parts. We first show that G satisfies (i). Then we study groups with that property and show that they satisfy the remaining conditions (ii) and (iii).

We start with a lemma. If λ is a linear character of G, then the order of λ, written $o(\lambda)$, is its order as an element of the dual group $\widehat{G/G'}$. If χ is any irreducible character of G we set $o(\chi)$ equal to $o(\lambda)$ where $\lambda = \det \chi$, the determinant of the representation associated with χ.

LEMMA 2.4. *Let p be a prime and let $U = \mathfrak{U}_p(G)$ be the minimal normal subgroup of G having a p-quotient group. Then*

$$|U| \equiv \sum_{p \nmid o(\chi)} \chi(1)^2 \mod p.$$

Proof. By induction on $|G|$. Suppose first that G has no normal subgroup of index p. Then $G = \mathfrak{U}_p(G)$ and G/G' is a p'-group. Hence for all $\lambda \in \widehat{G/G'}$ we have $p \nmid o(\lambda)$. Therefore the above congruence follows from the equation $|G| = \sum \chi(1)^2$.

Now let G have a normal subgroup H of index p. Clearly $\mathfrak{U}_p(G) = \mathfrak{U}_p(H)$ and thus by induction

$$|U| \equiv \sum_{p \nmid o(\chi)} \theta(1)^2 \mod p$$

where the sum runs over the irreducible characters θ of H. We show now that

$$\sum_{p \nmid o(\chi)} \chi(1)^2 \equiv \sum_{p \nmid o(\theta)} \theta(1)^2 \mod p\ .$$

In both sums we can of course discard those χ and θ with $p \mid \chi(1)$ and

$p \mid \theta(1)$. Also if $T(\theta) = H$, then θ has p conjugates $\theta_1, \theta_2, \cdots, \theta_\zeta$. Clearly $\theta_i(1) = \theta(1)$ and $o(\theta_i) = o(\theta)$. Thus the contribution of these conjugates to the right hand sum is a multiple of p. Hence we need only consider those θ with $T(\theta) = G$.

Let $\mathscr{S}_1 = \{\chi \mid \chi$ is an irreducible character of G, $p \nmid \chi(1)$, and $p \nmid o(\chi)\}$ and $\mathscr{S}_2 = \{\theta \mid \theta$ is an irreducible character of $H, p \nmid \theta(1)$, $p \nmid o(\theta)$, and $T(\theta) = G\}$. As we have shown above it suffices to prove that

$$\textstyle\sum_{\mathscr{S}_1} \chi(1)^2 \equiv \sum_{\mathscr{S}_2} \theta(1)^2 \qquad \bmod p \, .$$

We will in fact show that the map $\chi \rightarrow \chi \mid H$ is a one-to-one map of $\mathscr{S}_1$ onto $\mathscr{S}_2$ and this will yield the result since χ and $\chi \mid H$ have the same degree.

Let $\chi \in \mathscr{S}_1$. Since $[G:H] = p$ we have that either $\chi \mid H$ is irreducible or $\chi \mid H$ is the sum of p conjugates. Since $p \nmid \chi(1)$, the latter cannot occur so $\chi \mid H = \theta$ is irreducible. Clearly $\theta(1) = \chi(1)$, $o(\theta) \mid o(\chi)$ and $T(\theta) = G$ and hence $\theta \in \mathscr{S}_2$. Thus the restriction map sends $\mathscr{S}_1$ into $\mathscr{S}_2$.

Now let $\theta \in \mathscr{S}_2$ and let $\mu = \det \theta$. Since $T(\theta) = G$ we have $T(\mu) = G$ and thus $K = \ker \mu$ is normal in G. If χ is such that $\chi \mid H = \theta$, then χ is a constituent of θ^*. Thus to show that the restriction map is one-to-one and onto we must find a unique constituent χ of θ^* with $\chi \in \mathscr{S}_1$ and $\chi \mid H = \theta$. Let τ be a nonprincipal linear character of G/H and let χ be an irreducible constituent of θ^*. Since $[G:H] = p$ and $T(\theta) = G$ we see that $\chi \mid H = \theta$ and that all the constituents of θ^* are of the form $\chi_i = \tau^i \chi$ for $i = 0, 1, \cdots, p-1$. Let $\lambda = \det \chi$ so that $\lambda \mid H = \mu$. We have clearly

$$\det \chi_i = \det \tau^i \chi = \tau^{i\chi(1)} \lambda \, .$$

Since $\chi(1) = \theta(1)$ is prime to p we see that $\det \chi_i \neq \det \chi_j$ for $i \neq j$ and hence we obtain p distinct linear characters of G which extend μ.

Since $T(\mu) = G$ we see that H/K is central in G/K and since G/H is cyclic, G/K is abelian. Also H/K is a p'-group since $p \nmid o(\mu)$ and hence $G/K \simeq (H/K) \times (G/H)$. It follows easily from this that there are precisely p distinct linear characters of G which extend μ and that precisely one of these has order prime to p. Hence there is a unique i_0 with $o(\chi_{i_0})$ prime to p. Then $p \nmid o(\chi_{i_0})$ and $p \nmid \chi_{i_0}(1)$ since $\chi_{i_0}(1) = \theta(1)$. Thus $\chi_{i_0} \in \mathscr{S}_1$ and $\chi_{i_0} \mid H = \theta$. This completes the proof.

The first two parts of the following theorem are due to John Thompson. They generalize our original result, proved under more restrictive assumptions.

THEOREM 2.5. *Let p be a prime and π a set of primes.*

(i) *Suppose for every nonlinear irreducible character χ of G we have $p \mid \chi(1)$. Then G has a normal p-complement.*

(ii) *If the degrees of the irreducible characters of G are linearly ordered by divisibility, then G has a Sylow tower.*

(iii) *Suppose for every nonlinear irreducible charactor χ of G we have $\pi(\chi(1)) = \pi$. Then G has a normal abelian $\mathfrak{S}_{\pi'}$ subgroup H. Moreover if $|\pi| > 1$, then G/H is abelian.*

Proof. (i) Let U be as in the preceeding lemma. Since $p \mid \chi(1)$ if $\chi(1) \neq 1$ we see by Lemma 2.4 that

$$|U| \equiv \sum_{p \nmid o(\lambda)} \lambda(1)^2 \qquad \bmod p$$

where the sum runs over linear characters λ. Clearly $p \nmid o(\lambda)$ is equivalent to λ belonging to $\mathfrak{S}_{p'}(\widehat{G/G'})$. Hence

$$|U| \equiv |\mathfrak{S}_{p'}(\widehat{G/G'})| \qquad \bmod p$$

and so $p \nmid |U|$. Thus U is a normal p-complement of G.

(ii) By induction on $|G|$. If G is abelian the result is clear so assume that G is nonabelian. Let $d_0 = 1, d_1, \cdots, d_k$ be the distinct degrees of the irreducible characters of G with $d_i \mid d_{i+1}$. Since $k \geqq 1$, choose prime p with $p \mid d_1$. Then for all $i \geqq 1$, $p \mid d_i$. By (i), G has a normal p-complement H. Let χ be a character of G of degree d_i and say

$$\chi \mid H = a \sum_1^t \theta_i \,.$$

If $\theta = \theta_1$, then $at \deg \theta = \deg \chi = d_i$. As is well known $at \mid [G:H]$ and of course $\deg \theta \mid |H|$. Hence clearly $at = |d_i|_p$ and $\deg \theta = |d_i|_{p'}$. Thus the degrees of the irreducible characters of H are $|d_0|_{p'}, |d_1|_{p'}|, \cdots, |d_k|_{p'}$ and these are linearly ordered by divisibility. By induction H has a Sylow tower and thus the result follows here.

(iii) By (i), G has a normal p-complement for all $p \in \pi$. Hence G has a normal $\mathfrak{S}_{\pi'}$ subgroup H with G/H nilpotent. Let θ be an irreducible character of H and let χ be a constituent of θ^*. Then $\deg \theta \mid \deg \chi$ and $\deg \theta \mid |H|$ and so $\deg \theta = 1$. Thus H is abelian. Now let $\pi = \{p_1, p_2, \cdots, p_r\}$ and suppose $r > 1$. Let $G/H = P_1 \times P_2 \times \cdots \times P_r$ where $P_i = \mathfrak{S}_{p_i}(G/H)$. If P_i is nonabelian then G/H has a character χ with $\pi(\chi(1)) = \{p_i\} \neq \pi$, a contradiction. Hence for all i, P_i is abelian and thus G/H is abelian. This completes the proof.

Part (iii) of the above result yields (i) of Theorem 2.2. We now study groups satisfying this latter condition.

THEOREM 2.6. *Let π be a set of primes. Let G be a group with a normal abelian $\mathfrak{S}_{\pi'}$ subgroup H and with $G/H \cong \mathfrak{S}_\pi(G)$ abelian.*

Suppose the distinct degrees of the irreducible characters of G are $d_0, d_1, \cdots, d_k$ with $d_{i-1} \mid d_i$. Then

(i) $A = \mathfrak{C}(H)$ *is a normal abelian subgroup of* G.

(ii) *There exists a subset* $\{a_0, a_1, \cdots, a_r\}$ *of* $\{0, 1, \cdots, k\}$ *with* $0 = a_0 < a_1 < \cdots < a_r = k$ *such that* G/A *is abelian of type* $(d_{a_1}/d_{a_0}, d_{a_2}/d_{a_1}, \cdots, d_{a_r}/d_{a_{r-1}})$ *so* $[G:A] = d_k$ *and* $(d_{a_{i+1}}/d_{a_i}) \mid (d_{a_i}/d_{a_{i-1}})$ *for all* i.

Let K be a normal subgroup of G, maximal subject to G/K being nonabelian. If $G/K = E$ is solvable, then in the terminology of § 2 of [2], E is extra-special. By Proposition 2.2 of [2], E is either a Case P or Case Q group. We will refer to these as Case P and Case Q quotients of G.

Let G satisfy the hypotheses of the above theorem. Set $q_i = d_i/d_{i-1}$. These degree quotients will come into play in some later results.

LEMMA 2.7. *Let G satisfy the hypothesis of Theorem* 2.6. *Let* $K \triangle G$ *so that* G/K *is an extra-special group. Then* G/K *is a Case* Q *quotient. Let* Q/K *be the normal Sylow* q*-subgroup of* G/K *with* G/Q *cyclic of order* d. *Then* $| d |_\pi = d$ *and* $| q |_{\pi'} = q$. *Also there exists a subset* $\{b_0, b_1, \cdots, b_s\}$ *of* $\{0, 1, \cdots, k\}$ *with* $b_0 < b_1 < \cdots < b_s = k$ *such that the distinct degrees of the irreducible characters of* Q *are* d_{b_0}/d, $d_{b_1}/d, \cdots, d_{b_s}/d$.

Suppose further that $q_i \nmid d$ *for all* $i > 1$. *Then* $d = d_1$ *and the distinct degrees of the irreducible characters of* Q *are* $d_1/d_1, d_2/d_1, \cdots, d_k/d_1$. *Moreover if* θ *is an irreducible character of* Q, *then* θ^* *is either irreducible or it has all linear constituents.*

Proof. Let G/K be an extra-special quotient of G. If G/K is Case P, then G/K is a nonabelian p-group for some prime p. Since all Sylow subgroups of G are abelian, this cannot occur. Thus G/K is Case Q. By Ito's theorem we have $d_i \mid [G:H]$ for all i and hence $| d_i |_\pi = d_i$. Since d is the degree of an irreducible character of G we have $| d |_\pi = d$. Moreover since G/K is nonabelian and $\mathfrak{S}_\pi(G)$ is abelian, we see that G/K is not a π-group. Hence $| q |_{\pi'} = q$.

Let θ be an irreducible character of Q and let μ be a nonprincipal linear character of Q/K viewed as one of Q. Suppose $\theta = \theta\mu$. If L is the kernel of μ then $Q > L \supseteq K$ and θ vanishes off L. Say $\theta \mid L = a \sum_1^t \varphi_i$. Then $[\theta \mid L, \theta \mid L]_L = a^2 t$. On the other hand since θ vanishes off L, $[\theta \mid L, \theta \mid L]_L = [Q:L][\theta, \theta] = [Q:L]$. Hence $a^2 t$ is a proper power of q. Since $\deg \theta = at \deg \varphi_1$ we have $q \mid \deg \theta$. If χ is a constituent of θ^*, then $\deg \theta \mid \deg \chi$ and so $q \mid \deg \chi$. This is a contradiction since $| \deg \chi |_\pi = \deg \chi$ and $q \notin \pi$. Hence $\theta \neq \theta\mu$.

Now let λ, μ be two distinct characters of Q/K. We show that

$T(\theta\lambda) \cap T(\theta\mu) = Q$. If not we can find $x \in (T(\theta\lambda) \cap T(\theta\mu)) - Q$. Then

$$\theta\lambda = (\theta\lambda)^x = \theta^x\lambda^x \qquad \theta\mu = (\theta\mu)^x = \theta^x\mu^x$$

and hence

$$\theta^x = \theta\lambda\bar{\lambda}^x = \theta\mu\bar{\mu}^x$$

Now $\lambda\bar{\lambda}^x \neq \mu\bar{\mu}^x$ since x acts fixed point free on Q/K and $\lambda \neq \mu$. Thus $\theta = \theta\rho$ where $\rho = (\mu\bar{\mu}^x)(\bar{\lambda}\lambda^x) \neq 1$ and this contradicts the above. Let u be the number of minimal subgroups of G/Q. Then this says that there are at most u characters λ of Q/K with $T(\theta\lambda) > Q$. Clearly $u \leqq d - 1$ since each minimal subgroup is cyclic and has a nonidentity generator. On the other hand G/Q acts fixed point free on Q/K so there are at least $d + 1 > u$ linear characters of Q/K. Hence there exists λ with $T(\theta\lambda) = Q$.

Since $T(\theta\lambda) = Q$, it follows easily that $(\theta\lambda)^*$ is irreducible. Hence for some i

$$d_i = \deg(\theta\lambda)^* = d\deg(\theta\lambda) = d\deg\theta\ .$$

This implies that there exists a subset $\{b_0, b_1, \cdots, b_s\}$ of $\{0, 1, \cdots, k\}$ with $b_0 < b_1 < \cdots < b_s \leqq k$ such that the distinct degrees of the irreducible characters of Q are $d_{b_0}/d, d_{b_1}/d, \cdots, d_{b_s}/d$. We show now that $b_s = k$. Let χ be a character of G of degree d_k and let θ be an irreducible constituent of $\chi \mid Q$. Then certainly $\deg\theta \geqq d_k/d$. On the other hand by the above $\deg\theta = d_j/d$ for some j. Hence $d_j/d \geqq d_k/d$ so $j = k$ and $\deg\theta = d_k/d$. This completes the first half of the proof.

Now assume that $q_i \nmid d$ for all $i > 1$. Since $q_i \mid d_i$ and $d > 1$, it follows that $d = d_1$. Let χ be an irreducible character of G of degree d_i for $i > 0$ and let $\theta = \theta_1$ be an irreducible constituent of $\chi \mid Q$. We have $\chi \mid Q = a\sum_1^t \theta_i$ and thus if $b = at$ then $b \leqq d$ and $b\deg\theta = \deg\chi = d_i$. On the other hand we know that $\deg\theta = d_j/d$ for some j. Hence $d_i/b = d_j/d$. Since $d \geqq b$, it follows that $d_j \geqq d_i$. If $d_j > d_i$, then $d_{i+1} \mid d_j$ and we have

$$d = d_1 = b(d_j/d_i) = bq_{i+1}(d_j/d_{i+1})$$

and $q_{i+1} \mid d$, a contradiction for $i > 0$. Hence $i = j$ and $\deg\theta = d_i/d$. Moreover $b = at = d$ and since $a^2t \leqq d$, in general, we have $a = 1$, $t = d$ and $\chi = \theta^*$. Thus the distinct degrees of the irreducible characters of Q are $d_1/d_1, d_2/d_1, \cdots, d_k/d_1$.

Finally let θ be a character of Q and suppose that θ^* has a nonlinear irreducible constituent χ. Since θ is a constituent of $\chi \mid Q$, the above yields $\chi = \theta^*$ is irreducible. This completes the proof of the lemma.

Proof of Theorem 2.6. First $A = \mathfrak{S}_\pi(A)H$, $\mathfrak{S}_\pi(A)$ is abelian and H is central in A. Hence A is abelian and (i) follows. Note that G is solvable. If G is abelian, then (ii) is obvious. So assume G is nonabelian.

Let $K \triangle G$ with G/K an extra-special group. By the preceeding lemma, G/K is a Case Q quotient. Using the notation of that lemma we have $[G:Q] = d_i$ for some i. Moreover assume that K is so chosen that i is maximal with this occuring.

In G/K we have $\mathfrak{C}_{G/K}(Q/K) = Q/K$. This shows that $A = \mathfrak{C}_G(H) \subseteqq Q$ and hence $\mathfrak{C}_G(H) = \mathfrak{C}_Q(H) = A$. Let $x \in G/A$ be such that it generates the cyclic quotient G/Q. We show that $|\langle x \rangle| = d_i = d_i/d_0$. Clearly $d_i \mid |\langle x \rangle|$. If $d_i \neq |\langle x \rangle|$, then for some prime $p \in \pi$ we have $|\langle x \rangle|_p > |d_i|_p$. For this prime let J be the subgroup of $\langle x \rangle$ of order p. Now $\mathfrak{S}_\pi(A)$ centralizes H and some $\mathfrak{S}_\pi(G)$ and hence $\mathfrak{S}_\pi(A)$ is central in G. Thus by Lemma 1.2 we can write $A = D \times C$ where $D = \mathfrak{C}_A(J)$ and $D, C \triangle G$ and J acts fixed point free on C. Clearly $C \neq \langle 1 \rangle$. Let λ be a nonprincipal linear character of C viewed as one of A. Then $(T(\lambda)/A) \cap J = \langle 1 \rangle$ and hence $[G : T(\lambda)]_p \geqq |\langle x \rangle|_p > |d_i|_p$. Since $C \neq 1$ this implies that the distinct degrees of the irreducible characters of G/D are $1, d_j, \cdots$ with $j > i$. Hence G/D has a Case Q quotient with $[G:Q] = d_s > d_i$, a contradiction. Thus $|\langle x \rangle| = d_i$. Setting $a_1 = i$, we have by induction applied to Q, that G/A is abelian of type $(d_{a_1}/d_{a_0}, d_{a_2}/d_{a_1}, \cdots, d_{a_r}/d_{a_{r-1}})$ with $a_r = k$. Also $(d_{a_{j+1}}/d_{a_j}) \mid (d_{a_j}/d_{a_{j-1}})$ for $j > 1$ by induction. To obtain $(d_{a_2}/d_{a_1}) \mid (d_{a_1}/d_{a_0})$ we merely note that $|\langle x \rangle| = d_i$ for all such choices of x. This implies that the period of Q/A divides $d_i = d_{a_1}/d_{a_0}$. This completes the proof.

The proof of Theorem 2.2 is now immediate. Part (i) follows from Theorem 2.5 (iii) and from the assumption that if $\pi = \{p\}$, then $\mathfrak{S}_p(G)$ is abelian. Then Theorem 2.6 yields parts (ii) and (iii).

In the remainder of this section we assume that G satisfies the hypothesis of Theorem 2.6 and we will use the notation of the conclusion of that theorem. We first note a few simple facts about the characters of G.

LEMMA 2.8. *Let χ be an irreducible character of G. Then we have*

(i) $\chi \mid H = \sum_1^t \lambda_i$, *that is there is no ramification.*

(ii) *There exists a subgroup $L \supseteqq A$ and a linear character λ of L with $\chi = \lambda^*$.*

(iii) *If χ is faithful, then $L = A$ and* $\deg \chi = d_k$.

Proof. Let $\chi \mid H = a \sum_1^t \lambda_i$ and set $L = T(\lambda_1)$. Clearly $L \supseteqq \mathfrak{C}(H) =$

A. As is well known there exists a character θ of L with $\chi = \theta^*$ and $\theta \mid H = a\lambda_1$. Let K be the kernel of θ. Then clearly H is central modulo K. Since $\mathfrak{S}_\pi(L)$ is abelian this shows that L/K is abelian and hence $\deg \theta = 1$. Thus $a = 1$ and (i) and (ii) are proved.

If χ is faithful then since $L \triangle G$ we have that L is abelian. Hence $L \subseteqq \mathfrak{C}(H) = A$. This yields $\deg \chi = [G : A] = d_k$ and (iii) follows.

It is interesting to consider which subgroups L can occur in (ii) of the above lemma. Define a Galois connectivity between groups L with $G \supseteqq L \supseteqq A$ and groups B with $H \supseteqq B$ as follows.

$$L \xrightarrow{d} (L, H) \qquad B \xrightarrow{u} \{g \in G \mid (g, H) \subseteqq B\}$$

We say L is closed if $L^{du} = L$

PROPOSITION 2.9. Using the above notation, group L has a linear character λ with λ^* irreducible if and only if L is closed.

Proof. We note first that $(L, H) = L'$. This follows since $L/(L, H)$ has a central $\mathfrak{S}_{\pi'}$ subgroup and an abelian $\mathfrak{S}_\pi$ subgroup and hence is abelian.

Now let L have a linear character λ with $\lambda^* = \chi$ irreducible. Set $M = L^{du}$ so that $M \supseteqq L$. Suppose that $M > L$. Clearly $L \supseteqq \ker \chi$. Since G/A is abelian, $L \triangle G$ and hence $L/\ker \chi$ is abelian. Thus $L' = (L, H) \subseteqq \ker \chi$. Since $\deg \chi = [G : L]$ and $M > L$ it follows that $M' = (M, H) \not\subseteqq \ker \chi$. Thus $M^d = (M, H) \neq (L, H) = L^d$, a contradiction. Hence $M = L$ and L is closed.

Now assume L is closed. We consider $\bar{G} = G/(L, H)$ in which $\bar{L} = L/(L, H)$ is abelian. Since L is closed we see that $\bar{G}/\bar{L}$ acts faithfully on $\bar{D} = \mathfrak{S}_{\pi'}(\bar{L})$. Thus $\bar{G}/\bar{L}$ acts faithfully on $\hat{\bar{D}}$, the dual group of $\bar{D}$. Since these groups are abelian and have relatively prime orders, it follows by a trivial modification of Lemma 2.2 of [5] that there exists $\lambda \in \hat{\bar{D}}$ with $\mathfrak{C}_{\bar{G}/\bar{L}}(\lambda) = \langle 1 \rangle$. View λ as a character of $\bar{L}$ and then as one of L. We see that $T(\lambda) = L$ and hence that λ^* is irreducible. The result follows.

If G/A is cyclic we can obtain additional information.

THEOREM 2.10. *Suppose G/A is cyclic. Let L_i be the unique subgroup of G with $[G : L_1] = d_i$ and $L_i \supseteqq A$. Then we can write $A = B_0 \times B_1 \times \cdots \times B_k$ where each B_1 is characteristic in G, L_i centralizes B_i and G/L_i acts fixed point free on B_i. Here $B_i \neq \langle 1 \rangle$ for $i \neq 0$. Conversely a group with this structure has characters of degrees $d_0, d_1, \cdots, d_k$ only.*

Proof. Note that $A = \mathfrak{C}(H)$ and each L_i is characteristic in G.

Note also that $\mathfrak{S}_\pi(A)$ is central in G. We have

$$\langle 1\rangle \subseteqq \mathfrak{C}_A(L_0) \subseteqq \mathfrak{C}_A(L_1) \subseteqq \cdots \subseteqq \mathfrak{C}_A(L_k) = A$$

and each of these groups is characteristic in G. By Lemma 1.2 we can write for $i = 1, 2, \cdots, k$

$$\mathfrak{C}_A(L_i) = \mathfrak{C}_A(L_{i-1}) \times B_i$$

where each B_i is characteristic in G. Setting $B_0 = \mathfrak{C}_A(L_0) = \mathfrak{Z}(G)$ we have

$$A = B_0 \times B_1 \times \cdots \times B_k$$

where each B_i is characteristic in G and is centralized by L_i.

Let $i \geqq 1$ and let λ be a nonprincipal linear character of B_i viewed as one of A. Since L_i centralizes B_i we have $T(\lambda) \supseteqq L_i$. If $T(\lambda) > L_i$, then by Lemma 2.8 (i) we have $T(\lambda) \supseteqq L_{i-1}$. This implies that L_{i-1} centralizes an element of $B_i^\#$ which is not the case by definition of B_i. Hence $T(\lambda) = L_i$ and G/L_i acts fixed point free on $\hat{B}_i$ and hence on B_i.

We show now that $B_i \neq \langle 1\rangle$ for $i \neq 0$. Let χ be an irreducible character of G of degree d_i and let λ be an constituent of $\chi \mid A$. Since there is no ramification, $[G : T(\lambda)] = d_i$ and hence $T(\lambda) = L_i$. Write $\lambda = \lambda_0\lambda_1 \cdots \lambda_k$ where λ_j is a character of B_j viewed as one of A. As we showed above, L_i fixes no nonprincipal character of B_j for $j > i$. Hence $\lambda = \lambda_0\lambda_1 \cdots \lambda_i$. If $\lambda_i = 1$, then clearly $T(\lambda) \supseteqq L_{i-1}$ which is not the case. Hence $\lambda_i \neq 1$ and $B_i \neq 1$. The completes the forward half of the proof.

Conversely let G have the structure described above. Since A is abelian and G/A is cyclic we know that there is no ramification. Let χ be an irreducible character of G with $\lambda = \lambda_0\lambda_1 \cdots \lambda_k$ a constituent of $\chi \mid A$. Then $\deg \chi = [G : T(\lambda)]$. If $\lambda = 1$, then $\deg \chi = 1 = d_0$. If $\lambda \neq 1$ choose j maximal with $\lambda_j \neq 1$. Clearly $T(\lambda) = L_j$ and $\deg \chi = d_j$. This completes the proof.

We now seek sufficient conditions to guarantee that G/A is cyclic.

THEOREM 2.11. *Each of the following will guarantee that G/A is cyclic.*

(i) $d_k \nmid d_{k-1}^2$

(ii) *For all* $i < j$, $q_j \nmid q_i$.

(iii) *For all* i, $q_{i+1} > q_i$.

(iv) *There exists a prime* $p \in \pi$ *such that* $|q_{i+1}|_p > |q_i|_p$ *for all* i.

Proof. We consider (i) first. This is a simple corollary of Theorem 2.6. If G/A is not cyclic, then there exists $b < a < k$ with $(d_k/d_a) \mid (d_a/d_b)$.

Thus $d_k \mid d_a^2$ and since $d_a \mid d_{k-1}$ this yields $d_k \mid d_{k-1}^2$, a contradiction.

Now assume G satisfies condition (ii). We prove the result by induction on $|G|$. If $k = 1$, the result follows by (i) above so we assume $k > 1$. Let χ be a character of G of degree d_1. By Lemma 2.8 (iii), $G/\ker \chi$ has characters of degrees 1 and d_1 only. Choose $K \triangle G, K \supseteqq \ker \chi$ with G/K a Case Q quotient. Using the notation of Lemma 2.7, it is clear that $[G : Q] = d_1 = q_1$. Since $q_j \nmid d_1$ for all $j > 1$, it follows by Lemma 2.7 that the distinct degrees of the characters of Q are $d_1/d_1, d_2/d_1, \cdots, d_k/d_1$. Hence Q has degree quotients $q_2, q_3, \cdots, q_k$ and we can apply induction to Q. Thus Q/A is cyclic.

Theorem 2.10 applies to Q and thus A contains a characteristic subgroup B on which Q/A acts fixed point free. Then $B \triangle G$ and also $B \neq \langle 1 \rangle$. Let λ be a nonprincipal linear character of B viewed as one of A. Then $\tilde{\lambda}$ (induction to Q) is an irreducible nonlinear character of Q since $k > 1$. By Lemma 2.7, $\tilde{\lambda}^* = \lambda^*$ is irreducible. This shows that G/A acts fixed point free on $\hat{B}$ and hence G/A is cyclic.

Parts (iii) and (iv) follow immediately from (ii).

3. Groups with a. c. m.

In this section we study nonabelian groups G having the property that every nonlinear irreducible character has degree m for some fixed integer m. We say these groups have a.c. m (all characters m). As an immediate consequence of Theorems 2.2 and 2.10 we have the following.

THEOREM 3.1. *Let G have* a.c. m *with $\pi = \pi(m)$. Suppose that either $|\pi| > 1$ or $\pi = \{p\}$ and an $\mathfrak{S}_p$ subgroup of G is abelian. Then G has the following structure.*

(i) *G has a normal abelian $\mathfrak{S}_{\pi'}$ subgroup $H \neq \langle 1 \rangle$ with $G/H \cong \mathfrak{S}_\pi(G)$ abelian.*

(ii) *$A = \mathfrak{C}(H)$ is a normal abelian subgroup of G with G/A cyclic of order m.*

(iii) *$A = \mathfrak{Z}(G) \times B$ where $B \triangle G, B \neq \langle 1 \rangle$ and G/A acts fixed point free on B.*

Conversely any group G having this structure has a. c. m.

Therefore we need only consider the case in which $m = p^e$ for some prime p with $\mathfrak{S}_p(G)$ nonabelian. Actually the $e = 1$ case has already been studied in [2]. However there is little additional work involved in handling it so we will consider it again here. As we will see, the structure of those groups with $e > 1$ is much more restrictive than the structure with $e = 1$. We start with several lemmas.

LEMMA 3.2. *Let G have* a.c. m *with $m = p^e$ and $\mathfrak{S}_p(G)$ nonabelian. Then we have the following.*

(i) *G has a normal abelian* $\mathfrak{S}_{p'}$ *subgroup* H.

(ii) *G has a Case P quotient* E.

(iii) *If* E *is any Case P quotient of* G, *then* E *is a p-group with* $E/\mathfrak{Z}(E)$ *elementary abelian of order* m^2. *Also any abelian subgroup* B *of* E *satisfies* $[E:B] \geqq m$.

Proof. Since G has r.x. e, Proposition 3.4 of [2] yields (i). Now $G/H \cong \mathfrak{S}_p(G)$ is nonabelian. Thus we can choose $K \triangle G$, $K \supseteqq H$ and maximal with G/K nonabelian. Clearly G/K is a Case P quotient and (ii) follows. Now let E be any Case P quotient of G. Then E has a.c. m and hence E is a p-group and $E/\mathfrak{Z}(E)$ is abelian of order m^2 by Proposition 2.2 of [2]. Since E has an irreducible character of degree m, it follows that E has no abelian subgroup of index less than m. We need only show that $E/\mathfrak{Z}(E)$ is elementary abelian. Given $x, y \in E$. Since E has class 2 and E' has period p we have $(x^p, y) = (x, y)^p = 1$. Thus $x^p \in \mathfrak{Z}(E)$ and $E/\mathfrak{Z}(E)$ has period p.

LEMMA 3.3. *Let G have* a.c. m. *Then we have the following.*

(i) *Let* $\mathscr{P}$ *be a permutation representation of G with* $\deg \mathscr{P} \leqq m$. *Then* $G' \subseteqq \ker \mathscr{P}$.

(ii) *If* $|G'| \leqq m$, *then* $G' \subseteqq \mathfrak{Z}(G)$.

(iii) *Let* L *be a subgroup of* G *with* $[G:L] \leqq m$. *Then* $G' \subseteqq L$ *and hence* $L \triangle G$. *Moreover if* $[G:K] = m$ *and* $K < L \subseteqq G$, *then* $(K, L) = L' = G'$.

Proof. Let θ be the character corresponding to $\mathscr{P}$. Then $\deg \theta \leqq m$. We have $\theta = \sum a_i \chi_i$ where each χ_i is an irreducible character of G. Now if $\chi_1 = 1$, then $a_1 \geqq 1$ and hence for all i, $\deg \chi_i < m$. Since G has a.c. m, $\deg \chi_i = 1$ and $G' \subseteqq \ker \chi_i$ and hence $G' \subseteqq \ker \mathscr{P}$. This yields (i).

Suppose $|G'| \leqq m$. Let $x \in G$. Clearly $|Clx| \leqq |G'| \leqq m$, where Clx denotes the class of x. Now G permutes the elements of Clx by conjugation and this representation has degree $\leqq m$. Hence by (i) G' is in the kernel of the representation and thus G' centralizes x. Since x was arbitrary, $G' \subseteqq \mathfrak{Z}(G)$ and (ii) follows.

Now let $L \subseteqq G$ with $[G:L] \leqq m$. We see that G permutes the right cosets of L by right multiplication and this representation has degree $\leqq m$. Thus by (i), G' is in the kernel and hence $G' \subseteqq L$. Now let $[G:K] = m$ and $L > K$. Since both K and L are normal in G so is $H = (K, L)$. If $H < G'$, then G/H is nonabelian and thus has a.c. m. Since $K' \subseteqq H$, K/H is abelian and is centralized by a properly larger subgroup. Thus G/H has an abelian subgroup of index $< [G:K] = m$. This contradicts the existence of an irreducible character of G/H of degree m. Hence $H = G'$. Since $G' \supseteqq L' \supseteqq (K, L)$, the result follows.

LEMMA 3.4. *Let G have* a.c.m *where* $m = p^e$ *and* $\mathfrak{S}_p(G)$ *is nonabelian. Let* $A \supseteqq H = \mathfrak{S}_{p'}(G)$ *be a normal self-centralizing subgroup of* G. *If* G *has a faithful irreducible character* χ *then* G/A *is elementary abelian of order* m.

Proof. Since $A \supseteqq H, G/A$ is a p-group and hence there exists subgroup L and linear character λ of L with $G \supseteqq L \supseteqq A$ and $\chi = \lambda^*$. Since $\deg \chi = m, [G:L] = m$ and hence by Lemma 3.3 (iii), $L \triangle G$. Thus $\chi \mid L$ has only linear constituents. Since χ is faithful L is abelian and since A is self-centralizing $L = A$. Thus G/A has order m.

By Lemma 3.2(ii), G has a Case P quotient $E = G/K$. Let $Z/K = \mathfrak{Z}(E)$. Then $(AZ)/K$ is an abelian subgroup of E so $[G:AZ] \geqq m$ by Lemma 3.2 (iii). Since $[G:A] = m$, we have $A \supseteqq Z$ and hence G/A is elementary abelian.

We now reduce the study of these groups to a study of p-groups.

THEOREM 3.5. *Let G have* a.c. p^e *with* $P = \mathfrak{S}_p(G)$ *nonabelian. Let H be the normal abelian* $\mathfrak{S}_{p'}$ *subgroup.*

(i) *If* $e > 1$, *then H is central and hence* $G = H \times P$.

(ii) *If* $e = 1$, *then either H is central or G has a normal abelian subgroup of index* p.

Proof. We start with the case $e > 1$. Suppose first that G has a faithful irreducible character. By the preceeding lemma, G has a normal abelian subgroup A with G/A elementary abelian of order p^e. Then $A = B \times H$ where $B = \mathfrak{S}_p(A) \triangle G$. We consider G/B and show it is abelian. If not, then G/B has a.c. p^e. Now $\mathfrak{S}_p(G/B) \cong G/A$ is abelian and thus Theorem 3.1 applies. Hence since $A/B = \mathfrak{S}_{p'}(G/B)$ we see that $(G/B)/\mathfrak{C}(A/B)$ is cyclic of order p^e. This implies that $\mathfrak{C}(A/B) = A/B$ and therefore that G/A is cyclic of order p^e. Since G/A is elementary abelian this is a contradiction for $e > 1$. Thus G/B is abelian. Since $H \triangle G$ this yields $(G, H) \subseteqq B \cap H = \langle 1 \rangle$ and H is central.

Now let G be arbitrary with $e > 1$. We show that H is central. If not choose $x \in P', y \in (G, H)$ with $x, y \neq 1$. By Proposition 4.6 of [2] there exists an irreducible character χ of G with $x, y \notin \ker \chi$. Hence $G/\ker \chi$ has a. c. p^e, a nonabelian Sylow p-subgroup and a noncentral $\mathfrak{S}_{p'}$ subgroup. Since $G/\ker \chi$ has a faithful character this contradicts the above and (i) is proved.

Now let $e = 1$ and suppose that H is not central. Since $p > 1 = e$, G is imprimitive by Theorem 1.1. Thus there exists $A \triangle G$ with $[G:A] = p$ such that $e(A) = e(G) - 1 = 0$. Hence A is abelian and (ii) follows.

It is easy to construct examples to show that H need not be central in G if $e = 1$. For example, let Q be an abelian q-group ($q \neq p$) which has an automorphism of order p. Let $G = Q \times_\sigma P$ where P has order p^3 and P acts on Q in such a way that P_0 a subgroup of index p centralizes Q and P/P_0 corresponds to the automorphism of order p. Since $A = QP_0$ is an abelian subgroup of G of index p, G has a.c. p. Finally $Q = \mathfrak{S}_{p'}(G)$ is not central and $P = \mathfrak{S}_p(G)$ can be chosen to be nonabelian.

In the remainder of this section and in the next two sections we will consider only p-groups.

LEMMA 3.6. *Let G have* a.c. m. *Then we have the following.*

(i) *$\Phi(G)$, the Frattini subgroup of G, is abelian.*

(ii) *If G has two distinct abelian subgroups A and B of index m, then $|G'| \leqq m$ and hence G has class 2. Moreover if $|G'| = m$, then $\mathfrak{Z}(G) = A \cap B$ and $[G : \mathfrak{Z}(G)] = m^2$.*

(iii) *If G' is not central, then $\mathfrak{C}(G')$ is abelian.*

Proof. We consider (ii) first. Choose $x \in B - A$ with $x^p \in A$ and set $L = \langle A, x \rangle$. Then $[G : L] = p^{e-1}$ where $m = p^e$. Thus by Lemma 3.3 (iii), $L' = G'$. Clearly $L' = (A, x)$ and $|G'| = |L'| = [A : \mathfrak{C}_A(x)]$. Since $\mathfrak{C}_A(x) \supseteqq A \cap B$ and $[A : A \cap B] \leqq m$, we have $|G'| \leqq m$. By Lemma 3.3(ii), G has class 2. If $|G'| = m$, then $[A : A \cap B] = m$ and so $[AB : B] = m$. Thus $G = AB$ and since A and B are abelian $\mathfrak{Z}(G) \supseteqq A \cap B$. On the other hand A and B must be maximal abelian subgroups so $A, B \supseteqq \mathfrak{Z}(G)$. Thus $\mathfrak{Z}(G) = A \cap B$ and (ii) follows.

If $\Phi(G)$ is not abelian, then there exists an irreducible character χ of G with $\Phi' \nsubseteq \ker \chi$. Hence $\Phi(G/\ker \chi)$ is nonabelian. Now $\bar{G} = G/\ker \chi$ has a normal abelian subgroup A with $\bar{G}/\bar{A}$ elementary abelian by Lemma 3.4. Hence $\Phi(\bar{G}) \subseteqq \bar{A}$, a contradiction and (i) follows.

Now assume G' is not central. If $\mathfrak{C}(G')$ is not abelian we can choose $x, y \in \mathfrak{C}(G')$ with $(x, y) \neq 1$. Choose $z \in (G, G')$ with $z \neq 1$. Then there exists an irreducible character χ of G with $(x, y), z \notin \ker \chi$. Therefore it suffices to assume that G has a faithful irreducible character. Since G' is abelian, we can extend normal abelian subgroups $\langle x, G' \rangle$ and $\langle y, G' \rangle$ to normal self-centralizing subgroups A and B. By Lemma 3.4, $[G : A] = [G : B] = m$. Since $x \in A$, $y \in B$ and $(x, y) \neq 1$, we see that $A \neq B$. By (ii) above G has class 2, a contradiction. This completes the proof of the lemma.

THEOREM 3.7. *Let G be a p-group with* a.c.m. *Then either G has a normal abelian subgroup A with G/A elementary abelian of order m or G has class at most* 3.

Proof. By induction on $|G|$. If $\mathfrak{Z}(G)$ is cyclic, then G has a faithful irreducible character and the result follows by Lemma 3.4. Hence we can assume that $\mathfrak{Z}(G)$ is not cyclic and thus $\mathfrak{Z}(G)$ has at least three distinct subgroups J_1, J_2, J_3 of order p. We can clearly assume that G has class > 3. Since the subgroups J_i are disjoint it follows that at most one quotient G/J_i has class $\leqq 3$. Hence say G/J_1 and G/J_2 have class > 3. By induction, for $i = 1, 2, G/J_i$ has a normal abelian subgroup A_i/J_i with G/A_i elementary abelian of order m. Set $U = J_1J_2$ so that $U \subseteqq \mathfrak{Z}(G)$. If $A_1 \neq A_2$, then A_1/U and A_2/U are two distinct abelian subgroups of G/U of index m. By Lemma 3.6 (ii), G/U has class $\leqq 2$ and thus G has class $\leqq 3$, a contradiction. Therefore $A_1 = A_2 = A$. Since A_i/J_i is abelian, $A' \subseteqq J_1 \cap J_2 = \langle 1 \rangle$ and A is abelian. This completes the proof.

Let $\gamma^i G$ denote the ith term of the lower central series of G. Thus $\gamma^0 G = G, \gamma^{i+1} G = (\gamma^i G, G)$ and the class of G is the minimal c with $\gamma^c G = \langle 1 \rangle$.

LEMMA 3.8. (i) *Let G be an arbitrary p-group having a normal abelian subgroup A with G/A abelian of order m. Suppose for all subgroups H with $G \supseteqq H > A$ and $[H : A] = p$ we have $H' = G' \neq \langle 1 \rangle$. Then G has* a.c.m.

(ii) *Let G have* a. c. m. *and a normal abelian subgroup A with G/A abelian of order m. If G' is not central, then $K = G' \times_\sigma (G/A)$ has* a.c.m. *Moreover let $x \in G/A$ have order p. Then for all $i > 0$, $\gamma^i G = A^{(1-x)^i}$.*

Proof. We consider (i). Let χ be an irreducible character of G. Since $[G : A] = m$ we have $\deg \chi \leqq m$. We assume $\deg \chi < m$. Since A is normal and abelian, χ is induced from a linear character of some subgroup $L \supseteqq A$. Clearly $L > A$ and we can choose H with $L \supseteqq H > A$ and $[H : A] = p$. Since G/A is abelian, $L \triangle G$. Thus $\ker \chi \supseteqq L' \supseteqq H'$ and since $H' = G'$, $\ker \chi \supseteqq G'$. Thus $\deg \chi = 1$. Since $|G'| \neq 1$, G has a.c.m.

Now let G have a.c.m and a normal abelian subgroup A of index m. Set $K = G' \times_\sigma (G/A)$. Then K has a normal abelian subgroup G' with $K/G' \cong G/A$ abelian of order m. Let x have order p in G/A and set $H = \langle A, y \rangle$ where y is an inverse image of x in G. Then $H \triangle G$ and $(x, G') = (H, G') \triangle G$. Now $\bar{G} = G/(x, G')$ is nonabelian since $(x, G') < G'$ and thus $\bar{G}$ has a.c.m. Clearly $\mathfrak{C}(\bar{G}') \supseteqq H/(x, G')$ so $[\bar{G} : \mathfrak{C}(\bar{G}')] < m$. Now $\bar{G}$ cannot have an abelian subgroup of index $< m$ and hence by Lemma 3.6 (iii), $\bar{G}' \subseteqq \mathfrak{Z}(\bar{G})$. Therefore $(x, G') = (G, G') = (G/A, G')$. Thus we see that K satisfies the hypothesis of (i) above, since $(G, G') \neq \langle 1 \rangle$ by assumption. Thus K has a.c.m.

Let K be as above. We know that for $i \geqq 1$

$$\gamma^i G = (A, G/A, G/A, \cdots, G/A)$$

and

$$\gamma^i K = (G', G/A, G/A, \cdots, G/A)$$

where G/A occurs i times in each of the above. This follows using Lemma 3.3 (iii) for $i = 1$. Since $G' = (A, G/A)$ we have for $i \geqq 2$, $\gamma^i G = \gamma^{i-1} K$.

Let $x \in G/A$ have order p. We show that for $i \geqq 1, \gamma^i G = A^{(1-x)^i}$ by induction on i. If $i = 1$, the result follows from Lemma 3.3 (iii). Let $i \geqq 2$ so that $\gamma^i G = \gamma^{i-1} K$. By induction, since $i - 1 \geqq 1$, we have $\gamma^{i-1} K = (G')^{(1-x)^{i-1}}$. Since $G' = A^{(1-x)}$ the result follows.

EXAMPLE 3.9. Let D be an additive elementary abelian group of order $m = p^e$ and let $A_1, A_2, \cdots, A_p$ be p distinct groups isomorphic to D. Say $\alpha_i : D \rightarrow A_i$ is an isomorphism.

Let F be a field of endomorphisms of D with $|F| = m$. In fact F corresponds to the regular representation of $GF(p^e)$ on its additive group. For $\sigma \in F$ define $\sigma_i : A_i \rightarrow A_{i+1}$ by $\sigma_i(a_i) = \alpha_{i+1}\sigma\alpha_i^{-1}(a_i)$ for $i = 1, 2, \cdots, p-1$ and $\sigma_p : A_p \rightarrow \langle 0 \rangle$. Let $A = A_1 \dotplus A_2 \dotplus \cdots \dotplus A_p$ and define $\bar{\sigma}$ on A by $\bar{\sigma} = \sigma_1 + \sigma_2 + \cdots + \sigma_p$. Clearly $\bar{\sigma}^p = 0$.

Let $\sigma, \tau \in F$. We show that $\bar{\sigma}\bar{\tau} = \bar{\tau}\bar{\sigma}$. Let $a_i \in A_i$. If $i = p-1$ or p then $\bar{\sigma}\bar{\tau}(a_i) = 0 = \bar{\tau}\bar{\sigma}(a_i)$. Now let $i < p - 1$. Then

$$\bar{\sigma}\bar{\tau}(a_i) = \sigma_{i+1}\tau_i(a_i) = \alpha_{i+2}\sigma\alpha_{i+2}^{-1}\alpha_{i+1}\tau\alpha_i^{-1}(a_i) = \alpha_{i+2}\sigma\tau\alpha_i^{-1}(a_i) .$$

Since $\sigma\tau = \tau\sigma$ we have clearly $\bar{\sigma}\bar{\tau} = \bar{\tau}\bar{\sigma}$.

Now for $\sigma \in F$ set $x_\sigma = 1 + \bar{\sigma}$. Since $\bar{\sigma}^p = 0$ and A is elementary abelian we have $x_\sigma^p = 1$. Also for $\sigma, \tau \in F$ we have $x_\sigma x_\tau = x_\tau x_\sigma$.

Let $\sigma_1, \sigma_2, \cdots, \sigma_e$ be a basis of F over $GF(p)$ and set $x_i = x_{\sigma_i}$ for convenience. Let B be the elementary abelian group of automorphisms of A generated by the x_i. Clearly $|B| \leqq p^e$. Set $S = \{x_\sigma\}$. Since $x_\sigma x_\tau = 1 + \bar{\sigma} + \bar{\tau} + \bar{\sigma}\bar{\tau}$ it follows that when restricted to

$$\bar{A}_i = (A_i + A_{i+1} + \cdots + A_p)/(A_{i+2} + \cdots + A_p)$$

S is a group of order p^e. Here $i = 1, \cdots, p-1$ and if $i = p - 1$ then the denominator of the quotient is the group $\langle 0 \rangle$. Clearly B and S restricted to this quotient are isomorphic and hence $|B| = p^e$.

Now let $x \in B$ with $x \neq 1$. Then there exists $\sigma \in F$, $\sigma \neq 0$ such that x and x_σ act the same way on $\bar{A}_i$ above for all i. Since σ is an onto map we see that

$$\begin{aligned}[(1 - x)(A_i + A_{i+1} + \cdots + A_p)](A_{i+2} + \cdots + A_p) \\ = (A_{i+1} + \cdots + A_p)\end{aligned}$$

for $i = 1, 2, \cdots, p-1$. This clearly yields

$$(1 - x)A = (A_2 + A_3 + \cdots + A_p) .$$

Thus by Lemma 3.8(i), $G = A \times_\rho B$ has a.c.m. Moreover as is easily seen, G has class p.

The following result exhibits another difference between the $e = 1$ and $e > 1$ cases.

THEOREM 3.10. *Let G have* a.c.p^e *with $e > 1$. Then G has class at most p and G' is elementary abelian.*

Proof. By induction on $|G|$. If $\mathfrak{Z}(G)$ is not cyclic choose J_1, J_2 subgroups of $\mathfrak{Z}(G)$ with $J_1 \cap J_2 = \langle 1 \rangle$. By induction G/J_i has class $\leqq p$ so $\gamma^p G \subseteqq J_1 \cap J_2 = \langle 1 \rangle$. Also $(G/J_i)'$ has period p so clearly G' is elementary abelian.

Now assume $\mathfrak{Z}(G)$ is cyclic. By Lemma 3.4, G has a normal abelian subgroup A with G/A elementary abelian of order p^e. Let $H = G/A$ so that $I(H)$, the group ring of H over the rational integers I, acts on A. If S is a subset of H we let $\widetilde{S}$ denote the sum of the elements of S in $I(H)$. Let K be a nonidentity subgroup of H. Choose $x \in K$ with $x \neq 1$. By Lemma 3.8 (ii), $G' = A^{(1-x)}$. Hence

$$(G')^{\widetilde{K}} = A^{(1-x)\widetilde{K}} = \langle 1 \rangle$$

since $(1 - x)\widetilde{K} = 0$ in $I(H)$. Thus $\widetilde{K}$ annihilates G'.

Since $e \geqq 2$ we can choose K to be a subgroup of H of order p^2. Let $K_0, K_1, \cdots, K_p$ be the subgroups of K of order p. Note that K is elementary abelian. Now in $I(H)$

$$p = (\textstyle\sum_0^p \widetilde{K}_i) - \widetilde{K}$$

and hence p annihilates G'. Thus G' has period p.

Now let J be a subgroup of H of order p with $J = \langle x \rangle$. Then as is well known

$$\widetilde{J} = 1 + x + \cdots + x^{p-1} \equiv (1 - x)^{p-1} \bmod pI(H) .$$

By Lemma 3.8 (ii),

$$\gamma^p G = A^{(1-x)^p} = A^{(1-x)(1-x)^{p-1}} = (G')^{(1-x)^{p-1}} .$$

Since G' has period p we can take $(1 - x)^{p-1}$ modulo $pI(H)$ in the above. Therefore $\gamma^p G = (G')^{\widetilde{J}} = \langle 1 \rangle$ and G has class $\leqq p$. This completes the proof.

EXAMPLE 3.11. If $e = 1$, the above result is false. For example, let $A = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \cdots \times \langle \alpha_p \rangle$ where each α_i has order p^a. Let $J = \langle x \rangle$ be cyclic of order p and let J act on A by $\alpha_i^x = \alpha_{i+1}$ for $i = 1, 2, \cdots, p - 1$ and $\alpha_p^x = \alpha_1$. If $G = A \times_\sigma J$, then G has a.c.p.

Now $\alpha_2\alpha_1^{-1} \in G'$ and hence if $a > 1$ we see that G' is not elementary abelian. Moreover as we see below G has class $a(p-1)+1$. First in $I(J)$

$$(1-x)^{p-1} - \tilde{J} \equiv 0 \qquad \bmod pI(J)$$

and hence

$$[(1-x)^{p-1} - \tilde{J}]^a \equiv 0 \qquad \bmod p^a I(J)\,.$$

Since $(1-x)\tilde{J} = \tilde{J}(1-x) = 0$ and $(\tilde{J})^2 = p\tilde{J}$, the above yields

$$(1-x)^{a(p-1)} \equiv (-p)^{a-1}\tilde{J} \qquad \bmod p^a I(J)\,.$$

Now A has period p^a and hence $(1-x)^{a(p-1)}$ and $(-p)^{a-1}\tilde{J}$ act the same way on A. Since $\tilde{J}(1-x) = 0$ we see from the nature of the action of x on A that

$$\gamma^{a(p-1)}G = A^{(1-x)^{a(p-1)}} = A^{(-p)^{a-1}\tilde{J}} \neq \langle 1 \rangle$$

and

$$\gamma^{a(p-1)+1}G = A^{(-p)^{a-1}\tilde{J}(1-x)} = \langle 1 \rangle\,.$$

Hence G has class $a(p-1)+1$ and this can be arbitrarily large.

4. Special class 3 groups.

Let G be a p-group with a.c.p^e. We say that G is special if it does not have a normal abelian subgroup of index p^e. By Theorem 3.7 if G is special, then G has class 2 or 3. As is expected the structure of the special class 3 groups is quite restrictive. We study these latter groups in this section.

THEOREM 4.1. *Let G be a special class 3 group with* a.c.m. *Then we have the following.*

(i) $[G' : \gamma^2 G] = m$ *and* $\gamma^2 G = G' \cap \mathfrak{Z}(G)$.

(ii) $[G : \mathfrak{C}(G')] = m^2$ *and* $\mathfrak{C}(G')$ *is a normal self-centralizing subgroup.*

(iii) $[G : \mathfrak{Z}(G)] = m^3$.

(iv) *If* $H = G/\mathfrak{Z}(G)$, *then* $H' = \mathfrak{Z}(H)$ *is elementary abelian of order* m *and* H *has two normal abelian subgroups of index* m *whose intersection is equal to* H'.

We start with a lemma.

LEMMA 4.2. *Let G have* a.c.m *and class* 3.

(i) *If* $\gamma^2 G$ *is cyclic or if* $[G' : \gamma^2 G] > m$, *then* G *has an abelian subgroup* A *of index* m.

(ii) $[G' : G' \cap \mathfrak{Z}(G)] \geqq m$ *and* $[G : \mathfrak{Z}(G)] \geqq m^3$.

Proof. By induction on $|G|$. Suppose that $\gamma^2 G$ is cyclic. Then there exists an irreducible character χ of G with $\gamma^2 G \cap \ker \chi = \langle 1 \rangle$. By Lemma 3.4, $G/\ker \chi$ has an abelian subgroup $A/\ker \chi$ of index m with $A \supseteqq G'$. Then $(A, G') \subseteqq \gamma^2 G \cap \ker \chi = \langle 1 \rangle$ so $A \subseteqq \mathfrak{C}(G')$. Since G has class larger than 2, $\mathfrak{C}(G')$ is abelian by Lemma 3.6 (iii) and hence A is abelian.

Now suppose $[G' : \gamma^2 G] > m$. If $\gamma^2 G$ is cyclic, then the result follows by the above. Thus we can assume that $\gamma^2 G$ contains distinct subgroups J_1 and J_2 of order p. Since $\gamma^2 G \subseteqq \mathfrak{Z}(G)$, J_1 and J_2 are normal in G. By induction G/J_i has an abelian subgroup A_i/J_i of index m. Set $U = J_1 J_2 \subseteqq \gamma^2 G$. If $A_1 \neq A_2$, then A_1/U and A_2/U are two distinct abelian subgroups of G/U of index m. Hence $|(G/U)'| \leqq m$ by Lemma 3.6 (ii). Since $U \subseteqq \gamma^2 G$ this yields $[G' : \gamma^2 G] \leqq m$, a contradiction. Thus $A_1 = A_2 = A$ and hence $A' \subseteqq J_1 \cap J_2 = \langle 1 \rangle$. Therefore A is abelian and (i) follows.

We consider (ii). The result is obvious if $m = p$ and hence we assume $m = p^e$ with $e > 1$. By Theorem 3.10 G' is elementary abelian. If $G' \cap \mathfrak{Z}(G)$ is not cyclic, there exists subgroup J of $G' \cap \mathfrak{Z}(G)$ with $J \not\supseteqq \gamma^2 G$. Hence $\bar{G} = G/J$ has class 3. By induction $[\bar{G}' : \bar{G}' \cap \mathfrak{Z}(\bar{G})] \geqq m$. Now $\bar{G}' = G'/J$ and $\bar{G}' \cap \mathfrak{Z}(\bar{G}) \supseteqq (G' \cap \mathfrak{Z}(G))/J$. Thus the result follows in this case. Now let $G' \cap \mathfrak{Z}(G)$ be cyclic. Since G' is elementary abelian $|G' \cap \mathfrak{Z}(G)| = p$. Now G has class > 2 and thus by Lemma 3.3 (ii), $|G'| \geqq pm$. Hence $[G' : G' \cap \mathfrak{Z}(G)] \geqq m$.

Let $W/\gamma^2 G$ be the center of $G/\gamma^2 G$. Since $G/\gamma^2 G$ has a.c.m we see that $[G : W] \geqq m^2$. Clearly $\mathfrak{Z}(G) \subseteqq W$ and $G' \subseteqq W$. Hence

$$[W : \mathfrak{Z}(G)] \geqq [W \cap G' : \mathfrak{Z}(G) \cap G'] = [G' : \mathfrak{Z}(G) \cap G'] \geqq m .$$

Therefore $[G : \mathfrak{Z}(G)] = [G : W][W : \mathfrak{Z}(G)] \geqq m^3$ and the lemma is proved.

Proof of Theorem 4.1. We assume throughout that G is a special class 3 group with a.c.m. Since $\gamma^2 G \subseteqq G' \cap \mathfrak{Z}(G)$ we have $[G' : \gamma^2 G] \geqq m$ by Lemma 4.2 (ii). Moreover since G is special $[G' : \gamma^2 G] \leqq m$ by Lemma 4.2 (i). Hence $[G' : \gamma^2 G] = m$ and (i) follows.

Let $K_1, K_2, \cdots, K_s$ be all the proper subgroups of $\gamma^2 G$ with $\gamma^2 G/K_i$ cyclic. Clearly $\bigcap K_i = \langle 1 \rangle$. By the preceeding lemma, G/K_i has a normal abelian group B_i/K_i of index m. By Lemma 3.3 (iii) $B_i \supseteqq G'$. Since G/K_i has class 3, Lemma 3.6 (iii) yields $B_i/K_i = \mathfrak{C}(G'/K_i)$. Thus for all i, $B_i \supseteqq A = \mathfrak{C}(G')$. Set $B = \bigcap B_i$ so that $B \supseteqq A$. Since $(B, G') \subseteqq K_i$ we have $(B, G') \subseteqq \bigcap K_i = \langle 1 \rangle$. Thus $B = A$.

Choose $L \triangle G$ with G/L a Case P quotient. Let Z/L be the center of G/L so that $[G : Z] = m^2$. Clearly $L \supseteqq \gamma^2 G$. Since $B_i Z/L$ is an abelian subgroup of G/L of index $\leqq m$ we must have $B_i \supseteqq Z$ by Lemma 3.2 (iii). Thus $B \supseteqq Z$ and $[G : A] \leqq m^2$. Now if $B = B_1$, then clearly B is an abelian subgroup of G of index m and this does not occur.

Thus say $B_1 \neq B_2$. Then G/γ^2G has two distinct abelian subgroups B_1/γ^2G and B_2/γ^2G of index m. Since $|G'/\gamma^2G| = m$ we see that $[G : B_1 \cap B_2] = m^2$ by Lemma 3.6 (ii). Hence $[G : A] = m^2$. This proves (ii) and the part of (iv) concerning the existence of two abelian subgroups of H of index m.

We prove (iii) by induction on $|G|$. Say $|\gamma^2G| = p^r$. By the preceeding lemma γ^2G is not cyclic and hence $r \geqq 2$. Let J be a subgroup of γ^2G of order p. Suppose that G/J has an abelian subgroup B/J of index m. Then B is nonabelian so $B' = J$ and B has class 2. Clearly $B \supseteqq \mathfrak{C}(G')$ and $\mathfrak{C}(G')$ is a maximal normal abelian subgroup of B. Since $[B : \mathfrak{C}(G')] = m$, it follows that B has a.c.m and $[B : \mathfrak{Z}(B)] = m^2$ by Lemma 2.3 of [2]. Let $x \in B$ with $x \neq 1$. Then there exists an irreducible character χ of B with $x, y \notin \ker \chi$ where $J = \langle y \rangle$. Hence χ is nonlinear and $\deg \chi = m$. This says that B is character regular. Since $e(B) = e(G) = e$ where $m = p^e$, it follows by Proposition 1.3 (i) that $\mathfrak{Z}(B) \subseteqq \mathfrak{Z}(G)$. Since clearly $\mathfrak{Z}(G) \subseteqq \mathfrak{C}(G') \subseteqq B$ we have $\mathfrak{Z}(G) = \mathfrak{Z}(B)$ and thus $[G : \mathfrak{Z}(G)] = [G : B][B : \mathfrak{Z}(G)] = m^3$. Thus the result follows in this case. Note that if $r = 2$ the $\gamma^2(G/J)$ is cyclic so the result follows here.

We assume that $r \geqq 3$ and that for all subgroups J of γ^2G of order p the quotient G/J is a special class 3 group. Since γ^2G is not cyclic, let J_1 and J_2 be two such subgroups of order p and set $U = J_1J_2$. Thus $|U| = p^2 < p^r = |\gamma^2G|$ and $U < \gamma^2G$. By induction G/J_i has center Z_i/J_i of index m^3 If $Z_1 \neq Z_2$ then we see that $(Z_1Z_2)/U$ is central in G/U and has index $< m^3$. Since $U < \gamma^2G$, G/U has class 3 and a.c.m and this violates Lemma 4.2 (ii). Thus $Z_1 = Z_2 = Z$. Since $(Z_i, G) \subseteqq J_i$ it follows that $(Z, G) \subseteqq J_1 \cap J_2 = \langle 1 \rangle$ and hence $Z = \mathfrak{Z}(G)$. This yields (iii).

Finally we know that $|H| = m^3$, $[H : \mathfrak{Z}(H)] \geqq m^2$, $\mathfrak{Z}(H) \supseteqq H'$ and $|H'| \geqq m$. The latter follows since $[G' : G' \cap \mathfrak{Z}(G)] \geqq m$. Hence we must have equality throughout. Now H has a.c.m with $m = p^e$. If $e > 1$, then H' is elementary abelian by Theorem 3.10. If $e = 1$, then $|H'| = p$ and the result is clear here. Thus the theorem is proved.

We used simple facts about $GF(p^e)$ to obtain Example 3.9. In order to construct special class 3 groups we will need the following interesting fact about these fields. The authors would like to thank Walter Feit for his help with the proof of this result.

PROPOSITION 4.3. *Let E be a finite field of characteristic $p > 2$ and let F be a subfield. Then there exists a basis of E over F with respect to which every matrix of the regular representation of E over F is symmetric.*

Proof. Let $w = \{w_1, w_2, \cdots, w_n\}$ be a basis of E over F and let R_w be the matrix form of the regular representation with respect to this basis. Let $\theta \in E$ be a primitive element so that $E = F(\theta)$. Then the characteristic polynomial of $R_w(\theta)$ is irreducible over F. Note that all matrices below are over F.

By Theorem 1 of [8] there exists a matrix S with $S^{-1}R_w(\theta)S = R_w(\theta)'$. Here $'$ denotes the transpose operation. As is well known the norm map from E to F is onto and hence there exists $\alpha \in E$ with $\det S = N_{E/F}(\alpha) = \det R_w(\alpha)$. If $T = R_w(\alpha^{-1})S$, then $T^{-1}R_w(\theta)T = R_w(\theta)'$ since $R_w(\alpha)$ and $R_w(\theta)$ commute. Moreover $\det T = 1$. By Theorem 2 of [8], T is symmetric.

Now T is symmetric and $\det T = 1$, a square in F. Since F is a finite field of characteristic $p > 2$, there exists a matrix U with $T = UU'$. Let $A = U^{-1}R_w(\theta)U$. Then

$$\begin{aligned} A' &= U'R_w(\theta)'(U')^{-1} = U'T^{-1}R_w(\theta)T(U')^{-1} \\ &= U^{-1}R_w(\theta)U = A . \end{aligned}$$

Hence if we let U be a change of basis matrix, $U: w \longrightarrow v$, then $A = R_v(\theta)$ is symmetric. Since $E = F(\theta)$, the result follows.

THEOREM 4.4. *Special class 3 groups with* a.c.p^e *exist for all* $p > 2$ *and* e. *No such groups exist for* $p = 2$.

Proof. Let $p = 2$. If $e > 1$, then by Theorem 3.10 groups G with a.c.p^e have class $\leqq p = 2$. Hence no special class 3 groups exist. If $e = 1$ and G is a special class 3 group, then $[G : \mathfrak{Z}(G)] = 8$ by Theorem 4.1. Therefore $H = G/\mathfrak{Z}(G)$ is nonabelian of order 8. Such groups all have cyclic subgroups of order 4. Thus if $A < G$ with $A/\mathfrak{Z}(G)$ cyclic of order 4, then A is an abelian subgroup of G of index 2 and hence G is not special, a contradiction.

Now let $p > 2$ and let e be arbitrary. By the previous proposition there exists a basis $w_1, \cdots, w_e$ of $GF(p^e)$ over $GF(p)$ such that for all $\beta \in GF(p^e)$, $R_w(\beta)$ is symmetric. Let $\sigma_i = [\alpha_{rs}^{(i)}]$ be the matrix $\sigma_i = R_w(w_i)$. These e matrices of size $e \times e$ over $GF(p)$ have the following properties.

(1) $\alpha_{rs}^{(i)} = \alpha_{sr}^{(i)}$.

(2) $\alpha_{rj}^{(i)} = \alpha_{ri}^{(j)}$ and $\alpha_{is}^{(j)} = \alpha_{js}^{(i)}$.

(3) If $\sum f_i\sigma_i$ is singular for $f_i \in GF(p)$, then we must have $f_1 = f_2 = \cdots = f_e = 0$.

Condition (1) follows since $R_w(w_i)$ is symmetric and (3) follows since $w_1, w_2, \cdots, w_e$ are a basis of field $GF(p^e)$ over $GF(p)$. Finally $\sigma_i(w_j) = R_w(w_i)\cdot w_j = w_iw_j$ and hence $\sigma_i(w_j) = \sigma_j(w_i)$. This yields $\alpha_{rj}^{(i)} = \alpha_{ri}^{(j)}$. The remaining equality in (2) follows from this and symmetry.

Let

$$A = gp\langle x_1, x_2, \cdots, x_e, y_1, y_2, \cdots, y_e, u, v \mid$$
$$x_i^p = y_i^p = u^p = v^p = 1 ,$$
$$u \text{ and } v \text{ are central}$$
$$(x_i, x_j) = (y_i, y_j) = 1$$
$$(y_i, x_j) = u^{\delta_{ij}}\rangle$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Clearly $|A| = m^2p^2$ where $m = p^e$ and $A' = \langle u\rangle$. Let τ_i act on A by

$$u^{\tau_i} = u \qquad v^{\tau_i} = v$$
$$y_j^{\tau_i} = y_j v^{\delta_{ij}}$$
$$x_j^{\tau_i} = x_j\Big(\prod_r y_r^{\alpha_{jr}^{(i)}}\Big)w_{ij}$$

where

$$w_{ij} = (u^{-1}v)^{(1/2)\alpha_{ii}^{(j)}} .$$

Here division in the exponent is performed modulo p.

We show first that τ_i defines an automorphism of A. To do this it suffices to show the following.

$$(x_j^{\tau_i})^p = (y_j^{\tau_i})^p = (u^{\tau_i})^p = (v^{\tau_i})^p = 1$$
$$u^{\tau_i} \text{ and } v^{\tau_i} \text{ are central in } A$$
$$(y_j^{\tau_i}, y_k^{\tau_i}) = 1$$
$$(y_j^{\tau_i}, x_k^{\tau_i}) = (u^{\tau_i})^{\delta_{jk}} = u^{\delta_{jk}}$$
$$(x_j^{\tau_i}, x_k^{\tau_i}) = 1 .$$

Now A has class 2 and $p > 2$ so A is regular. Since it is generated by elements of order p, it has period p. Hence the first equation holds. Since A has class 2, the next three equations are obvious. We consider the last one now. We have

$$x_j^{\tau_i} = x_j\Big(\prod\nolimits_r y_r^{\alpha_{jr}^{(i)}}\Big)w_{ij} \qquad x_k^{\tau_i} = x_k\Big(\prod\nolimits_r y_r^{\alpha_{kr}^{(i)}}\Big)w_{ik}$$

so

$$(x_j^{\tau_i}, x_k^{\tau_i}) = (x_j, y_j^{\alpha_{kj}^{(i)}})(y_{jk}^{\alpha_{jk}^{(i)}}, x_k) = u^{\alpha_{jk}^{(i)} - \alpha_{kj}^{(i)}} = 1$$

by (1). Thus τ_i is an automorphism of A.

We show now that as an automorphism τ_i has order p. Clearly $\tau_i \neq 1$. Now τ_i fixes u and v and $y_j^{\tau_i^n} = y_j v^{n\delta_{ij}}$. Thus τ_i^p fixes y_j. Finally

$$x_j^{\tau_i^n} = x_j(\prod y_r^{\alpha_{jr}^{(i)}})^n v^{\alpha_{ji}^{(i)} n(n-1)/2} w_{ij}^n .$$

So for $p > 2$, τ_i^p fixes x_j and hence τ_i has order p.

We know that $\tau_i, \tau_j \in \operatorname{Aut} A$. We show that $\tau_i\tau_j = \tau_j\tau_i$. Clearly $u = u^{\tau_i\tau_j} = u^{\tau_j\tau_i}$, $v = v^{\tau_i\tau_j} = v^{\tau_j\tau_i}$ and $y_k^{\tau_i\tau_j} = y_k v^{\delta_{ik}+\delta_{jk}} = y_k^{\tau_j\tau_i}$. Finally

$$\begin{aligned} x_k^{\tau_i\tau_j} &= [x_k(\prod y_r^{\alpha_{kr}^{(i)}})w_{ik}]^{\tau_j} \\ &= x_k (\prod y_r^{\alpha_{kr}^{(j)}})w_{jk}(\prod y_r^{\alpha_{kr}^{(i)}})w_{ik}v^{\alpha_{kj}^{(i)}} \end{aligned}$$

and

$$\begin{aligned} x_k^{\tau_j\tau_i} &= [x_k(\prod y_r^{\alpha_{kr}^{(j)}})w_{jk}]^{\tau_i} \\ &= x_k(\prod y_r^{\alpha_{kr}^{(i)}})w_{ik}(\prod y_r^{\alpha_{kr}^{(j)}})w_{jk}v^{\alpha_{ki}^{(j)}} . \end{aligned}$$

These two expressions are equal since $\alpha_{ki}^{(j)} = \alpha_{kj}^{(i)}$ by (2).

Let $E = gp\langle z_1, z_2, \cdots, z_e \mid z_i^p = (z_i, z_j) = 1\rangle$ and set $G = A \times_\tau E$, the semidirect product of A by E, where $\tau : E \to \operatorname{Aut} A$ is the map induced by $z_i \to \tau_i$. We note some elementary properties of G. Clearly $|G| = m^3p^2$, $G' = \langle y_1, y_2, \cdots, y_e, u, v\rangle$ and $\gamma^2 G = \langle u, v\rangle$ so that G has class 3. We show that $\mathfrak{C}(G') = G'$ so that $[G : \mathfrak{C}(G')] = m^2$. Since G' is abelian we have $\mathfrak{C}(G') \supseteqq G'$. On the other hand if

$$h = g\Big(\prod_i x_i^{a_i})(\prod_i z_i^{b_i}) \in \mathfrak{C}(G')$$

with $g \in G'$, then for all j, $1 = (y_j, h) = u^{a_j}v^{b_j}$ and hence $a_j = b_j = 0$. Thus $\mathfrak{C}(G') = G'$.

Since $[G : A] = m$, we see easily by (3) and Lemma 3.8 (i) that $G/\gamma^2 G$ has a.c.m. Set $B = \langle y_1, y_2, \cdots, y_e, z_1, z_2, \cdots, z_e, u, v\rangle$. We see that $\mathfrak{Z}(A) = \mathfrak{Z}(B) = \langle u, v\rangle$, $A' = \langle u\rangle$ and $B' = \langle v\rangle$. Since $[A : \mathfrak{Z}(A)] = [B : \mathfrak{Z}(B)] = m^2$ we conclude by Lemma 2.3 of [2] that both A and B have a.c.m. Let χ be an irreducible character of G with $\gamma^2 G \not\subseteq \ker \chi$. Then either $u \notin \ker \chi$ or $v \notin \ker \chi$ or both. If say $u \notin \ker \chi$, then $\chi \mid A$ is faithful on A' and hence $\deg \chi \geqq m$. Similarly if $v \notin \ker \chi$. Thus in either case $\deg \chi \geqq m$.

For each integer t set $E_t = \langle z_1x_1^t, z_2x_2^t, \cdots, z_ex_e^t\rangle$ and $J_t = \langle vu^t\rangle$ so that $E_0 = E$. We show that $E_t' \subseteqq J_t$. Now

$$\begin{aligned} (z_ix_i^t, z_jx_j^t) &= x_i^{-t}z_i^{-1}(z_ix_i^t)^{z_jx_j^t} \\ &= x_i^{-t}z_i^{-1}[z_i(x_i(\prod y_r^{\alpha_{ir}^{(j)}})w_{ji})^t]^{x_j^t} \\ &= x_i^{-t}z_i^{-1}[z_ix_i^t(\prod y_r^{\alpha_{ir}^{(j)}})^t u^{\alpha_{ii}^{(j)} t(t-1)/2} w_{ji}^t]^{x_j^t} \\ &= x_i^{-t}z_i^{-1}x_j^{-t}z_ix_j^tx_i^t \cdot \{\quad\} \end{aligned}$$

where

$$\{\quad\} = [(\prod y_r^{\alpha_{ir}^{(j)}})^t u^{\alpha_{ii}^{(j)} t(t-1)/2} w_{ji}^t]^{x_j^t}$$
$$= (\prod y_r^{\alpha_{ir}^{(j)}})^t u^{\alpha_{ij}^{(j)} t^2} u^{\alpha_{ii}^{(j)} t(t-1)/2} w_{ji}^t .$$

Hence

$$\begin{aligned}(z_i x_i^t, z_j x_j^t) &= x_i^{-t}(x_j^{-t})^{z_i} x_j^t x_i^t \cdot \{\quad\} \\ &= x_i^{-t}[x_j(\prod y_r^{\alpha_{jr}^{(i)}}) w_{ij}]^{-t} x_j^t x_i^t \cdot \{\quad\} \\ &= x_i^{-t} w_{ij}^{-t} (\prod y_r^{\alpha_{jr}^{(i)}})^{-t} u^{-\alpha_{jj}^{(i)} t(t-1)/2} x_j^{-t} x_j^t x_i^t \cdot \{\quad\} \\ &= w_{ij}^{-t} (\prod y_r^{\alpha_{jr}^{(i)}})^{-t} u^{-\alpha_{jj}^{(i)} t(t-1)/2} u^{-d_{ji}^{(i)} t^2} \cdot \{\quad\} .\end{aligned}$$

Since $\alpha_{jr}^{(i)} = \alpha_{ir}^{(j)}$ by (2) and since all terms in the last line above commute, it follows that the y_r terms drop out. Thus

$$(z_i x_i^t, z_j x_j^t) = w_{ji}^t w_{ij}^{-t} u^{(\alpha_{ii}^{(j)} - \alpha_{jj}^{(i)}) t(t-1)/2} u^{(\alpha_{ij}^{(j)} - \alpha_{ji}^{(i)}) t^2} .$$

Now $\alpha_{ij}^{(j)} = \alpha_{jj}^{(i)}$ and $\alpha_{ji}^{(i)} = \alpha_{ii}^{(j)}$ by (2) and so

$$(z_i x_i^t, z_j x_j^t) = [w_{ji}^2 w_{ij}^{-2} u^{(t+1)(\alpha_{jj}^{(i)} - \alpha_{ii}^{(j)})}]^{t/2}$$

where $t/2$ is viewed as division in $GF(p)$. Finally using $w_{ij}^2 = (u^{-1}v)^{\alpha_{ii}^{(j)}}$ we obtain

$$(z_i x_i^t, z_j x_j^t) = (vu^t)^{(\alpha_{jj}^{(i)} - \alpha_{ii}^{(j)}) t/2} \in J_t .$$

Thus $E_t' \subseteqq J_t$.

Set $B_t = \langle G', E_t \rangle$. Then $[G : B_t] = m$ and $B_t' = J_t$. The latter follows since G' is abelian, $E_t' \subseteqq J_t$ and $(y_i, z_j x_j^t) = (vu^t)^{\delta_{ij}}$. We show now that G is a special class 3 group with a.c.m. Let χ be an irreducible nonlinear character of G. If $\ker \chi \supseteqq \gamma^2 G$, then χ is a character of $G/\gamma^2 G$ and hence has degree m. Assume $\ker \chi \not\supseteqq \gamma^2 G$. As we showed above $\deg \chi \geqq m$. Let $J = \ker \chi \cap \gamma^2 G$ so that G is a character of G/J. If $J = \langle u \rangle$, then G/J has an abelian subgroup A/J of index m and if $J = \langle vu^t \rangle = J_t$ then G/J has an abelian subgroup B_t/J of index m. Thus $\deg \chi \leqq m$ and hence $\deg \chi = m$. This shows that G has a.c.m. Since G has class 3 and $[G : \mathfrak{C}(G')] = m^2$, we see that G is a special class 3 group. This completes the proof of the theorem.

5. Special class 2 groups. In this section we study special class 2 groups with a.c.m ($m = p^e$). As is to be expected, the structure of these groups is less restrictive than in the class 3 case. Let G have a.c.p^e. We say G is imprimitive if it has a normal subgroup H of index p with a.c.p^{e-1}. Otherwise G is primitive. We first note the following. Let G have a.c.p^e and let $H \triangle G$ with $[G : H] = p$. If H has a.c.p^{e-1}, then certainly $e(H) = e(G) - 1$. Suppose now that $e(H) = e(G) - 1$. If φ is a nonlinear irreducible character of H and if χ is a constituent of φ^*, then $\deg \chi = p^e$ so $\deg \varphi = p^{e-1}$. Hence H has a.c.p^{e-1}. Thus the concepts of imprimitivity as an r.x.e group and as an a.c.p^e group are entirely equivalent. We now relate this idea to a certain characteristic subgroup of G.

PROPOSITION 5.1. Let G have a.c.m and class 2. Set

$$S = S(G) = \langle g \in G \mid (g, G) < G' \rangle .$$

Then we have the following.

(i) If $[G : S(G)] < m$, then G is a special class 2 group.

(ii) G is primitive if and only if $G = S(G)$.

Proof. (i) Suppose that G has a normal abelian subgroup A of index m. If $x \in G - A$, then by Lemma 3.3 (iii), $(x, A) = G'$ and hence $(x, G) = G'$. Thus $S(G) \subseteqq A$ and $[G : S(G)] \geqq m$, a contradiction.

(ii) We show that $G > S(G)$ if and only if G is imprimitive. Suppose first that $G > S(G)$. Choose subgroup H with $G > H \supseteqq S(G)$ and $[G : H] = p$. Let φ be a nonlinear irreducible character of H and let χ be constituent of φ^*. If $x \in G - H$, then $(x, G) = G' \not\subseteqq \ker \chi$ and thus x is not in the center of the representation associated with χ. Since $x \in \mathfrak{Z}_2(G)$, this yields $\chi(x) = 0$. Thus χ vanishes off H and so $\chi = \varphi^*$. This yields $p \deg \varphi = \deg \varphi^* = \deg \chi = p^e$ and thus H has a.c.p^{e-1} and G is imprimitive.

Now let G be imprimitive so that G has a normal subgroup H of index p with a.c.p^{e-1}. We show that $S(G) \subseteqq H$. If not, there exists $x \in G - H$ with $W = (x, G) < G'$. Note that x is central modulo W and we have $G/W = (H/W)\mathfrak{Z}(G/W)$. Since $W < G'$ we see that G/W has a.c.p^e and that H/W is either abelian or has a.c.p^{e-1}. Let χ be a nonlinear irreducible character of G/W. The above implies that $\chi = \varphi^*$ for some irreducible character φ of H/W. Thus

$$H/W = T(\varphi) \supseteqq (H/W)\mathfrak{Z}(G/W) = G/W ,$$

a contradiction. Therefore $G > H \supseteqq S(G)$.

We now consider an example.

EXAMPLE 5.2. Let Z be an elementary abelian group of order p^{s+1} with $s > 0$. Set $k = (p^{s+1} - 1)/(p - 1)$ and suppose that $E_1, E_2, \cdots, E_k$ are k nonabelian groups of order p^3. Let $Z_i = \langle z_i \rangle$ be the center of E_i. We define a homomorphism

$$\tau : = Z_1 \times Z_2 \times \cdots \times Z_k \longrightarrow Z$$

by sending each z_i onto a generator of the k distinct subgroups of Z of order p. Let N be the kernel of τ. Then N is central and hence normal in $E = E_1 \times E_2 \times \cdots \times E_k$. Set $G = E/N$.

It is clear that $Z = \mathfrak{Z}(G) = G'$. Let χ be a nonlinear irreducible character of G so that $\chi \mid Z = (\deg \chi)\lambda$ with $\lambda \neq 1$. By way of the homomorphism $E \to G, \chi$ can be viewed as a character of E and as such $\chi = \theta_1\theta_2 \cdots \theta_k$ where θ_i is a character of E_i and hence has degree 1 or p. Moreover $\deg \theta_i = p$ if and only if $Z_iN/N \not\subseteq \ker \lambda$. Thus there are precisely $(p^{s+1} - p^s)/(p - 1) = p^s$ such θ_i with $\deg \theta_i = p$ and hence $\deg \chi = p^{p^s}$. Thus G has a.c.p^{p^s}.

Now since $|Z| > p$ we have clearly $E_iN/N \subseteq S(G)$ for all i. Thus $S(G) = G$ and by Proposition 5.1, G is primitive and therefore special. Note finally that if $e = p^s$, then G has a.c.p^e and $[G : \mathfrak{Z}(G)] = p^{2k}$ with $k > e$.

The above example shows that special class 2 groups with arbitrarily large commutator subgroups and central quotients do in fact exist. However the above construction required that we let m get arbitrarily large. We will show in Theorem 5.5 that this is typical of the general situation. We first obtain a generalization of Theorem B of [2].

THEOREM 5.3. *Let G be a p-group with $e(G) = e$. Then either G has a normal abelian subgroup of index p^e or G has a subgroup H of index p^{e-1} with $[H : \mathfrak{Z}(H)] \leqq p^{6e(e+3)}$.*

Proof. By Theorom B of [2], there exists subgroups N and A of G with $[G : N] = p^e, A = \mathfrak{Z}(N)$ and $[N : A] \leqq p^{3e(e+2)}$. If $\mathfrak{C}(A) > N$, then we can choose subgroup H with $\mathfrak{C}(A) \supseteq H > N$ and $[H : N] = p$. With this H the result follows. So assume $\mathfrak{C}(A) = N$.

Suppose now that N is not normal in G. Let $N^x \neq N$. If $B = A \cap A^x$, then $\mathfrak{C}(B) \supseteq \langle N, N^x \rangle > N$. Since $[N : B] \leqq p^{6e(e+2)+e}$, the result follows also in this case.

Thus we can assume that $N \triangle G$ and hence that $A \triangle G$. If $N = A$, then G has a normal abelian subgroup of index p^e. Hence we can assume that N is nonabelian. Since N is a p-group and $A = \mathfrak{Z}(N)$ we can choose subgroup J with $J \subseteq N' \cap A$ and $|J| = p$. Set $n = \min\{[A : \mathfrak{C}_A(x)] \mid x \notin N\}$. We compute as in Lemma 4.4 of [2].

Clearly

$$r(A) \leqq |N| + (|G| - |N|)/n \, .$$

Now let λ be a character of A. If λ has p^e conjugates, then since $T(\lambda) \supseteq N$ we have $T(\lambda) = N$. Thus if χ is a constituent of λ^*, then there exists character η of N with $\chi = \eta^*$ and λ a constituent of $\eta \mid A$. Since $\deg \chi \leqq p^e$, we see that $\deg \eta = 1$. Hence $\lambda = \eta \mid A$ and $\ker \lambda \supseteqq N' \cap A \supseteqq J$. Thus we see that

$$s(A) \geqq [A : J]/p^e + (|A| - [A : J])/p^{e-1} .$$

By Lemma 4.3 of [2], $r(A) = [G : A]s(A)$. Thus

$$p^{-e} + (1 - p^{-e})/n \geqq p^{-e-1} + (1 - p^{-1})p^{1-e} .$$

Hence

$$p^{e+1} > p(p^e - 1)/(p - 1)^2 \geqq n .$$

Choose $x \in G - N$ with $[A : \mathfrak{C}_A(x)] = n$ and set $K = \langle N, x\rangle > N$. Then $\mathfrak{Z}(K) = \mathfrak{C}_A(x)$ so

$$[N : \mathfrak{Z}(K)] = [N : A][A : \mathfrak{C}_A(x)] \leqq p^{3e(e+2)}p^{e+1} < p^{6e(e+3)-1} .$$

If H is chosen with $K \supseteqq H > N$ and $[H : N] = p$, then the result follows.

We now return to our study of class 2 p-groups with a.c.m.

LEMMA 5.4. *Let G have* a.c.m *and class* 2. *Then G' and $G/\mathfrak{Z}(G)$ are both elementary abelian.*

Proof. We show that G' is elementary abelian by induction on $|G|$. Of course G' is abelian since $G' \subseteqq \mathfrak{Z}(G)$. If $\mathfrak{Z}(G)$ is not cyclic, let J_1 and J_2 be two distinct subgroups of $\mathfrak{Z}(G)$ order p. By induction $G'J_i/J_i$ has period p and hence so does G'. Now let $\mathfrak{Z}(G)$ be cyclic. By Lemma 3.4, G has a normal abelian subgroup A with G/A elementary abelian of order m. If $x \in G - A$, then $G' = (x, A)$ by Lemma 3.3 (iii). Let $y \in A$. Since $x^p \in A$ and G has class 2, we have $(x, y)^p = (x^p, y) = 1$ and thus G' is elementary abelian.

We show now that $G/\mathfrak{Z}(G)$ is elementary abelian. The quotient is of course abelian. Let $x, y \in G$. Since G has class 2 and G' has period p, we have $(x^p, y) = (x, y)^p = 1$. Thus $x^p \in \mathfrak{Z}(G)$ and $G/\mathfrak{Z}(G)$ has period p. This completes the proof.

We will use the following notation throughout this section. Let W be a subgroup of G'. Set

$$Z_W = \{g \in G \mid (g, G) \subseteqq W\}$$

so that $Z_W/W = \mathfrak{Z}(G/W)$. We let T denote a hyperplane in G' and J denote a line (that is, $[G' : T] = p$ and $|J| = p$). We have $[G : Z_T] = m^2$ by Lemma 2.3 of [2] and $S(G) = \langle Z_T \mid \text{all } T\rangle$.

THEOREM 5.5. *Let G have* a.c.p^e *and class* 2. *Suppose that* $|G'| = p^s$ *and* $[G : \mathfrak{Z}(G)] = p^z$. *Then*

(i) $z \leqq e(s+1)$ *and* $s \leqq \frac{1}{2}z(z-1)$

(ii) *if G is special, then* $z \leqq 18e^3(e+3)^2$ *and* $s < 18e^2(e+3)^2$.

Proof. (i) Let T_1 and T_2 be two hyperplanes in G'. We show first that $[Z_{T_1} : Z_{T_1} \cap Z_{T_2}] \leqq p^e$. Let χ_i be a nonlinear irreducible character of G/T_i for $i = 1, 2$. Since χ_i vanishes off Z_{T_i}, we see that $\chi_1\chi_2$ vanishes off $N = Z_{T_1} \cap Z_{T_2}$. Also $\deg \chi_i = p^e$ so that $\chi_1\chi_2 \mid N = p^{2e}\lambda$ where λ is a linear character of N. Now let θ be an irreducible constituent of $\chi_1\chi_2$ so that $\theta \mid N = (\deg \theta)\lambda$. Then

$$1 \leqq [\chi_1\chi_2, \theta] = (1/[G:N])[\chi_1\chi_2 \mid N, \theta \mid N]_N = p^{2e}(\deg \theta)/[G:N] .$$

Since $\deg \theta \leqq p^e$ we have $[G: N] \leqq p^{3e}$ and hence $[Z_{T_1} : Z_{T_1} \cap Z_{T_2}] \leqq p^e$.

Let $T_1, T_2, \cdots, T_u$ be hyperplanes. We show that $[G: \bigcap_1^u Z_{T_i}] \leqq p^{e(u+1)}$ by induction on u. For $u = 1, 2$ we have result by the above so let $u \geqq 3$. Set $U = \bigcap_1^{u-1} Z_{T_i}$ so that by induction $[G : U] \leqq p^{eu}$. Hence since $U \subseteqq Z_{T_1}$ we have

$$\begin{aligned}[G : U \cap Z_{T_u}] &= [G : U][U : U \cap Z_{T_u}] \leqq [G : U][Z_{T_1} : Z_{T_1} \cap Z_{T_u}] \\ &\leqq p^{eu}p^e = p^{e(u+1)}\end{aligned}$$

and this follows.

Since $|G'| = p^s$, we can find s hyperplanes $T_1, T_2, \cdots, T_s$ with $\bigcap T_i = \langle 1 \rangle$. Clearly $\bigcap_1^s Z_{T_i} = \mathfrak{Z}(G)$. By the above

$$p^z = [G : \mathfrak{Z}(G)] = [G : \bigcap_1^s Z_{T_i}] \leqq p^{e(s+1)}$$

and hence $z \leqq e(s+1)$. Now let $x_1, x_2, \cdots, x_z \in G$ generate the quotient $G/\mathfrak{Z}(G)$. We see easily that the commutators (x_i, x_j) with $i < j$ generate G'. Since G' is abelian and has period p, this yields $s \leqq \frac{1}{2}z(z-1)$ and (i) follows.

(ii) We apply Theorem 5.3. Since G is special we see that G has a subgroup H of index p^{e-1} with $[H : \mathfrak{Z}(H)] \leqq p^{6e(e+3)}$. By Lemma 3.3 (iii), $H' = G'$. If $[H : \mathfrak{Z}(H)] = p^t$ and $|H'| = |G'| = p^s$, then as above we have

$$s \leqq \tfrac{1}{2}t(t-1) < \tfrac{1}{2}t^2 \leqq 18e^2(e+3)^2 .$$

Finally by (i) we obtain

$$z \leqq e(s+1) \leqq 18e^3(e+3)^2$$

and the theorem is proved.

The above result is of course qualitative in nature. The bounds are no where near best possible. If G has an abelian subgroup of index m, the following example shows that $|G'|$ and $[G:\mathfrak{Z}(G)]$ can be arbitrarily large for fixed m.

EXAMPLE 5.6. Let group G have a.c.m and a normal abelian subgroup A of index m. Given integer k, set $B = A_1 + A_2 + \cdots + A_k$, the direct sum of k copies of A. Set $H = B \times_\sigma (G/A)$ where G/A acts on B in the natural way. If $x \in G/A$ with $x \neq 1$, then $(x, A) = G'$. Hence clearly $(x, B) = H'$. By Lemma 3.8 (i) we see that H has a.c.m. Moreover $|H'| = |G'|^k$ and $[B:\mathfrak{Z}(H)] = [A:\mathfrak{Z}(G)]^k$.

If we now take G to be an extra-special Case P group with $[G:\mathfrak{Z}(G)] = m^2$, then G has a.c.m and $|G'| = p$. Also G is nonspecial so the above construction yields nonspecial groups H with $|H'|$ and $[H:\mathfrak{Z}(H)]$ arbitrarily large.

LEMMA 5.7. *Let G and H have class* 2 *with* $|G'| = |H'|$. *Suppose that G has* a.c.m *and H has* a.c.n. *Let K be the product of G and H with G' and H' identified. Then K has class* 2 *and* a.c.mn. *Also with G and H naturally embedded in K we have* $\mathfrak{Z}(K) = \mathfrak{Z}(G)\mathfrak{Z}(H)$ *and* $S(K) \supseteqq S(G)S(H)$.

Proof. By Lemma 5.4, $G' \cong H'$ and so K clearly exists. Let χ be a nonlinear irreducible character of K. By way of the map $E = G \times H \rightarrow K$ we can view χ as a character of E. As such $\chi = \theta\varphi$ where θ is a character of G and φ is one of H. In K, $\ker \chi \not\supseteqq K'$ and thus in E, $\ker \chi \not\supseteqq G'$ and $\ker \chi \not\supseteqq H'$. Hence both θ and φ are nonlinear. Thus $\deg \theta = m$, $\deg \varphi = n$ and $\deg \chi = mn$. Therefore K has a.c.mn. The remaining results are obvious.

The following proposition considers minimal special groups.

PROPOSITION 5.8. *Let G be a primitive group with* a.c.p^e *and class* 2. *Suppose that for all $J \subseteqq G'$ with $|J| = p$ the quotient G/J has an abelian subgroup of index p^e. Then either $|G'| = p^2$ and $p \mid e$ or $|G'| = p^3$. Moreover for all p, e (with $p \mid e$ in the first case) such groups exist.*

Proof. We show first that $|G'| \leqq p^3$. Suppose by way of contradiction that $|G'| \geqq p^4$. Let T_1 and T_2 be two not necessarily distinct hyperplanes in G'. Since $|G'| \geqq p^4$ we have $|T_1 \cap T_2| \geqq p^2$. Let J_1 and J_2 be two distinct subgroups of $T_1 \cap T_2$ of order p. By assumption G/J_i has an abelian subgroup A_i/J_i of index $m = p^e$. This implies that $S(G/J_i) \subseteqq A_i/J_i$ and so $Z_{T_1/J_i} = Z_{T_1}/J_i$ and $Z_{T_2/J_i} = Z_{T_2}/J_i$ are both contained in A_i/J_i. This yields $(Z_{T_1}, Z_{T_2}) \subseteqq J_1 \cap J_2 = \langle 1 \rangle$. Now G is primitive so $G = S(G) = \langle Z_T \rangle$. Since $(Z_{T_1}, Z_{T_2}) = \langle 1 \rangle$ for all T_1 and T_2 we see that G is abelian, a contradiction. Thus $|G'| \leqq p^3$.

By Lemma 3.4, we must have $|G'| = p^2$ or p^3. We consider the case $|G'| = p^2$ now. Let $T_0, T_1, \cdots, T_p$ be the subgroups of G' of order p and set $Z_i = Z_{T_i}$. Since G is primitive, $G = S(G) = Z_0 Z_1 \cdots Z_p$. Consider

$$W = Z_i \cap (Z_0 Z_1 \cdots Z_{i-1} Z_{i+1} \cdots Z_p) .$$

Note that for $i \neq j$, $(Z_i, Z_j) \subseteqq T_i \cap T_j = \langle 1 \rangle$ so Z_i and Z_j commute elementwise. Since $W \subseteqq Z_i$ we see that $\mathfrak{C}(W) \supseteqq Z_0 Z_1 \cdots Z_{i-1} Z_{i+1} \cdots Z_p$ and since $W \subseteqq Z_0 Z_1 \cdots Z_{i-1} Z_{i+1} \cdots Z_p$ we see that $\mathfrak{C}(W) \supseteqq Z_i$. Thus $\mathfrak{C}(W) = G$. Since clearly $W \supseteqq \mathfrak{Z}(G)$ we have $W = \mathfrak{Z}(G)$. This says that

$$G/\mathfrak{Z}(G) = \dot{\coprod}_0^p Z_i/\mathfrak{Z}(G) .$$

Now $[G : Z_i] = p^{2e}$ so that $|Z_i| = |Z_j|$. If $[Z_i : \mathfrak{Z}(G)] = p^f$ then the above direct product yields

$$p^{f(p+1)} = [G : \mathfrak{Z}(G)] = [G : Z_0][Z_0 : \mathfrak{Z}(G)] = p^{2e} p^f$$

and hence $2e = pf$. If $p \neq 2$, then clearly $p \mid e$. If $p = 2$, then $f = e \geqq 1$. Clearly $\mathfrak{Z}(Z_0) = \mathfrak{Z}(G)$ and $Z_0' = T_0$. Hence by Lemma 2.3 of [2], $[Z_0 : \mathfrak{Z}(Z_0)] = p^f$ is a square. Thus $2 \mid f$, $f = e$ and the result follows.

We show now that all such groups exist. Consider first $|G'| = p^2$ and $e/p = f$, Example 5.2 with $s = 1$ yields a group H with a.c.p^p, $S(H) = H$ and $|H'| = p^2$. Let G be the product of f copies of H with their commutator subgroups indentified. By Lemma 5.7 and induction, G has a.c.p^{fp} = a.c.p^e, $S(G) = G$ and $|G'| = p^2$. If $J \subseteqq G'$ with $|J| = p$, then G/J has a cyclic commutator subgroup and hence an abelian subgroup of index p^e. Thus G is the required example.

Now we consider $|G'| = p^3$. Let F be the group of Example 3.7 of [2]. Then $|F'| = p^3$, $|F| = p^6$ and $S(F) = F$. Also it is easy to see that if J is a subgroup of F' of order p, then F/J has an abelian subgroup of index p. Let G be the product of e copies of F with their commutator subgroups identified. Since F has a.c.p, Lemma 5.7 and induction show that G has a.c.p^e, $|G'| = p^3$ and $S(G) = G$. Let J be a subgroup of G' of order p. Then each factor in G/J has an abelian subgroup of index p so G/J has an abelian subgroup of index p^e. This completes the proof.

We now apply the above results to improve the bounds in Theorem 5.5 in case $p > e$.

THEOREM 5.9. *Let G be a special class 2 group with* a.c.p^e. *Suppose that* $|G'| = p^s$ *and* $[G : \mathfrak{Z}(G)] = p^z$. *If* $p > e$, *then* $z \leqq 4e^2$ *and* $s \leqq 4e - 1$.

Proof. Let T be a hyperplane in G'. We show first that $p > e$ implies that Z_T is abelian. This of course a consequence of Theorem 1.5 and Proposition 1.7. However we can give an alternate inductive proof as follows. Suppose first that $|T| \geqq p^2$. Then we can choose distinct subgroups J_1 and J_2 of T of order p. By induction $Z_{T/J_i} = Z_T/J_i$ is abelian and hence $Z_T' \subseteqq J_1 \cap J_2 = \langle 1 \rangle$. Thus we need only consider $|T| = 1, p$. If $|T| = 1$, then certainly $Z_T = \mathfrak{Z}(G)$ is abelian. Now let $|T| = p$ so that $|G'| = p^2$. Note that groups G with a.c.p^e and $|G'| = p^2$ have the property that if J is any subgroup of G' of order p, then G/J is nonspecial. Hence since $p > e$, Proposition 5.8 and induction easily imply that G is nonspecial. Therefore Z_T is contained in an abelian subgroup of G of index p^e and thus Z_T is abelian.

We show now that $s \leqq 4e - 1$. Suppose first that G is imprimitive. Choose $H \triangle G$ with $[G : H] = p$ and such that H has a.c.p^{e-1}. Since G is special, H is special and hence $e > 1$. By Lemma 3.3 (iii), $|H'| = |G'|$. By induction $|H'| \leqq p^{4(e-1)-1}$ and so the result follows here. Now let G be primitive so that $G = S(G) = \langle Z_T \rangle$. We assume that $|G'| \geqq p^{4e}$ and derive a contradiction. Let T_1 and T_2 be two not necessarily distinct hyperplanes and let $x \in Z_{T_1}$ and $y \in Z_{T_2}$. We show that x and y commute. Since each Z_{T_i} is abelian of index p^{2e} we see that $|(x, G)| \leqq p^{2e}$ and $|(y, G)| \leqq p^{2e}$. If $(x, G) \cap (y, G) = \langle 1 \rangle$, then certainly $(x, y) = 1$. Thus we can suppose that $(x, G) \cap (y, G) > \langle 1 \rangle$. This yields $|(x, G)(y, G)| \leqq p^{4e-1} < |G'|$ and thus we can choose hyperplane T with $T \supseteqq (x, G)(y, G)$. Clearly $x, y \in Z_T$ and so x and y commute. Since $G = \langle Z_T \rangle$, the above shows that G is abelian, a contradiction. Hence $|G'| < p^{4e}$. Finally by Theorem 5.5 (i) $z \leqq e(s + 1) \leqq 4e^2$ and the result follows.

6. Additional results. We generalize our r.x.1 results in another direction now.

THEOREM 6.1. *Let G be a group with the property that every nonlinear irreducible character has prime degree. Suppose further that at least two distinct primes occur. Then there exists primes $p \neq q$ such that G has one of the following two normal series.*

(i) $G \underset{p}{>} Q \underset{q^2}{>} \mathfrak{Z}(G)$

with $G/\mathfrak{Z}(G)$ and Q both nonabelian.

(ii) $G \underset{p}{>} Q \underset{q}{>} A = \mathfrak{Z}(G) \times R$

with both G/A and Q nonabelian. Here R is elementary abelian of order r^m for some prime r and Q/A acts irreducibly on it. Also $(r^m - 1)/(r^{m/p} - 1) = q$.

Conversely if group G has either of the above structures and if χ is an irreducible character of G, then $\deg \chi = 1, p$ *or* q *and all three degrees occur.*

We start with two lemmas.

LEMMA 6.2. *Let G be a group with the property that every nonlinear irreducible character has prime degree. Then every normal subgroup and quotient group of G has this property.*

Proof. The result is clear for quotient groups. Let $N \triangle G$ and let φ be a nonlinear irreducible character of N. If χ is a constituent of φ^*, then $\chi \mid N = a \sum_1^t \varphi_i$ and hence $\deg \chi = at \deg \varphi$. Since $\deg \chi$ is a prime, $at = 1$ and $\deg \varphi = \deg \chi$ is a prime.

LEMMA 6.3. *Let G satisfy the hypothesis of Lemma* 6.2. *Then G is solvable.*

Proof. Since this property is inherited by normal subgroups and quotient groups, it suffices to show that G cannot be a nonabelian simple group. Thus suppose G is nonabelian and simple. Let χ be a nonlinear irreducible character of minimal degree p. Since G is simple, χ is faithful. If $p = 2$ and if $x \in G$ is a nonidentity involution, then since $\det \chi = 1$ we see that the eigenvalues of x in this representation are both -1. Hence $\mathfrak{Z}(G) \neq \langle 1 \rangle$, a contradiction. Thus $p > 2$.

Let $\pi = \{\deg \varphi \mid \varphi$ is irreducible and $\deg \varphi > p\}$. Then π is a set of primes and $q \in \pi$ implies that $q > p + 1$. If π is empty, then G has r.x.1 for prime p and is therefore solvable. Hence we have $|\pi| \geqq 1$. Since χ is faithful, a result of Blichfeldt ([7] Satz 196) shows that G has an abelian $\mathfrak{S}_\pi$ subgroup $H \neq \langle 1 \rangle$.

Let $x \in H^\#$. Then $|Cl\, x|$ is prime to the degree of every irreducible character φ of degree different from p. By Burnside's Lemma ([7] Satz 168) since G is simple we have $\varphi(x) = 0$. If ρ is the regular character of G, then we have $0 = \rho(x) = \sum \chi_i(1)\chi_i(x) = 1 + p\alpha$ where α is an algebraic integer. This is impossible and the result follows.

We now proceed to prove the theorem.

Proof of Theorem 6.1. We know that G is solvable. Choose $A \triangle G$ with G/A extra-special. We show first that A is abelian. If not let φ be a nonlinear irreducible character of A and let χ be a constituent of φ^*. Then $\chi \mid A = a \sum_1^t \varphi_i$ and $\deg \chi = at \deg \varphi$. Since $\deg \chi$ is a prime we must have $a = t = 1$. Let θ be a nonlinear irreducible character of G/A viewed as one of G. Then $\chi\theta$ is irreducible (see Lemma 5.5 of [2]) and $\deg \chi\theta = (\deg \chi)(\deg \theta)$ is not a prime, a contradiction.

If G/A is a Case P quotient for prime p, then by Ito's Theorem the degrees of the irreducible characters of G are powers of p which is not the case. Thus G/A is Case Q. Let Q/A be the normal Sylow q-subgroup of G/A. Since G/A has an irreducible character of degree $[G:Q]$ we see that G/Q is cyclic of prime order $p \neq q$.

Note that Q is nonabelian. Otherwise G would have r.x.1 for prime p. Now G/A has trivial center so $\mathfrak{Z}(G) \subseteqq A \subseteqq Q$. We show that $\mathfrak{Z}(Q) = \mathfrak{Z}(G)$. Clearly $\mathfrak{Z}(G) \subseteqq \mathfrak{Z}(Q)$. If $x \in \mathfrak{Z}(Q) - \mathfrak{Z}(G)$, then there exists $y \in G$ with $(x, y) \neq 1$. Now Q is nonabelian and $z = (x, y) \neq 1$ so there exists a nonlinear irreducible character φ of Q with $z \notin \ker \varphi$. As above, there exists an irreducible character χ of G with $\chi \mid G = \varphi$. Since x is in the center of the representation associated with φ and since $\chi \mid Q = \varphi$ we see that $(x, y) \in \ker \chi \cap Q = \ker \varphi$, a contradiction. Hence $\mathfrak{Z}(G) = \mathfrak{Z}(Q)$.

Case 1. $|Q/A| \geqq q^2$. Let $\lambda \in \hat{A}$. Then clearly $[G : T(\lambda)]$ is 1 or a prime. If $|Q/A| \geqq q^2$, then the only subgroup of G/A having prime index is Q/A. Hence $T(\lambda) \supseteqq Q$. This implies that $\mathfrak{C}_G(A) \supseteqq Q$ and hence Q is nilpotent of class 2. Let φ be a nonlinear irreducible character of Q and let $W = W(\varphi)$ denote the subgroup of Q mapping into the center of the representation. Since clearly $\deg \varphi = q$ we have $[Q : W] = q^2$ by Lemma 2.3 of [2] and also $W \supseteqq A$. Now $T(\varphi) = G$ so $W/A \triangle G/A$. Hence $W(\varphi) = A$ and $[Q : A] = q^2$. We saw above that $A \subseteqq \mathfrak{Z}(Q)$. This clearly implies that $A = \mathfrak{Z}(Q) = \mathfrak{Z}(G)$ and G satisfies (i).

We assume now that $|Q/A| = q$. Since Q is nonabelian, $\mathfrak{C}_G(A) \not\supseteqq Q$ and hence $\mathfrak{C}_G(A) = A$. Suppose $A = M \times N$ with $M \triangle G$ and $N \triangle G$ and $N, M \neq \langle 1 \rangle$. We show that either M or N is central in G. Say $N \not\subseteqq \mathfrak{Z}(G) = \mathfrak{Z}(Q)$. Choose $\lambda \in \hat{N}$ so that $T(\lambda) \cap Q = A$. If $\rho \in \hat{M}$, then $T(\lambda\mu) = T(\lambda) \cap T(\mu)$ and $[G : T(\lambda\mu)]$ is a prime. Hence $T(\mu) \supsetneqq T(\lambda)$ and so $\mathfrak{C}_G(M) \supseteqq T(\lambda)$. Since $\mathfrak{C}_G(M) \triangle G$, this implies that $\mathfrak{C}_G(M) = G$ and $M \subseteqq \mathfrak{Z}(G)$. In particular we see that precisely one Sylow subgroup of A is noncentral. Hence $A/\mathfrak{Z}(G)$ in an r-group for some prime r.

Case 2. $q \neq r$. Since Q/A is cyclic of prime order, we can write $A = \mathfrak{Z}(Q) \times R$ where Q/A acts fixed point free on R by Lemma 1.2. Also $\mathfrak{Z}(Q) = \mathfrak{Z}(G)$ and $R \triangle G$ since $Q \triangle G$. Let λ be a nonprincipal linear character of R. Then $T(\lambda) \cap Q = A$ and $[G : T(\lambda)]$ is a prime. Hence $|T(\lambda)/A| = p$. Thus G/A acts half transitively but non fixed point free on $\hat{R}$. By Theorem I of [3], $\hat{R}$ is elementary abelian and G/A acts irreducibly on it. Let $\bar{G} = G/A$, $\bar{Q} = Q/A$ and let $\bar{P} = \mathfrak{S}_p(\bar{G})$. Let W be an nonidentity irreducible $\bar{Q}$-submodule of $\hat{R}$. If $\lambda \in W^{\#}$, then $\bar{G} = \bar{Q}(T(\lambda)/A)$ and thus W is a $\bar{G}$-module. Hence $\bar{Q}$ acts irreducibly on $\hat{R}$.

We view $\hat{R}$ as a vector space over $GF(r)$ of dimension m and we find $\dim \mathfrak{C}_{\hat{R}}(\bar{P})$. This dimension is clearly invariant under field extension so we can extend to the algebraic closure F of $GF(r)$. If $\bar{Q} = \langle x \rangle$, then since $\hat{R}$ is an irreducible $\bar{Q}$-module, all eigenvalues of x are distinct and not equal to 1. Let S be an irreducible $\bar{G}$-submodule of $F \otimes \hat{R}$. By Clifford's theorem, this representation restricted to $\bar{Q}$ breaks up into either p distinct conjugates or all equivalent representations. If the latter occured then since all eigenvalues of x are distinct, $\dim S = 1$ and hence $\bar{Q} = \bar{G}'$ is in the kernel. This contradicts the fact that x has no eigenvalue equal to 1. Thus the former case must always occur. From this we see easily that $p \mid m$ and $\dim \mathfrak{C}_{\hat{R}}(\bar{P}) = m/p$.

Now $\bar{G}$ contains q conjugate subgroups $\bar{P}_1, \cdots, \bar{P}_q$ of order p. We have $\overline{\mathfrak{C}}_{\hat{R}}(P_i) \cap \overline{\mathfrak{C}}_{\hat{R}}(P_j) = \langle 1 \rangle$ for $i \neq j$ and $\hat{R} = \bigcup \mathfrak{C}_{\hat{R}}(\bar{P})$. Since $|\mathfrak{C}_{\hat{R}}(\bar{P}_i)| = r^{m/p}$ we obtain from this disjoint union $(r^m - 1) = q(r^{m/p} - 1)$. Finally since $\hat{R}$ is elementary abelian and $\bar{Q}$ acts irreducibly, we see that the same is true for R. Thus G satisfies (ii).

Case 3. $q = r$. Here Q is clearly nilpotent. Let $R = \mathfrak{S}_r(A)$. As above we have

$$\hat{R} = \mathfrak{C}_{\hat{R}}(\bar{Q}) \cup \bigcup_1^q \mathfrak{C}_{\hat{R}}(\bar{P}_i) .$$

Let $W = \mathfrak{C}_{\hat{R}}(\bar{G})$ and set

$$[\hat{R} : W] = r^m, [\mathfrak{C}_{\hat{R}}(\bar{Q}) : W] = r^a \text{ and } [\mathfrak{C}_{\hat{R}}(\bar{P}_i) : W] = r^b .$$

Note since all the $\bar{P}_i$ are conjugate this is well defined. Now

$$(\hat{R} - W) = (\mathfrak{C}_{\hat{R}}(\bar{Q}) - W) \cup \bigcup_1^q (\mathfrak{C}_{\hat{R}}(\bar{P}_i) - W)$$

is a disjoint union so

$$r^m - 1 = r^a - 1 + q(r^b - 1)$$

and since $r = q$, $r^m - r^a = r^{b+1} - r$. Again since the union is disjoint, we have $a + b \leqq m$ and $2b \leqq m$. Finally $m > a$ since $\mathfrak{C}_G(A) = A$ and hence the above equation yields $m = b + 1, a = 1$. Since $2b \leqq m$ we have $m = 2$ and $b = 1$.

Since $m = 2, a = 1$ we have $[\hat{R} : \mathfrak{C}_{\hat{R}}(\bar{Q})] = q$. Thus $|(Q, R)| = q$

and Q' is cyclic of order q. This shows that $[Q:\mathfrak{Z}(Q)] = q^2$ by Lemma 2.3 of [2]. Thus G satisfies (i). (Note, the difference between Cases 1 and 3 is that in the former G/Q acts irreducibly on $Q/\mathfrak{Z}(G)$ and in the latter it does not.)

We show now that groups with structure (i) or (ii) have characters of degree 1, p and q only. Let G satisfy (i) and let χ be an irreducible character of G. By Ito's Theorem $\deg \chi \mid pq^2$ and also $(\deg \chi)^2 \leqq [G:\mathfrak{Z}(G)] = pq^2$. Since $G/\mathfrak{Z}(G)$ is nonabelian we see easily that $p \leqq q + 1$. This yields $\deg \chi = 1, p$ or q. Since $G/\mathfrak{Z}(G)$ is nonabelian, it has a character of degree p and since Q is nonabelian it has a character of degree q. Thus G does not have a.c.p or a.c.q and hence G has characters of degree 1, p and q.

Now let G satisfy (ii) and let χ be an irreducible character of G. By Ito's theorem, $\deg \chi \mid pq$ and hence $\deg \chi = 1, p, q$ or pq. We show that the latter cannot occur. If $\deg \chi = pq$ and $\chi \mid A = a \sum_1^t \lambda_i$ then $at = pq$ and also $a^2 t \leqq pq$. Thus $a = 1$ and $t = pq$. Let $\lambda = \lambda_1$ and write $\lambda = \eta\varepsilon$ where $\eta \in \widehat{\mathfrak{Z}(G)}$ and $\varepsilon \in \hat{R}$. This implies that $A = T(\lambda) = T(\varepsilon)$. As in our Case 2 computation above, we see that $\bigcup_1^q \mathfrak{C}_{\hat{R}}(\bar{P}_i)$ is a disjoint union and $|\mathfrak{C}_{\hat{R}}(\bar{P}_i)| = r^{m/p}$. Hence $|\bigcup_1^q \mathfrak{C}_R(P_i)| = q(r^{m/p}) + 1 = r^m$. Thus for every $\varepsilon \in \hat{R}$ we have $T(\varepsilon) > A$, a contradiction and $\deg \chi \neq pq$. Now G/A being nonabelian has a character of degree p and Q has a character of degree q. Thus G has characters of degree 1, p and q. This completes the proof of the theorem.

The following are essentially canonical examples of the above.

EXAMPLE 6.4. First let Q be a nonabelian group of order q^3. If $q = 2$, let Q be the quaternion group and if $q > 2$, let Q have period q. As is well known, the group of automorphisms of Q, fixing $\mathfrak{Z}(Q)$, is isomorphic to $SP(2, q) = SL(2, q)$ and hence has order $q(q-1)(q+1)$. If we choose prime p with $p \mid (q-1)(q+1)$ then we can find an appropriate automorphism group P of Q of order p. Clearly $G = Q \times_\sigma P$ satisfies (i).

Now suppose we are given primes p, q, r with $p \neq q$ and $(r^m - 1)/(r^{m/p} - 1) = q$. Let R be the additive group of $GF(r^m)$. Since $q \mid (r^m - 1)$ we see that the multiplicative group of $GF(r^m)$ has an element ζ of order q. Since $p \mid m$ we see that $GF(r^m)$ has a field automorphism σ of order p. Let $\bar{G}$ be the set of automorphisms of R given by $x \to \zeta^i \cdot \sigma^j(x)$. We see easily that $\bar{G}$ is a group of order pq with a normal subgroup of order q. It is nonabelian since the fixed field of σ has size $r^{m/p}$ and clearly $q > r^{m/p}$. Thus $G = R \times_\rho \bar{G}$ satisfies (ii).

An interesting corollary to Theorem 6.1 is the following.

COROLLARY 6.5. *Let G have* r.b.3, *that is every irreducible character of G has degree at most* 3. *Then either G has a normal abelian subgroup of index $\leqq 3$ or $G/\mathfrak{Z}(G)$ is isomorphic to one of the following groups.*

(i) *the elementary abelian group of order* 8

(ii) *the two groups of order* 27 *and period* 3

(iii) *the symmetric and alternating groups on* 4 *letters*

(iv) *the dihedral group of order* 18 *having an elementary abelian Sylow* 3-*subgroup.*

Proof. If G is abelian, the result is clear. If G has a.c.2 or a.c.3, then by Theorem C of [2] either G has a normal abelian subgroup of index $\leqq 3$ or $G/\mathfrak{Z}(G)$ has order 8 or 27. Since we can assume that $G/\mathfrak{Z}(G)$ has no cyclic subgroup of index $\leqq 3$, we obtain (i) and (ii).

We assume now that G has characters of degree 2 and 3 and thus Theorem 6.1 applies. If $p = 3, q = 2$, then case (ii) of that theorem cannot occur since G/A is nonabelian. Since Q is nonabelian in case (i) we see that $Q/\mathfrak{Z}(G)$ is type (2, 2) and hence $G/\mathfrak{Z}(G)$ is isomorphic to the alternating group A_4.

Now let $p = 2, q = 3$. If G is case (i), then as above $Q/\mathfrak{Z}(G)$ is type (3, 3). Let $x, y \in Q$ generate $Q/\mathfrak{Z}(G)$. Then $(x, y) \in \mathfrak{Z}(G)$ and $(x, y) \neq 1$. Since the action of G/Q on $G/\mathfrak{Z}(G)$ is nontrivial and preserves this commutator, we see easily that the action must be dihedral and we obtain (iv). If G is case (ii), then $(r^m - 1)/(r^{m/2} - 1) = 3$ and so $r^{m/2} = 2$. Thus $G/\mathfrak{Z}(G)$ is the extension of a (2, 2) group by the nonabelian group of order 6 acting faithfully. Since this group has no normal 3-complement, Burnside's transfer theorem implies that the normalizer of a Sylow 3-subgroup contains an element of order 2. Hence the extension is split and $G/\mathfrak{Z}(G) \cong S_4$, the symmetric group on 4 letters.

We close with a result which generalizes Theorem 3.5(i).

THEOREM 6.6. *Let p^e be a fixed power of p with $e > 1$ and let G be a group with a nonabelian Sylow p-subgroup. Suppose further that if χ is a nonlinear irreducible character of G, then $p^e \mid \deg \chi$ and $p^{e+1} \nmid \deg \chi$. Then G is the direct product of $\mathfrak{S}_p(G)$ with an abelian p'-group.*

Proof. By induction on $|G|$. By Theorem 2.5 (i), G has a normal p-complement K. Let P be a Sylow p-subgroup of G. If $P \triangle G$, then $G = P \times K$ and clearly K must be abelian. Suppose G has a proper normal subgroup H with $p \nmid [G:H]$. Let φ be a nonlinear irreducible character of H and let χ be a constituent of φ^*. Then $\chi \mid H = a \sum_1^t \varphi_i$ and $\deg \chi = at \deg \varphi$. Since $at \mid [G:H]$ we have $|\deg \varphi|_p = |\deg \chi|_p = p^e$. By induction, $P \triangle H$ and since P is characteristic in H, $P \triangle G$ and the result follows.

We assume now that $K \neq \langle 1 \rangle$ and that G has no proper normal subgroups of p' index and we obtain a contradiction. Let λ be a nonprincipal linear character of K which has a linear extension μ on G. Then $G/\ker \mu$ is abelian and not a p-group and thus some H as above exists. Since this cannot happen, we see that if $\varphi \neq 1$ is any irreducible character of K, then φ^* has no linear constituents. We show now that $T(\varphi) \triangle G$ and that $G/T(\varphi)$ is elementary abelian of order p^e.

Note that $G/K \cong P$ is nonabelian and has a.c.p^e. Let χ be a constituent of φ^*. Then $\chi \mid K = a \sum_1^t \varphi_i$ and so $t \mid p^e$. This yields $[G/K : T(\varphi)/K] \leqq p^e$ and $T(\varphi) \triangle G$ by Lemma 3.3 (iii). Now let ξ be an irreducible character of $T(\varphi)$ with $\xi \mid K = b(\xi) \cdot \varphi$. Clearly $T(\xi) = T(\varphi)$ and hence ξ^* is irreducible. Since ξ^* is a constituent of φ^*, it is nonlinear and thus $tb(\xi) = p^e$. In particular, for all such choices of ξ, $b(\xi)$ is the same. Now by Theorem 6 of [1], there exists ξ_0 with $b(\xi_0) = 1$. Thus $t = p^e$ and for all such ξ, $b(\xi) = 1$. Let β be an irreducible character of $T(\varphi)/K$ viewed as one of $T(\varphi)$. Then $\xi = \xi_0 \beta$ is irreducible and $\xi \mid K = \beta(1) \cdot \varphi$. Therefore $\beta(1) = 1$ and $T(\varphi)/K$ is abelian. As in the latter part of the proof of Lemma 3.4, we see that $G/T(\varphi)$ is elementary abelian of order p^e.

Now let $x \in K$ with $x \neq 1$ and suppose that $[P : \mathfrak{C}_P(x)] \leqq p^e$. We show that $\mathfrak{C}_P(x) \triangle P$ and $P/\mathfrak{C}_P(x)$ is elementary abelian of order p^e. Let τ be a nonprincipal linear character of $\langle x \rangle$. Clearly $\mathfrak{C}_P(x)$ fixes τ and hence $\mathfrak{C}_P(x)$ fixes $\tilde{\tau}$ (induction to K). Since the degree of $\tilde{\tau}$ is prime to p we see that $\mathfrak{C}_P(x)$ fixes some irreducible constituent φ of $\tilde{\tau}$. Clearly $\varphi \neq 1$ so $T(\varphi) \supseteqq K\mathfrak{C}_P(x)$ and $[G : T(\varphi)] = p^e$. Hence $T(\varphi) = K\mathfrak{C}_P(x)$ and $G/T(\varphi) \cong P/\mathfrak{C}_P(x)$ is elementary abelian of order p^e.

Let K have k nonprincipal irreducible characters and hence k nonidentity classes. We have shown that in the action of P on the characters of K we have $1 + k/p^e$ orbits. Hence by Brauer's Lemma, the same is true for the action of P on the classes of K. In particular there must exist a class, say $Cl\,y$, belonging to an orbit of size $\leqq p^e$ with $y \neq 1$. Let S be the subgroup of P fixing this class so that $[P : S] \leqq p^e$. Since $|Cl\,y|$ is prime to p, there exists $x \in Cl\,y$ with $S \subseteqq \mathfrak{C}_P(x)$. Thus $[P : \mathfrak{C}_P(x)] \leqq p^e$ and by the above $P/\mathfrak{C}_P(x)$ is elementary abelian of order p^e. Clearly $S = \mathfrak{C}_P(x)$. Since $S \triangle P$ we see that P/S acts on $\mathfrak{C}_K(S) \neq \langle 1 \rangle$. As above, if $z \in \mathfrak{C}_K(S)$ with $z \neq 1$, then $\mathfrak{C}_P(z) = S$. Hence P/S acts fixed point free on $\mathfrak{C}_K(S)$, a contradiction since P/S is elementary abelian of order $p^e \geqq p^2$. This completes the proof.

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POINT-LIKE 0-DIMENSIONAL DECOMPOSITIONS OF S^3

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This paper is concerned with upper semicontinuous decompositions of the 3-sphere which have the property that the closure of the sum of the nondegenerate elements projects onto a set which is 0-dimensional in the decomposition space. It is shown that such a decomposition is definable by cubes with handles if it is point-like. This fact is then used to obtain some properties of point-like decompositions of the 3-sphere which imply that the decomposition space is a topological 3-sphere. It is also shown that decompositions of the 3-sphere which are definable by cubes with one hole must be point-like if the decomposition space is a 3-sphere.

In this paper we consider upper semicontinuous decompositions of S^3, the Euclidean 3-sphere. In particular, we shall restrict ourselves to those decompositions G of S^3 which have the property that the union of the nondegenerate elements of G projects onto a set whose closure is 0-dimensional in the decomposition space of G. We shall refer to such decompositions as 0-dimensional decompositions of S^3. Numerous examples of such decompositions appear in the literature. (One should note that some of the examples and results to which we refer are in E^3, Euclidean 3-space, but the corresponding examples and results for S^3 will be obvious in each case.)

In § 3, a technique of McMillan [10] is used to show that point-like 0-dimensional decompositions of S^3 are definable by cubes with handles. Armentrout [2] has shown this in the case where the decomposition space is homeomorphic with S^3. The proof of this theorem shows that compact proper subsets of S^3 with point-like components are definable by cubes with handles.

In § 4 we give some properties of point-like 0-dimensional decompositions of S^3 which imply that the decomposition space is homeomorphic with S^3. These properties were suggested by Bing in § 7 of [6].

It is not known whether monotone 0-dimensional decompositions of S^3 which yield S^3 must have point-like elements. Partial results in this direction have been obtained by Armentrout [2], Bean [5], and Martin [9]. Bing, in § 4 of [6], has presented an example of a decomposition of S^3 which yields S^3 even though it is not a point-like decomposition, but this example is not 0-dimensional. In § 5 we show that a 0-dimensional decomposition of S^3 that yields S^3 must have point-like elements if it is definable by cubes with one hole.

2. Definitions and notation. Let G be an upper semicontinuous decomposition of S^3, the 3-sphere. We denote the decomposition space of G by S^3/G, the union of the nondegenerate elements of G by H_G, and the projection map from S^3 onto S^3/G by P.

The decomposition G is said to be *monotone* if each element of G is a continuum. If cl $P(H_G)$ is 0-dimensional in S^3/G, then G is a *0-dimensional decomposition* of S^3. If each element of G has a complement in S^3 which is homeomorphic with E^3, Euclidean 3-space, then G is a *point-like decomposition* of S^3.

The sequence $M_1, M_2, M_3, \cdots$ is a *defining sequence* for G if and only if $M_1, M_2, M_3, \cdots$ is a sequence of compact 3-manifolds with boundary in S^3 such that (1) for each positive integer i, $M_{i+1} \subset \operatorname{Int} M_i$, and (2) g is a nondegenerate element of G if and only if g is a nondegenerate component of $\bigcap_{i=1}^{\infty} M_i$. Here, as in the remainder of the paper, subsets of S^3 which are manifolds will be assumed to be polyhedral subsets of S^3. It is well known that if G is a 0-dimensional decomposition of S^3, a defining sequence exists for G. If a defining sequence $M_1, M_2, M_3, \cdots$ exists for G such that for each positive integer i, each component of M_i is a cube with handles, G is said to be *definable by cubes with handles*. If a defining sequence $M_1, M_2, M_3, \cdots$ exists for G such that for each positive integer i, each component of M_i is a cube with one hole, G is said to be *definable by cubes with one hole*.

3. Some consequences of a result of McMillan. The following lemma is a special case of Lemma 1 of [11]. Its proof follows from the very useful technique used by McMillan to prove Theorem 1 of [10].

LEMMA 1. (*McMillan*). *In S^3, let M' be a compact polyhedral 3-manifold with boundary such that BdM' is connected, and let M be a compact polyhedral 3-manifold with boundary such that $M \subset Int\, M'$, and each loop in M can be shrunk to a point in $Int\, M'$. Then there is a cube with handles C such that $M \subset IntC \subset C \subset Int\, M'$.*

LEMMA 2. *If G is a point-like 0-dimensional decomposition of S^3, then there is a defining sequence $M_1, M_2, M_3, \cdots$ for G such that for each positive integer i, each component of M_i has a connected boundary.*

Proof. Let $M_1', M_2', M_3', \cdots$ be a defining sequence for G, let n be a positive integer, and let K be a component of M_n'. Let g be a component of $\bigcap_{i=1}^{\infty} M_i'$ which lies in K and let U be an open subset of K containing g such that cl $U \cap BdK = \varnothing$. Since g is point-like, there is a 3-cell C such that $g \subset \operatorname{Int} C \subset C \subset U$. There is an integer j such that L, the component of M_j' containing g, lies in $\operatorname{Int} C$. Since

C separates no points of BdK in K, L separates no points of BdK in K.

Using compactness of $\bigcap_{i=1}^{\infty} M_i'$, one obtains a finite collection $L_1, \cdots, L_k$ of mutually exclusive defining elements whose interiors cover $(\bigcap_{i=1}^{\infty} M_i') \cap K$ and so that no L_i separates points of BdK in K. It follows easily that $\bigcup_{i=1}^{k} L_i$ separates no points of BdK in K. By suitable relabeling, we suppose then, that if i is a positive integer and K is a component of M_i', $K \cap M_{i+1}'$ does not separate points of BdK in K. We construct disjoint arcs in K-M_{i+1}' connecting the boundary components of K and "drill-out" these arcs to replace K by a compact 3-manifold with connected boundary. Doing this for each component of each M_i', we obtain a defining sequence $M_1, M_2, M_3, \cdots$ as required by the conclusion of the lemma.

THEOREM 1. *If G is a point-like* 0-*dimensional decomposition of* S^3, *then G is definable by cubes with handles.*

Proof. Using Lemma 2, there is a defining sequence $M_1', M_2', M_3', \cdots$ for G such that each component of each M_i' has a connected boundary. Let n be a positive integer and N a component of M_n'. Since G is point-like, there is no loss of generality in supposing that each loop in $M_{n+1}' \cap N$ can be shrunk to a point in $\mathrm{Int}\, N$. From Lemma 1, there is a cube with handles, C, such that $(M_{n+1}' \cap N) \subset \mathrm{Int}\, C \subset C \subset \mathrm{Int}\, N$. Hence, there is a sequence $M_1, M_2, M_3, \cdots$ of compact 3-manifolds with boundary such that (1) for each positive integer i, $M_{i+1}' \subset \mathrm{Int}\, M_i \subset M_i \subset \mathrm{Int}\, M_i'$, and (2) each component of M_i is a cube with handles. The sequence $M_1, M_2, M_3, \cdots$ is a defining sequence for G and so G is definable by cubes with handles.

The proof of the next theorem follows from the proof of Theorem 1.

THEOREM 2. *If M is a closed subset of S^3 such that each component of M is point-like, then there exists a sequence $M_1, M_2, M_3, \cdots$ of compact* 3-*manifolds with boundary such that* (1) *for each positive integer i, $M_{i+1} \subset \mathrm{Int}\, M_i$,* (2) *each component of M_i is a cube with handles, and* (3) $M = \bigcap_{i=1}^{\infty} M_i$.

The concept of equivalent decompositions of S^3 was introduced in [4] and the following theorem follows immediately from Theorem 1 of this paper and Theorem 8 of [4].

THEOREM 3. *If G is a point-like* 0-*dimensional decomposition of* S^3, *then G is equivalent to a point-like* 0-*dimensional decomposition of S^3 each of whose nondegenerate elements is a* 1-*dimensional continuum.*

In the remaining two sections, we shall utilize some of the above results to investigate certain properties of 0-dimensional decompositions of S^3.

4. Properties of point-like 0-dimensional decompositions of S^3. In this section we give two properties, each of which is both necessary and sufficient to imply S^3/G is homeomorphic to S^3.

A space X will be said to have the *Dehn's Lemma property* if and only if the following condition holds: If D is a disk and f is a mapping of D into X such that on some neighborhood of $f(\mathrm{Bd}D)$, f^{-1} is a function, and U is neighborhood of the set of singular points of $f(D)$, then there is a disk D' in $f(D) \cup U$ such that $\mathrm{Bd}D' = f(\mathrm{Bd}D)$.

A space X will be said to have the *map separation property* if and only if the following condition holds: If D is a disk and $f_1, \cdots, f_n$ are maps of D into X such that (1) for each i, on some neighborhood of $f_i(BdD), f_i^{-1}$ is a function, (2) if $i \neq j$, $f_i(\mathrm{Bd}D) \cap f_j(D) = \varnothing$, and (3) U is a neighborhood of $f_1(D) \cup \cdots \cup f_n(D)$, then there exist maps $f_1', \cdots, f_n'$ of D into X such that (1) for each i, $f_i' \mid \mathrm{Bd}D = f_i \mid \mathrm{Bd}D$, (2) $f_1'(D) \cup \cdots \cup f_n'(D) \subset U$, and (3) if $i \neq j$, $f_i'(D) \cap f_j'(D) = \varnothing$.

It is a well known (and useful) fact that S^3 has the Dehn's Lemma property and the map separation property.

THEOREM 4. *If G is a point-like 0-dimensional decomposition of S^3, then S^3/G is homeomorphic with S^3 if and only if S^3/G has the Dehn's Lemma property.*

Proof. The "if" portion of the theorem is the only part that requires proof. Let U be an open set containing $\mathrm{cl}\, H_G$ and $\varepsilon > 0$. We shall construct a homeomorphism $h_\varepsilon : S^3 \to S^3$ such that if $x \in S^3 - U$, $h_\varepsilon(x) = x$ and if $g \in G$, $\operatorname{diam} h_\varepsilon(g) < \varepsilon$. It will follow from Theorem 3 of [2] that S^3/G is homeomorphic with S^3.

By Theorem 1, G is definable by cubes with handles. Hence, there exist disjoint cubes with handles $C_1, \cdots, C_n$ such that $\mathrm{cl}\, H_G \subset \bigcup_{i=1}^n \mathrm{Int}\, C_i \subset \bigcup_{i=1}^n C_i \subset U$. Let $W_1, \cdots, W_n$ be pairwise disjoint neighborhoods of $C_1, \cdots, C_n$ respectively such that $\bigcup_{i=1}^n W_i \subset U$. Since C_1 is a cube with (possibly 0) handles, there is a homeomorphism h_0 of S^3 onto S^3 such that $h_0(x) = x$ for $x \in S^3 - W_1$ and $h_0(C_1)$ can be written as the union of a finite number of cubes such that (1) each cube has diameter less than $\varepsilon/2$, (2) no three cubes have a point in common, and (3) the intersection of any two cubes is empty or a disk on the boundary of each. The homeomorphism h_0 can be thought of as pulling C_1 towards a 1-dimensional spine of C_1. Let $D_1, D_2, \cdots, D_k$ be the inverse images under h_0 of the disks obtained by intersecting the various cubes making up $h_0(C_1)$. We note that if a continuum in

C_1 intersects at most one D_i, then its image under h_0 has diameter less than ε. For each $i = 1, \cdots, k$, let D_i' be a subdisk of D_i such that $D_i' \subset \mathrm{Int}\, D_i$ and $D_i \cap \mathrm{cl}\, H_G = \mathrm{Int}\, D_i' \cap \mathrm{cl}\, H_G$. Let D be a disk in S^3 such that $\mathrm{Bd}\, D \cap (\bigcup_{i=1}^n C_i) = \varnothing$ and $\bigcup_{i=1}^k D_i = D \cap (\bigcup_{i=1}^n C_i) = D \cap C_1$. Denote the punctured disk $\mathrm{cl}\,(D - \bigcup_{i=1}^k D_i')$ by D'. Now $P_1 = P \mid D$ is a map of D into S^3/G and P_1^{-1} is a homeomorphism on a neighborhood of $P_1(\mathrm{Bd}\, D)$. The singular set of $P_1(D)$ is contained in $P_1(\bigcup_{i=1}^k \mathrm{Int}\, D_i')$. Let V be an open set in S^3/G containing the singular set of $P_1(D)$ and such that $P^{-1}(V) \subset (\mathrm{Int}\, C_1) - D'$. By hypothesis there exists a disk E in $P_1(D) \cup V$ bounded by $P_1(\mathrm{Bd}\, D)$. Let $E_1, \cdots, E_k$ be the subdisks of E bounded by $P_1(\mathrm{Bd}\, D_1'), \cdots, P_1(\mathrm{Bd}\, D_k')$ respectively, and let $U_1, \cdots, U_k$ be open sets whose closures lie in $P(\mathrm{Int}\, C_1)$ such that for each $i = 1, \cdots, k$, $E_i \subset U_i$, and if $i \neq j$, $\mathrm{cl}\, U_i \cap \mathrm{cl}\, U_j = \varnothing$. By the proof of Theorem 2.1 of [12], each $\mathrm{Bd}\, D_i'$ can be shrunk to a point in $P^{-1}(U_i)$. Each map can be "glued" to the annulus $\mathrm{cl}\,(D_i - D_i')$ to obtain a map from D_i into $D_i \cup P^{-1}(U_i)$ with no singularities on $D_i - P^{-1}(\mathrm{cl}\, U_i)$. We now apply Dehn's Lemma in S^3 to these maps to obtain disjoint disks $F_1, \cdots, F_k$ such that (1) for each i, $\mathrm{Bd}\, D_i = \mathrm{Bd}\, F_i$, (2) $\mathrm{Int}\, F_i \subset \mathrm{Int}\, C_1$, and (3) if $g \in G$, g intersects no more than one of the disks $F_1, \cdots, F_k$. Let h_1' be a homeomorphism of S^3 onto itself fixed on S^3-$\mathrm{Int}\, C_1$ such that for each i, $h_1'(F_i) = D_i$. Let $h_1 = h_0 h_1'$. Note that if $g \in G$ and $g \subset C_1$, diam $h_1(g) < \varepsilon$. Let $h_2, \cdots, h_n$ be homeomorphisms such as h_1 for the sets $C_2, \cdots, C_n$. We define h_ε: $S^3 \to S^3$ by $h_\varepsilon(x) = h_1 h_2 \cdots h_n(x)$.

REMARK. If G is the upper semicontinuous decomposition of S^3 whose only nondegenerate element is a polyhedral 2-sphere, then S^3/G has the Dehn's Lemma property but S^3/G is not homeomorphic with S^3.

The essential ideas of the proof of the following theorem are so like those of the proof of Theorem 4 that we shall not include the proof here.

THEOREM 5. *If G is a point-like 0-dimensional decomposition of S^3, then S^3/G is homeomorphic with S^3 if and only if S^3/G has the map separation property.*

5. Decompositions of S^3 which yield S^3. Let S, T be polyhedral solid tori such that $S \subset \mathrm{Int}\, T$ and let J be a polygonal center curve of S. Following a definition of Schubert [13] which was used in [7], we let $N(S, T)$ be the $\min_D\{N(J \cap D)$: where D is a polyhedral meridional disk of T and $N(J \cap D)$ is the number of points in $J \cap D\}$.

THEOREM 6. *If G is definable by cubes with one hole and S^3/G*

is homeomorphic to S^3, then G is point-like.

Proof. Let $M_1, M_2, \cdots,$ be the defining sequence for G and let T_0 be a component of some M_n. By hypothesis, T_0 is a cube with one hole. Let g be a component of $\bigcap_{i=1}^{\infty} M_i$ contained in T_0. We first show that there is a defining stage M_{n+m} such that each loop in the component of M_{n+m} containing g can be shrunk to a point in T_0.

For $i = 1, 2, 3, \cdots$, let T_i be the component of M_{n+i} that contains g. Then each T_i is a cube with one hole, $T_{i+1} \subset \operatorname{Int} T_i$, and $\bigcap_{i=1}^{\infty} T_i = g$. Suppose that there is a positive integer s such that each $T_j, j \geqq s$, is a solid torus. If the center curve of each T_{j+1} cannot be shrunk to a point in T_j, then g has nontrivial Cech cohomology, and it follows from Corollary 2 of [8] that S^3/G is not homeomorphic to S^3, contradicting our hypothesis. Hence there is an m such that the center curve of T_m can be shrunk to a point in T_0 and hence each loop in T_m can be shrunk to a point in T_0.

Suppose then that infinitely many of the T_i are not solid tori. We may suppose for convenience that each T_i is not a solid torus. By [1], each $T_i' = S^3 - \operatorname{Int} T_i$ is a solid torus. We now have three cases.

Case I. Suppose there is an m such that $N(T'_{m-1}, T'_m) = 0$. This implies that there is a meridional disk D of T'_m such that $D \cap T'_{m-1} = \emptyset$. Then there is a cube K in T'_m such that $T'_{m-1} \subset \operatorname{Int} K$. It then follows that each loop in $T_m (= S^3 - \operatorname{Int} T'_m)$ can be shrunk to a point in T_0.

We now show that the remaining two cases cannot occur.

Case II. Suppose that there is a positive integer s such that $N(T'_j, T'_{j+1}) = 1$ for $j \geqq s$. Since $P(\bigcap_{i=1}^{\infty} M_i)$ is 0-dimensional there is a positive integer t and a cube K such that $P(T_{s+t}) \subset \operatorname{Int} K \subset K \subset P$ $(\operatorname{Int} T_s)$. Let D'_{s+t} be a meridional disk of T'_{s+t}. Using Dehn's Lemma we may adjust $P(D'_{s+t})$ in $P(\operatorname{Int} T'_{s+t})$ so that it is polyhedral, and it follows that $P(T'_{s+t})$ is a solid torus with the adjusted $P(D'_{s+t})$ as a meridional disk. Let J be a longitudinal simple closed curve of T'_{s+t} such that $J \subset \operatorname{Bd} T'_{s+t}$ and J intersects $\operatorname{Bd} D'_{s+t}$ at just one point. Let A be an annulus with boundary components A_1 and A_2. By [13], $N(T'_s, T'_{s+t}) = 1$. Hence there is a mapping f of A into T'_{s+t} such that $f \mid A_1$ is a homeomorphism, $f(A_1) = J$, and $f(A_2) \subset T'_s$. Now $P(f(A_2))$ can be shrunk to a point missing K since it is contained in $S^3 - K$; hence $P(f(A_2))$ can be shrunk to a point in $P(T'_{s+t})$. But this implies that the longitudinal simple closed curve $P(J)$ of $P(T'_{s+t})$ can be shrunk to a point in $P(T'_{s+t})$. Hence Case II cannot occur.

Case III. Now assume there is a positive integer s such that $N(T'_j, T'_{j+1}) > 1$ for $j \geqq s$. Since each T'_j is knotted in S^3, we may use an argument similar to that used in [7] to conclude that Case III cannot occur.

These three cases now imply that there is a defining stage M_{n+m} such that each loop in the component of M_{n+m} containing g can be shrunk to a point in T_0. Since $T_0 \cap (\bigcap_{i=1}^{\infty} M_i)$ is compact, there is a defining stage $M_p (p \geqq n+m)$ such that each loop in $T_0 \cap M_p$ can be shrunk to a point in T_0. By Lemma 1 there is a cube with handles C such that $T_0 \cap M_p \subset \text{Int}\, C \subset C \subset \text{Int}\, T_0$. It then follows that G is definable by cubes with handles. By Bean's result [5], G is a point-like decomposition, and the proof of Theorem 6 is complete.

COROLLARY. *Let f be a mapping of S^3 onto S^3 and let $H = \text{cl}$ $(\{x : x \in S^3$ and $f^{-1}(x)$ is nondegenerate$\})$. If H is a 0-dimensional set which is definable by cubes with one hole, then for each $x \in S^3$, $S^3 - f^{-1}(x)$ is homeomorphic to E^3.*

Proof. Let $G = \{f^{-1}(x) : x \in S^3\}$. It is not hard to show that G is an upper semicontinuous decomposition of S^3 and that S^3/G is homeomorphic to S^3. Since H is definable by cubes with one hole, it follows that G is definable by cubes with one hole. By Theorem 6, G is a point-like decomposition of S^3; hence if $x \in S^3$, then $S^3 - f^{-1}(x)$ is homeomorphic to E^3.

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SUBDIRECT DECOMPOSITIONS OF LATTICES OF WIDTH TWO

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The class of nontrivial distributive lattices is the class of subdirect products of two-element chains. Lattices of width one are distributive and hence are subdirect products of two element chains. Below it is shown that lattices of width two are subdirect products of two element chains and nonmodular lattices of order five (N_5). (width = greatest number of pairwise incomparable elements.)

The statement follows from several lemmas. Throughout we shall assume that a, b are arbitrary noncomparable elements of a lattice L of width two.

LEMMA 1. $x \cdot (a + b) + y \cdot (a + b) = (x + y) \cdot (a + b)$ *and*

$$(x + a \cdot b) \cdot (y + a \cdot b) = x \cdot y + a \cdot b$$

for any $x, y \in L$.

Proof. In any lattice

$$x \cdot (a + b) + y \cdot (a + b) \leqq (x + y) \cdot (a + b) \,. \tag{1}$$

Trivially, if x and y are related, the identity holds. Thus, assume that x and y are unrelated. There are three possibilities:

(i) Suppose $x \leqq a$ and $y \leqq b$. Then

$$x \cdot (a + b) + y \cdot (a + b) = x + y = (x + y) \cdot (a + b) \,.$$

(ii) In case $a \leqq x$ and $b \leqq y, a + b \leqq x + y$. If $a + b \leqq x$ or y, it is easy to verify that the identity holds. If $a + b \nleqq x$ or y, then x or $y \leqq a + b$. Suppose $x \leqq a + b$. Then

$$(x + y) \cdot (a + b) = a + b \leqq x + y \cdot (a + b) = x \cdot (a + b) + y \cdot (a + b) \,.$$

This relation and (1) yield the equality.

(iii) Now suppose $a \leqq x$ and $y \leqq b$. $b \leqq x$ implies that x and y are comparable while $x \leqq b$ implies that a and b are comparable. Thus, x and b are unrelated. Since L is of width two, $a + y$ is related to either x or b, $a + y \leqq x$ and $a + y \leqq b$ imply that $y \leqq x$ and $a \leqq b$ respectively. Thus, either $x \leqq a + y$ or $b \leqq a + y$. In case $x \leqq a + y, x \leqq a + y \leqq a + b$ and $y \leqq b \leqq a + b$. Hence

$$x \cdot (a + b) + y \cdot (a + b) = x + y = (x + y)(a + b) \,.$$

In case $b \leqq a + y, y \leqq b \leqq a + y \leqq a + b$. Thus,

$$(x + y) \cdot (a + b) \leqq a + b \leqq a + y \leqq x \cdot (a + b) + y$$
$$= x \cdot (a + b) + y \cdot (a + b)$$

and the identity holds in all cases. A dual argument yields the other identity.

By Lemma 1, if s and t are unrelated elements of a lattice of width two, the mappings $x \rightarrow x \cdot (s + t)$ and $x \rightarrow x + s \cdot t$ determine congruence relations θ_{s+t} and $\psi_{s \cdot t}$.

LEMMA 2. $\theta_{a+b} \cap \psi_{a \cdot b} = 0$.

Proof. If $x \equiv y(\theta_{a+b} \cap \psi_{a \cdot b})$, $x \cdot (a + b) = y \cdot (a + b)$ and $x + a \cdot b = y + a \cdot b$. x and y are each related to either a or b. Thus $x \leqq a + b$ or $a \cdot b \leqq x$. Similarly, $y \leqq a + b$ or $a \cdot b \leqq y$. If

$$x, y \leqq a + b, x = x \cdot (a + b) = y \cdot (a + b) = y \; .$$

If $a \cdot b \leqq x, y; x = x + a \cdot b = y + a \cdot b = y$. Finally, if $x \leqq a + b$ and $a \cdot b \leqq y, a \cdot b \leqq y \cdot (a + b) = x \cdot (a + b) = x$, i.e., $a \cdot b \leqq x, y$ again. Thus $x = y$ in every case, and $\theta_{a+b} \cap \psi_{a \cdot b} = 0$.

LEMMA 3. *If* $\theta_{a+b} = 0, a + b = 1$; *and if* $\psi_{a \cdot b} = 0, a \cdot b = 0$.

Proof. By definition $x \cdot (a + b) \equiv x(\theta_{a+b})$. Thus $\theta_{a+b} = 0$ implies that

$$x \cdot (a+ b) = x$$

for all x, and consequently that $a + b = 1$. Similarly, $\psi_{a \cdot b} = 0$ implies that $a \cdot b = 0$.

LEMMA 4. *If L is subdirectly irreducible,* $\theta_{a+b} = 0$, *and* $a \cdot b \neq 0$, *then there exists $p \in L$ such that p and $a \cdot b$ are noncomparable.*

Proof. If $\theta_{a+b} = 0, a \cdot b \neq 0$, and there exists no p as above, than it is easy to verify that $\theta_{a \cdot b} \cap \psi_{a \cdot b} = 0$. (Note that $x \equiv y \, (\theta_{a \cdot b})$ if and only if $x = y$ or $a \cdot b \leqq x, y$, and that $x \equiv y(\psi_{a \cdot b})$ if and only if $x = y$ or $x, y \leqq a \cdot b$). Since $a \cdot b \neq 0$, neither $\theta_{a \cdot b}$ nor $\psi_{a \cdot b} = 0$. Thus L is reducible. This contradiction implies that p must exist.

If L and p are as in Lemma 4, p must be related to a or b, but $a \leqq p$ or $b \leqq p$ implies that p and $a \cdot b$ are comparable. Thus we can assume that p is less than one of a, b; assume $p < a$.

LEMMA 5. *If L and p are as in Lemma* 4, $p + a \cdot b$ *and b are* *noncomparable.*

Proof. Clearly $a \cdot b \leqq b \cdot (p + a \cdot b)$. Since $p < a, p + a \cdot b \leqq a$, and hence $b \cdot (p + a \cdot b) \leqq a \cdot b$. Thus $b \cdot (p + a \cdot b) = a \cdot b$. Since a and b are noncomparable, $a \cdot b \neq b$; and since $a \cdot b, p$ are noncomparable, $p + a \cdot b \neq a \cdot b$. Thus b and $p + a \cdot b$ are noncomparable.

LEMMA 6. *If L and p are as in Lemma 4,*

$$L = \{x \mid x \leqq p + a \cdot b\} \cup \{x \mid a \cdot b \leqq x\} .$$

Proof. Trivially, if z is related to $a \cdot b, z$ is in one of the sets. Thus suppose that $z, a \cdot b$ are unrelated. Since $a \cdot b, p$ are noncomparable and L is of width two, z must be related to p. If $z \leqq p$, $z \leqq p + a \cdot b$. If z is also related to $p + a \cdot b, z$ is in one of the sets. Thus, suppose that $p \leqq z$ and that z and $p + a \cdot b$ are unrelated. By Lemma 5, $p + a \cdot b$ and b are unrelated. Thus, z must be related to b. If $b \leqq z, a \cdot b \leqq z$; and if $z \leqq b, p \leqq z < b(p + a \cdot b \leqq b)$. But both conclusions are impossible. Thus L is the union of the two sets.

LEMMA 7. *If L and p are as in Lemma 4, $\theta_{p+a \cdot b} \cap \psi_{a \cdot b} = 0$.*

Proof. If $x \equiv y(\theta_{p+a \cdot b} \cap \psi_{a \cdot b}), x \cdot (p + a \cdot b) = y \cdot (p + a \cdot b)$ and $x + a \cdot b = y + a \cdot b$. If

$$x, y \leqq p + a \cdot b, x = x \cdot (p + a \cdot b) = y \cdot (p + a \cdot b) = y;$$

and if $a \cdot b \leqq x, y, x = x + a \cdot b = y + a \cdot b = y$. Thus suppose

$$x \leqq p + a \cdot b$$

and $a \cdot b \leqq y$ (By Lemma 6, we can assume that this is the only remaining possibility.) Then $x = x \cdot (p + a \cdot b) = y \cdot (p + a \cdot b) \leqq y$, i.e., $x \leqq y$. Also, $x + a \cdot b = y + a \cdot b = y$. Since

$$x \leqq p + a \cdot b, y = x + a \cdot b \leqq p + a \cdot b .$$

Thus, $x \leqq y \leqq p + a \cdot b$, and $x = x \cdot (p + a \cdot b) = y \cdot (p + a \cdot b) = y$.

LEMMA 8. *If L is a subdirectly irreducible lattice of width two and a, b are noncomparable elements of $L, a + b = 1$ and $a \cdot b = 0$.*

Proof. By Lemma 2, $\theta_{a+b} \cap \psi_{a \cdot b} = 0$. Since L is irreducible, $\theta_{a+b} = 0$ or $\psi_{a \cdot b} = 0$. Suppose $\theta_{a+b} = 0$. Then $a + b = 1$ by Lemma 3. If $a \cdot b \neq 0, \psi_{a \cdot b} \neq 0$ by Lemma 3. Also, by Lemma 4, there is an element p of L which is noncomparable to $a \cdot b$. For this p,

$$\theta_{p+a\cdot b} \cap \psi_{a\cdot b} = 0$$

by Lemma 7. Hence $\theta_{p+a\cdot b} = 0$. But this is impossible since it implies that $1 = p + a \cdot b \leqq a$ or b. Hence $a \cdot b = 0$. If $\psi_{a\cdot b} = 0$, a dual argument completes the proof.

(Note that Lemma 8 implies that a subdirectly irreducible lattice of width two has a zero and a one.)

Let L be a subdirectly irreducible lattice of width two. If there were an element z of $L - \{0, 1\}$ which was comparable to each element of L, $\theta_z \cap \psi_z = 0$ with $\theta_z \neq 0$ and $\psi_z \neq 0$. Thus, since L is irreducible, it must be the union of the pairwise disjoint sets $\{0, 1\}$, C_1, C_2 where C_1, C_2 are chains such that the sum of elements from different chains is 1 and the product, 0. If each chain has at least two elements, then one can define two congruence relations R_1, R_2 as follows:

$x \equiv y(R_i)$ if and only if $x = y$ or $x, y \in C_i$ $(i = 1, 2)$. Clearly, $R_1 \cap R_2 = 0$, but $R_1, R_2 \neq 0$ since each chain contains at least two elements. Thus, one chain must contain exactly one element. If both chains consist of a single element, L is a direct product of two-element chains, and hence is reducible. Thus, L consists of $\{0, 1\}$, C_1, C_2 where C_1 contains only one element and C_2 contains at least two elements. Suppose C_2 contains at least three elements $p < q < r$. Define relations S_1, S_2 on L by

$$x \equiv y\,(S_1) \text{ if and only if } x = y \text{ or } 0 < x, y \leqq q,$$
$$x \equiv y\,(S_2) \text{ if and only if } x = y \text{ or } q \leqq x, y < 1\,.$$

It is easy to show that these are congruence relations. Clearly $S_1 \cap S_2 = 0$. Thus $S_1 = 0$ or $S_2 = 0$. But $p \equiv q(S_1)$ and $q \equiv r(S_2)$, a contradiction. Thus C_2 consists of exactly two elements, and $L \cong N_5$. Hence

THEOREM. *Every lattice of width two is a subdirect product of two-element chains and* N_5.

COROLLARY. *The only subdirectly irreducible lattice of width two is* N_5.

For each $n \geqq 3$, one can exhibit a lattice to show that it is false that all lattices of width n are subdirect products of lattices from some class of finite lattices. For a fixed n, it would be of interest to find a lattice property P such that if L were of width n and had property P, that L would be a subdirect product of finite lattices.

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INTEGRALS WHICH ARE CONVEX FUNCTIONALS

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This paper examines numerical functionals defined on function spaces by means of integrals having certain convexity properties. The functionals are themselves convex, so they can be analysed in the light of the theory of conjugate convex functions, which has recently undergone extensive development. The results obtained are applicable to Orlicz space theory and in the study of various extremum problems in control theory and the calculus of variations.

In everything that follows, let T denote a measure space with a σ-finite measure dt. Let L be a particular real vector space of measurable functions u from T to R^n (for a fixed n). For instance, one could take L to be the space $L_n^p(T)$ consisting of all R^n-valued measurable functions u on T such that $\Phi_p(u) < +\infty$, where

$$\Phi_p(u) = \int_T \varphi_p(u(t))dt \quad \text{and} \quad \varphi_p(x) = (1/p)\,|\,x\,|^p,\ \ 1 \leqq p < +\infty$$

with $|\cdot|$ denoting the Euclidean norm on R^n. No matter which L is chosen, one can regard Φ_p as a functional from L to $(-\infty, +\infty]$. Then Φ_p is convex, in consequence of the fact that the function φ_p is convex on R^n. (A function F from a real vector space to $(-\infty, +\infty]$ is said to be *convex* if

$$F(\lambda x + (1-\lambda)y) \leqq \lambda F(x) + (1-\lambda)F(y)$$

always holds when $0 < \lambda < 1$.) Notice that, if φ_∞ is the convex function defined by

$$\varphi_\infty(x) = \lim_{p\to\infty} \varphi_p(x) = \begin{cases} 0 & \text{if} \quad |\,x\,| \leqq 1\,, \\ +\infty & \text{if} \quad |\,x\,| > 1\,, \end{cases}$$

the corresponding integral $\Phi_\infty(u)$ is finite if and only if u belongs to the unit ball of the space $L_n^\infty(T)$ of essentially bounded measurable functions.

Here we propose to study a much broader class functionals than the Φ_p, $1 \leqq p \leqq \infty$. These functionals are of the form

$$I_f(u) = \int_T f(t, u(t))dt \quad \text{for} \quad u \in L\,,$$

where f is a function from $T \times R^n$ to $(-\infty, +\infty]$, such that $f(t, x)$ is a convex function of $x \in R^n$ for each $t \in T$. Such a function f we call a *convex integrand* for convenience.

As a preliminary task, we must come up with conditions on f

ensuring that various functions such as $f(t, u(t))$ be measurable in t. The well-known condition of Carathéodory is no help, because we do not want to assume that $f(t, x)$ is continuous in x. That would prevent us from considering most of the cases were f can be infinity-valued. We have already encountered one such case, namely $f = \varphi_\infty$. Generally speaking, the device of allowing f to have the value $+\infty$ has the effect of constraining $u(t)$ to lie in a certain convex subset of R^n, *depending perhaps on* t. Indeed, a necessary condition for $I_f(u)$ to be finite is that

$$u(t) \in \operatorname{dom} f_t \qquad \text{for almost all } t ,$$

where f_t denotes the convex function $f_t(x) = f(t, x)$. (For any convex function F, the set of points where F does not have the value $+\infty$ is a convex set, which we call the *effective domain* of F and denote by $\operatorname{dom} F$.)

In order that $I_f(u)$ be an unambiguous number in $(-\infty, +\infty]$, a further condition besides measurability, is usually needed, since $f(t, x)$ is not required to be nonnegative. The important thing, however, is that I_f turns out to be a convex function on L when it is well-defined.

The Φ_p have already been cited as examples of convex functionals of type I_f which have received close attention from functional analysts. If the integrands φ_p are generalized to those of the form $f(t, x) = N(|x|)$ where N is a finite nonnegative convex function on the real line such that $N(\lambda) > 0$ for $\lambda > 0$,

$$\lim_{\lambda \uparrow 0} N(\lambda)/\lambda = 0 \quad \text{and} \quad \lim_{\lambda \uparrow \infty} N(\lambda)/\lambda = \infty ,$$

one gets convex functionals I_f defining generalized L^p spaces, called *Orlicz spaces*. These spaces are very useful in dealing with integral equations. We refer the reader to [5] for an excellent account.

Possible applications along the lines suggested by the theory of Orlicz spaces are one motivation for looking at the convex functionals I_f in the general case. Another motivation is that such functionals arise naturally in the calculus of variations. For example, suppose that $T = [0, 1]$, with dt as the ordinary Lebesgue measure. Regarding R^n as $R^k \oplus R^k$, write each vector x as a pair (y, z), where y and z have k components. Then I_f may be interpreted as a functional defined for pairs of measurable functions from $[0, 1]$ to R^k. Now let

$$J(q) = I_f(q, \dot{q}) = \int_T f(t, q(t), \dot{q}(t))dt ,$$

where q is a differentiable function from $[0, 1]$ to R_k (a *curve*) and $\dot{q} = dq/dt$. Inasmuch as differentiation is a linear operation, J will be a convex function on the space of curves q. Problems which involve minimizing J can hopefully be tackled therefore by convexity methods, such as the existence and duality theory in [11]. Note that infinite values of f correspond to constraints on the values of $q(t)$ and $\dot{q}(t)$ for the curves q such that $J(q) < +\infty$. Nonclassical convex constrained minimization problems of this sort abound in control theory. We plan elsewhere to take up applications of our results to such areas.

The main question treated in this paper is whether the conjugate of a convex functional I_f is another such functional I_g. The question is significant, because the present theory of convex functions is so extensively concerned with conjugates. The notion of conjugacy, due to Fenchel [2], may be formulated in a general way as follows. Let E and E^* be real vector spaces, and let $\langle x, x^*\rangle$ be a (real) bilinear function of $x \in E$ and $x^* \in E^*$. Let F be a *proper* convex function on E (i.e. a convex function with values in $(-\infty, +\infty]$ which is not identically $+\infty$). The function F^* on E^* defined by

$$F^*(x^*) = -\inf\{F(x) - \langle x, x^*\rangle \mid x \in E\}$$

is called the *conjugate* of F (with respect to the given pairing of E and E^* by $\langle\cdot, \cdot\rangle$). It is a convex function on E^* with values in $(-\infty, +\infty]$. Furthermore, F^* is always lower semi-continuous with respect to the weak topology induced on E^* by E. (Lower semi-continuity means that the set $\{x^* \mid F^*(x^*) \leqq \mu\}$, which incidentally is always convex, is closed for every real μ.) The conjugate of F^* is in turn the function F^* on E defined by

$$F^{**}(x) = -\inf\{F^*(x^*) - \langle x, x^*\rangle \mid x^* \in E^*\} .$$

In order that F^* be proper and $F^{**} = F$, it is necessary and sufficient that F itself be lower semi-continuous with respect to the weak topology induced on E by E^*. General proofs of these result are given in [1] and [6].

Two conjugacy contexts will mostly concern us here. In the first case, $E = E^* = R^n$ with $\langle x, x^*\rangle$ as the ordinary inner product. The weak topologies are then the ordinary topologies on R^n. In the second case we take $E = L$ and $E^* = L^*$, where L^* is *any* space of R^n-valued measurable functions, such that the inner product $\langle u(t), u^*(t)\rangle$ is summable as a function of t for every $u \in L$ and $u^* \in L^*$. The pairing is given by

$$\langle u, u^*\rangle = \int_T \langle u(t), u^*(t)\rangle dt .$$

Any topologies compatible with the duality between E and E^* could be invoked in place of the weak topologies, for instance the norm topologies if $E = E^* = L_n^2(T)$.

Suppose that $f(t, x)$ is a convex integrand which is proper and lower semi-continuous in x for each t. Define $f^*(t, x^*)$ by taking conjugates in x, i.e. $f_t^* = (f_t)^*$ for each t. Then, according to the results described above, f^* is another convex integrand, proper and lower semi-continuous in its convex argument. We call it the *integrand conjugate to* f. The conjugate of the conjugate is the original integrand f. The principal fact brought out in this paper (Theorem 2) is that conjugate integrands f and f^* usually furnish conjugate functionals of L and L^*. This generalizes the fact that φ_p is conjugate to φ_q, and Φ_p on $L_n^p(T)$ is conjugate to Φ_q on $L_n^q(T)$ (with $(1/p) + 1/q) = 1$). The resulting class of "best inequalities" of the type

$$\langle u, u^* \rangle \leqq I_f(u) + I_{f^*}(u^*)$$

is likewise a generalization of certain classical inequalities.

2. Normal integrands and measurability. Before we can proceed, we must established that various technical constructions result in functions which are measurable. To this end, some regularity conditions must be imposed. We shall call a convex integrand f *normal* if $f(t, x)$ is proper and lower semi-continuous in x for each t, and if further there exists a *countable* collection U of measurable functions u from T to R^n having the following properties:

(a) for each $u \in U$, $f(t, u(t))$ is measurable in t;

(b) for each t, $U_t \cap \operatorname{dom} f_t$ is dense in $\operatorname{dom} f_t$, where

$$U_t = \{u(t) \mid u \in U\} .$$

The latter conditions, which seem offhand to be rather complicated, are automatically satisfied in some notable cases, as we shall now indicate.

LEMMA 1. *Suppose $f(t, x) = F(x)$ for all t, where F is a lower semi-continuous proper convex function on R^n. Then f is a normal convex integrand.*

Proof. Let D be a countable dense subset of the effective domain of F ($= \operatorname{dom} f_t$ for all t). (Such a D exists, of course, because $\operatorname{dom} F$ is a nonempty convex set in R^n.) Let U consist of the constant functions on T with values in D. Then conditions (a) and (b) are satisfied in a trivial way.

LEMMA 2. *Suppose f is a convex integrand such that $f(t, x)$ is measurable in t for each fixed x, and such that, for each t, $f(t, x)$ is lower semi-continuous in x and has interior points in its effective domain $\{x \mid f(t, x) < +\infty\}$. Then f is a normal convex integrand.*

Proof. Let D be a countable dense subset of R^n, and let U be the constant functions with values in D. The measurability condition for normality is satisfied in virtue of the present measurability hypothesis. The density condition is satisfied, because D has a dense intersection with the interior of $\operatorname{dom} f_t$, and $\operatorname{dom} f_t$ is the closure of its interior by convexity.

COROLLARY. *Suppose f is a convex integrand having only finite values, such that $f(t, x)$ is measurable in t for each x. Then f is a normal convex integrand.*

Proof. Here $\operatorname{dom} f_t = R^n$ for every t. The lower semi-continuity of f_t is then automatic, since a finite convex function on an open convex set in R^n is always continuous.

An intermediate fact about the consequences of normality will now be deduced.

LEMMA 3. *Let f be a normal convex integrand with conjugate f^*. Then, for every measurable function u^* from T to R^n, the function $f^*(t, u^*(t))$ is measurable in t.*

Proof. By definition,

$$-f^*(t, u^*(t)) = \inf \{f(t, x) - \langle x, u^*(t)\rangle \mid x \in R^n\} .$$

We shall show that, for each t, the infimum can actually be taken over $x \in U_t$ instead, where U_t is the set in the definition of normality. Since $f(t, x) = +\infty$ for $x \notin \operatorname{dom} f_t$, the question is whether any value of $f(t, x) = \langle x, u^*(t)\rangle$ with $x \in \operatorname{dom} f_t$ can be approximated by one with $x \in U_t \cap \operatorname{dom} f_t$. Now $U_t \cap \operatorname{dom} f_t$ is dense in $\operatorname{dom} f_t$ by hypothesis. Furthermore $\operatorname{dom} f_t$, being a nonempty convex set, is the closure of its relative interior (its interior relative to the affine manifold it generates). The intersection of U_t with this relative interior must be dense in $\operatorname{dom} f_t$. According to familiar results about lower semi-continuous convex functions (e.g. in [3], [13]), f_t is continuous with respect to the relative interior of $\operatorname{dom} f_t$ and its values at relative boundary points can be obtained as limits of the relative interior values. Therefore the values of $f(t, x)$ for $x \in \operatorname{dom} f_t$ are limits of those for $U_t \cap \operatorname{dom} f_t$, as we wanted to show. The upshot is that

$$-f^*(t, u^*(t)) = \inf \{f(t, u(t)) - \langle u(t), u^*(t)\rangle \mid u \in U\} .$$

This formula expresses $f^*(t, u^*(t))$ as the pointwise infimum of a collection of functions on T. Each of the functions in the collection is measurable, in view of the hypotheses, and the collection is countable.

The pointwise infimum is consequently another measurable function on T.

Moreau's proximation mappings, whose properties are elucidated in [9], will be very useful to us. Here is how they are defined. Let F be any lower semi-continuous closed proper convex function on R^n. It can be proved that, for each $z \in R^n$, there exist unique vectors x and x^* such that

$$z = x + x^* \quad \text{and} \quad F(x) + F^*(x^*) = \langle x, x^* \rangle .$$

We write

$$x = \operatorname{prox}(z \mid F) \quad \text{and} \quad x^* = \operatorname{prox}(z \mid F^*) .$$

The mapping $\operatorname{prox}(\cdot \mid F)$ from R^n into itself is called the *proximation* associated with F. It is continuous (a metric contraction as a matter of fact), and its range is dense in $\operatorname{dom} F$. If F is the indicator function of a closed convex set K (in other words $F(x) = 0$ when $x \in K$ and $F(x) = +\infty$ when $x \notin K$), then $\operatorname{prox}(z \mid F)$ is the point of K nearest to z. In general, $\operatorname{prox}(z \mid F)$ is the unique x for which

$$F(x) + \frac{1}{2} | x - z |^2$$

achieves its minimum.

LEMMA 4. *Let f be a normal convex integrand. Let z be a measurable function from T to R^n. Then the functions* $\operatorname{prox}(z(t) \mid f_t)$ *and* $\operatorname{prox}(z(t) \mid f_t^*)$ *are measurable in t.*

Proof. Set

$$g(t, x) = f(t, x) + \frac{1}{2} | x - z(t) |^2 .$$

It is easily verified that g is another normal convex integrand. We shall be concerned with the conjugate integrand $g^*(t, x^*)$. By Moreau's theory, g_t^* is differentiable at 0 for each t, and $\nabla g_t^*(0) = \operatorname{prox}(z(t) \mid f_t)$. Now, for an arbitrary $a \in R^n$,

$$\langle a, \nabla g_t^*(0) \rangle = \lim_{\lambda \downarrow 0} [g^*(t, \lambda a) - g^*(t, 0)]/\lambda .$$

The difference quotient is a measurable function of t for each λ by Lemma 3 and the normality of g. The limit can be taken over a countable sequence in λ, so $\langle a, \nabla g_t^*(0) \rangle$ is measurable in t. It follows that $\operatorname{prox}(z(t) \mid f_t)$ is measurable in t, and likewise $\operatorname{prox}(z(t) \mid f_t^*)$ because

$$\operatorname{prox}(z(t) \mid f_t^*) = z(t) - \operatorname{prox}(z(t) \mid f_t)$$

for every t.

We can now prove that normality is preserved when one passes to the conjugate.

LEMMA 5. *If f is a normal convex integrand, then f^* is a normal convex integrand, too.*

Proof. We already know from the theory of conjugates that $f^*(t, x^*)$ is a lower semi-continuous proper convex function of x^* for each t. The problem is to produce a collection U satisfying condition (b) of normality (with f^* in place of f). Condition (a) will then hold by virtue of Lemma 3. Let D be any countable dense subset of R^n. Let U consist of the functions of the form $u(t) = \text{prox}\,(z \mid f_t^*)$ with z ranging over D. Each $u \in U$ is measurable by Lemma 4. The set U_t is the image of D under $\text{prox}\,(\cdot \mid f_t^*)$. Since the proximation is continuous and its range is dense in $\text{dom}\, f_t^*$, U_t is dense in $\text{dom}\, f_t^*$.

COROLLARY. *If f is a normal convex integrand, then $f(t, u(t))$ is measurable in t for every measurable function u from T to R^n.*

Proof. This is immediate from Lemma 3, since $f_t = f_t^{**}$.

Our final lemma guarantees the existence of enough measurable functions for one to minimize a normal convex integrand pointwise in a measurable fashion.

LEMMA 6. *Let f be a normal convex integrand. Let α be a measurable real-valued function on T such that*

$$\inf_x f(t, x) < \alpha(t) \qquad \text{for every } t\,.$$

Then there exists a measurable function u from T to R^n such that

$$f(t, u(t)) \leqq \alpha(t) \qquad \text{for every } t\,.$$

Proof. Set $K_t = \{x \mid f(t, x) \leqq \alpha(t)\}$ for each t. According to the general theory of convex functions on R^n, each K_t is a nonempty closed convex subset of $\text{dom}\, f_t$ having the same dimension as $\text{dom}\, f_t$, inasmuch as $f_t(x) < \alpha(t)$ for at least one x. Therefore $U_t \cap K_t$ is dense in K_t, where U_t is the set in the definition of the normality of f. Let $g(t, x) = 0$ when $x \in K_t$ and $g(t, x) = +\infty$ when $x \notin K_t$. Evidently g is another convex integrand satisfying the normality conditions with the same collection U as invoked for f. Let $u(t)$ be the point of K_t nearest to the origin, i.e.

$$u(t) = \text{prox}\,(0 \mid g_t) \quad \text{for each } t\,.$$

This u is a measurable function by Lemma 4 (applied to g), and $f(t, u(t)) \leqq \alpha(t)$ by definition of K_t.

3. Conjugate convex integrals. The stage is now set for proving our chief results. We assume throughout that L^* is a space of measurable functions paired with L in the manner described in the introduction. (When L is a Banach space, L^* does not have to be its dual.)

THEOREM 1. *Let f be a normal convex integrand. Suppose there exists at least one $u^* \in L^*$ such that $f^*(t, u^*(t))$ is a summable function of t. Then*

$$I_f(u) = \int_T f(t, u(t))dt \,, \qquad u \in L \,,$$

is a well-defined convex function on L with values in $(-\infty, +\infty]$.

Proof. The measurability prerequisite to considering I_f is ensured by the corollary to Lemma 5. Let u^* be one of the functions in L^* whose existence is provided for in the hypothesis. Since f_t and f_t^* are conjugate to each other

$$f(t, u(t)) \geqq \langle u(t), u^*(t)\rangle - f^*(t, u^*(t))$$

for every t. The right side is a summable function of t by the hypothesis. Thus there can be no question of $I_f(u)$ being $-\infty$: either $f(t, u(t))$ is summable or its integral is unambiguously $+\infty$. As for the convexity of I_f, that is immediate from the inequality

$$f(t, \lambda u(t) + (1 - \lambda)v(t)) \leqq \lambda f(t, u(t)) + (1 - \lambda)f(t, v(t)) \,,$$

which holds for every t when $0 < \lambda < 1$ by the convexity of f_t.

We shall say that L is *decomposable* when it satisfies the following conditions:

(a) L contains every bounded measurable function from T to R^n which vanishes outside a set of finite measure;

(b) if $u \in L$ and E is a set of finite measure in T, then L contains $\chi_E \cdot u$, where χ_E is the characteristic function of E.

These conditions guarantee that one can alter functions in L arbitrarily in a bounded manner on sets of finite measure. (Subtract $\chi_E \cdot u$ from u, and then add any bounded measurable function vanishing outside E.) The first condition also implies that the functions in L^* are summable on sets of finite measure. The $L_n^p(T)$ are examples of of spaces decomposable in this sense.

THEOREM 2. *Suppose L and L^* are decomposable. Let f be a* *normal convex integrand such that $f(t, u(t))$ is summable in t for at least one $u \in L$, and $f^*(t, u^*(t))$ is summable in t for at least one $u^* \in L^*$. Then I_f on L and I_{f^*} on L^* are proper convex functions conjugate to each other.*

Proof. I_f and I_{f^*} are well-defined and convex by Theorem 1 and Lemma 5, and they are proper by the hypothesis. For any x^* in R^n we have

$$f(t, x) + f^*(t, x^*) \geqq \langle x, x^* \rangle$$

by conjugacy. Hence, for any $u \in L$ and $u^* \in L^*$,

$$\begin{aligned} I_f(u) + I_{f^*}(u^*) &= \int_T f(t, u(t))dt + \int_T f^*(t, u^*(t))dt \\ &\geqq \int_T \langle u(t), u^*(t) \rangle dt = \langle u, u^* \rangle . \end{aligned}$$

It follows that

$$\begin{aligned} I_{f^*}(u^*) &\geqq \sup \{\langle u, u^* \rangle - I_f(u) \mid u \in L\} \\ &= -\inf \{I_f(u) - \langle u, u^* \rangle \mid u \in L\} = (I_f)^*(u^*) . \end{aligned}$$

Verification of the opposite inequality will establish that I_{f^*} is the conjugate of I_f. Fix any $u^* \in L^*$ and any $\beta < I_{f^*}(u^*)$. Select any real summable function μ on T such that

$$\mu(t) < f^*(t, u^*(t)) \quad \text{for all } t, \text{ and } \quad \int_T \mu(t)dt > \beta .$$

Since by conjugacy

$$f^*(t, x^*) = -\inf \{f(t, x) - \langle x, x^* \rangle \mid x \in R^n\} ,$$

we have

$$-\mu(t) > \inf \{f(t, x) - \langle x, u^*(t) \rangle \mid x \in R^n\}$$

for all t. We now apply Lemma 6 to $\alpha(t) = -\mu(t)$ and g, where

$$g(t, x) = f(t, x) - \langle x, u^*(t) \rangle .$$

(The normality of f carries over to g.) The function u we obtain from Lemma 6 satisfies

$$-\mu(t) \geqq f(t, u(t)) - \langle u(t), u^*(t) \rangle .$$

Since T is σ-finite by our underlying assumption, we can choose an increasing sequence of measurable sets E_k of finite measure with union T, such that the $u(t)$ we have constructed is bounded in $t \in E_k$ for each k. Let $\bar{u}$ be any particular function in L for which the integrand in I_f is summable. (Such a function exists by hypothesis.) For each k let

$$u_k(t) = \begin{cases} u(t) & \text{if } t \in E_k , \\ \bar{u}(t) & \text{if } t \in E_k' , \end{cases}$$

where E_k' denotes the complement of E_k in T. These functions u_k belong to L by the decomposability hypothesis. For each k we have

$$\int_{E_k} \mu(t)dt \leqq \int_{E_k} [\langle u(t), u^*(t)\rangle - f(t, u(t))]dt$$
$$= \langle u_k, u^*\rangle - I_f(u_k) - \int_{E_k'} [\langle \bar{u}(t), u^*(t)\rangle - f(t, \bar{u}(t))]dt .$$

The boundedness assumption on E_k is used here to ensure that $\langle u_k(t), u^*(t)\rangle$ be summable, so that

$$\int_T [\langle u_k(t), u^*(t)\rangle - f(t, u_k(t))]dt = \langle u_k, u^*\rangle - I_f(u_k)$$

unambiguously. The integral over E_k' in the calculation above can be made arbitrarily small by choosing k sufficiently large. On the other hand

$$\lim_{k\to\infty} \int_{E_k} \mu(t)dt = \int_T \mu(t)dt > \beta$$

by our assumptions. Thus

$$\langle u_k, u^*\rangle - I_f(u_k) > \beta$$

when k is large, implying that $(I_f)^*(u^*) > \beta$. Inasmuch as β was any number less than I_{f^*}, we may now conclude that $I_{f^*}(u^*) = (I_f)^*(u^*)$. The fact that $I_f = (I_{f^*})^*$ follows dually.

COROLLARY. *Suppose that L and L^* are decomposable, and that T is of finite measure. Let f be of the form $f(t, x) = F(x)$, where F is a lower semi-continuous proper convex function on R^n. Then I_f on L and I_{f^*} on L^* are conjugate to each other.*

Proof. Such an f is normal by Lemma 1. The existence of summable function for I_f and I_{f^*} is elementary in this case. Namely, take any x for which $F(x)$ is finite, and let u be the constant function whose sole value is x. Since T is of finite measure, u is summable. By decomposability (b), $u \in L$. Similary for I_{f^*}. The hypothesis of Theorem 2 is therefore satisfied.

The next theorem furnishes a different way of establishing the conjugacy of I_{f^*} and I_f in certain situations. It also yields a continuity property.

THEOREM 3. *Let T be of finite measure. Suppose that L^* is decomposable, and that L is actually $L_n^\infty(T)$. Let f be a normal convex integrand satisfying the following condition: there exists some $a \in L$ and $\varepsilon > 0$ such that, for each $x \in R^n$ with $|x| < \varepsilon$, the function $f(t, a(t) + x)$ is finite and bounded in t. Then I_f on L and I_{f^*} on L^* are convex functions conjugate to each other. Moreover, I_f is continuous at a in the norm topology of $L = L_n^\infty(T)$.*

Proof. Replacing f by g if necessary, where

$$g(t, x) = f(t, a(t) + x) - f(t, a(t))$$

(evidently another normal convex integrand), we can reduce everything to the case where $a(t) \equiv 0$ and $f(t, 0) \equiv 0$. Then $I_f(0) = 0$. We must show that I_f is norm-continuous at 0, and that $f^*(t, u^*(t))$ is summable in t for some $u^* \in L^*$. The conjugacy of I_f and I_{f^*} will then follow from the last theorem. Define

$$F(x) = \sup \{f(t, x) \mid t \in T\} .$$

As a pointwise supremum of lower semi-continuous convex functions on R^n, F is itself lower semi-continuous and convex. By hypothesis, $F(x)$ is finite on the open convex set $\{x \mid \; |x| < \varepsilon\}$. As is well-known a finite convex function on a finite-dimensional open convex set is automatically continuous. Hence F is continuous when $|x| < \varepsilon$. Fix a positive δ less than ε, and let

$$k = \max \{F(x) \mid |x| \leqq \delta\} < +\infty .$$

Now $F(0) = 0$, so that we have $F(x) \leqq (k/\delta)|x|$ when $|x| \leqq \delta$ by convexity. (Consider the values of F along the line segment from 0 to αx, where $\alpha = \delta/|x|$.) Hence for every t

$$f(t, x) \leqq (k/\delta)|x| \quad \text{when} \quad |x| \leqq \delta .$$

This inequality also implies that $f(t, x) \geqq -(k/\delta)|x|$ for every x. (To verify this, one expresses 0 as a convex combination of x and μx, where $\mu = -\delta/|x|$, namely $0 = \lambda x + (1 - \lambda)\mu x$ with $\lambda = -\mu/(1 - \mu)$. Then by the convexity of f

$$\begin{aligned} 0 = f(t, 0) &\leqq \lambda f(t, x) + (1 - \lambda) f(t, \mu x) \\ &\leqq \lambda f(t, x) + (1 - \lambda)(k/\delta)|\mu x| \\ &= -(\mu/(1 - \mu))[f(t, x) + (k/\delta)|x|] . \end{aligned}$$

The first inequality has been applied here to μx, which is permissible because $|\mu x| = \delta$ by the choice of μ. One concludes that $f(t, x) + (k/\delta)\,|x|$ is always non-negative.) In particular, therefore

$$|f(t, x)| \leqq (k/\delta)\,|x| \quad \text{when} \quad |x| \leqq \delta \, .$$

If $u \in L$ satisfies $\|u\| \leqq \delta$, where $\|\cdot\|$ is the $L_n^\infty(T)$ norm, then

$$\int_T |f(t, u(t))|\,dt \leqq (k/\delta)\,\|u\| \operatorname{meas} T < +\infty \, .$$

Thus $I_f(u)$ is well-defined when $\|u\| \leqq \delta$, and it approaches $0 = I_f(0)$ as $\|u\|$ approaches 0. This establishes the continuity. We must still construct a $u^* \in L^*$ for which $f^*(t, u^*(t))$ is summable. It suffices to find such a u^* in $L_n^\infty(T)$, for L^* contains $L_n^\infty(T)$ in consequence of the hypothesis that L^* is decomposable and T is of finite measure. Let

$$\bar{u}(t) = \operatorname{prox}(0 \,|\, f_t) \, .$$

The measurability of $\bar{u}$ is asserted by Lemma 4. For each t, $\bar{u}(t)$ is the point which minimizes $f(t, x) + 1/2\,|x|^2$ on R^n. Since the minimand vanishes at $x = 0$,

$$0 \geqq f(t, \bar{u}(t)) + \frac{1}{2}\,|\bar{u}(t)|^2 \geqq -(k/\delta)\,|\bar{u}(t)| + \frac{1}{2}\,|\bar{u}(t)|^2 \, .$$

It follows that $|\bar{u}(t)| \leqq 2k/\delta$ for all t, so $\bar{u} \in L_n^\infty(T)$. It now follows further that

$$0 \geqq f(t, \bar{u}(t)) \geqq -(k/\delta)\,|\bar{u}(t)| \geqq -2(k/\delta)^2 \, ,$$

so $f(t, \bar{u}(t))$ is bounded in t (and hence summable). Now take

$$u^*(t) = -\bar{u}(t) = 0 - \operatorname{prox}(0 \,|\, f_t) = \operatorname{prox}(0 \,|\, f_t^*) \, .$$

Again $u^* \in L_n^\infty(T)$. According to the basic property of proximations,

$$f(t, \bar{u}(t)) + f^*(t, u^*(t)) = \langle \bar{u}(t), u^*(t) \rangle$$

for every t. The first and last terms in this equation yield summable functions, so we can conclude that $f^*(t, u^*(t))$ is summable, too.

THEOREM 4. *Let T be of finite measure. Let $f(t, x)$ be a finite convex function of x for each t and a bounded measurable function of t for each x. Then I_f is a well-defined finite convex function on $L_n^\infty(T)$ which is everywhere continuous with respect to the uniform norm. Moreover, the conjugate $(I_f)^*$ of I_f on $L_n^\infty(T)^*$, the space of all linear function as on $L_n^\infty(T)$ continuous with respect to the uniform norm, is given by I_{f*} in the following sense: if $v \in L_n^\infty(T)^*$ is of the form*

$$v(u) = \int_T \langle u(t), u^*(t)\rangle\, dt\ , \qquad u^* \in L_n^1(T)\ ,$$

one has $(I_f)^*(v) = I_{f*}(u)$, *whereas otherwise one has* $(I_f)^*(v) = +\infty$.

Proof. We note that f is a normal convex integrand by the corollary to Lemma 2. The finiteness and continuity of I_f are asserted by Theorem 3. Fix any $v \in L_n^\infty(T)^*$ such that $(I_f)^*(v) < +\infty$. We shall show that v corresponds to some $u^* \in L_n^1(T)$ as above, whence it will follow from Theorem 3 that

$$(I_f)^*(v) = \sup_u\{v(u) - I_f(u)\} = I_{f*}(u^*)\ .$$

For each measurable $E \subset T$, let $\mu(E)$ denote the unique vector in R^n such that

$$\langle x, \mu(E)\rangle = v(x\cdot\chi_E) \text{ for every } x \in R^n\ ,$$

where $x\cdot\chi_E$ is the function which has the value x on E but the value 0 elsewhere on T. Then μ is a finitely additive set function. We have

$$\begin{aligned}\langle x, \mu(E)\rangle &\leqq I_f(x\cdot\chi_E) + (I_f)^*(v)\\ &= \int_E f(t, x)dt + \int_{T/E} f(t, 0)dt + (I_f)^*(v)\\ &\leqq F(x) \text{ meas } E + \alpha\ ,\end{aligned}$$

where

$$\begin{aligned}F(x) &= \sup\{f(t, x) \mid t \in T\} < +\infty\ ,\\ \alpha &= \max\{0, F(0) \text{ meas } T\} + (I_f)^*(v) < +\infty\ .\end{aligned}$$

The function F is convex, and hence continuous, so that the quantity

$$k(r) = \sup\{F(x) \mid |x| \leqq r\}$$

is finite for every $r > 0$. For every measurable $E \subset T$ and every $r > 0$, we have

$$\begin{aligned}r\,|\mu(E)| &= \sup\{< x, \mu(E) > \mid |x| \leqq r\}\\ &\leqq k(r) \text{ meas } E + \alpha < +\infty\ .\end{aligned}$$

It follows that, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that meas $E < \delta$ implies $|\mu(E)| < \varepsilon$. Thus μ is absolutely continuous with respect to dt, and μ must be countably additive. By the Radon-Nikodym Theorem, there exists some $u^* \in L_n^1(T)$ such that

$$\int_E \langle x, u^*(t)\rangle\, dt = < x, \mu(E) > = v(x\cdot\chi_E)$$

for every $x \in R^n$ and every measurable E. The formula

$$v(u) = \int_T \langle u(t), u^*(t) \rangle \, dt$$

then holds for every u which is a linear combination of functions of the form $x \cdot \chi_E$, and since such linear combinations are dense in $L_n^\infty(T)$ the formula must actually hold for every $u \in L_n^\infty(T)$ by continuity.

COROLLARY 1. *Under the hypothesis of Theorem 4, the convex set*

$$\{u^* \in L_n^1(T) \mid (I_{f*})(u^*) + \langle a, u^* \rangle + \alpha \leqq 0\}$$

is weakly compact (with respect to the pairing between $L_n^1(T)$ and $L_n^\infty(T)$) for any $a \in L_n^\infty(T)$ and any real number α.

Proof. Since I_f norm-continuous, the set

$$\{v \in L_n^\infty(T)^* \mid (I_f)^*(v) + v(a) + \alpha \leqq 0\}$$

is weak* compact in $L_n^\infty(T)^*$ for any a and α, according to a theorem proved independently by Moreau [7] and the author [12, Theorem 7A].

COROLLARY 2. *Let D be a subspace of $L_n^\infty(T)$ supplied with a locally convex topology at least as strong os the uniform norm topology, and let D^* be the space of continuous linear functionals on D. Suppose that no nonzero linear functional on $L_n^\infty(T)$ of the form*

$$u \longrightarrow \int_T \langle u(t), u^*(t) \rangle \, dt \;, \qquad u^* \in L_n^1(T) \;,$$

vanishes throughout D. Then, under the hypothesis of Theorem 4, I_f is a continuous finite convex function on D, and the conjugate $(I_f)^*$ of I_f on D^* is given by I_{f*}, in the sense that if $v \in D^*$ corresponds to some $u^* \in L_n^1(T)$ as above one has $(I_f)^*(v) = I_{f*}(u^*)$, whereas otherwise $(I_f)^*(v) = +\infty$.

Proof. Let J be the convex functional on D^* such that $J(v) = I_{f*}(u^*)$ if v corresponds to a $u^* \in L_n^1(T)$, whereas otherwise $J(v) = +\infty$. This J is well-defined, in view of the hypothesis about linear functionals which vanish on D, and the conjugate of J on D with respect to the natural pairing of D and D^*, is just the restriction of I_f to D. By Corollary 1, the convex sets

$$\{v \in D^* \mid J(v) \leqq \mu\} \;, \qquad \mu \in R \;,$$

are compact in the weak topology on D^* induced by D, so that J is lower semi-continuous in this topology. It follows that $J = J^{**} = (I_f)^*$.

REMARK. Corollary 2 is applicable, of course, to various situations where T has topological or differentiable structure, and D is a space of continuous or differentiable functions on T (with D^* a corresponding space of measures of distributions).

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REFLECTION LAWS OF SYSTEMS OF SECOND ORDER ELLIPTIC DIFFERENTIAL EQUATIONS IN TWO INDEPENDENT VARIABLES WITH CONSTANT COEFFICIENTS

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In this paper we shall consider the reflection of solutions of systems of equations

$$u_{xx} + u_{yy} + Au_x + Bu_y + Cu = 0 \,, \tag{1·1}$$

where $u = (u_1, u_2, \cdots, u_n)^T, A, B, C$ are constant, pairwise commutative $n \times n$ matrices, across an analytic arc κ on which the solutions satisfy n analytic linear differential boundary conditions. If the boundary conditions have coefficients which are analyiic in a specific preassigned geometrical region cantaining κ, then we shall show that the solution of (1.1) satisfying such boundary conditions can be extended across κ, provided certain inequalities are satisfied. Moreover, the region into which u can be extended will depend only on the analytic arc κ, the original region, and the coefficients of the boundary conditions; i.e., we shall have reflection "in the large" and the region will not be restricted by the equation.

There are two basically different situations considered, the results of which are stated in Theorem 1, Theorem 2, and Theorem 3.

Theorem 1 treats the reflection of solutions of a system (1.1) initially given on an open set Ω for which the boundary conditions on an arc κ adjacent to Ω are

$$\sum_{\beta=1}^{n} p_{\alpha\beta}(D)u_\beta = f_\alpha(z) \,, \qquad \alpha = 1, 2, \cdots, n \text{ or } \kappa$$

where

$$p_{\alpha\beta}(D) = \sum_{r+s \leqq k < 2n} < p_{\alpha\beta}^{rs}(z) D_x^r D_y^r$$

with $p_{\alpha\beta}^{rs}(z)$ and $f_\alpha(z)$ analytic in $\Omega \cup \kappa \cup \hat{\Omega}$, where $\hat{\Omega}$ is an open set determined by κ adjacent to κ and disjoint from Ω. In the event that two inequalities involving the $p_{\alpha\beta}^{rs}(z)(r + s = k)$ are satisfied, then we can reflect the solution of the system across κ into $\kappa \cup \hat{\Omega}$, so that the original solution is extended into all of $\Omega \cup \kappa \cup \hat{\Omega}$.

In Theorems 2 and 3 the reflection of solutions given in Ω, of the special system (1.1)

$$\Delta u + Eu = 0 \,, \qquad E = n \times n \text{ constant matrix} \,,$$

is treated. In these cases boundary conditions of the form

$$\sum_{r+s\leqq k<2n} p_{\nu 1}^{rs}(z)D_x^r D_y^s u_1 = f_\nu(z) , \qquad \nu = 1, 2, \cdots, n$$

are assumed prescribed on κ, in which $p_{\nu 1}^{rs}(z)$ and $f_\nu(z)$ are analytic in $\Omega \cup \kappa \cup \hat{\Omega}$. For Theorem 2, $k \geqq n$, and for Theorem 3, $k = n-1$. There are five conditions which must be satisfied in Theorem 2 to insure reflection. Aside from two inequalities involving the $p_{\nu 1}^{rs}(z)$ that must be satisfied as in Theorem 1, there is an additional determinental inequality on the arc $(z = \overline{G(z)})$

$$(1.2) \qquad | D^\nu[G(c) - G(z)]^j | \neq 0 , \quad 1 \leqq \nu \leqq n-1 , \quad 1 \leqq j \leqq n-1 ,$$

which must be satisfied as well as two additional inequalities which depend on the constants of the differential equations.*

In Theorem 3 it is unnecessary to assume (1) one of the differential equation conditions, and (2) condition (1.2). Moreover, in Theorem 3 the reflection is reduced in quadratures whereas in Theorem 2, for the general case, we have only an existential proof.

Finally, we shall give equations and boundary conditions to which the theorems apply. Theorem 1 is applicable when the boundary conditions are $u = (\varphi_1(z), \varphi_2(z), \cdots, \varphi_n(z))$.

Theorem 2 and Theorem 3 are suitable for systems of differential equations of the form

$$\sum_{j=1}^{m} P_{ij}(\Delta)u_j = 0 , \qquad i = 1, 2, \cdots, m$$

where the P_{ij} are polynomials with constant coefficients and Δ is the Laplacian. Two inequalities involving the coefficients of the P_{ij} must be satisfied. A special example is the metaharmonic equation

$$\Delta^n u + a_1\Delta^{n-1}u + \cdots + a_n u = 0 , \qquad a_j = \text{constant} .$$

In this case it is only necessary to check one inequality for the differential equation in the case of Theorem 2. A special example of the metaharmonic equation is the polyharmonic equation

$$\Delta^n u = 0 .$$

In this case there are no inequalities for the differential equation that must be checked for Theorem 2 or 3. Also in this case there is a particularly simple representation of the solution in terms of analytic functions in Ω and analytic functions in $\hat{\Omega}$ which is a generalization of the representation in [8].

In the special case of equations

* J. Leray kindly pointed out to me that (62) and one of the d.e. inequalities are always satisfied and that the 3 holds for $k<2n$.

$$\Delta u_j = a_{j2}u_1 + a_{j2}u_2 \qquad j = 1, 2 , \qquad a_{ji} = \text{constants} ,$$

a special case of which is the metaharmonic equation

$$\Delta^2 u + a\Delta u + bu = 0 , \qquad a, b \text{ constant} ,$$

the condition on the arc κ is automatically satisfied. Moreover, the conditions of Theorem 2, for the biharmonic equation, reduce to the conditions given in [8] with the exception that Theorem 2 requires $u \in C^{2+k}(\Omega \cup \kappa) \cap C^4(\Omega)$ whereas [8] requires only that

$$u \in C^k(\Omega \cup \kappa) \cap C^4(\Omega) \cap C^2(\Omega \cup \kappa) .$$

Finally, it is noted that in the case when the analytic arc is a portion of the x axis then the condition (1.2) is automatically satisfied.

Restricting ourselves to equations of the type (1.1) we get explicit representations for the solutions in terms of the zero order matrix Bessel function. For purposes of brevity we shall consider homogeneous equations (1.1) since the treatment of nonhomogeneous equations involves only obvious changes.

In his beautiful paper [6], Lewy thoroughly considered the question of a single elliptic equation in two independent variables for which the coefficients are analytic functions in a neighborhood of κ. Brown [1] considered the reflection laws for a general fourth order elliptic equation, with constant coefficients, in two independent variables across a straight line segment on which he assumes the solution satisfies two boundary conditions of the form

$$\sum_{r+s\leqq 3} p_\nu^{rs} D_x^r D_y^s u(x, 0) = f_\nu(x) , \qquad \nu = 1, 2 ,$$

where the line of reflection is $y = 0$ and p_ν^{rs} are constants. Assuming the original domain is convex then he achieves reflection in the large, i.e., the domain of reflection is determined initially by the differential equation. Filipenko [2] investigated reflection for the harmonic equation in more than two independent variables across the plane $x_1 = 0$ and has shown that reflection in the large for certain initial domains is possible provided boundary conditions of the form

$$\frac{\partial u}{\partial x_1} + P(x_2, x_3, \cdots, x_n)u = 0$$

are prescribed on the plane, where P is a polynomial. Lewy [7] has given an example to show that the modification of P from a polynomial to an analytic function is not possible. Garabedian [3], [4] has also investigated certain reflection laws in the small for a nonlinear elliptic equation and for quasilinear equations with special boundary conditions. J. Leray [5] has, in a very interesting paper, used reflection for the explicit determination of the Greens function for an M-harmonic equation in a band, when differential boundary conditions are given on the boundary of the band.

2. Geometric reflection across an analytic arc. Let κ be an open analytic arc defined by the real analytic function $F(x, y) = 0$ with $F_x^2 + F_y^2 \neq 0$. As shown in [8], this defines a function $\zeta = G(z)$, of the complex variable $z = x + iy$ which is analytic in a neighborhood of κ and for which κ is described by $z = \overline{G(z)}$. $\hat{z} = \overline{G(z)}$ is called the reflection of z across κ. $\hat{z} = z$ on κ. Let Ω be a semi-neighborhood of κ, with $G(z)$ analytic and univalent on Ω and thus $G'(z) \neq 0$ on Ω. Let $\hat{\Omega} = \overline{G(\Omega)}$ and assume $\hat{\Omega} \cap \Omega = \varnothing$. Then it can be shown that for z in $\Omega \cup \kappa \cup \hat{\Omega}$, $G(z)$ is univalent, $\hat{\hat{z}} = z$ and $G'(z) \neq 0$. Moreover

$$\overline{G'(z)} = [G'(\hat{z})]^{-1}$$

and

$$\overline{G''(z)} = -G''(\hat{z})G'(\hat{z})^{-3} .$$

3. Representation of the solutions. In this section we shall derive a representation for the solutions of (1.1) which are in $C'(\Omega \cup \kappa)$. This will be done by a slight variant of the very elegant method developed by Lewy [6]. The solutions are expressed by means of a complex Riemann function, which can be found explicitly in our case.

First we consider the transformation

$$u = e^{-(1/2)(Ax+By)}w(x, y) \tag{3.1}$$

where the exponential matrix is defined as usual by its McLaurin expansion. Due to the pairwise commutativity of A, B and C we get (1.1) becomes

$$e^{-(1/2)(Ax+By)}\left\{w_{xx} + w_{yy} + \frac{1}{4}[4C - A^2 - B^2]w\right\} = 0 , \tag{3.2}$$

which is equivalent to:

$$w_{xx} + w_{yy} + Dw = 0 \tag{3.3}$$

where

$$D = \frac{1}{4}[4C - A^2 - B^2] . \tag{3.3.1}$$

Note that (3.3) can be written for z in Ω as

$$4\overline{G'(z)}w_{z\hat{z}} + Dw = 0 \tag{3.4}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right], \quad \frac{\partial}{\partial \hat{z}} = \frac{1}{2}\left[\overline{G'(z)}\right]^{-1}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]. \tag{3.5}$$

Let

$$w(x, y) = w\left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right] = w\left[\frac{z + G(\hat{z})}{2}, \frac{z - G(\hat{z})}{2i}\right] = W(z, \hat{z}) \tag{3.6}$$

for $z = x + iy \in \Omega \cup \kappa$ and $\hat{z} = \overline{G(z)} \in \hat{\Omega} \cup \kappa$.

With the idea of finding a representation of the solution of (3.3), we seek the complex Riemann function; viz. the solution of

$$L[v] = v_{z\zeta} + \frac{1}{4}DG'(\zeta)v = 0 \tag{3.7}$$

which is a function $R[z^0, \zeta^0, z, \zeta]$ of four complex arguments each ranging independently over $\Omega \cup \kappa \cup \hat{\Omega}$ with

$$L_{z,\zeta}[R] = 0 \tag{3.8}$$

and

$$\begin{aligned} R[z^0, \zeta^0; z^0, \zeta^0] &= I \\ R_z[z^0, \zeta^0; z, \zeta^0] &= 0\ . \\ R_\zeta[z^0, \zeta^0; z^0, \zeta] &= 0 \end{aligned} \tag{3.9}$$

We claim that such a matrix function is given by

$$R[z^0, \zeta^0; z, \zeta] = J_0[\sqrt{D((z - z^0)(G(\zeta) - G(\zeta^0))}] \tag{3.10}$$

where if Q is an $n \times n$ matrix, we define

$$J_0[\sqrt{Q}] = I - \frac{Q}{2^2} + \frac{Q^2}{2^4(2!)^2} - \frac{Q^3}{2^6(3!)^2} + \cdots .$$

With any norm for Q we get

$$\| J_0[\sqrt{Q}] \| \leqq J_0[i\sqrt{\| Q \|}]$$

where J_0, on the right, is the zero order Bessel function and thus the matrix series converges for all Q. Thus $R[z^0, \zeta^0; z, \zeta]$, as defined, is analytic in z^0, ζ^0, z and ζ for z^0, ζ^0, z and ζ in $\Omega \cup \kappa \cup \hat{\Omega}$. Moreover it is easy to see that (3.9) are satisfied and by direct computation, we see that (3.8) is satisfied.

Our next aim is to find a representation of $W(z, \hat{z})$ as defined by (3.6). This will be done by finding a function $W^*(z, \zeta)$ which is analytic for z in Ω and analytic for ζ in $\hat{\Omega}$ and for which

$$W^*(z, \hat{z}) = W(z, \hat{z})\ .$$

We consider now the Cartesian product

$$S \times \hat{S} = \{(z, \zeta) : z \in \Omega \cup \kappa, \zeta \in \hat{\Omega} \cup \kappa\} .$$

Let z_1 and z_2 be arbitrary points of $\Omega \cup \kappa$ and let p be a path joining z_1 to z_2 which lies in $\Omega \cup \kappa$. Then let $\hat{p}$ be the reflection of p joining $\hat{z}_1$ to $\hat{z}_2$ in $\hat{\Omega} \cup \kappa$. Let

$$S^2(z_1, z_2, p) = \{(z, \zeta) \in S \times \hat{S} : z \in p \text{ and } \zeta \in \hat{p}\} .$$

Note that

$$\begin{aligned} &RL[W^*] - L[R]W^* \\ (3.11) \qquad &= RW^*_{z\zeta} + \frac{1}{4}G'(\zeta)RDW^* - R_{z\zeta}W^* - \frac{1}{4}G'(\zeta)DRW^* \\ &= (RW^*)_{z\zeta} - (R_z W^*)_\zeta - (R_\zeta W^*)_z \end{aligned}$$

since, as is clear from (3.10), $RD = DR$. We define

$$W^*(z, \hat{z}) = W(z, \hat{z}) = w(x, y),\ W^*_z(z, \zeta)\,|_{\zeta=\hat{z}} = W_z(z, \hat{z}),$$
$$W^*_\zeta(z, \zeta)\,|_{\zeta=\hat{z}} = W_{\hat{z}}(z, \hat{z}) ,$$

i.e. the solution to equation (3.3), and let this be the "initial condition" for the extension of W^* as an analytic function in (z, ζ). Let W^* be assumed to be a solution of $L[W^*] = 0$ for $(z, \zeta) \in S^2(z_1, z_2, p)$. We shall want to integrate (3.11), when W^* is such a solution and R is a Riemann function, over "triangles" Δ_1 of $S^2(z_1, z_2, p)$ with vertices $(\hat{\zeta}^0, \zeta^0)$, (z^0, ζ^0) and $(z^0, \hat{z}^0)$, over "triangles" Δ_2 with vertices $(\hat{\zeta}^0, \zeta^0)$, $(\hat{\zeta}^0, \hat{z}^0)$ and $(z^0, \hat{z}^0)$ and over "squares" with vertices $(c, \hat{c})$, $(z, \hat{c})$, $(z, \hat{z})$, $(c, \hat{z})$, c being a point of κ. Over such "regions" as these, we have:

$$(3.12) \qquad 0 = -\oint (RW^*)_t dt + \oint R_t W^* dt - \oint R_\sigma W^* d\sigma .$$

Consider $R[z^0, \zeta^0, t, \sigma]$ and $W^*(t, \sigma)$ in the above, where the region is the "triangle" $\subset S^2(z_1, z_2, p)$ with vertices $(\hat{\zeta}^0, \zeta^0)$, (z^0, ζ^0), $(z^0, \hat{z}^0)$. We get, due to the nature of $R[z^0, \zeta^0; t, \sigma]$,

$$\begin{aligned} W^*(z^0, \zeta^0) &= W^*(\hat{\zeta}^0, \zeta^0) \\ (3.13) \qquad &+ \int_{(\hat{\zeta}^0,\zeta^0)}^{(z^0,\hat{z}^0)} \{(R[z^0, \zeta^0; t, \hat{t}\,]W^*[t, \hat{t}\,])_t - R_t W^*\} dt \\ &+ \int_{(\hat{\zeta}^0,\zeta^0)}^{(z^0,\hat{z}^0)} R_\sigma[z^0, \zeta^0; \hat{\sigma}, \sigma] W^*(\hat{\sigma}, \sigma) d\sigma . \end{aligned}$$

Next we consider $R[\hat{\zeta}^0, \hat{z}^0; t, \sigma]$ and $W^*(t, \sigma)$ in (3.12) and integrate over the triangle $\subset S^2(z_1, z_2, p)$ with vertices $(\hat{\zeta}^0, \zeta^0)$, $(z^0, \hat{z}^0)$, $(\hat{\zeta}^0, \hat{z}^0)$ and get, making use of the special character of the Riemann function

$$
\begin{aligned}
W^*(\hat{\zeta}^0, \hat{z}^0) = W^*(z^0, \hat{z}^0) & \\
& - \int_{(\hat{\zeta}^0\, \zeta^0)}^{(z^0, \hat{z}^0)} \{(R[\hat{\zeta}^0, \hat{z}^0; t, \hat{t}]W^*(t, \hat{t}))_t - R_t W^*\} dt \\
& - \int_{(\hat{\zeta}^0\, \zeta^0)}^{(z^0, \hat{z}^0)} R_\sigma[\hat{\zeta}^0, \hat{z}^0; \hat{\sigma}, \sigma] W^*(\hat{\sigma}, \sigma) d\sigma\ .
\end{aligned} \tag{3.14}
$$

Finally we shall integrate (3.12) with $R[z, \hat{z}; t, \sigma]$ over the rectangle $\subset S^2(z_1, z_2, p)$ with vertices (c, c), (z, c), $(z, \hat{z})$, $(c, \hat{z})$ where c is assumed to be a point of κ, and thus, $(c = \hat{c})$:

$$
\begin{aligned}
W^*(z, \hat{z}) = W^*(c, \hat{z}) + W^*(z, c) - R[z, \hat{z}; c, c]W^*(c, c) & \\
& - \int_{(c,c)}^{(z,c)} R_t[z, \hat{z}; t, c] W^*(t, c) dt \\
& - \int_{(c,c)}^{(c,\hat{z})} R_\sigma[z, \hat{z}; c, \sigma] W^*(c, \sigma) d\sigma\ .
\end{aligned} \tag{3.15}
$$

This gives the representation of the solution of (3.3) for which we were looking. The integrals entering (3.13), (3.14) and (3.15) are independent of the path p since in (3.13) and (3.14)

$$
\begin{aligned}
[(RW^*)_t - (R_t W^*)]_\sigma - [R_\sigma W^*]_t &= R_\sigma W_t^* + RW_{t\sigma}^* - R_{\sigma t} W^* - R_\sigma W_t^* \\
&= RL_{t\sigma}[W^*] - L_{\sigma t}[R] W^* = 0
\end{aligned}
$$

by (3.11).

Next we show that $W^*(z, \zeta)$ as defined by (3.13) is an analytic function of z and ζ for z in Ω and $\zeta \in \hat{\Omega}$. This is done by showing $\partial W^*(z, \zeta)/\partial \bar{z} = 0$ and $\partial W^*(z, \zeta)/\partial \bar{\zeta} = 0$; i.e. the Cauchy Riemann equations are satisfied. Since R is an analytic function of its arguments, $\hat{\hat{z}} = z$, $\hat{z} = \overline{G(z)}$, $d\hat{z}/d\bar{z} = \overline{G'(z)}$,

$$
\overline{G'(z)} W^*(z, \hat{z}) R_\sigma(z, \zeta; \hat{\sigma}, \sigma) |_{\hat{\sigma}=z, \sigma=\hat{z}} = 0
$$

by the nature of R. Next we check analyticity in ζ.

$$
\begin{aligned}
\overline{G'(\zeta)} W_{\hat{\zeta}}^*(\hat{\zeta}, \zeta) - \overline{G'(\zeta)}[(R(z, \zeta; t, \hat{t}) W^*(t, \hat{t}))_t - R_t W^*(t, \hat{t})]_{t=\hat{\zeta}\, \hat{t}=\zeta} & \\
= \overline{G'(\zeta)}[W_{\hat{\zeta}}^*(\hat{\zeta}, \zeta) - R(z, \zeta; t, \hat{t}) W_t^*(t, \hat{t}) |_{t=\hat{\zeta}\, \hat{t}=\zeta}] = 0 &
\end{aligned}
$$

by the nature of $R = J_0$. Note that (3.14) can be got from (3.13) simply by substitution. The representation of the solutions of (1.1), which we shall use, is given with the aid of (3.1) and (3.15) by:

$$
\begin{aligned}
U^*(z, \hat{z}) = \exp\left\{A^* z + B^* G(\hat{z})\right\}\Big\{ W^*(z, c) + W^*(c, \hat{z}) & \\
& - R[z, \hat{z}; c, c] W^*(c, c) \\
& - \int_c^z R_t[z, \hat{z}; t, c] W^*(t, c) dt \\
& - \int_c^{\hat{z}} R_\sigma[z, \hat{z}; c, \sigma] W^*(c, \sigma) d\sigma \Big\}
\end{aligned} \tag{3.16}
$$

where

$$A^* = -\frac{1}{4}(A - iB)\ , \qquad B^* = -\frac{1}{4}(A + iB)\ .$$

4. Reflection of solutions of (1.1) across analytic boundary conditions. Before proving the reflection theorems we shall need to prove two lemmas.

LEMMA 1. *Let $\mu = \mu_1 + \mu_2$, μ_1 and μ_2 nonnegative integers,*

$$D_z = \frac{1}{2}(D_x - iD_y)\ , \qquad D_{\hat z} = \frac{1}{2}G'(\hat z)(D_x + iD_y) \tag{4.1}$$

then for functions

$$M(x, y) = M\left[\frac{z + G(\hat z)}{2}, \frac{z - G(\hat z)}{2i}\right] = M^*(z, \hat z)$$

that are analytic in x and y, the following operator relation holds:

$$\begin{aligned} D_x^{\mu_1} D_y^{\mu_2} = (i)^{\mu_2}\Big\{ & \sum_{j=0}^{\mu} \alpha_j^{\mu_1\mu_2}(G'(\hat z))^{-j} D_z^{\mu-j} D_{\hat z}^{j} - \delta(2, \mu)\,\alpha_\mu^{\mu_1\mu_2}\binom{\mu}{\mu - 2} \\ & \cdot (G'(\hat z))^{-(\mu+1)}(G''(\hat z)) D_{\hat z}^{\mu-1} \\ & + \text{terms of order} \leqq \mu - 1 \text{ in } D_z \text{ and } D_{\hat z} \text{ all} \\ & \text{of which contain terms of order at least} \\ & \text{one and not greater than } \mu - 2 \text{ in } D_{\hat z}\text{; i.e.} \\ & \text{all of order} \leqq \mu - 2 \text{ in } D_z \Big\}\ , \end{aligned} \tag{4.2}$$

where $\alpha_0^{\mu_1\mu_2}, \alpha_1^{\mu_1\mu_2}, \alpha_2^{\mu_1\mu_2}, \cdots, \alpha_\mu^{\mu_1\mu_2}$ are the coefficients of $a^\mu, a^{\mu-1}b, \cdots, b^\mu$ in $(a + b)^{\mu_1}(a - b)^{\mu_2}$ (Note $\alpha_0^{\mu_1\mu_2} = 1$, $\alpha_\mu^{\mu_1\mu_2} = (-1)^{\mu_2}$) and

$$\begin{aligned} \delta(\nu, \mu) &= 0 \text{ if } \nu > \mu\ , \\ &= 1 \text{ if } \nu \leqq \mu\ . \end{aligned}$$

Proof. By induction on μ. For $\mu = 0$ clearly true. For $\mu = 1$ we have from (4.1) for $\mu_1 = 1$, $\mu_2 = 0$

$$D_x = D_z + (G'(\hat z))^{-1} D_{\hat z} \qquad \text{which is (4.2) for } \mu_1 = 1,\ \mu_2 = 0$$

and for $\mu_1 = 0$, $\mu_2 = 1$

$$D_y = iD_z - i(G'(\hat z))^{-1} D_{\hat z} \qquad \text{which is (4.2) for } \mu_1 = 0,\ \mu_2 = 1\ .$$

Assume (4.2) true for μ, we must show it is ture for $\mu + 1$. i.e., assume (4.2) and then consider

$$D_x^{\mu_1+1}D_y^{\mu_2} = (i)^{\mu_2}\Big\{\sum_{j=0}^{\mu} \alpha_j^{\mu_1\mu_2}[(G'(\hat{z}))^{-j}D_z^{\mu+1-j}D_{\hat{z}}^j + (G'(\hat{z}))^{-(j+1)}D_z^{\mu-j}D_{\hat{z}}^{j+1}]$$

$$- \alpha_\mu^{\mu_1\mu_2}(G'(\hat{z}))^{-(\mu+2)}G''(\hat{z})\Big[\mu + \binom{\mu}{\mu-2}\Big]D_{\hat{z}}^\mu$$

(A) $+$ terms of order $\leqq \mu$ in D_z and $D_{\hat{z}}$ all containing terms of order at least one and not greater than $\mu - 1$ in $D_{\hat{z}}\Big\}$. i.e.,

$$D_x^{\mu_1+1}D_y^{\mu_2} = (i)^{\mu_2}\Big\{D_z^{\mu+1} + \sum_{j=1}^{\mu} (G'(\hat{z}))^{-j}D_z^{\mu+1-j}D_{\hat{z}}^j[\alpha_j^{\mu_1\mu_2} + \alpha_{j-1}^{\mu_1\mu_2}]$$

$$+ \alpha_\mu^{\mu_1\mu_2}(G'(\hat{z}))^{-(\mu+1)}D_{\hat{z}}^{\mu+1}$$

$$- \alpha_\mu^{\mu_1\mu_2}(G'(\hat{z}))^{-(\mu+2)}G''(\hat{z})\binom{\mu+1}{\mu-1}D_{\hat{z}}^\mu + \cdots\Big\}.$$

But

$$(a + b)(a + b)^{\mu_1}(a - b)^{\mu_2}$$

$$= (a + b)\{a^\mu + \alpha_1^{\mu_1\mu_2}a^{\mu-1}b + \alpha_2^{\mu_1\mu_2}a^{\mu-2}b^2 + \cdots + \alpha_\mu^{\mu_1\mu_2}b^\mu\}$$

$$= a^{\mu+1} + (\alpha_1^{\mu_1\mu_2} + 1)a^\mu b + (\alpha_1^{\mu_1\mu_2} + \alpha_2^{\mu_1\mu_2})a^{\mu-1}b^2$$

$$+ \cdots + (\alpha_{\mu-1}^{\mu_1\mu_2} + \alpha_\mu^{\mu_1\mu_2})ab^\mu$$

$$+ \alpha_\mu^{\mu_1\mu_2}b^{\mu+1}$$

and thus we see

$$D_x^{\mu_1+1}D_y^{\mu_2} = (i)^{\mu_2}\Big\{\sum_{j=0}^{\mu+1} \alpha_j^{\mu_1+1\mu_2}(G'(\hat{z}))^{-j}D_z^{\mu+1-j}D_{\hat{z}}^j$$

$$- \alpha_{\mu+1}^{\mu_1+1\mu_2}(G'((\hat{z}))^{-(\mu+2)}G''(\hat{z})\binom{\mu+1}{\mu-1}D_{\hat{z}}^\mu + \cdots\Big\}$$

where $\alpha_j^{\mu_1+1\mu_2}$ are the coefficients of $a^{\mu+1}, a^\mu b, \cdots, b^{\mu+1}$ in $(a+b)^{\mu_1+1}(a-b)^{\mu_2}$. Now consider

$$D_x^{\mu_1}D_y^{\mu_2+1} = (i)^{\mu_2+1}\Big\{D_z^{\mu+1} + \sum_{j=1}^{\mu} (G'(\hat{z}))^{-j}D_z^{\mu+1-j}D_{\hat{z}}^j[\alpha_{j+1}^{\mu_1\mu_2} - \alpha_j^{\mu_1\mu_2}]$$

$$- \alpha_\mu^{\mu_1\mu_2}(G'(\hat{z}))^{-(\mu+1)}D_{\hat{z}}^{\mu+1}$$

(B)
$$+ \alpha_\mu^{\mu_1\mu_2}(G'(\hat{z}))^{-(\mu+2)}G''(\hat{z})\Big[\mu + \binom{\mu}{\mu-2}\Big]D_{\hat{z}}^\mu + \cdots\Big\}$$

$$= (i)^{\mu_2+1}\Big\{\sum_{j=0}^{\mu+1} \alpha_j^{\mu_1\mu_2+1}(G'(\hat{z}))^{-j}D_z^{\mu+1-j}D_{\hat{z}}^j$$

$$- \alpha_{\mu+1}^{\mu_1\mu_2+1}(G'(\hat{z}))^{-(\mu+2)}G''(\hat{z})\binom{\mu+1}{\mu-1}D_{\hat{z}}^\mu + \cdots\Big\}$$

where since

$$(a - b)(a + b)^{\mu_1}(a - b)^{\mu_2} = a^{\mu+1} + (\alpha_1^{\mu_1\mu_2} - 1)a^\mu b + (\alpha_2^{\mu_1\mu_2} - \alpha_1^{\mu_1\mu_2})a^{\mu-1}b^2$$
$$+ \cdots + (\alpha_\mu^{\mu_1\mu_2} - \alpha_{\mu-1}^{\mu_1\mu_2})ab^\mu - \alpha_\mu^{\mu_1\mu_2}b^{\mu+1}$$

the $\alpha_j^{\mu_1\mu_2}$ are the coefficients of $a^{\mu+1}$, $a^\mu b, \cdots b^{\mu+1}$ in $(a + b)^{\mu_1}(a - b)^{\mu_2+1}$. Thus the lemma is proved.

LEMMA 2. *Given the operators*

$$p_{i1}(D) + p_{i2}(D) + \cdots + p_{in}(D) \qquad i = 1, 2, \cdots, n$$

where

$$p_{i1}(D) = \sum_{r+s\leqq k} p_{i1}^{rs}(z) D_x^r D_y^s,\ p_{i2}(D) = \sum_{r+s\leqq k} p_{i2}^{rs}(z) D_x^r D_y^s,\ \cdots$$

then for

$$D_z = \frac{1}{2}(D_x - iD_y),\ D_{\hat{z}} = \frac{1}{2}G'(\hat{z})(D_x + iD_y)$$

$$\begin{aligned}(p_{ij}(D)) = M = {} & M_{k,0}(z)D_z^k + M_{k-1,1}(z)(G'(\hat{z}))^{-1}D_z^{k-1}D_{\hat{z}} + M_{k-1,0}(z)D_z^{k-1} \\ & + M_{0,k}(z)(G'(\hat{z}))^{-k}D_{\hat{z}}^k + M_{1,k-1}(z)(G'(\hat{z}))^{-(k-1)}D_z D_{\hat{z}}^{k-1} \\ & + M_{0,k-1}(z)D_{\hat{z}}^{k-1} + T_{k-2}(z)\end{aligned}$$

where $T_{k-2}(z)$ *is a matrix of terms in* D_z *and* $D_{\hat{z}}$ *and of order* $\leqq k-2$ *in* D_z *and of order* $\leqq k-2$ *in* $D_{\hat{z}}$.

$$M_{k,0}(z) = \sum_{r+s=k} (i)^s \begin{pmatrix} p_{11}^{rs} & p_{12}^{rs} \cdots p_{1n}^{rs} \\ \vdots & \\ p_{n1}^{rs} & \cdots p_{nn}^{rs}\end{pmatrix} = \sum_{r+s=k} (i)^s (p_{\alpha\beta}^{rs})$$

$$M_{k-1,1}(z) = \sum_{r+s=k} (i)^s \alpha_1^{rs}(p_{\alpha\beta}^{rs}),\ M_{k-1,0}(z) = \sum_{r+s=k-1} (i)^s (p_{\alpha\beta}^{rs})$$

$$M_{0,k}(z) = \sum_{r+s=k} (-i)^s (p_{\alpha\beta}^{rs}),\ M_{1,k-1}(z) = \sum_{r+s=k} (i)^s \alpha_{k-1}^{rs}(p_{\alpha\beta}^{rs})$$

$$\begin{aligned}M_{0,k-1}(z) = {} & \sum_{r+s=k-1} (i)^s \alpha_{k-1}^{rs}(G'(\hat{z}))^{-(k-1)}(p_{\alpha\beta}^{rs}) - \delta(2, k) \\ & \times \sum_{r+s=k} (i)^s \alpha_k^{rs} \binom{k}{k-2} (G'(\hat{z}))^{-(k+1)}(G''(\hat{z}))(p_{\alpha\beta}^{rs})\end{aligned}$$

where α_l^{rs} *are the same as in Lemma* 1.

Proof. By Lemma 1

$$\begin{aligned}p_{\alpha 1}(D) = {} & \sum_{r+s=k} p_{\alpha 1}^{rs}\Big\{(i)^s\Big[\sum_{j=0}^{k} \alpha_j^{rs}(G'(\hat{z}))^{-j}D_z^{k-j}D_{\hat{z}}^j \\ & - \delta(2, k)\alpha_k^{rs}\binom{k}{k-2}(G'(\hat{z}))^{-(k+1)}(G''(\hat{z}))D_{\hat{z}}^{k-1}\Big]\Big\} \\ & + \sum_{r+s=k-1} (i)^s p_{\alpha 1}^{rs} D_z^{k-1} + \sum_{r+s=k-1} (i)^s p_{\alpha 1}^{rs}\alpha_{k-1}^{rs}(G'(\hat{z}))^{-(k-1)}D_{\hat{z}}^{k-1} \\ & + q_{i1}(D_z, D_{\hat{z}}) \text{ where } q_{i1} \text{ is a polynomial in } D_z \text{ and } D_{\hat{z}}\end{aligned}$$

of degree $\leqq k - 1$ with coefficients analytic in $\Omega \cup \kappa \cup \hat{\Omega}$ and which contains terms of order $\leqq k - 2$ in D_z and of order $\leqq k - 2$ in $D_{\hat{z}}$. Similar results hold for $p_{\alpha\beta}(D)$, $\beta = 2, \cdots, n$. Combining these results with the fact that $\alpha_k^{rs} = (-1)^s$, we get the conclusion of the lemma.

We are now in a position to prove

THEOREM 1. *Let* $u = (u_1, u_2, \cdots, u_n) \in C^2(\Omega)$ *and satisfy in* Ω

$$u_{xx} + u_{yy} + Au_x + Bu_y + Cu = 0 \tag{4.3}$$

where A, B, C *are pairwise commutative constant* $n \times n$ *matrices. Moreover let* $u \in C^k(\Omega \cup \kappa) \cap C'(\Omega \cup \kappa)$ *and satisfy on* κ *the boundary conditions*:

$$\sum_{\beta=1}^{n} p_{\alpha\beta}(z, D_x, D_y)u_\beta = f_\alpha(z) , \qquad \alpha = 1, 2, \cdots, n ,$$

where

$$p_{\alpha\beta}(z, \xi, \eta) = \sum_{r+s \leq k < 2n} p_{\alpha\beta}^{rs}(z)\xi^r\eta^s ,$$

with $p_{\alpha\beta}^{rs}(z)$ *and* $f_\alpha(z)$ *analytic in* $\Omega \cup \kappa \cup \hat{\Omega}$. *Moreover, if* $P_{\alpha\beta}(z, \xi, \eta)$ *is the principal part of* $p_{\alpha\beta}(z, \xi, \eta)$, *we assume in* $\Omega \cup \kappa \cup \hat{\Omega}$

$$0 \neq |M_{k,0}(z)| = |P_{\alpha\beta}(z, 1, -i)|^1 \tag{4.4}$$

and

$$0 \neq |M_{0,k}(z)| = |P_{\alpha\beta}(z, 1, -i)|^1 .$$

Then we can reflect u *across* κ *into* $\hat{\Omega}$; *i.e., there exists a unique function* u *which is a solution of* (1.1) *in* $\Omega \cup \kappa \cup \hat{\Omega}$ *and which agrees with the given solutions* u *of* (1.1) *in* $\Omega \cup \kappa$.

Proof. We apply M of Lemma 2 to the representation (3.16) and evaluate on κ. For simplicity let

$$g(\hat{z}) = W^*(c, \hat{z}) , \qquad h(z) = W^*(z, c) .$$

Then we get on κ, remembering that $z = \hat{z}$ there:

$$e^{A^*z+B^*G(z)}\{M_{k,0}(z)h^{(k)}(z) + M_{0,k}(z)(G'(z))^{-k}g^{(k)}(z) + T_{k-1}[g, h, z]\} = f(z) , \tag{4.5}$$

where $f(z) = (f_1(z), \cdots, f_n(z))^T$ and $T_{k-1}[g, h, z]$ is an expression of the form:

$$\begin{aligned} T_{k-1}[g, h, z] = {} & a_{k-1}(z)h^{(k-1)}(z) + a_{k-2}(z)h^{(k-2)}(z) + \cdots + a_0(z)h(z) \\ & + \int_c^z a_{-1}(z, t)h(t)dt + b_{k-1}(z)g^{(k-1)}(z) + \cdots + b_0(z)g(z) \\ & + \int_c^z b_{-1}(z, t)g(t)dt + E(z)W^*(c, c) \end{aligned} \tag{4.6}$$

[1] As Professor Jean Leray has kindly pointed out to me, these statements concern the behavior of the boundary conditions in the characteristic directions.

where E and the a's and b's are matrices, analytic in $\Omega \cup \kappa \cup \hat{\Omega}$. Note that for $j \geqq 1$,

$$(4.7) \qquad h(z) = \frac{1}{(j-1)!}\int_c^z (z-t)^{j-1}h^{(j)}(t)dt + \sum_{\sigma=0}^{j-1}\frac{1}{\sigma!}(z-c)^\sigma h^{(\sigma)}(c)$$

and similarly for $g(z)$ where $h^{(\sigma)}(c)$ and $g^{(\sigma)}(c)$ are known via (3.13) and (3.14). Moreover if $K(z, t)$ is a matrix function known and analytic in $\Omega \cup \kappa \cup \hat{\Omega}$ then for $k \geqq 1$

$$\int_c^z K(z, t)h(t)dt = \frac{1}{(k-1)!}\int_c^z dt_1 K(z, t_1)\int_c^{t_1}(t_1 - t_2)^{k-1}h^{(k)}(t_2)dt_2 + K_1(z)$$

where $K_1(z)$ is a known matrix function analytic in $\Omega \cup \kappa \cup \hat{\Omega}$ and thus

$$(4.8) \qquad \int_c^z K(z, t)h(t)dt = \int_c^z K_2(z, t)h^{(k)}(t)dt + K_1(z)$$

where

$$K_2(z, t) = \frac{1}{(k-1)!}\int_{t_1=t}^z dt_1 K(z, t_1)(t_1 - t)^{k-1} .$$

Thus (4.5) becomes on κ with the aid of (4.7) and (4.8) and the significance of (4.6).

$$(4.9) \qquad \begin{aligned} e^{A^*z+B^*G(z)}\Big\{&M_{k,0}(z)h^{(k)}(z) + M_{0,k}(z)(G'(z))^{-k}g^{(k)}(z) \\ &+ \int_c^z K^*(z, t)h^{(k)}(t)dt + \int_c^z K^{**}(z, t)g^{(k)}(t)dt + H(z)\Big\} = f(z) \end{aligned}$$

where $H(z)$ is a known vector function of z, analytic in $\Omega \cup \kappa \cup \hat{\Omega}$, and K^* and K^{**} are known matrix functions of z, t analytic in $\Omega \cup \kappa \cup \hat{\Omega}$. Thus, since $|M_{k,0}(z)| \neq 0$ and $|M_{0,k}(z)| \neq 0$ and $G'(z) \neq 0$ in $\Omega \cup \kappa \cup \hat{\Omega}$, we can solve for $h^{(k)}(z)$ and $g^{(k)}(z)$ and get:

$$(4.10) \qquad h^{(k)}(z) = \int_c^z \tilde{K}(z, t)h^{(k)}(t)dt + \tilde{H}(z) \text{ on } \kappa$$

and

$$(4.11) \qquad g^{(k)}(z) = \int_c^z \tilde{\tilde{K}}(z, t)g^{(k)}(t)dt + \tilde{\tilde{H}}(z) \text{ on } \kappa$$

where $\tilde{K}(z, t)$ and $\tilde{\tilde{K}}(z, t)$ are known matrix functions analytic for z and t in $\Omega \cup \kappa \cup \hat{\Omega}$ and $\tilde{H}(z)$ is analytic in $\hat{\Omega}$ and continuous in $\hat{\Omega} \cup \kappa$ $\tilde{\tilde{H}}(z)$ is analytic in Ω and continuous in $\Omega \cup \kappa$.

Treating (4.10) as a system of Volterra integral equations in $\hat{\Omega} \cup \kappa$ and treating (4.11) as a system of Volterra integral equations in $\Omega \cup \kappa$, we get the analytic extension of $h^{(k)}(z)$ into $\Omega \cup \kappa \cup \hat{\Omega}$ and

$g^{(k)}(z)$ into $\Omega \cup \kappa \cup \hat{\Omega}$. By integration, since h and g and their derivatives of order $\leqq k$ are known and continuous on κ and specifically at c at the outset, we get the unique extension of $h(z)$ and $g(z)$. By means of (3.15) we get the unique analytic extension of $W^*(z, \hat{z})$ into $\Omega \cup \kappa \cup \hat{\Omega}$ and thus the extension of $u(x, y)$ into $\Omega \cup \kappa \cup \hat{\Omega}$.

We shall next concern ourselves with a system which is particularly useful when certain higher order equations are reduced to a system of equations. With this in mind, we shall consider a more restricted class of equations, since the inequality becomes very unwieldy.

Notation. Let

$$E = \begin{pmatrix} e_1^1 \\ e_2^1 \\ \vdots \\ e_n^1 \end{pmatrix}, E^2 = \begin{pmatrix} e_1^2 \\ e_2^2 \\ \vdots \\ e_n^2 \end{pmatrix}, \cdots, E^k = \begin{pmatrix} e_1^k \\ e_2^k \\ \vdots \\ e_n^k \end{pmatrix}$$

where $e_m^j = (e_{m1}^j, e_{m2}^j, \cdots, e_{mn}^j)$ is the m^{th} row of E^j.

Before stating the theorem, we shall prove

LEMMA 3.

$$\begin{aligned} R[z, \zeta; t, \sigma] &= p_{n-1}\{E, (t - z)[G(\sigma) - G(\zeta)]\} \\ &= \sum_{j=0}^{n-1} a_j\{(t - z)[G(\sigma) - G(\zeta)]\}E^j \end{aligned}$$

where $p_{n-1}(x, s)$ is the polynomial of degree $\leqq n - 1$ in x that interpolates $J_0[\sqrt{xs}]$ at the eigenvalues of E (s held fixed); $a_j(s)$ are entire functions of s and $E^0 = I$. In the event that some or all of the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_j$ of E are multiple, i.e.,

$$|E - \lambda I| = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_j)^{n_j}(-1)^n ,$$

$\lambda_i \neq \lambda_k$ if $i \neq k$, $n_1 + n_2 + \cdots + n_j = n$, then we use Hermite interpolation to determine $p_{n-1}(x)$ such that if $J(\lambda s) = J_0[\sqrt{\lambda s}]$

$$\begin{aligned} &p_{n-1}(\lambda_1, s) = J(\lambda_1 s), \frac{\partial}{\partial\lambda} p_{n-1}(\lambda_1, s) = sJ^{(1)}(\lambda_1 s), \cdots, \frac{\partial^{n_1-1}}{\partial\lambda^{n_1-1}}, p_{n-1}(\lambda_1, s) \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad = s^{n_1-1}J^{(n_1-1)}(\lambda_1 s) \\ &p_{n-1}(\lambda_j, s) = J(\lambda_j s), \frac{\partial}{\partial\lambda} p_{n-1}(\lambda_j, s) = sJ^{(1)}(\lambda_j s), \cdots, \frac{\partial^{n_j-1}}{\partial\lambda^{n_j-1}}, p_{n-1}(\lambda_j, s) \\ &\qquad\qquad\qquad\qquad\qquad\qquad = s^{n_j-1}J^{(n_j-1)}(\lambda_j s) . \end{aligned}$$

Proof. The unique Hermite interpolation polynomial $p_{n-1}(\lambda, s)$ is of the form:

$$p_{n-1}(\lambda, s) = \sum_{i=1}^{j} J(\lambda_i s) l_{i0}(\lambda) + \sum_{i=1}^{j} s J^{(1)}(\lambda_i s) l_{i1}(\lambda) + \cdots + \sum_{i=1}^{j} s^{n_i - 1} J^{(n_i - 1)}(\lambda_i s) l_{i(n_i - 1)}(\lambda)$$

where the $l_{ik}(\lambda)$ are polynomials in λ of degree $\leqq n - 1$.

Consider for $Q(\lambda)$ the characteristic polynomial of E:

$$f(\lambda, s) = \frac{J(\lambda s) - p_{n-1}(\lambda, s)}{Q(\lambda)} .$$

$J(\lambda s)$ and $p_{n-1}(\lambda, s)$ are entire functions of λ and s, moreover the polynomial $Q(\lambda)$ has the same zeros in λ (multiplicity included) as $J(\lambda s) - p_{n-1}(\lambda, s)$. Thus $f(\lambda, s)$ is an entire function of λ and s. Rearranging, we get

$$J(\lambda s) = Q(\lambda) f(\lambda, s) + p_{n-1}(\lambda, s) .$$

But $Q(\lambda)$ is the characteristic polynomial of E. Thus by the Cayley-Hamilton theorem $Q(E) = 0$ and

$$J(Es) = p_{n-1}(E, s)$$

which gives the result since

$$J\{E(t - z)[G(\sigma) - G(\zeta)]\} = J_0\{\sqrt{E(t - z)[G(\sigma) - G(\zeta)]}\} = R\{z, \zeta; t, \sigma\} .$$

Now we are in a position to state the theorem.

THEOREM 2. (H1) *Let κ be an analytic arc of the type described in § 2 for which the determinant of the $(n - 1) \times (n - 1)$ matrix*

(4.12) $\Delta_4(z) = \| D_z^{\nu}[G(c) - G(z)]^j \| \neq 0 , \quad 1 \leqq \nu \leqq n - 1, 1 \leqq j \leqq n - 1$

for z on $\Omega \cup \kappa \cup \hat{\Omega}$ (arc condition).

(H2) $$det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ & a_j^{(1)}(0) & & \\ & \vdots & & \\ & a_j^{(n-1)}(0) & & \end{pmatrix} \neq 0$$

where a_j are those of Lemma 3 (differential equation condition).

(H3) *Let $u = (u_1, u_2, \cdots, u_n)^T \in C^k(\Omega \cup \kappa) \cap C^2(\Omega)$, $(n \leqq k \leqq 2n - 1, n \geqq 2)$ and satisfy in Ω*

(4.13) $$u_{xx} + u_{yy} + Eu = 0$$

where E is a constant $n \times n$ matrix for which

$$(4.14)\qquad \Delta_1 \equiv \begin{vmatrix} 1, & 0, & \cdots, 0 \\ (E)_{11}, & (E)_{12}, & \cdots, (E)_{1n} \\ (E^2)_{11}, & (E^2)_{12}, & \cdots, (E^2)_{1n} \\ \vdots & & \\ (E^{n-1})_{11}, & (E^{n-1})_{12}, & \cdots, (E^{n-1})_{1n} \end{vmatrix} \neq 0 \qquad \textit{(differential equation condition)}$$

where $(E^k)_{ij}$ *is the ij component of the* k^{th} *power of* E.

(H4) *Let* u *satisfy on the analytic arc* κ *the boundary conditions*

$$(4.15)\qquad p_{\alpha 1}(z, D_x, D_y)u_1 = f_\alpha(z)\ , \qquad \alpha = 1, 2, \cdots, n\ ,$$

where

$$p_{\alpha 1}(z, \xi, \eta) = \sum_{r+s \leqq k < 2n} p_{\alpha 1}^{rs}(z)\xi^r \eta^s\ ,$$

with $p_{\alpha 1}^{rs}(z)$ *and* $f_\alpha(z)$ *analytic in* $\Omega \cup \kappa \cup \hat{\Omega}$. *Moreover if* $P_{\alpha 1}(z, \xi, \eta)$ *is the principal part of* $p_{\alpha 1}(z, \xi, \eta)$ *(as polynomial in* ξ *and* η*), we assume for*

$$\Delta_2(z, \xi, \eta) = \begin{vmatrix} D_\xi^{n-1}P_{11}(z, \xi, \eta), & D_\xi^{n-2}D_\eta P_{11}, & \cdots, & D_\eta^{n-1}P_{11} \\ D_\xi^{n-1}P_{12}(\qquad), & D^{n-2}D_\eta P_{21}, & \cdots, & D_\eta^{n-1}P_{21} \\ \vdots & & & \\ D_\xi^{n-1}P_{n1}(\qquad), & D_\xi^{n-2}D_\eta P_{n1}, & \cdots, & D_\eta^{n-1}P_{n1} \end{vmatrix}$$

that for $z \in \Omega \cup \kappa \cup \Omega$

$$(4.16)\qquad \Delta_2 = \Delta_2(z, \xi, \eta)_{\xi=1, \eta=i} \neq 0^2$$

and

$$(4.17)\qquad \Delta_3 = \Delta_2(z, \xi, \eta)_{\xi=1, \eta=-i} \neq 0^2\ .$$

Then $u = (u_1, u_2, \cdots, u_n)^T$ *can be reflected across the boundary conditions* (4.15) *into* $\Omega \cup \kappa \cup \hat{\Omega}$.

Before proceeding to the proof of Theorem 2, we shall state Theorem 3, which deals with the case $k = n - 1$, since the proof of Theorem 3 follows the same lines as the proof of Theorem 2. Only in the proof of Theorem 3, Lemmas 4, 5, 4A and 5A are unnecessary.

THEOREM 3. *Let* κ *be an analytic arc of the type described in* § 2.

(H1*) *Let* $u = (u_1, u_2, \cdots, u_n)^T \in C^{n-1}(\Omega \cup \kappa) \cap C^2(\Omega)$ *and satisfy in* Ω

[2] These, as Professor Jean Leray has kindly pointed out to me, are conditions on the behavior of the boundary conditions in the characteristic direction. He has also proved that Conditions (H1) and (H2) are always satisfied, i.e. they are unnecessary restrictions.

$$u_{xx} + u_{yy} + Eu = 0$$

where E is a constant $n \times n$ matrix for which

$$(4.14^*) \qquad \Delta_1 \neq 0 \qquad \text{(see Th. 2)}.$$

(H2*) *Let u satisfy on the analytic arc κ the boundary conditions*

$$(4.15^*) \qquad p_\nu(D)u_1 = \sum_{r+s \leqq n-1} p_{\nu 1}^{rs}(z) D_x^r D_y^s u_1 = f_\nu(z) , \qquad \nu = 1, 2, \cdots, n ,$$

where $p_{\nu 1}^{rs}(z)$ and $f_\nu(z)$ are analytic in $\Omega \cup \kappa \cup \hat{\Omega}$. Also assume in $\Omega \cup \kappa \cup \hat{\Omega}$

$$(4.16)^* \qquad \Delta_2(z, \xi, \eta)_{\xi=1,\eta=i} \neq 0 \qquad \text{(see Th. 2 with } k = n-1).$$

$$(4.17)^* \qquad \Delta_2(z, \xi, \eta)_{\xi=1,\eta=-i} \neq 0 \qquad \text{(see Th. 2 with } k = n-1).$$

Then $u = (u_1, u_2, \cdots, u_n)^T$ can be reflected across the boundary conditions (4.15*) *into $\Omega \cup \kappa \cup \hat{\Omega}$. Moreover the reflection can be reduced to quadratures.*

Proof of Theorem 2. We first consider (3.16) with $A^* = B^* = 0$ and

$$g(\hat{z}) = W^*(c, \hat{z}) , \qquad h(z) = W^*(z, c)$$

and get:

$$(4.18) \qquad \begin{aligned} U^*(z, \hat{z}) &= h(z) + g(\hat{z}) - R[z, \hat{z}; c, c]h(c) \\ &\quad - \int_c^z R_t[z, \hat{z}; t, c]h(t)dt - \int_c^{\hat{z}} R_\sigma[z, \hat{z}; c, \sigma]g(\sigma)d\sigma \end{aligned}$$

where, since it was shown that $W^*(z, \zeta)$ is an analytic function of z for z in Ω and an analytic function of ζ for ζ in $\hat{\Omega}$, then $h(z)$ is an analytic function of z for z in Ω and $g(\zeta)$ is an analytic function of ζ for ζ in $\hat{\Omega}$. From (3.13) and (3.14) and (H.3) we see that $h(z) \in C^k(\Omega \cup \kappa)$ and $g(z) \in C^k(\hat{\Omega} \cup \kappa)$.

With the aid of Lemma 3 we get

$$(4.19) \qquad R_t[z, \hat{z}; t, c] = \sum_{j=0}^{n-1} [G(c) - G(\hat{z})] a_j^{(1)} \{(t - z)[G(c) - G(\hat{z})]\} E^j$$

$$(4.20) \qquad R_\sigma[z, \hat{z}; c, \sigma] = \sum_{j=0}^{n-1} G'(\sigma)(c - z) a_j^{(1)} \{(c - z)[G(\sigma) - G(\hat{z})]\} E^j .$$

Let

$$(e_m^j, h) = e_{m1}^j h_1 + e_{m2}^j h_2 + \cdots + e_{mn}^j h_n .$$

Then the first component of (4.18) becomes

(4.21) $U_1^*(z, \hat{z}) = h_1^*(z, \hat{z}) + g_1^*(z, \hat{z}) - \{R[z, \hat{z}; c, c]h(c)\}_{\text{1st component}}$

where

$$h_1^*(z, \zeta) = h_1(z) - G_1(\zeta) \sum_{j=0}^{n-1} \int_c^z a_j^{(1)}\{(t - z)G_1(\zeta)\}(e_1^j, h(t))dt \tag{4.22}$$

with

$$G_1(\zeta) = G(c) - G(\zeta)$$

and

$$\begin{aligned} g_1^*(z, \zeta) = g_1(\zeta) - (c - z) \sum_{j=0}^{n-1} \int_c^\zeta a_j^{(1}\{(c - z)[G(\sigma) - G(\zeta)]\} \\ \times G'(\sigma)(e_1^j, g(\sigma))d\sigma \, . \end{aligned} \tag{4.23}$$

Note that $h_1^*(z,\zeta)$ is analytic for (z,ζ) on $\Omega \times \Omega$ and $\in C^k[(\Omega \cup \kappa) \times (\Omega \cup \kappa)]$ and that $g_1^*(z,\zeta)$ is analytic for (z,ζ) on $\hat{\Omega} \times \hat{\Omega}$ and $\in C^k[(\hat{\Omega} \cup \kappa) \times (\hat{\Omega} \cup \kappa)]$.

With the aid of Lemma 1, the boundary conditions can be written:

$$\begin{aligned} f_\nu(z) = \sum_{r+s=k} p_{\nu 1}^{rs}(z)(i)^s \left\{ \sum_{m=0}^{k} \alpha_m^{rs}[G'(\hat{z})]^{-m} D_z^{k-m} D_{\hat{z}}^m \right\} \cdot U_1^*(z, \hat{z}) \\ + \text{terms of order } l + m \leqq k - 1 \text{ in } D_z^l D_{\hat{z}}^m U_1^* \, . \end{aligned} \tag{4.24}$$

Apply the boundary conditions to (4.21), evaluate on the boundary κ, remembering that on κ, $z = \hat{z}$, and substitute the new functions $\tilde{h}_j(z, z)$ and $\tilde{g}_j(z, z)$, $(0 \leqq j \leqq 2n - 1)$ where

$$\tilde{h}_j(z, \zeta) = D_\zeta^j h_1^*(z, \zeta) \, , \qquad \tilde{g}_j(z, \zeta) = D_z^j g_1^*(z, \zeta) \tag{4.25}$$

with $D^\circ f = f$, $D^\circ g = g$. Thus the boundary conditions become, since $k \geqq n - 1$, for z on κ

$$\begin{aligned} \tilde{f}_\nu(z) = \sum_{r+s=k} p_{\nu 1}^{rs}(z)(i)^s \left\{ \sum_{m=0}^{n-1} + \sum_{m=n}^{k}{}' \right\} \alpha_m^{rs}[G'(z)]^{-m} D_z^{k-m} \tilde{h}_m(z, \zeta) \Big]_{\zeta=z} \\ + \text{terms of order } l + m \leqq k - 1 \text{ in } D_z^l \tilde{h}_m(z, z) \\ + \sum_{r+s=k} p_{\nu 1}^{rs}(z)(i)^s \left\{ \sum_{m=0}^{n-1} + \sum_{m=n}^{k}{}' \right\} \alpha_{k-m}^{rs}[G'(z)]^{m-k} D_\zeta^{k-m} \tilde{g}_m(z, \zeta) \Big]_{\zeta=z} \\ + \text{terms of order } l + m \leqq k - 1 \text{ in } D_\zeta^l \tilde{g}_m(z, z) \\ \nu = 1, 2, \cdots, n \end{aligned} \tag{4.26}$$

(Σ' indicates we sum when $k \geqq n$),

where $\tilde{f}_\nu(z)$ is known and analytic in $\Omega \cup \kappa \cup \hat{\Omega}$. It should be remembered that the first two terms (involving only $\tilde{h}_m$ and their derivatives) are analytic functions of z for z in Ω and that the last two terms (involving only $\tilde{g}_m$ and their derivatives) are analytic functions of z for z in $\hat{\Omega}$

Rearranging terms in (4.26) yields for z on κ:

$$\sum_{m=0}^{n-1}\beta^k_{\nu m}(z)[G'(z)]^{-m}D_z^{k-m}\tilde{h}_m(z,\zeta)\Big]_{\zeta=z}$$

$$+\sum_{m=0}^{n-1}\beta^k_{\nu,k-m}(z)[G'(z)]^{m-k}D_\zeta^{k-m}\tilde{g}_m(z,\zeta)\Big]_{\zeta=z} \tag{4.27}$$

$$= A_\nu(z)+\hat{A}_\nu(z)+\tilde{f}_\nu(z)$$

where

$$A_\nu(z) = -\sum_{m=n}^{k\geqq n}\beta^k_{\nu m}(z)[G'(z)]^{-m}D_z^{k-m}\tilde{h}_m(z,\zeta)\Big]_{\zeta=z}$$

$$+ \text{ terms of order } l+m\leqq k-1 \text{ in } D_z^l\tilde{h}_m(z,\zeta)\Big]_{\zeta=z} \tag{4.28}$$

with coefficients analytic in $\Omega\cup\kappa\cup\hat{\Omega}$.

$$\hat{A}_\nu(z) = -\sum_{m=n}^{k\geqq n}\beta^k_{\nu,k-m}(z)[G'(z)]^{m-k}D_\zeta^{k-m}\tilde{g}_m(z,\zeta)\Big]_{\zeta=z}$$

$$+ \text{ terms of order } l+m\leqq k-1 \text{ in } D_\zeta^l\tilde{g}_m(z,\zeta)\Big]_{\zeta=z}$$

with coefficients analytic in $\Omega\cup\kappa\cup\hat{\Omega}$.

$$\beta^k_{\nu\mu}(z) = \sum_{r+s=k}p^{rs}_{\nu 1}(z)(i)^s\alpha^{rs}_\mu \tag{4.29}$$

with $A_\nu(z)$ analytic in Ω, $\hat{A}_\nu(z)$ analytic in $\hat{\Omega}$, and $\beta^k_{\nu m}(z)$ and $\tilde{f}_\nu(z)$ analytic in $\Omega\cup\kappa\cup\hat{\Omega}$.

Our next goal is to convert (4.27) into a system of Volterra integral equations for the n functions $D_z^{k-m}\tilde{h}_m(z,z)$, $0\leqq m\leqq n-1$. On κ, the system is to be satisfied and we shall see that they also have an analytic solution for z in $\hat{\Omega}\cup\kappa$. With this in mind we state and prove two lemmas.

LEMMA 4. *Let* $k\geqq n$

$$\alpha_{k-m}(z) = D_z^{k-m}D_\zeta^m h_1^*(z,\zeta)\Big]_{\zeta=z} = D_z^{k-m}\tilde{h}_m(z,\zeta)\Big]_{\zeta=z}, \qquad 0\leqq m\leqq k \tag{4.30}$$

and the hypotheses (H1), (H2), *and* (H3) *of the theorem hold. Then*

$$\alpha_{k-m}(z) = \sum_{r=0}^{n-1}\int_c^z K_{k-m,r}(t,z)\alpha_{k-r}(t)dt + C_{k-m}(z), \qquad n\leqq m\leqq k \tag{4.31}$$

where $K_{k-m,r}(t,z)$ *are analytic for* t *and* z *in* $\Omega\cup\kappa\cup\hat{\Omega}$ *and* $C_{k-m}(z)$ *is analytic for* z *in* $\Omega\cup\kappa\cup\hat{\Omega}$.

Proof. Since the $a_j(\sigma)$ occuring in (4.22) are entire functions, the following Taylor's expansion with remainder is valid:

$$a_j^{(1)}[(t-z)G_1] = p_{n-2}(t-z)+R_{j,n-2}(t,z)$$

where $p_{n-2}(\sigma)$ is a polynomial of degree $n-2$ in σ and

$$R_{j,n-2}(t, z) = \frac{G_1^{n-1}}{(n-2)!}\int_z^t (t-\sigma)^{n-2} a_j^{(n)}[(\sigma - z)G_1]d\sigma .$$

Introducing this into (4.22) and interchanging the order of integration for the remainder yields:

$$\begin{aligned} h_1^*(z, \zeta) = h_1(z) &- G_1(\zeta)\sum_{j=0}^{n-1} a_j^{(1)}(0)\int_c^z (e_1^j, h(t))dt \\ &- G_1^2(\zeta)\sum_{j=0}^{n-1} a_j^{(2)}(0)\int_c^z (t-z)(e_1^j, h(t))dt \\ &- \frac{1}{2!}G_1^3(\zeta)\sum_{j=0}^{n-1} a_j^{(3)}(0)\int_c^z (t-z)^2(e_1^j, h(t))dt - \cdots \\ &- \frac{1}{(n-2)!}G_1^{n-1}(\zeta)\sum_{j=0}^{n-1} a_j^{(n-1)}(0)\int_c^z (t-z)^{n-2}(e_1^j, h(t))dt \\ &+ \frac{1}{(n-2)!}G_1^n(\zeta)\sum_{j=0}^{n-1}\int_c^z a_j^{(n)}[(s-z)G_1(\zeta)] \\ &\cdot\left(\int_c^s (\sigma - s)^{n-2}(e_1^j, h(\sigma))d\sigma\right)ds . \end{aligned}$$

Next let

$$B_r(z) = \frac{1}{(r-1)!}\sum_{j=0}^{n-1} a_j^{(r)}(0)(e_1^j, h(z)) , \qquad 1 \leqq r \leqq n-1$$

$$B_0(z) = h_1(z) .$$

Then, since by assumption

$$\det\begin{pmatrix} 1 & 0 \cdots\cdots 0 \\ & a_j^{(1)}(0) \\ & \vdots \\ & a_j^{(n-1)}(0) \end{pmatrix} \neq 0 , \qquad 0 \leqq j \leqq n-1$$

we can invert the system of equations and get

(4.31.A) $$(e_1^j, h(z)) = \sum_{r=0}^{n-1} b_{jr}B_r(z) ,$$

where (b_{jr}) is a constant matrix. Thus the expression for h_1^* can be written:

$$
\begin{aligned}
h_1^*(z, \zeta) = B_0(z) &- G_1(\zeta)\int_c^z B_1(t)dt \\
&- G_1^2(\zeta)\int_c^z (t-z)B_2(t)dt - \cdots \\
&- G_1^{n-1}(\zeta)\int_c^z (t-z)^{n-2}B_{n-1}(t)dt \\
&+ \frac{1}{(n-2)!}G_1^n(\zeta)\sum_{j=0}^{n-1}\int_c^z a_j^{(n)}[(s-z)G_1(\zeta)] \\
&\cdot\left(\int_c^s (\sigma-s)^{n-2}\sum_{r=0}^{n-1} b_{jr}B_r(\sigma)d\sigma\right)ds .
\end{aligned} \tag{4.32}
$$

Next we let

$$
\begin{aligned}
F_0(z) &= B_0(z) \\
F_r(z) &= -\int_c^z (t-z)^{r-1}B_r(t)dt
\end{aligned}
$$

and thus

$$
F_r^{(r)}(z) = (-1)^r(r-1)!\, B_r(z) . \tag{4.32A}
$$

Introducing these into (4.32) gives

$$
\begin{aligned}
h_1^*(z, \zeta) = F_0(z) &+ G_1(\zeta)F_1(z) + G_1^2(\zeta)F_2(z) + \cdots + G_1^{n-1}(\zeta)F_{n-1}(z) \\
&+ \frac{1}{(n-2)!}G_1^n(\zeta)\sum_{j=0}^{n-1}\sum_{r=0}^{n-1} b_{jr}\frac{(-1)^r}{(r-1)!}\int_c^z a_j^{(n)}[(s-z)G_1(\zeta)] \\
&\cdot\left(\int_c^s (\sigma-s)^{n-2}F_r^{(r)}(\sigma)d\sigma\right)ds .
\end{aligned}
$$

Consider now for $0 \leqq m \leqq k$

$$
\begin{aligned}
D_z^{k-m}D_\zeta^m h_1^*(z, \zeta)\Big]_{\zeta=z} &= \delta_{m0}F_0^{(k-m)}(z) + D_z^m[G_1(z)]F_1^{(k-m)}(z) \\
&+ D_z^m[G_1^2(z)]F_2^{(k-m)}(z) + \cdots + D_z^m[G_1^{n-1}(z)]F_{n-1}^{(k-m)}(z) \\
&+ \frac{1}{(n-2)!}\sum_{j=0}^{n-1}\sum_{r=0}^{n-1} b_{jr}\frac{(-1)^r}{(r-1)!}D_z^{k-m}\int_c^z D_\zeta^m\{G_1^n(\zeta) \\
&\cdot a_j^{(n)}[(s-z)G_1(\zeta)]\}\left(\int_c^s F_r^{(r)}(\sigma)(\sigma-s)^{n-2}d\sigma\right)ds\Big]_{\zeta=z} ,
\end{aligned} \tag{4.33}
$$

where δ_{m0} is the Kronecker delta. Since

$$
\begin{aligned}
k-m+r-(n-2)-2 &\leqq k-m+(n-1)-(n-2)-2 \\
&= k-m-1 < k-m , \\
k-m-r+(n-2)+1 &\geqq k-m-(n-1)+(n-2)+1 = k-m ,
\end{aligned}
$$

the last term involving the integrals is of the form:

$$\sum_{r=0}^{n-1}\int_c^z K_r(t, z)F_r^{(k-m)}(t)dt + \tilde{C}_{k-m}(z) \tag{4.34}$$

where the $K_r(t, z)$ are analytic for t and z in $\Omega \cup \kappa \cup \hat{\Omega}$ and $\tilde{C}_{k-m}(z)$ is analytic for z in $\Omega \cup \kappa \cup \hat{\Omega}$. This follows since $r \leqq n-1$ and 1.

$$\frac{1}{(n-2)!}\int_c^s F_r^{(r)}(\sigma)(\sigma - s)^{n-2}d\sigma$$
$$= \frac{(-1)^{r-k+m}}{[k-m-r+n-2]!}\int_c^s F_r^{(k-m)}(\sigma)(\sigma - s)^{k-m-r+n-2}d\sigma + C_r^*(s)$$

if $k - m > 0$ or, if $k - m = 0$ and $r < n - 1$, where $C_r^*(s)$ is a polynomial in s. The only difference in the case $k - m = 0, r = n - 1$ is that the integral on the right side is replaced by $F_{n-1}(s)$. And since 2.

$$\int_c^z K^*(s, z)\left(\int_c^s F_r^{(k-m)}(\sigma)(\sigma - s)^l d\sigma\right)ds = \int_c^z K^{**}(s, z)F_r^{(k-m)}(\sigma)d\sigma$$

where $K^{**}(s, z)$ is an analytic function of s and z (since $K^*(s, z)$ is) for s and z in $\Omega \cup \kappa \cup \hat{\Omega}$. The last integral follows from integration by parts $l + 1$ times.

Thus by the definition of $\alpha_{k-m}(z)$, (4.30), (4.33), and (4.34) we get for $0 \leqq m \leqq k$

$$\begin{aligned} \alpha_{k-m}(z) = {} & \delta_{m0}F_0^{(k-m)}(z) + D_z^m[G_1(z)]F_1^{(k-m)}(z) \\ & + D_z^m[G_1^2(z)]F_2^{(k-m)}(z) + \cdots + D_z^m[G_1^{n-1}(z)]F_{n-1}^{(k-m)}(z) \\ & + \sum_{r=0}^{n-1}\int_c^z \tilde{L}_{r,k-m}(t, z)F_r^{(k-m)}(t)dt \\ & + \tilde{C}_{k-m}(z) , \end{aligned} \tag{4.35}$$

where for $t, z \in \Omega \cup \kappa \cup \hat{\Omega}$, $\tilde{L}_{r,k-m}(t, z)$ and $C_{k-m}(z)$ are analytic.

Next we consider for $0 \leqq m \leqq n - 2 < k$

$$\begin{aligned} A_{k-m}^*(z) \equiv {} & \frac{1}{(n-m-2)!}\int_c^z (z - t)^{n-m-2}\alpha_{k-m}(t)dt \\ = {} & \delta_{m0}F_0^{(k-(n-1))}(z) + D_z^m[G_1(z)]F_1^{(k-(n-1))}(z) \\ & + D_z^m[G_1^2(z)]F_2^{(k-(n-1))}(z) + \cdots \\ & + D_z^m[G_1^{n-1}(z)]F_{n-1}^{(k-(n-1))}(z) \\ & + \sum_{r=0}^{n-1}\int_c^z L_{k-m,r}(t, z)F_r^{(k-(n-1))}(t)dt + R_{k-m}(z) \end{aligned} \tag{4.36}$$

where we have integrated by parts $n - m - 1$ times. $R_{k-m}(z)$ and $L_{k-m,r}$ are functions analytic for z, t in $\Omega \cup \kappa \cup \hat{\Omega}$. For consistency let $A_{k-(n-1)}^*(z) = \alpha_{k-(n-1)}(z)$ and $R_{k-(n-1)} = \tilde{C}_{k-(n-1)}(z)$, $L_{k-(n-1),r} = L_{r,k-(n-1)}$.

Since by hypothesis we have for z in $\Omega \cup \kappa \cup \hat{\Omega}$

$$|\Delta_4(z)| = \begin{vmatrix} 1 & G_1 \cdots G_1^{n-1} \\ 0 & \\ \vdots & D_z^m G_1^\nu(z) \\ 0 & \end{vmatrix} \neq 0 \,, \qquad 1 \leqq m \leqq n-1,\ 1 \leqq \nu \leqq n-1$$

the above system, coupled with $\alpha_{k-(n-1)}(z)$ in (4.35) can be written as:

$$F^{(k-(n-1))}(z) = \Delta_4^{-1}(z)A^*(z) - \Delta_4^{-1}(z)\int_c^z L(t, z)F^{(k-(n-1))}(t)dt - \Delta_4^{-1}(z)R(z) \tag{4.37}$$

with:

$$F^{(k-(n-1))}(z) = \begin{pmatrix} F_0^{(k-(n-1))}(z) \\ F_1^{(k-(n-1))}(z) \\ \vdots \\ F_{n-1}^{(k-(n-1))}(z) \end{pmatrix},$$

$$A^*(z) = \begin{pmatrix} A_k^*(z) \\ A_{k-1}^*(z) \\ \vdots \\ A_{k-(n-1)}^*(z) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{(n-2)!}\displaystyle\int_c^z (z-t)^{n-2}\alpha_k(t)dt \\ \dfrac{1}{(n-3)!}\displaystyle\int_c^z (z-t)^{n-3}\alpha_{k-1}(t)dt \\ \vdots \\ \alpha_{k-(n-1)}(z) \end{pmatrix}$$

$$L(t, z) = (L_{k-m,r}(t, z)) \qquad \begin{cases} 0 \leqq r \leqq n-1 \text{ columns} \\ k-(n-1) \leqq k-m \leqq k \text{ rows} \end{cases}$$

$$R(z) = (R_k(z), R_{k-1}(z), \cdots, R_{k-(n-1)}(z))^T \,.$$

We consider (4.37) as a system of Volterra integral equations and obtain its solution in the form:

$$\begin{aligned} F^{(k-(n-1))}(z) &= \Delta_4^{-1}(z)[A^*(z) - R(z)] \\ &\quad - \Delta_4^{-1}(z)\int_c^z \Gamma(t, z)\Delta_4^{-1}(t)[A^*(t) - R(t)]dt \end{aligned} \tag{4.38}$$

where $\Gamma(t, z)$ is the resolvent matrix which is an analytic function of t and z for t and z in $\Omega \cup \kappa \cup \hat{\Omega}$.

Now we are in a position to express $\alpha_{k-m}(z)$, $n \leqq m \leqq k$ in terms of $\alpha_{k-m}(z)$, $0 \leqq m \leqq n-1$. To this end we consider (4.35), which is valid for $0 \leqq m \leqq k$ and combine with it the integrated expression of (4.38):

$$F^{(k-m)}(z) = \int_c^z \Gamma^*_{k-m}(t, z)A^*(t)dt + R^*(z) , \qquad n \leqq m \leqq k$$

where $\Gamma^*_{k-m}(t, z)$ is analytic for t and z in $\Omega \cup \kappa \cup \hat{\Omega}$ and $R^*(z)$ is analytic for z in $\Omega \cup \kappa \cup \hat{\Omega}$. This combination gives:

$$\alpha_{k-m}(z) = \sum_{r=0}^{n-1} \int_c^z \Gamma^{**}_{r,k-m}(t, z)A^*_{k-r}(t)dt + R^{**}_{k-m}(z) , \qquad n \leqq m \leqq k \tag{4.39}$$

where $R^{**}_{k-m}(z)$ is analytic for z in $\Omega \cup \kappa \cup \hat{\Omega}$ and $\Gamma^{**}_{r,k-m}(t, z)$ is analytic for t and z in $\Omega \cup \kappa \cup \hat{\Omega}$. But from the definition of $A^*_{k-r}(z)$ in (4.36) for $0 \leqq r \leqq n-2$ and $A^*_{k-(n-1)}(z) = \alpha_{k-(n-1)}(z)$, we get the result upon integrating (4.39) by parts if necessary.

Thus Lemma 4 is proved.

With the notation and assumptions of Lemma 4, we next state and prove:

LEMMA 5. *For $l + m \leqq k - 1$*

$$D^l_z \tilde{h}_m(z, \zeta)\Big]_{\zeta=z} = \sum_{j=0}^{n-1} \int_c^z \tilde{K}^*_{k-m,j}(t, z)\alpha_{k-j}(t)dt + \tilde{C}^*_{k-m}(z) \tag{4.40}$$

where $\tilde{K}^*_{k-m,j}(t, z)$ are analytic for t and z in $\Omega \cup \kappa \cup \hat{\Omega}$ and $C^*_{k-m}(z)$ are analytic for z in $\Omega \cup \kappa \cup \hat{\Omega}$.

Proof. In the notation of Lemma 4, since $l + m \leqq k - 1$

$$\begin{aligned} D_t^{(k-l-m)}[D^l_z \tilde{h}_m(z, \zeta)]_{z=\zeta=t} &= \sum_{j=0}^{k-l-m} \binom{k-l-m}{j} D^j_\zeta D_z^{k-l-m-j} D^l_z \tilde{h}_m(z, \zeta)\Big]_{z=\zeta=t} \\ &= \sum_{j=0}^{k-l-m} \binom{k-l-m}{j} D_z^{k-m-j} \tilde{h}_{m+j}(z, \zeta)\Big]_{z=\zeta=t} \\ &= \sum_{j=0}^{k-l-m} \binom{k-l-m}{j} \alpha_{k-m-j}(t) . \end{aligned}$$

Thus

$$\begin{aligned} D^l_z \tilde{h}_m(z, \zeta)_{z=\zeta=t} &= \frac{1}{(k-l-m-1)!} \int_c^t (t-s)^{k-l-m-1} \\ &\quad \cdot \left[\sum_{j=0}^{k-l-m} \binom{k-l-m}{j} \alpha_{k-m-j}(s) \right] ds \\ &\quad + \text{polynomial in } (t-c) \\ &= \sum_{j=0}^{n-1} \int_c^t \tilde{K}^*_{k-m,j}(s, t)\alpha_{k-j}(s)dt + \tilde{C}^*_{k-m}(t) \end{aligned}$$

with $\tilde{C}^*_{k-m}(z)$ analytic for z in $\Omega \cup \kappa \cup \hat{\Omega}$, $\tilde{K}^*_{k-m,j}(t, z)$ is analytic for t and z in $\Omega \cup \kappa \cup \hat{\Omega}$ and where we have made use of Lemma 4 and the fact that if $r \geqq 0$ then

$$\int_c^z dt(z-t)^r \int_c^t K(s,t)\alpha_{k-j}(s)ds = \frac{1}{r+1}\int_c^z (z-t)^{r+1}K(t,t)\alpha_{k-j}(t)dt$$

for the case when $m + j \geqq n$. Thus the Lemma is proved.

We are now in a position to continue the proof of the theorem. Combining (4.28) with the results (4.31) of Lemma 4 yields:

$$\begin{aligned}A_\nu(z) = &\sum_{j=0}^{n-1}\int_c^z K_{\nu j}^{**}(t,z)\alpha_{k-j}(t)dt \\ &+ \text{terms of order } l+m \leqq k-1 \text{ in} \\ &D_z^l \tilde{h}_m(z,\zeta)_{\zeta=z} \text{ with coefficients analytic in } \Omega \cup \kappa \cup \hat{\Omega}\,,\end{aligned} \tag{4.41}$$

where $K_{\nu j}^{**}(t, z)$ are analytic for t and z in $\Omega \cup \kappa \cup \hat{\Omega}$. With the aid of Lemma 5 applied to the second term on the right we get:

$$A_\nu(z) = \sum_{m=0}^{n-1}\int_c^z \tilde{K}_{\nu m}(t,z)\alpha_{k-m}(t)dt + \tilde{C}_\nu(z) \tag{4.42}$$

where $\tilde{K}_{\nu j}(t, z)$ are analytic in t and z for t and z in $\Omega \cup \kappa \cup \hat{\Omega}$, and $\tilde{C}_\nu(z)$ is analytic for z in $\Omega \cup \kappa \cup \hat{\Omega}$.

Finally, we combine (4.27) and (4.42) and get for z on κ:

$$\beta_0(z)\Phi(z) = \int_c^z \tilde{K}(t,z)\Phi(t)dt + g^*(z)\,, \tag{4.43}$$

where $g^*(z)$ is an analytic vector function for z in $\hat{\Omega}$ and in $C(\Omega \cup \kappa)$

$$\begin{aligned}\tilde{K}(t,z) &= (\tilde{K}_{\nu m}(t,z))\,,\\ \Phi(z) &= (\alpha_k(z), \alpha_{k-1}(z), \cdots, \alpha_{k-n+1}(z))^T\,,\\ \beta_0(z) &= (\beta_{\nu m}^k(z)[G'(z)]^{-m})\,,\\ &\quad m \text{ designates the column, } \nu \text{ the row.}\\ &0 \leqq m \leqq n-1,\ 1 \leqq \nu \leqq n\,.\end{aligned}$$

But from (4.29) we see that

$$|\beta_{\nu\mu}^k(z)| = \begin{vmatrix} \sum_{r+s=k}(i)^s\alpha_0^{rs}p_{11}^{rs}(z)\,, & \sum_{r+s=k}(i)^s\alpha_1^{rs}p_{11}^{rs}, & \cdots, & \sum_{r+s=k}(i)^s\alpha_{n-1}^{rs}p_{11}^{rs} \\ \sum_{r+s=k}(i)^s\alpha_0^{rs}p_{21}^{rs}(z)\,, & \sum_{r+s=k}(i)^s\alpha_1^{rs}p_{21}^{rs}, & \cdots, & \sum_{r+s=k}(i)^s\alpha_{n-1}^{rs}p_{21}^{rs} \\ \vdots & & & \\ \sum_{r+s=k}(i)^s\alpha_0^{rs}p_{n1}^{rs}(z)\,, & \sum_{r+s=k}(i)^s\alpha_1^{rs}p_{n1}^{rs}, & \cdots, & \sum_{r+s=k}(i)^s\alpha_{n-1}^{rs}p_{n1}^{rs} \end{vmatrix}$$

where by definition of α_μ^{rs}

$$(a+b)^r(a-b)^s = \sum_{\mu=0}^{r+s}\alpha_\mu^{rs}a^{r+s-\mu}b^\mu$$

and by definition

$$P_{\nu 1}(z, \xi, \eta) = \sum_{r+s=k} p_{\nu 1}^{rs}(z)\xi^r\eta^s .$$

Thus if $\zeta = \xi + i\eta$, $\bar{\zeta} = \xi - i\eta$ then

$$P_{\nu 1}(z, \xi, \eta) = 2^{-k} \sum_{r+s=k} (i)^s p_{\nu 1}^{rs}(z)(\bar{\zeta} + \zeta)^r(\bar{\zeta} - \zeta)^s$$

and

$$D_{\zeta}^j P_{\nu 1}(z, \xi, \eta)\big|_{\substack{\xi=1\\ \eta=i}} = j!2^{-j} \sum_{r+s=k} (i)^s \alpha_j^{rs} p_{\nu 1}^{rs}(z) .$$

$$|\beta_{\nu\mu}^k(z)| = \\ = C_n \begin{vmatrix} P_{11}(z, \xi, \eta), (D_\xi - iD_\eta)P_{11}(z, \xi, \eta), \cdots, (D_\xi - iD_\eta)^{n-1}P_{11}(z, \xi, \eta) \\ P_{21}(z, \xi, \eta), (D_\xi - iD_\eta)P_{21}(z, \xi, \eta), \cdots, (D_\xi - iD_\eta)^{n-1}P_{21}(z, \xi, \eta) \\ \vdots \\ P_{n1}(z, \xi, \eta), (D_\xi - iD_\eta)P_{n1}(z, \xi, \eta), \cdots, (D_\xi - iD_\eta)^{n-1}P_{n1}(z, \xi, \eta) \end{vmatrix}_{\substack{\xi=1\\ \eta=i}}$$

with $C_n = 1!\,2! \cdots (n-1)!$

Since $P_{\nu 1}(z, \xi, \eta)$ are homogeneous polynomials in (ξ, η) of degree k, we see from Euler's formula that

$$(\xi D_\xi + \eta D_\eta)^l P_{\nu 1}(z, \xi, \eta) = k^l P_{\nu 1}(z, \xi, \eta) .$$

Thus

$$|\beta_{\nu\mu}^k(z)| = (k)^{\{-n(n-1)\}/2} C_n \,|\,(\xi D_\xi + \eta D_\eta)^{n-l-1}(\xi D_\xi - \eta D_\eta)^l P_{\nu 1}(z, \xi, \eta)\,|_{\substack{\xi=1\\ \eta=i}} \neq 0$$

if and only if $\Delta_2(z, \xi, \eta)\,|_{\xi=1, \eta=i} \neq 0$. This follows immediately upon writting the determinant as a sum of determinants. Thus we have

$$|\beta_0(z)| = [G'(z)]^{\{-n(n-1)\}/2}\,|\beta_{\nu\mu}^k(z)| \neq 0^3$$

for z in $\Omega \cup \kappa \cup \hat{\Omega}$ by assumption (4.16). Thus

$$\Phi(z) = \beta_0^{-1}(z)\int_c^z \tilde{K}(t, z)\Phi(t) + \beta_0^{-1}(z)g^*(z) . \tag{4.44}$$

We now consider (4.43) as a system of Volterra integral equations in $\Phi(z)$ for z in $\hat{\Omega} \cup \kappa$. As such, this system has a unique solution vector $\Phi_e(z)$ which is analytic for z in $\hat{\Omega}$ and continuous for z in $\hat{\Omega} \cup \kappa$ and moreover agrees with $\Phi(z)$ for z on κ. Thus $\Phi_e(z)$ furnishes the analytic continuation of $\Phi(z)$ into $\hat{\Omega} \cup \kappa$. Thus $\Phi(z)$ is analytic for z in $\Omega \cup \kappa \cup \hat{\Omega}$. From the definition of $\Phi(z)$ in (4.43.A) we see this yields the analytic continuation of $\alpha_k(z), \alpha_{k-1}(z), \cdots, \alpha_{k-n+1}(z)$ into $\Omega \cup \kappa \cup \hat{\Omega}$. But by (4.36), the definition of $A^*(z)$ and (4.38) we get the analytic continuation of $F^{(k-(n-1))}(z)$ into $\Omega \cup \kappa \cup \hat{\Omega}$. By integration, we get the analytic continuation of $F(z)$ into $\Omega \cup \kappa \cup \hat{\Omega}$. (We adjust the

[3] Professor J. Leray pointed out to me the relation between $\beta_{\nu\mu}^k$ and $\Delta_2(z, \xi, \eta)_{\eta=i}^{\xi=1}$.

constants of integration to agree with $F(z)$ and its derivatives at the point c of the boundary; this gives uniqueness.)

Upon differentiating $F_j(z)$ and using (4.32A) and (4.31A) we get the analytic continuation of $(e_1^j, h(z))$ into $\Omega \cup \kappa \cup \hat{\Omega}$. However by assumption (4.14)

$$\Delta_1 = \begin{vmatrix} 1 & 0 \cdots 0 \\ & e_1^j \end{vmatrix} \neq 0 , \qquad 1 \leqq j \leqq n-1 .$$

Thus we get the analytic continuation of $h_1(z), h_2(z), \cdots, h_n(z)$ into $\Omega \cup \kappa \cup \hat{\Omega}$.

In a completely analogous way, we can show how to analytically extend $g_k(z), g_{k-1}(z), \cdots, g_{k-(n-1)}(z)$ into $\Omega \cup \kappa \cup \hat{\Omega}$, knowing initially only that they are analytic in $\hat{\Omega}$ and continuous on $\hat{\Omega} \cup \kappa$. In this direction we first note that we have:

LEMMA 4-B. *Let* $k \geqq n$

$$\text{(4.30B)} \qquad \alpha_{k-m}^*(z) = D_\zeta^{k-m} D_z^m g_1^*(z, \zeta) = D_\zeta^{k-m} \tilde{g}_m(z, z) , \qquad 0 \leqq m \leqq k$$

and the hypotheses of the theorem hold. Then

$$\text{(4.31B)} \qquad \alpha_{k-m}^*(z) = \sum_{r=0}^{n-1} \int_c^z \hat{K}_{k-m,r}(t, z) \alpha_{k-r}^*(t) dt + \hat{C}_{k-m}(z)$$

where $\hat{K}_{k-m,r}(t, z)$ *are analytic for* t *and* z *in* $\Omega \cup \kappa \cup \hat{\Omega}$ *and* $\hat{C}_{k-m}(z)$ *is analytic for* z *in* $\Omega \cup \kappa \cup \hat{\Omega}$.

Proof. The proof of this lemma is the same as that of Lemma 4, with only obvious modifications. In place of the expression for the Taylor's expansion for a_j about $t = z$ we start with the expression for the Taylor's expansion for a_j about $G_1(\sigma) = G(\zeta)$ and integrate viz:

$$\begin{aligned} g_1^*(z, \zeta) = g_1(\zeta) &- (c - z) \sum_{j=0}^{n-1} a_j^{(1)}(0) \int_c^\zeta G'(\sigma)(e_1^j, g(\sigma)) d\sigma \\ &- (c - z)^2 \sum_{j=0}^{n-1} a_j^{(2)}(0) \int_c^\zeta [G(\sigma) - G(\zeta)] G'(\sigma)(e_1^j, g(\sigma)) d\sigma \\ &- \frac{(c - z)^3}{2!} \sum_{j=0}^{n-1} a_j^{(3)}(0) \int_c^\zeta [G(\sigma) - G(\zeta)]^2 G'(\sigma)(e_1^j, g(\sigma)) d\sigma - \cdots \\ &- \frac{1}{(n-2)!} (c - z)^n \sum_{j=0}^{n-1} \int_c^\zeta a_j^{(n)} [(G(s) - G(\zeta))(c - z)] \\ &\cdot \left(\int_c^s [G(\sigma) - G(s)]^{n-2} G'(\sigma)(e_1^j, g(\sigma)) d\sigma \right) ds . \end{aligned}$$

In place of $B_r(z)$ we introduce:

$$\tilde{B}_r(\zeta) = \frac{1}{(r-1)!}\sum_{j=0}^{n-1} a_j^{(r)}(0)(g_1^j, e(\zeta)) \, ,$$
$$\tilde{B}_0(\zeta) = g_1(\zeta)$$

and the expression (4.32) becomes replaced by:

$$\begin{aligned} g_1^*(z, \zeta) = \tilde{B}_0(\zeta) &- (c-z)\int_c^\zeta \tilde{B}_1(\sigma)G'(\sigma)d\sigma \\ &- (c-z)^2\int_c^\zeta \tilde{B}_2(\sigma)[G(\sigma) - G(\zeta)]G'(\sigma)d\sigma \\ &- (c-z)^3\int_c^\zeta \tilde{B}_3(\sigma)[G(\sigma) - G(\zeta)]^2G'(\sigma)d\sigma - \cdots . \end{aligned}$$

Also $F_r(z)$ is replaced by

$$\tilde{F}_r(\zeta) = -\int_c^\zeta [G(\sigma) - G(\zeta)]^{r-1}G'(\sigma)\tilde{B}_r(\sigma)d\sigma$$

so

$$\begin{aligned} \tilde{F}_r^{(r)}(\zeta) = (-1)^r(r-1)!\,&(G'(\zeta))^r\tilde{B}_r(\zeta) \\ &- \int_c^\zeta D_\zeta^{(r)}[G(\sigma) - G(\zeta)]^{r-1}G'(\sigma)\tilde{B}_r(\sigma)d\sigma \, . \end{aligned}$$

Considering these as Volterra integral equations for $\tilde{B}_r(\zeta)$, we can solve, since $G'(\zeta) \neq 0$ in $\Omega \cup \kappa \cup \hat{\Omega}$, and get:

$$\begin{aligned} \tilde{B}_r(\zeta) = &(-1)^r\frac{1}{(r-1)!}[G'(\zeta)]^{-r}\tilde{F}_r^{(r)}(\zeta) \\ &+ (-1)^r\frac{1}{(r-1)!}[G'(\zeta)]^{-r}\int_c^\zeta \tilde{Q}_r(\sigma, \zeta)F_r^{(r)}(\sigma)d\sigma \end{aligned}$$

where $\tilde{Q}_r(\sigma, \zeta)$ is the resolvent which is analytic for σ and ζ in $\Omega \cup \kappa \cup \hat{\Omega}$. Thus

$$\begin{aligned} g_1^*(z, \zeta) = \tilde{F}_0(\zeta) &+ (c-z)\tilde{F}_1(\zeta) + (c-z)^2\tilde{F}_2(\zeta) \\ &+ \cdots + (c-z)^{n-1}\tilde{F}_{n-1}(\zeta) \\ &- \frac{1}{(n-2)!}(c-z)^n\sum_{j=0}^{n-1}\sum_{r=0}^{n-1}\frac{(-1)^r}{(r-1)!}b_{jr} \\ &\cdot \int_c^\zeta a_j^{(n)}[(G(s) - G(\zeta))(c-z)] \\ &\cdot \left(\int_c^s [G(\sigma) - G(s)]^{n-2}[G'(\sigma)]^{-r+1}\right. \\ &\cdot \left.\left\{\tilde{F}_r^{(r)}(\sigma) + \int_c^\sigma \tilde{Q}_r(\tau, \sigma)\tilde{F}_r^{(r)}(\tau)d\tau\right\}d\sigma\right)ds \end{aligned} \tag{4.33*}$$

and for $0 \leqq m \leqq n-1$

$$
\begin{aligned}
\alpha^*_{k-m}(z) &= D_\zeta^{k-m} D_z^m g_1^*(z, \zeta)\Big]_{z=\zeta} \\
&= (-1)^m m!\, \widetilde{F}_m^{(k-m)}(\zeta) + (-1)^m \frac{(m+1)!}{1!}(c-z)\widetilde{F}_{m+1}^{(k-m)}(\zeta) \\
&\quad + \cdots + (-1)^m \frac{(n-1)!}{[n-1-m]!}(c-z)^{n-1-m}\widetilde{F}_{n-1}^{(k-m)}(\zeta) \qquad (4.45) \\
&\quad + \sum_{j=0}^{n-1}\sum_{r=0}^{n-1} D_\zeta^{k-m}\int_c^\zeta ds D_z^m\{(c-z)^m a_j^{(n)}[(G(s)-G(\zeta))(c-z)]\} \\
&\quad \cdot \Big(\int_c^s [G(\sigma)-G(s)]^{n-2}G'(\sigma) \\
&\quad \cdot \Big\{\widetilde{F}_r^{(r)}(\sigma) + \int_c^\sigma \widetilde{Q}_r(\tau,\sigma)\widetilde{F}_r^{(r)}(\tau)d\tau\Big\}d\sigma\Big]_{\zeta=z}\Big)\,.
\end{aligned}
$$

Since

$$r - [(n-2) - (k-m-1)] \leqq k - m\,,$$

the last term can be written

$$\sum_{r=0}^{n-1}\int_c^z \widetilde{F}_r^{(k-m)}(\sigma)\widetilde{Q}^*_{k-m,r}(\sigma, z)d\sigma \qquad (4.46)$$

where $\widetilde{Q}^*_{k-m,r}(\sigma, \zeta)$ are analytic functions of σ and ζ for σ and ζ in $\Omega \cup \kappa \cup \hat{\Omega}$. Introducing (4.46) into (4.45) gives an expression of the same form as (4.35). We now proceed exactly as in Lemma 4 and find that:

$$\widetilde{F}^{(k-m)}(z) = \int_c^z \widetilde{\Gamma}^*_{k-m}(t, z)\widetilde{A}^*(t)dt + \widetilde{R}^*(z)\,, \qquad n \leqq m \leqq k\,, \qquad (4.47)$$

where

$$
\begin{aligned}
\widetilde{F}^{(k-m)}(z) &= (\widetilde{F}_0^{(k-m)}(z), \widetilde{F}_1^{(k-m)}(z), \cdots, \widetilde{F}_{n-1}^{(k-m)}(z)) \\
\widetilde{A}^*(z) &= \Big(\int_c^z (z-t)^{n-2}\alpha_k^*(t)dt, \int_c^z (z-t)^{n-3}\alpha^*_{k-1}(t)dt, \cdots, \alpha^*_{k-(n-1)}(z)\Big)
\end{aligned}
$$

and $\widetilde{\Gamma}^*(t, z)$ is an analytic function of t and z for t and z in $\Omega \cup \kappa \cup \hat{\Omega}$ and $\widetilde{R}^*(z)$ is an analytic function of z for z in $\Omega \cup \kappa \cup \hat{\Omega}$. But for $n \leqq m \leqq k$, $0 \leqq r \leqq n-1$, we have $r - [(n-2) - (k-m-1)] \leqq k - m$, thus (4.45) becomes replaced in this case by

$$
\begin{aligned}
\alpha^*_{k-m}(z) &= D_\zeta^{k-m} D_z^m g_1^*(z, \zeta)\Big]_{\zeta=z} \\
&= \sum_{r=0}^{n-1}\int_c^\zeta \widetilde{F}_r^{(k-m)}(\sigma)\widetilde{Q}^{**}_{k-m,r}(\sigma, z)d\sigma\,, \qquad n \leqq m \leqq k\,,
\end{aligned}
$$

where $\widetilde{Q}^{**}_{k-m,r}(\sigma, z)$ are analytic functions of σ and z for σ and z in $\Omega \cup \kappa \cup \hat{\Omega}$. Combining this with (4.47) gives the result.

The condition $\Delta_4(z) \neq 0$ is unnecessary in this lemma since the corresponding condition is:

$$\Delta_4 = (-1)^{\{n(n-1)\}/2} 2!\, 3! \cdots (n-1)! \neq 0 .$$

Next we note that we have

LEMMA 5-B. *For $l + m \leqq k - 1$*

$$D_\zeta^l \tilde{g}_m(z, \zeta)\Big]_{\zeta=z} = \sum_{j=0}^{n-1} \int_c^z \tilde{\tilde{K}}{}^*_{k-m,j}(z, t)\alpha^*_{k-j}(t)dt + \tilde{\tilde{C}}{}^*_{k-m}(z) \tag{4.38}$$

where $\tilde{\tilde{K}}{}^*_{k-m,j}(z, t)$ are analytic for z and t in $\Omega \cup \kappa \cup \hat{\Omega}$ and $\tilde{\tilde{C}}{}^*_{k-m}(z)$ is analytic for $\Omega \cup \kappa \cup \hat{\Omega}$.

Proof. Same as Lemma 5 using Lemma 4-B instead of Lemma 4.

As in the case of $(\alpha_k(z), \alpha_{k-1}(z), \cdots, \alpha_{k-n+1}(z))^T$, we get analytic extension of $(\alpha^*_k(z), \alpha^*_{k-1}(z), \cdots, \alpha^*_{k-n+1}(z))^T$ into $\hat{\Omega} \cup \kappa \cup \hat{\Omega}$ which are analytic initially only in $\hat{\Omega}$, and continuous on $\hat{\Omega} \cup \kappa$. The only difference is that we use the fact that $\Delta_2(z, \xi, \eta)\,|_{\xi=1,\eta=i} \neq 0$ on $\Omega \cup \kappa \cup \hat{\Omega}$ whereas in the extension of the $\alpha'_j s$ we used the fact that $\Delta_2(z, \xi, \eta)\,|_{\xi=1,\eta=i} \neq 0$ on $\Omega \cup \kappa \cup \hat{\Omega}$.

In an analogous way we get the analytic extension of $\tilde{F}_r^{(k-n-1)}(z)$ into $\Omega \cup \kappa \cup \hat{\Omega}$, which in turn gives the analytic extension of $\tilde{B}_r(z)$ into $\Omega \cup \kappa \cup \hat{\Omega}$, which finally gives the extension of $(e_1^j, g(z))$. Since by assumption $\Delta_1 \neq 0$ this system yields the analytic continuation of $g_1(\zeta), g_2(\zeta), \cdots, g_n(\zeta)$ into $\Omega \cup \kappa \cup \hat{\Omega}$.

Upon introducing the extended vector functions $h(z)$ and $g(z)$ into (4.18) we get the extension of $U^*(z, \hat{z})$ for z in $\Omega \cup \kappa \cup \hat{\Omega}$, which was given originally only for z in $\Omega \cup \kappa$. And thus, the solution of (4.13) has been extended across the boundary conditions on κ into $\Omega \cup \kappa \cup \hat{\Omega}$. This completes the proof of the theorem.

5. Applications. (A.1) Consider the situation where we are given a solution to the differential equations

$$u_{xx} + u_{yy} + Au_x + Bu_y + Cu = 0 \tag{1.1}$$

where $(u_1, u_2, \cdots, u_n)^T$, A, B and C are pairwise commutative constant $n \times n$ matrices in a simply connected open region Ω of the type described in 2, part of whose boundary is the analytic arc κ, and on κ satisfies

$$u\,|_\kappa = (\varphi_1(z), \varphi_2(z), \cdots, \varphi_n(z))$$

where $\varphi_1(z), \cdots, \varphi_n(z)$ are functions analytic in $\Omega \cup \kappa \cup \hat{\Omega}$. Moreover let $u \in C'(\Omega \cup \kappa)$. Then by Theorem 1 we can uniquely extend the solution u into $\Omega \cup \kappa \cup \hat{\Omega}$ so that it is a solution in this large region and is the only one that satisfies the given conditions provided $0 \neq |M_{0,0}|$ where

$$M_{0,0} = \begin{pmatrix} p_{11}^{00} & p_{12}^{00} \cdots p_{1n}^{00} \\ \vdots & \\ p_{n1}^{00} & \cdots\cdots \ p_{nn}^{00} \end{pmatrix}.$$

In this case

$$p_{ij}^{00} = \begin{matrix} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{matrix}.$$

Thus $|M_{0,0}| = 1 \neq 0$ and reflection is possible.

(A.II) Theorem 2 is suitable for systems of equations of the form:

$$\sum_{j=1}^{m} P_{ij}(\Delta)u_j = 0 \qquad i = 1, 2, \cdots, m$$

where the P_{ij} are polynomials with constant coefficients and Δ is the Laplacian, e.g.,

$$\begin{aligned} \Delta^2 u_1 + a\Delta u_1 + bu_2 &= 0 \\ \Delta^3 u_2 + c\Delta u_2 + d\Delta u_1 &= 0 \end{aligned}$$

for which if

$$(w_1, w_2, w_3, w_4, w_5)^T = (u_1, \Delta u_1, u_2, \Delta u_2, \Delta^2 u_2)^T$$

then

$$E = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & d & 0 & c & 0 \end{pmatrix}.$$

(A.III) When the arc κ is a portion of the x axis, then condition (H.1) of Theorem 2 is automatically satisfied since then $G(z) = z$ and

$$\Delta_4(z) = \begin{vmatrix} -1 & -2(c-z) & -3(c-z)^2 \cdots & & -(n-1)(c-z)^{n-2} \\ 0 & 2! & 3\cdot 2(c-z) & \cdots & (n-1)(n-2)(c-z)^{n-3} \\ 0 & 0 & -3! & \cdots & (n-1)(n-2)(n-3)(c-z)^{n-4} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \pm(n-1)! \end{vmatrix}$$
$$= \pm 1!2!3! \cdots (n-1)! \neq 0 .$$

(A.IV) When we consider systems of the form:

$$\begin{aligned} \Delta u_1 &= a_{11}u_1 + a_{12}u_2 \\ \Delta u_2 &= a_{21}u_1 + a_n u_2 \end{aligned} \qquad a_{ij} \text{ constants}$$

then

$$E = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$$

and condition (H.1) of Theorem 2 becomes $G'(z) \neq 0$ for $\Omega \cup \kappa \cup \hat{\Omega}$ which is automatically satisfied because of our initial restrictions on $G(z)$.

(A.V) Given that u_1 is a solution of the metaharmonic equation

$$\Delta^n u_1 + a_1 \Delta^{n-1} u_1 + \cdots + a_{n-1}\Delta u_1 + a_n u_1 = 0 \tag{5.1}$$

in Ω where $a_1, a_2, \cdots, a_n$ are constants and $u_1(x, y)$ is a single function, $u_1 \in C^{2n-2+k}(\Omega \cup \kappa) \cap C^{2n}(\Omega)$, $n - 1 \leqq k \leqq 2n$, $n \geqq 2$ and u_1 satisfies on κ:

$$p_\alpha(D)u_1 = \sum_{r+s \leqq k} p_{\alpha 1}^{rs}(z) D_x^r D_y^s u_1 = f_\alpha(z) \qquad \alpha = 1, 2, \cdots, n \tag{5.2}$$

where the $p_{\alpha 1}^{rs}(z)$ and $f_\alpha(z)$ are analytic in $\Omega \cup \kappa \cup \hat{\Omega}$. Assume that κ is such that (H.1) is satisfied. This equation can be written as a system by letting $u_2 = \Delta u_1$, $u_3 = \Delta^2 u_1, \cdots, u_n = \Delta^{n-1} u_1$ and equation (5.1) is equivalent to the system

$$\left[\Delta + \begin{pmatrix} 0 & -1 & 0 & 0 \cdots 0 & 0 \\ 0 & 0 & -1 & 0 \cdots 0 & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & 0 \cdots 0 & -1 \\ a_n & a_{n-1} & & \cdots\cdots a_2 & a_1 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_n \end{pmatrix} = 0 .$$

Thus

$$E^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 0 & 1 & 0 \cdots 0 \\ \vdots & & & & \\ -a_n & -a_{n-1} & \cdots & & -a_1 \\ * & * & \cdots & & * \end{pmatrix}^{n \times n}, \cdots, E^{n-1} = \begin{pmatrix} 0 \cdots 0 & (-1)^{n-1} \\ \pm a_n \pm a_{n-1} \cdots & \pm a_1 \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix}$$

and in this case (4.14) becomes.

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 & 0 \cdots 0 \\ 0 & -1 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 \cdots 0 \\ \vdots & & & \\ 0 & 0 \cdots & (-1)^{n-1} & \end{vmatrix} = \pm 1$$

which means $\Delta_1 \neq 0$ is not a condition in the case of the metaharmonic equation. Thus if $\Delta_2 \neq 0$ and $\Delta_3 \neq 0$ for z in $\Omega \cup \kappa \cup \hat{\Omega}$, as given by (4.16) and (4.17), and if the a_j are such that (H.2) is satisfied then we get that u_1 can be extended into $\Omega \cup \kappa \cup \hat{\Omega}$.

To get some idea of how we check condition (H.2) consider the example

$$\Delta^2 u + 3\Delta u + 2u = 0 .$$

To determine $a_0(s)$ and $a_1(s)$ of Lemma 3 where

$$p_1(\lambda, s) = a_0(s) + a_1(s)\lambda ,$$

note that

$$| E - \lambda I | = \lambda^2 - 3\lambda + 2\lambda = (\lambda - 1)(\lambda - 2)$$

and thus

$$p_1(1, s) = a_0(s) + a_1(s) = J_0[\sqrt{s}] ,$$
$$p_1(2, s) = a_0(s) + 2a_1(s) = J_0[\sqrt{2s}] .$$

Thus

$$a_1(s) = J_0[\sqrt{2s}] - J_0[\sqrt{s}] ,$$
$$a_0(s) = 2J_0[\sqrt{s}] - J_0[\sqrt{2s}] .$$

Thus

$$R[z, \zeta; t, \sigma] = a_0\{(t - z)[G(\sigma) - G(\zeta)]\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1\{(t - z)[G(\sigma) - G(\zeta)]\}\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$$

and the representation of the solution (4.18) becomes:

$$\begin{aligned} U_1^*(z, \hat{z}) = {} & h_1(z) + g_1(\hat{z}) - a_0\{(c - z)[G(c) - G(\hat{z})]\}h_1(c) \\ & + a_1\{(c - z)[G(c) - G(\hat{z})]\}h_2(c) \\ & - [G(c) - G(\hat{z})]\int_c^z [a_0^{(1)}\{(t - z)[G(c) - G(\hat{z})]\}h_1(t) \\ & \quad - a_1^{(1)}\{(t - z)[G(c) - G(\hat{z})]\}h_2(t)]dt \\ & - (t - z)\int_c^{\hat{z}} [a_0^{(1)}\{(c - z)[G(\sigma) - G(\hat{z})]\}g_1(\sigma) \\ & \quad - a_1^{(1)}\{(c - z)[G(\sigma) - G(\hat{z})]\}]g_2(\sigma)G'(\sigma)d\sigma . \end{aligned}$$

In this case, the condition (H.2) becomes:

$$\begin{vmatrix} 1 & 0 \\ a_0^{(1)}(0) & a_1^{(1)}(0) \end{vmatrix} = a_1^{(1)}(0) = -\frac{1}{4} \neq 0 ,$$

and is thus satisfied. Note that in this example (H.1) is also satisfied, since as a special case of (A.IV) it is simply $G'(z) \neq 0$ for z in $\Omega \cup \kappa \cup \hat{\Omega}$.

Note that the polyharmonic equation is a special case of the metaharmonic equation.

(A.VI) It is of interest to note that in the case of the polyharmonic equation viz. $\Delta^n u = 0$, E is of the form

$$E = \begin{pmatrix} 0 & -1 & 0 & 0 \cdots & 0 \\ 0 & 0 & -1 & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & 0 \cdots & -1 \\ 0 & 0 & 0 & 0 \cdots & 0 \end{pmatrix} .$$

Thus $E^n = 0$ and the Riemann function is only the finite sum:

$$R[z^0, \zeta^0, z, \zeta] = J_0[2\sqrt{E(z - z^0)(G(\zeta) - G(\zeta^0))}] ,$$

where

$$J_0[\sqrt{Q}] = I - \frac{Q}{2^2} + \frac{Q^2}{2^4(2!)^2} - \cdots + (-1)^{n-1} \frac{Q^{n-1}}{2^{2(n-1)}[(n-1)!]^2} .$$

Note that the $a_j(s)$ of Lemma 3 are given in this case by

$$a_j(s) = (-1)^j 2^{-2j}(j!)^{-2} s^j , \qquad 0 \leqq j \leqq n - 1 .$$

Thus condition (H.2) is clearly satisfied automatically. Thus for the representation (4.20) of the first component

$$\begin{aligned} \omega(z, \hat{z}) = h_1(z) &- \sum_{j=1}^{n-1} j b_j [G(c) - G(\hat{z})]^j \int_c^z (t - z)^{j-1} h_{j+1}(t) dt \\ &+ g_1(\hat{z}) - \sum_{j=1}^{n-1} j b_j (c - z)^j \int_c^{\hat{z}} G'(\sigma)[G(\sigma) - G(\hat{z})]^{j-1} g_{j+1}(\sigma) d\sigma \end{aligned} \tag{5.3}$$

where

$$b_j = \frac{1}{2^{2j}(j!)^2} .$$

Let

$$\begin{aligned} \varphi_j(z) &= -j b_j \int_c^z (t - z)^{j-1} h_{j+1}(t) dt , & j = 1, 2, \cdots, n - 1 \\ \Psi_j(\hat{z}) &= -j b_j \int_c^{\hat{z}} G'(\sigma)[G(\sigma) - G(\hat{z})]^{j-1} g_{j+1}(\sigma) d\sigma , & j = 1, 2, \cdots, n - 1 \end{aligned} \tag{5.4}$$

and

$$\varphi_0(z) = h_1(z) , \qquad \Psi(\hat{z}) = g_1(\hat{z}) ,$$

then the representation (5.3) becomes:

$$\omega_1(z, \hat{z}) = \sum_{j=0}^{n-1}[G(c) - G(\hat{z})]^j \varphi_j(z) + \sum_{j=0}^{n-1}(c - z)^j \Psi_j(\hat{z}) \tag{5.5}$$

which is an equally good representation since the $h's$ and $g's$ can be obtained simply by differentiation of the $\varphi's$ and $\Psi's$ if we utilize (H.1). This is a generalization of the representation formula of the author [8] for the biharmonic equation.

(A.VII) Next we shall check that the results of [8] for the biharmonic equation are a special case of Theorem 2. In this case $\Delta^2 u = 0, 1 \leqq k \leqq 3, \alpha_0^{rs} = 1, \alpha_1^{0k} = -k, \alpha_1^{1,k-1} = -(k-2), \alpha_1^{2,k-2} = -(k-4)$, $\alpha_1^{rs} = (k - 2r) = r - s$. Thus condition (4.16) and (4.17) become the same; viz.:

$$0 \neq \Delta_2 = \Delta_3 = \begin{vmatrix} \sum_{r+s=k}(i)^s p_{11}^{rs}(z) & \sum_{r+s=k}(i)^s(r - s)p_{11}^{rs}(z) \\ \sum_{r+s=k}(i)^s p_{21}^{rs}(z) & \sum_{r+s=k}(i)^s(r - s)p_{21}^{rs}(z) \end{vmatrix}$$

which is precisely the condition of [8]. As seen in (A.IV), (H.1) is satisfied and as seen in (A.VI), (H.2) is automatically satisfied and as seen in (A.V) $\Delta_1 \neq 0$. And in this special case our theorem reduces to the theorem of [8], but with the continuity requirements strengthened by insisting that $u \in C^{k+2}(\Omega \cup \kappa) \cap C^4(\Omega)$ instead of only

$$u \in C^k(\Omega \cup \kappa) \cup C^4(\Omega) \cap C^2(\Omega \cup \kappa)$$

as in [8].

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NONLINEAR ELLIPTIC CONVOLUTION EQUATIONS OF WIENER-HOPF TYPE IN A BOUNDED REGION

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The existence of a solution of a nonlinear perturbation of an elliptic convolution equation of Wiener-Hopf type in a bounded region G of $\mathbf{R}^n$ is proved. More explicitly, let A be an elliptic convolution operator on G of order $\alpha, \alpha > 0$; A_j the principal part of A in a local coordinate system and $\tilde{A}_j(x^j, \xi)$ be the symbol of A_j with a factorization with respect to ξ_n of the form: $\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$ for $x_n^j = 0$. $\tilde{A}_j^+, \tilde{A}_j^-$ are homogeneous of orders $0, \alpha$ in ξ respectively; the first admitting an analytic continuation in $\operatorname{Im} \xi_n > 0$, the second in $\operatorname{Im} \xi_n \leqq 0$. Let T_k, $k = 0, \cdots, [\alpha] - 1$ be bounded linear operators from $H_+^k(G)$ into $L^2(G)$ where $H_+^k(G)$, $k \geqq 0$ are the Sobolev-Slobo detskii spaces of generalized functions.

The purpose of the paper is to prove the solvability of: $Au_+ + \lambda^\alpha u_+ = f(x, T_0 u_+, \cdots, T_{[\alpha]-1} u_+)$ on G; u_+ in $H_+^\alpha(G)$ for large $|\lambda|$ and on a ray $\arg \lambda = \theta$ such that $\tilde{A}_j + \lambda^\alpha \neq 0$ for $|\xi| + |\lambda| \neq 0$ and for all j. $f(x, \zeta_0, \cdots, \zeta_{\alpha-1})$ has at most a linear growth in $(\zeta_0, \cdots, \zeta_{\alpha-1})$ and is continuous in all the variables.

Linear elliptic convolution equations in a bounded region for arbitrary α and with symbols having the above type of factorization ($\lambda = 0$) have been considered recently by Visik-Eskin [3]. Those equations are similar to integral equations since no boundary conditions are required.

The notation and terminology are those of Visik-Eskin and are given in § 1. The theorems are proved in § 2.

1. Let s be an arbitrary real number and $H^s(R^n)$ be the Sobolev-Slobodetskii space of (generalized) functions f such that:

$$||f||_s^2 = \int_{E^n} (1 + |\xi|^2)^s |\tilde{f}(\xi)|^2 d\xi < +\infty$$

where $\tilde{f}(\xi)$ is the Fourier transform of f.

We denote by $H^s(R_+^n)$, the space consisting of functions defined on $R_+^n = \{x: x_n > 0\}$ and which are the restrictions to R_+^n of functions in $H^s(R^n)$. Let lf be an extension of f to R^n, then:

$$||f||_s^+ = ||f||_{H^s(R_+^n)} = \inf ||lf||_s .$$

The infimum is taken over all extensions lf of f.

The $\overset{\circ}{H}{}_0^+ = \{f_+; f_+(x) = f(x) \text{ if } x_n > 0, f \in L^2(R^n), f_+(x) = 0 \text{ if } x_n \leqq 0\}$

and similarly for $\overset{\circ}{H}{}_0^-$.

We denote by H_s^+, the space of functions f_+ with f_+ in $\overset{\circ}{H}{}_0^+$ and $f_+ \in H^s(R_+^n)$ on R_+^n.

$\overset{\circ}{H}{}_s^+$ is the subspace of $H^s(R^n)$ consisting of functions with supports in $\mathrm{cl}\,(R_+^n)$. $\widetilde{H}_s^+$, $\widetilde{H}_s$, $\widetilde{\overset{\circ}{H}}{}_s^+$ denote respectively the spaces which are the Fourier images of H_s^+, $H^s(R^n)$, $\overset{\circ}{H}{}_s^+$.

Let $\tilde{f}(\xi)$ be a smooth decreasing (i.e., $|\tilde{f}(\xi)| \leqq M|\xi_n|^{-1-\varepsilon}$ for large $|\xi_n|$ and for some $\varepsilon > 0$) function. The operator Π^+ is defined as:

$$\Pi^+ \tilde{f}(\xi) = \frac{1}{2}\tilde{f}(\xi) + i(2\pi)^{-1}\,\text{v.p.}\int_{-\infty}^{\infty} \tilde{f}(\xi', \eta_n)(\xi_n - \eta_n)^{-1} d\eta_n$$

where $\xi' = (\xi_1, \cdots, \xi_{n-1})$.

For any $\tilde{f}$, then the above relation is understood as the result of the closure of the operator Π^+ defined on the set of smooth and decreasing functions.

Π^+ is a bounded mapping from $\widetilde{H}_s$ into $\widetilde{\overset{\circ}{H}}{}_s^+$ if $0 \leqq s < 1/2$ and is a bounded mapping from $\widetilde{H}_s$ into $\widetilde{H}_s^+$ if $s \geqq 1/2$.

Set: $\xi_- = \xi_n - i|\xi'|$; $(\xi_- - i)^s$ is analytic for any s if $\operatorname{Im}\xi_n \leqq 0$ and:

$$||f||_s^+ = ||\Pi^+(\xi_- - i)^s l\tilde{f}(\xi)||_0$$

where lf is any extension of f to R^n (Cf. [3], p. 93 relation (8.1)).

Let G be a bounded open set of R^n with a smooth boundary. $H^s(G)$ denotes the restriction to G of functions in $H^s(R^n)$ with the norm:

$$||u||_s = \inf ||v||_{H^s(R^n)}; \qquad v = u \text{ on } G\,.$$

By $H_+^s(G)$, we denote the space of functions f defined on all of R^n, equal to 0 on $R^n/\mathrm{cl}\,(G)$ and coinciding in $\mathrm{cl}\,G$ with functions in $H^s(G)$.

DEFINITION 1. $\tilde{A}(\xi)$ is in 0_α if and only if:
(i) $\tilde{A}(\xi)$ is a homogeneous function of order α in ξ.
(ii) $\tilde{A}$ is continuous for $\xi \neq 0$.

DEFINITION 2. $\tilde{A}_+(\xi)$ is in 0_α^+ if and only if:
(i) $\tilde{A}_+(\xi)$ is in 0_α.
(ii) $\tilde{A}_+(\xi', \xi_n)$ has an analytic continuation with respect to ξ_n in the half-plane $\operatorname{Im}\xi_n > 0$ for each ξ'.

Similar definition for 0_α^-:

DEFINITION 3. $\tilde{A}$ is in E_α if and only if:
(i) $\tilde{A}$ is in 0_α.

(ii) $\widetilde{A}(\xi) \neq 0$ for $\xi \neq 0$.

(iii) $\widetilde{A}(\xi)$ has, for $\xi' \neq 0$, continuous first order derivatives, bounded if $|\xi| = 1$, $\xi' \neq 0$.

DEFINITION 4. $\widetilde{A}(x, \xi', \xi_n)$ is in D^0_α if and only if:

(i) $\widetilde{A}(x, \xi)$ is infinitely differentiable with respect to x and ξ; $\xi \neq 0$.

(ii) $\widetilde{A}(x, \xi)$ is in 0_α for x in R^n.

(iii) $a_{k2}(x) = \dfrac{\partial^k}{(\partial\xi')^k}\widetilde{A}(x, 0, -1) = (-1)^k \exp(-i\alpha\pi)\dfrac{\partial^k}{(\partial\xi')^k}\widetilde{A}(x, 0, 1)$

x in R^n; $0 \leqq |k| < \infty$; $k = (k_1, \cdots, k_n)$.

DEFINITION 5. Let A be a bounded linear operator from H^+_s into $H^{s-\alpha}(R^n_+)$. Then any bounded linear operator T from H^+_{s-1} into $H^{s-\alpha}(R^n_+)$, (or from H^+_s into $H^{s-\alpha+1}(R^n_+)$) is called a right (left) smoothing operator with respect to A.

T is a smoothing operator with respect to A if it is both a left ane right smoothing operator.

Let $\widetilde{A}(\xi)$ be in 0_α for $\alpha > 0$. For u_+ in H^+_s, $s \geqq 0$, with support in $\mathrm{cl}\,(R^n_+)$, set: $Au_+ = F^{-1}(\widetilde{A}(\xi)\widetilde{u}_+(\xi))$ where F^{-1} is the inverse Fourier transform. It is well defined in the sense of generalized functions. A is a bounded linear operator from H^+_s into $H^{s-\alpha}(R^n)$.

Let $\widetilde{A}(x, \xi)$ be an element of E_α for each x in $\mathrm{cl}\,G$ and $\widetilde{A}(x, \xi)$ be infinitely differentiable with respect to x and ξ. Since G is a bounded set of R^n, we may assume that G is contained in a cube of side $2p$ centered at 0. We extend $\widetilde{A}(x, \xi)$ with respect to x to all of R^n by setting $\widetilde{A}(x, \xi) = 0$ if $|x| \geqq p - \varepsilon$ for $\varepsilon > 0$. We get a finite function, homogeneous of order α with respect to ξ.

We take the expansion into Fourier series of $\widetilde{A}(x, \xi)$:

$$\widetilde{A}(x, \xi) = \sum_{k=-\infty}^{\infty} \psi_0(x) \exp[(i\pi kx)/p]\widetilde{L}_k(\xi) \; ; \qquad k = (k_1, \cdots, k_n)$$

where:

$$\widetilde{L}_k(\xi) = (2p)^{-n}\int_{-p}^{p} \exp[(-i\pi kx)/p]\widetilde{A}(x, \xi)dx$$

$\psi_0(x) = 1$ for $|x| \leqq p - \varepsilon$; $\psi_0(x) = 0$ for $|x| \geqq p$; $\psi_0(x) \in C^\infty_c(R^n)$. We have: $|\widetilde{L}_k(\xi)| \leqq C|\xi|^\alpha(1 + |k|)^{-M}$ for arbitrary positive M. Let u_+ be in $H^s_+(G)$, we define:

$$Au_+ = \sum_{-\infty}^{\infty} \psi_0(x)[\exp((ikx\pi)/p)]L_k * u_+ \tag{1.1}$$

where $L_k * u_+ = L_k u_+$ is defined as before since $\widetilde{L}_k(\xi)$ is independent of x.

Denote by P^+, the restriction operator of functions defined on R^n to G. We consider an elliptic convolution equation of order α, on G of the form:

$$P^+Au_+ = \sum_j P^+\varphi_j A\psi_j u_+ + Tu_+ \tag{1.2}$$

T is a smoothing operator. The φ_j is a finite partition of unity corresponding to a covering N_j of $\mathrm{cl}\, G$ with $\mathrm{diam}\,(N_j)$ sufficiently small. The ψ_j are in $C_c^\infty(R^n)$ with $\varphi_j\psi_j = \varphi_j$ and $\mathrm{supp}\,(\psi_j) \subseteq N_j$.

Suppose $\tilde{A} \in D_\alpha^0$, then the operator $\varphi_j A\psi_j$ taken in local coordinates may be written as:

$$\varphi_j A\psi_j = \varphi_j A_j \psi_j + T_j$$

where A_j is a convolution operator of the form (1.1) and T_j is a smoothing operator (Cf. [3] Appendix 2).

2. The main result of the paper is the following theorem:

THEOREM 1. *Let A be an elliptic convolution operator on G, of order $\alpha > 0$, and of the form* (1.2). *Suppose that*:

(i) $\tilde{A}_j(x^j, \xi) \in E_\alpha \cap D_\alpha^0$.

(ii) $\tilde{A}_j(x^j, \xi)$ *has for $x_n^j = 0$ a factorization of the form*:

$$\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$$

where $\tilde{A}_j^+ \in 0_0^+$; $\tilde{A}_j^- \in 0_\alpha^-$ for all $x^j \in N_j \cap G$.

(iii) *There exists a ray $\arg \lambda = \theta$ such that $\tilde{A}_j(x^j, \xi) + \lambda^\alpha \neq 0$ for $|\xi| + |\lambda| \neq 0$, $x^j \in N_j \cap G$.*

Let $f(x, \zeta_0, \cdots, \zeta_{[\alpha]-1})$ be a function measurable in x on G, continuous in all the other variables. Suppose there exists a positive constant M such that:

$$|f(x, \zeta_0, \cdots, \zeta_{[\alpha]-1})| \leqq M\left\{1 + \sum_{j=0}^{[\alpha]-1} |\zeta_j|\right\}.$$

Let T_k; $k = 0, \cdots, [\alpha] - 1$ be bounded, linear operators from $H_+^k(G)$ into $L^2(G)$. Then for $|\lambda| \geqq \lambda_0 > 0$; $\arg \lambda = \theta$; there exists a solution u in $H_+^\alpha(G)$ of:

$$P^+(A + \lambda^\alpha)u_+ = f(x, T_0u_+, \cdots, T_{[\alpha]-1}u_+) \quad \text{on } G.$$

The solution is unique if f satisfies a Lipschitz condition in $(\zeta_0, \cdots, \zeta_{[\alpha]-1})$.

To prove the theorem, we shall do as in [2]. First, following Visik-Agranovich [4], we establish an *a priori* estimate and show the existence and the uniqueness of a solution of a linear elliptic convolution equation depending on a large parameter in a bounded region. Then we use the Leray-Schauder fixed point theorem to prove Theorem 1.

We have:

THEOREM 2. *Let A be an elliptic convolution operator, of order $\alpha > 0$, of the form* (1.2). *Suppose that all the hypotheses of Theorem* 1 *are satisfied. Let $f \in L^2(G)$; then there exists a unique solution u_+ in $H_+^\alpha(G)$ of*:

$$P^+(A + \lambda^\alpha)u_+ = f \text{ on } G; |\lambda| \geqq \lambda_0 > 0 \qquad \arg\lambda = \theta .$$

Moreover:

$$\| u_+ \|_\alpha + |\lambda|^\alpha \| u_+ \|_0 \leqq M \| f \|_0$$

where M is independent of λ, u_+.

Proof of Theorem 1. Let v be an element of $H_+^\alpha(G)$ and $0 \leqq t \leqq 1$. Consider the linear elliptic convolution equation:

$$P^+(Au_+ + \lambda^\alpha u_+) = f(x, tT_0v, \cdots, tT_{[\alpha]-1}v) .$$

With the hypotheses of the theorem, $f(x, tT_0v, \cdots, tT_{[\alpha]-1}v)$ is in $L^2(G)$. It follows from Theorem 2 that there exists a unique solution u_+ in $H_+^\alpha(G)$ of the problem.

Let $\mathscr{A}(t)$ be the nonlinear mapping from $[0, 1] \times H_+^\alpha(G)$ into $H_+^\alpha(G)$ defined by $\mathscr{A}(t)v = u_+$ where u_+ is the unique solution of the above problem.

The theorem is proved if we can show that $\mathscr{A}(1)$ has a fixed point.

PROPOSITION 1. $\mathscr{A}(t)$ is a completely continuous mapping from $[0, 1] \times H_+^\alpha(G)$ into $H_+^\alpha(G)$.

Proof. (i) $\mathscr{A}(t)$ is continuous. Suppose that $t_n \to t$; t_n, t in $[0, 1]$ $v_n \to v$ in $H_+^\alpha(G)$. Set: $u_n = \mathscr{A}(t_n)v_n$. Then from Theorem 2, we get:

$$\| u_n - u \|_\alpha \leqq M \| f(\cdot, t_nT_0v_n, \cdots, t_nT_{[\alpha]-1}v_n) \\ - f(\cdot, tT_0v, \cdots, tT_{[\alpha]-1}v) \|_0 .$$

It follows from Lemmas 3.1 and 3.2 of [1] that $u_n \to u$ in $H_+^\alpha(G)$.

(ii) $\mathscr{A}(t)$ is compact. Suppose that $\| v_n \|_\alpha \leqq M$. Then from the weak compactness of the unit ball in a Hilbert space and from the generalized Sobolev imbedding theorem, we get:

$$v_{n_j} \to v \text{ weakly in } H_+^\alpha(G) \text{ and strongly in } H_+^{\alpha-\varepsilon}(G); 0 < \varepsilon, \alpha - \varepsilon \geqq 0 .$$

Applying the argument of the first part, we get the compactness of $\mathscr{A}(t)$.

PROPOSITION 2. $I - \mathscr{A}(0)$ is a homeomorphism of $H_+^\alpha(G)$ into itself. If $v = \mathscr{A}(t)v$, for $0 < t \leqq 1$; then: $\| v \|_\alpha \leqq M$ where M is independent of t.

Proof. The first assertion is trivial.

Suppose that $v = \mathscr{A}(t)v$. It follows from Theorem 2 that:

$$\| v \|_\alpha + |\lambda|^\alpha \| v \|_0 \leqq M \| f(\cdot, tT_0v, \cdots, tT_{[\alpha]-1}v) \|_0$$
$$\leqq M\{1 + \| v \|_{[\alpha]-1}\} .$$

It is well-known that:

$$\| v \|_{[\alpha]-1} \leqq 1/2M \| v \|_\alpha + C \| v \|_0 .$$

Taking $|\lambda|$ sufficiently large, we have: $\| v \|_\alpha \leqq M_2$. $\mathscr{A}(t)$ satisfies the hypotheses of the Leray-Schauder fixed point theorem (the uniform continutiy condition as in [2] is not necessary). So $\mathscr{A}(1)$ has a fixed point, i.e. $\mathscr{A}(1)u_+ = u_+$.

The uniqueness of the solution in the case $f(x, \zeta_0, \cdots, \zeta_{[\alpha]-1})$ satisfies a Lipschitz condition in $(\zeta_0, \cdots, \zeta_{[\alpha]-1})$ follows trivially from the estimate of Theorem 2. We shall not reproduce it.

Proof of Theorem 2. As usual, we consider first the case of the positive half-space R_+^n with the convolution operator A having a constant symbol.

LEMMA 1. *Let $\tilde{A}(\xi)$ be an element of E_α, $(\alpha > 0)$. Suppose that: $\tilde{A}(\xi) = \tilde{A}_+(\xi)\tilde{A}_-(\xi)$ with $\tilde{A}_+(\xi)$ in 0_0^+, $\tilde{A}_-(\xi)$ in 0_α^-. Let P^+ be the restriction operator of functions in R^n to R_+^n and A be the convolution operator with symbol $\tilde{A}(\xi)$. Suppose there exists a ray* $\arg \lambda = \theta$ *such that: $\tilde{A}(\xi) + \lambda^\alpha \neq 0$ for $|\xi| + |\lambda| \neq 0$. If f is in $H^0(R_+^n)$, then there exists a unique solution u in H_α^+ of:*

$$P^+(A + \lambda^\alpha)u_+ = f \text{ on } R_+^n; |\lambda| \geqq \lambda_0 > 0 .$$

Moreover:

$$\| u_+ \|_\alpha^+ + |\lambda|^\alpha \| u_+ \|_0^+ \leqq M \| f \|_0^+$$

where M is independent of λ, u_+, f.

Proof. Set $\tilde{A}(\xi, \lambda) = \tilde{A}(\xi) + \lambda^\alpha$. It is homogeneous of order α in (ξ, λ). Since $\tilde{A}(\xi)$ is in E_α, we have the following factorization with respect to ξ_n, which is unique up to a constant multiplier:

$$\tilde{A}(\xi) = \tilde{A}_+(\xi)\tilde{A}_-(\xi)$$

(Cf. Theorem 1.2 of [3], p. 95). The same proof with $\xi_+ = \xi_n + i\,|\,\xi'\,|$ replaced by $\xi_+^\lambda = \xi_n + i(|\,\lambda\,| + |\,\xi'\,|)$ and ξ_- replaced by:

$$\xi_-^\lambda = \xi_n - i(|\,\lambda\,| + |\,\xi'\,|)$$

gives:

$$\tilde{A}(\xi, \lambda) = \tilde{A}_+(\xi, \lambda)\tilde{A}_-(\xi, \lambda)\ .$$

Moreover:

If $\tilde{A}_+(\xi)$ is in 0_0^+, then $\tilde{A}_+(\xi, \lambda)$ is also in O_0 (is homogeneous of order 0 in (ξ, λ)). Similarly for $\tilde{A}_-(\xi, \lambda)$.

Let $lf(x)$ be an extension of f to R^n. Consider:

$$\tilde{u}_+(\xi) = (\tilde{A}_+(\xi, \lambda))^{-1} \prod{}^+ \, l\tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}\ .$$

For $|\,\lambda\,| \neq 0$, $\tilde{u}_+(\xi)$ has an analytic continuation in $\operatorname{Im}\xi_n > 0$ and:

$$\int |\,\tilde{u}_+(\xi', \xi_n + i\tau)\,|^2\, d\xi' d\xi_n \leqq C\ ,$$

C is independent of $\tau > 0$. So: $\tilde{u}_+(\xi) \in \overset{\sim}{\overset{\circ}{H}}{}_0^+$. (Cf. [3], p. 91).

We get:

$$\begin{aligned} ||\,u_+\,||_\alpha^+ &= ||\prod{}^+ (\xi_- - i)^\alpha \tilde{u}_+(\xi)\,||_0^+ \\ &\leqq ||\,(\xi_- - i)^\alpha(\tilde{A}_+(\xi, \lambda))^{-1} \prod{}^+ \, l\tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}\,||_0\ . \end{aligned}$$

Since $\tilde{A}_+(\xi, \lambda)$ is homogeneous of order 0 in (ξ, λ), we have:

$$\tilde{A}_+(\xi, \lambda) = \tilde{A}_+(\xi/(|\,\xi\,| + |\,\lambda\,|), \lambda/(|\,\xi\,| + |\,\lambda\,|))\ .$$

Let $c = \operatorname{Min}|\,\tilde{A}_+(\xi, \lambda)\,|$ for $|\,\xi\,| + |\,\lambda\,| = 1, \arg\lambda = \theta$. Then $c > 0$ and is independent of λ.

So:

$$\begin{aligned} ||\,u_+\,||_\alpha^+ &\leqq c^{-1}\,||\,(\xi_- - i)^\alpha \prod{}^+ \, l\tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}\,||_0 \\ &\leqq C\,||\,l\tilde{f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}\,||_\alpha\ . \end{aligned}$$

We may write:

$$\tilde{A}_-(\xi, \lambda) = (|\,\xi\,| + |\,\lambda\,|)^\alpha\, \tilde{A}_-(\xi/(|\,\xi\,| + |\,\lambda\,|), \lambda/(|\,\xi\,| + |\,\lambda\,|))\ .$$

Let $C = \operatorname{Min}|\,\tilde{A}_-(\xi, \lambda)\,|$ for $|\,\xi\,| + |\,\lambda\,| = 1, \arg\lambda = \theta$. Then $C > 0$ and is independent of λ.

We obtain:

$$||\,u_+\,||_\alpha^+ \leqq C\,||\,l\tilde{f}(\xi)\,||_0 \leqq C_2\,||\,f\,||_0^+\ .$$

A similar argument gives:

$$\| u_+ \|_0^+ \leqq C |\lambda|^{-\alpha} \| f \|_0^+ .$$

So:

$$\| u_+ \|_\alpha^+ + |\lambda|^\alpha \| u_+ \|_0^+ \leqq C \| f \|_0^+ .$$

C is independent of λ, f, u_+.

A direct verification shows that u_+ is a solution of the equation. It remains to show that the solution is unique. Let v_+ be an element of H_α^+. Suppose that v_+ is also a solution of the equation. Then as in [3], $\tilde{v}_+(\xi)$, its Fourier transform is given by an expression of the same form as $\tilde{u}_+(\xi)$ with $\widetilde{lf}(\xi)$ replaced by $\widetilde{l_1 f}(\xi)$. $l_1 f$ being an extension of f to R^n.

Set $l_2 f = lf - l_1 f$. Then $l_2 f \in H_0^-$, so $\widetilde{l_2 f} \in \overset{\circ}{\tilde{H}}{}_0^-$. $\widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}$ is analytic in $\operatorname{Im} \xi_n \leqq 0$ for $|\lambda| \neq 0$ and moreover:

$$\int |\widetilde{l_2 f}(\xi', \xi_n + i\tau)|^2 |\tilde{A}_-(\xi', \xi_n + i\tau)|^{-2} d\xi' d\xi_n \leqq C$$

where C is independent of $\tau \leqq 0$.

Hence $\widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}$ is in $\overset{\circ}{\tilde{H}}{}_0^-$ (Cf. [3], p. 91), so:

$$\Pi^+ \widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1})) = 0 .$$

Therefore: $\tilde{A}_+(\xi, \lambda)(\tilde{u}_+(\xi) - \tilde{v}_+(\xi)) = 0$.

But $\tilde{A}_+(\xi, \lambda) \neq 0$ for $|\lambda| \neq 0$, we get $\tilde{u}_+ = \tilde{v}_+$. Q.E.D.

Set:

$$A_1 u = \sum_{k=-\infty}^{\infty} \psi_0(x) \exp [(ik\pi x)/p] L_k * u$$

$$A_0 u = \sum_{k=-\infty}^{\infty} \psi_0(x_0) \exp [(ik\pi)/p] L_k * u$$

where L_k, ψ_0 are as in § 1.

LEMMA 2. *Let A_1, A_0 be as above and $\psi(x)$ be in $C_c^\infty(R^n)$ with $\psi(x) = 0$ for $|x - x_0| > \delta$; $|\psi(x)| \leqq K$ where K is independent of δ. Then*:

$$\| \psi(A_1 - A_0) u \|_{s-\alpha}^+ \leqq C\delta \| u \|_s^+ + C(\delta) \| u \|_{s-1}^+$$

for all u in H_s^+, $s \geqq 0$.

Proof. Cf. Lemma 4.7 of [3], p. 119.

Proof of Theorem 2 (continued). (1) First, we establish an *a-priori* estimate of the solutions.

Consider:

$$P^+\varphi_j A\psi_j u_+ + \lambda^\alpha P^+(\varphi_j u_+) = P^+(\varphi_j f) - Tu_+$$

where T is a smoothing operator with respect to $\varphi_j A\psi_j$.

It has been shown in [3] (Appendix 2) that in a local coordinates system, the operator $\varphi_j A\psi_j$ becomes: $\varphi_j A_j\psi_j + T_j$ where A_j has for symbol $\tilde{A}_j(x^j, \xi)$ and T_j is a smoothing operator.

So, we have:

$$P^+\varphi_j A_j(\psi_j u_+) + \lambda^\alpha P^+(\varphi_j u_+) = P^+(\varphi_j f) + T_j^2 u_+$$

where T_j^2 is again a smoothing operator.

Let A_{j0} be the convolution operator with symbol $\tilde{A}_j(x_0^j, \xi)$ evaluted at the point x_0^j. We write:

$$P^+\varphi_j A_{j0}(\psi_j u_+) + \lambda^\alpha P^+(\varphi_j u_+) = P^+(\varphi_j f) \\ + T_j^2 u_+ + P^+\varphi_j(A_{j0} - A_j)\psi_j u^+ .$$

Applying Lemma 4.D.1 of [3] (p. 145), we have:

$$P^+\varphi_j A_{j0}(\psi_j u_+) = P^+A_{j0}(\varphi_j u_+) + T_j^3 u_+$$

where T_j^3 is a smoothing operator.

Therefore:

$$(A_{j0} + \lambda^\alpha)\varphi_j u_+ = \varphi_j f + T_j^4 u_+ + \varphi_j(A_{j0} - A_j)(\psi_j u_+) .$$

The symbols $\tilde{A}_{j0}$ satisfy the hypotheses of Lemma 1. Applying Lemma 1; 2, we obtain:

$$\| \varphi_j u_+ \|_\alpha^+ + |\lambda|^\alpha \|\varphi_j u_+ \|_0^+ \leqq M\{\| \varphi_j f \|_0^+ + \| u_+ \|_0 \\ + 1/2M \| u_+ \|_\alpha + \| \psi_j u_+ \|_\alpha^+ + \| \varphi_j u_+ \|_0^+\}$$

where we have used the well-known inequality:

$$\| u_+ \|_{\alpha-1} \leqq \varepsilon \| u_+ \|_\alpha + C(\varepsilon) \| u_+ \|_0 .$$

On the other hand: $\| \psi_j u_+ \|_\alpha^+ \leqq M \| u_+ \|_\alpha$. Summing with respect to j, we get:

$$\| u_+ \|_\alpha + |\lambda|^\alpha \| u_+ \|_0 \leqq M\{\| f \|_0 + 1/2M \| u_+ \|_\alpha \\ + \delta \| u_+ \|_\alpha + K \| u_+ \|_0\} .$$

Taking δ small and $|\lambda|$ sufficiently large, we have:

$$\| u_+ \|_\alpha + |\lambda|^\alpha \| u_+ \|_0 \leqq M \| f \|_0 .$$

So, if there exists a solution, then the solution is unique.

(2) It remains to show the existence of a solution. From Lemma 1, we know that $P^+(A_{j0} + \lambda^\alpha)$ has an inverse R_{j0}. Let $\hat{R}_{j0}$ be the operator R_{j0} expressed in the global system of coordinates of G. Consider:

$$Rf = \sum_j \varphi_j \hat{R}_{j0}(\psi_j f) .$$

R is a bounded linear mapping from $L^2(G)$ into $H_+^\alpha(G)$.

We show that: $\mathscr{A}Rf = P^+(A + \lambda^\alpha)Rf = f + \mathscr{C}f$ with $\|\mathscr{C}\| \leqq 1/2$.

We have:

$$\mathscr{A}Rf = \sum_j P^+(A + \lambda^\alpha)\varphi_j \psi_j \hat{R}_{j0}(\psi_j f) .$$

Applying Lemma 4.D.1. of [3], we may write:

$$\mathscr{A}Rf = \sum_j P^+\varphi_j(A + \lambda^\alpha)\psi_j \hat{R}_{j0}(\psi_j f) + TRf$$

where T is a smoothing operator.

We express $\varphi_j(A + \lambda^\alpha)\psi_j \hat{R}_{j0}(\psi_j f)$ in local coordinates. We get:

$$\varphi_j(A_{j0} + \lambda^\alpha)\psi_j R_{j0}(\psi_j f) + \varphi_j(A_j - A_{j0})\psi_j R_{j0}(\psi_j f) + T_j^1 R_{j0}(\psi_j f) .$$

Using Lemma 4.D.1 of [3] again, we obtain:

$$\begin{aligned} &\varphi_j(A_{j0} + \lambda^\alpha)R_{j0}(\psi_j f) + \varphi_j(A_j - A_{j0})\psi_j R_{j0}(\psi_j f) + T_j^2 R_{j0}(\psi_j f) \\ &= T_j^2 R_{j0}(\psi_j f) + \varphi_j f + \varphi_j(A_j - A_{j0})\psi_j R_{j0}(\psi_j f) = \varphi_j f + \mathscr{C}_j(\psi_j f) . \end{aligned}$$

The T_j are all smoothing operators.

Applying Lemma 1, we have:

$$\| T_j^2 R_{j0}(\psi_j f) \|_0^+ \leqq C \| R_{j0}(\psi_j f) \|_{\alpha-1}^+ \leqq \varepsilon \| f \|_0 + C |\lambda|^{-\alpha} \| f \|_0 .$$

From Lemmas 1 and 2, we get:

$$\begin{aligned} \| \varphi_j(A_j - A_{j0})\psi_j R_{j0}(\psi_j f) \|_0^+ &\leqq \delta \| \psi_j R_{j0}(\psi_j f) \|_\alpha^+ \\ &\quad + C(\delta) \| \psi_j R_{j0}(\psi_j f) \|_{\alpha-1}^+ \\ &\leqq \delta \| f \|_0 + C(\delta) \| \hat{R}_{j0}(\psi_j f) \|_{\alpha-1} \\ &\leqq \delta \| f \|_0 + \varepsilon C(\delta) \| R_{j0}(\psi_j f) \|_\alpha \\ &\quad + C(\delta)M(\varepsilon) \| \hat{R}_{j0}(\psi_j f) \|_0 \\ &\leqq \{\delta + \varepsilon C(\delta)\} \| f \|_0 \\ &\quad + |\lambda|^{-\alpha} M(\varepsilon)C(\delta) \| f \|_0 . \end{aligned}$$

Taking ε, δ small, $|\lambda|$ large enough, we have:

$$\| \mathscr{C}_j(\psi_j f) \|_0^+ \leqq \frac{1}{4N} \| f \|_0 .$$

We obtain:

$$Rf = f + TRf + \sum_j \hat{\mathscr{C}}_j(\psi_j f) = f + \mathscr{C}f$$

where $\hat{\mathscr{C}}_j$ is the operator $\mathscr{C}_j$ expressed in the global coordinates system of G. We obtain: $\|\mathscr{C}f\|_0 \leqq 1/4 \|f\|_0 + 1/4 \|f\|_0$ for large $|\lambda|$.

Hence $\|\mathscr{C}\| \leqq 1/2$; therefore $(I + \mathscr{C})^{-1}$ exists. We define:

$$\mathscr{A}^{-1} = R(I + \mathscr{C})^{-1} .$$

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SOME COMPLEMENTED FUNCTION SPACES IN $C(X)$

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et X and Z be compact Hausdorff spaces, and let P be a linear subspace of $C(X)$ which is isometrically isomorphic to $C(Z)$. In this paper conditions, some necessary and some sufficient, are presented which insure that P is complemented in $C(X)$. For example if X is metrizable, P contains a strictly positive function, and the decomposition induced on X by P is lower semi-continuous then P is complemented in $C(X)$.

D. Amir has shown that not all such spaces P are complemented when X is metrizable ([1], see also R. Arens, [4]). However, R. Arens [4] has constructed a class of subspaces of $C(X)$ which are complemented. In § 2 we present classes of complemented subspaces which extend the class exhibited by R. Arens [Theorem 4, Lemma 5, Theorem 8]. A comparison of these results preceds Theorem 8.

Suppose that X is the Stone-Čech compactification of a locally compact completely regular space Y, Z is a compactification of Y which has first countable remainder, and P is the natural embedding of $C(Z)$ in $C(X)$. In § 3 we show that if P is complemented in $C(X)$, then Y is pseudo-compact. This theorem was proved by J. Conway [6] for the case in which Z is the one point compactification of Y.

By introducing the concept of weakly separating in § 2, we are paralleling the concept of a Choquet boundary. Related results and definitions are found in [22].

1. If A and B are subsets of a topological space, $\mathrm{cl}\, A$ will denote the closure of A, and A-B will denote the set of points which are in A but not in B. If E is a normed linear space, $S(E)$ and E^* denote the unit ball in E and the dual of E respectively. If K is a convex subset of a topological vector space, $\mathrm{ext}\, K$ will represent the set of extreme points of K. If g and h are functions such that the range of g is contained in the domain of h, the composite of g and h will be written $h \circ g$. Finally, if X is a topological space and x is in X, the point evaluation functional associated with x is the linear functional x' defined on $C(X)$ by $x'(f) = f(x)$ for each f in $C(X)$. In this paper $C(X)$ will denote the Banach space of all bounded real-valued continuous functions on X normed with the supremum norm.

2. Let P be a subspace of a normed linear space E. We define $D(P) = \{b \text{ in } S(E^*)\colon b \text{ restricted to } P \text{ is in } \mathrm{ext}\, S(P^*)\}$. We say that P is *weakly separating* (with respect to E) if P separates the points

of $D(P)$ intersect ext $S(E^*)$, that is, if g and h are distinct points in this intersection, then there is a p in P such that $g(p) \neq h(p)$. Although we have stated the definition for an arbitrary normed linear space, we are mainly interested in the space $E = C(X)$, where X is a compact Hausdorff space. It follows readily from the definition that a subspace P of $C(X)$ is weakly separating if for any two distinct point evaluation functionals x' and y' whose restrictions to P have norm one, there is a p in P such that $|p(x)| \neq |p(y)|$. In particular, a subspace of $C(X)$ which contains the constants and separates the points of X, or a closed ideal in $C(X)$ is weakly separating.

LEMMA 1. *Let P be a subspace of E. The following are equivalent:*

(i) *P separates the members of $D(P)$*

(ii) *P separates the members of $D(P)$ intersect* ext $S(E^*)$

(iii) ext $S(E^*)$ *contains $D(P)$.*

Proof. (iii) *implies* (i). If P does not separate the elements of $D(P)$, then there must exist distinct elements g and h in $D(P)$ such that the restriction of $g - h$ to P is the zero functional. It follows that $b = (1/2)(g + h)$ agrees with g and h on P. Hence b is in $D(P)$ but not in ext $S(E^*)$.

(ii) *implies* (iii). Now suppose that P separates the elements of $D(P)$ intersect ext $S(E^*)$. Let b be a point in $D(P)$. We are to prove that b is in ext $S(E^*)$. Let $K = \{k$ in $S(E^*)$: k agrees with b on $P\}$. Clearly K is a convex set containing b. Also K is closed, and hence compact, in the weak* topology on E^*. By the Krien-Milman theorem, K has extreme points. We will show that ext K is contained in ext $S(E^*)$. Suppose $k = (1/2)(g + h)$ where k is in ext K and g and h are in $S(E^*)$. Thus for each p in P, $1/2h(p) + 1/2g(p) = k(p) = b(p)$. The restrictions of g and h to P both belong to $S(P^*)$, and the restriction of b is in ext $S(P^*)$. Therefore g and h agree with b on P and both must belong to K. Since k was assumed to be an extreme point of K, we have $g = h = k$. We conclude that ext $S(E^*)$ contains ext K. If b is the only point in K, then b must be in ext $S(E^*)$. Otherwise K must contain two distinct extreme points. Clearly P can not separate these two points of $D(P)$ intersect ext $S(E^*)$. This proves that (ii) implies (iii).

Since the fact that (i) implies (ii) is obvious, the proof is complete.

LEMMA 2. *If P is weakly separating in E, then the weak topology on $D(P)$ induced by P is equivalent to the weak topology induced by E.*

Proof. Clearly, the weak topology induced by P is coarser than the one induced by E. To prove the converse, suppose that g_i is a net of functionals in $D(P)$ which converge with respect to the weak topology induced by P to a functional g which is also in $D(P)$. If g_i does not converge to g with respect to the weak topology induced by E, there will exist a subnet which never intersects some neighborhood (in topology induced by E) of g. Since by Alaoglu's theorem $S(E^*)$ is compact, we may assume the existence of a further subset g_j which converges to a functional h distinct from g. Since g_j is a subset of g_i, h must agree with g on P. Since the norm of h is less than or equal to one, h is in $D(P)$. Since P does not distinguish between g and h, the previous lemma contradicts the hypothesis that P is weakly separating. The lemma is proved.

In the following let X be a compact Hausdorff space.

LEMMA 3. *Let P be a weakly separating subspace of $C(X)$. The following are equivalent:*

(i) *There is a projection of norm one of $C(X)$ onto P,*

(ii) *P is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z,*

(iii) *There exist a closed subset Y of X such that P is isometrically isomorphic to $C(Y)$ via the restriction mapping.*

Furthermore, if P is weakly separating there can exist at most one projection of norm one of $C(X)$ onto P.

Proof. (i) *implies* (iii). Let L be a projection of norm one of $C(X)$ onto P. If x' is an evaluation functional in $D(P)$, then $x' \circ L$ is a functional in $S(C(X)^*)$ which agrees with x' on P. Since P is weakly separating in $C(X)$, $x' \circ L = x'$. Hence for each f in $C(X)$, Lf agrees with f on $\{x \text{ in } X: x' \text{ is in } D(P)\}$, and therefore on the closure Y of this set. With a simple application of the Tietze Extension Theorem, we see that the restriction map carries P onto $C(Y)$. Furthermore, this restriction mapping does not decrease the norm of points in P. For by Lemma 1 every functional in $D(P)$ can be expressed as either an evaluation functional of a point in Y or as the negative of such a functional, and for p in P, $\|p\| = \sup\{h(p): h \text{ in } D(P)\}$. We have shown that the restriction mapping is an isometric isomorphism of P onto $C(Y)$.

(ii) *implies* (i). Let Z be a compact Hausdorff space, and let L be an isometric isomorphism of P onto $C(Z)$. Let L' denote the adjoint of L. Since L is an isometric isomorphism, L' is an isometric isomorphism of $C(Z)^*$ onto P^*. Furthermore, L' restricted to $\operatorname{ext} S(C(Z)^*)$ is a homeomorphism onto $\operatorname{ext} S(P^*)$ with the weak topologies induced by $C(Z)$ and P respectively. Now for x in $\operatorname{ext} S(P^*)$, let

$H(x)$ be the unique element in $\operatorname{ext} S(C(X)^*)$ which agrees with x on P. For z in Z let $E(z)$ denote the evaluation functional of z. Now for f in $C(X)$ consider the function $f \circ H \circ L' \circ E(\cdot)$ defined on Z. By Lemma 2 this function is continuous. The map Q which carries f in $C(X)$ onto $L^{-1}(f \circ H \circ L' \circ E(\cdot))$ is a mapping of norm one of $C(X)$ into P. Furthermore, if p is in P, then $p \circ H \circ L' \circ E(z) = Lp(z)$, for all z in Z. Thus $p \circ H \circ L' \circ E(\cdot) = Lp$, and Q is a projection of $C(X)$ onto P.

It is evident that (iii) implies (ii).

To prove the second part of the lemma, suppose that H and L are two projections from $C(X)$ onto P, both of which have norm one. Let Y be the subset of X constructed in the proof that (i) implies (iii). For any f in $C(X)$, we have shown that Lf, Hf and f all agree on Y. It of course follows that $(H - L)(f)$ vanishes on Y. However, we have shown that the restriction mapping carries P isometrically onto $C(Y)$. Therefore, $(H - L)(f)$ must be the zero function, and $Hf = Lf$ for all f in $C(X)$. This completes the proof.

We will say that a subspace P of $C(X)$ has a *weakly separating quotient* if it has the property that for any two distinct points x and y in X such that $p(x) = -p(y)$ for every p in P, then the evaluation functional of x (or equivalently the evaluational functional of y) restricted to P is not an extreme point of $S(P^*)$.

REMARK. Each of the following properties on a subspace P of $C(X)$ imply that P has a weakly separating quotient:

(i) P is weakly separating in $C(X)$,

(ii) P contains a function which is strictly positive,

(iii) for each p in P, $|p|$ is also in P.

A proof for the above remark is straightforward. In particular, any closed ideal in $C(X)$, or any subspace of $C(X)$ which contains the constants has a weakly separating quotient.

In order to state the next theorem we make a few more definitions. Let X be a Hausdorff space and let M be a partition of X into closed subsets. For x in X let $M(x)$ denote the member of M which contains x. Corresponding to the standard definitions we say that M is *lower semi-continuous* if $\{x \text{ in } X\colon M(x) \text{ intersect } U \text{ is nonempty}\}$ is an open set in X for every open set U in X.

If P is a linear space of bounded, continuous functions, then the *P-partition* of X is the partition associated with the following equivalence relation R. A couple (x, y) is in R if and only if $p(x) = p(y)$ for every p in P. Now let $K(P) = \cup\ \{K \text{ contained in } X\colon K$ is a member of the P-partition of X, and K contains more than one point of $X\}$. We will say that P has a *lower semi-continuous quotient* if the restriction of the P-partition to $\operatorname{cl} K(P)$ is lower semicontinuous.

In the following let X denote a compact Hausdorff space, and let

P be a linear subspace of $(C(X)$ which has a weakly separating quotient.

THEOREM 4. *If there is a projection of norm one of $C(X)$ onto P, then P is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z. Conversely, suppose that X is metrizable, and that P has a lower semi-continuous quotient. If P is isometrically isomorphic to $C(Z)$, for some compact Hausdorff space Z, then there is a projection of $C(X)$ onto P which has norm less than or equal three.*

Proof. Let M denote the P-partition of X. Let X/M have the quotient topology, and let $M(\cdot)$ denote the natural mapping of X onto X/M. We observe that X/M is a compact Hausdorff space. Now let Q denote the linear subspace of $C(X)$ consisting of all functions that are constant on each closed subset of X which is a member of M. One can verify that P is contained in Q, and that the mapping which carries q in Q onto the function $q \circ M^{-1}(\ )$ in $C(X/M)$ is an isometric isomorphism of Q onto $C(X/M)$. The image P' of P under this mapping is a weakly separating subspace of $C(X/M)$ since P has a weakly separating quotient. If there is a projection of norm one from $C(X)$ onto P, then there certainly is a projection of norm one from $C(X/M)$ onto P. By the preceding lemma, we conclude that P', and hence P, is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z.

To prove the second part of the theorem, we assume that X is metrizable, P has a lower semi-continuous quotient, and that there is a compact Hausdorff space Z such that P is isometrically isomorphic to $C(Z)$. We maintain the same notation used directly above. Since P' is weakly separating in $C(X/M)$, and P is isometrically isomorphic to $C(Z)$, it follows from the preceding lemma that there is a projection of norm one from Q onto P. To complete the proof it will suffice to show that there is a projection from $C(X)$ onto Q which has norm less than or equal to three. We will prove a stronger result.

Let Y be a metric space. Let K be a partition of Y such that every member of K is a complete subset of Y. A member of K will be called a *plural set* if it contains two distinct points of Y. Let the restriction K' of K to the subset of Y,

$$B = \text{cl} \cup \{A \text{ contained in } Y: A \text{ a plural set in } K\}$$

be lower semi-continuous. Assume also that B/K' is paracompact. Let Q denote the subspace of $C(Y)$ consisting of the functions which are constant on each member of K. We recall that by the notation we adopted, $C(Y)$ is the Banach space of all bounded continuous functions on Y. The following lemma establishes the theorem.

LEMMA 5. *There is a projection of $C(Y)$ onto Q which has norm less than or equal to three.*

Proof. In the usual manner we can embed B into the unit ball of $C(B)^*$. With the weak topology on $C(B)^*$ induced by $C(B)$, $C(B)^*$ is a locally convex space, B is embedded onto a homeomorphic image of itself, say B', and the closed convex hull of compact subsets of B' are again compact. Let s denote the composite of the quotient mapping of B onto B/K' with the homeomorphism, h, between B and B'.

We now show that s^{-1} is a lower semi-continuous function carrying points in B/K' onto closed subsets of B'. Let U be an open set in B'. Let

$$W = \{y \text{ in } B/K' \colon s^{-1}(y) \text{ intersect } U \text{ is not empty}\} .$$

To show that s^{-1} is lower semi-continuous we must show that W is open in B/K'. We note that $W = s(U)$. Now since K' is lower semi-continuous and $h^{-1} \circ s^{-1} \circ s \circ h(\cdot)$ carries a point b in B onto the member of K' which contains b, the set

$$V = \{b \text{ in } B \colon h^{-1} \circ s^{-1} \circ s \circ h(b) \text{ intersect } h^{-1}(U) \text{ is not empty}\}$$

is open in B. Hence $h(V) = \{b' \text{ in } B \colon h^{-1} \circ s^{-1} \circ s(b') \text{ intersect } h^{-1}(U) \text{ is not empty}\}$ is open in B'. Since this last set is $s^{-1} \circ s(U)$, $s^{-1} \circ s(U)$ is open. Since B/K' has the quotient topology induced by s, this implies that $s(U)$—and hence W—is open in B/K'. Therefore s^{-1} is lower semi-continuous.

Now since B/K' is paracompact, and since there is a metric on B' (which induces an equivalent topology for B') for which the set $s^{-1}(y)$ is complete for each y in B/K', we have satisfied the hypothesis for a selection theorem proved by E. Michael [20]. This theorem proves the existence of a continuous function t which carries B/K' into $C(B)^*$, and has property that $t(y)$ is contained in the closed convex hull of $s^{-1}(y)$ for each y in B/K'.

We now define a projection from $C(B)$ onto Q' the subspace of functions in $C(B)$ which are constant on members of K'. For f in $C(B)$, let Lf denote the function such that for each b in B,

$$(Lf)(b) = [t(s \circ h(b)](f) .$$

Since t is continuous on B/K', Lf is a continuous function. Since $t(s \circ h(b))$ is in the closed convex hull of $s^{-1} \circ s \circ h(b)$, the norm of $t(s \circ h(b))$ does not exceed one. Thus the maximum of Lf over B does not exceed the maximum of f over B. Finally, one can verify that if q is in Q', $Lq = q$, and that for each f in $C(B)$, Lf is in Q'. We have shown that L is a projection of norm one of $C(B)$ onto Q'.

Since Y is a metric space, there is an operator E of norm one from $C(B)$ into $C(Y)$ such that $R \circ Ef = f$ for every f in $C(B)$. Here R denotes the operator which assigns to each function in $C(Y)$ its restriction to B (R. Arens [3], also Dugundji [8]). Following a construction due to Arens [4], we define an operator J by $Jf = f + E(LRf - Rf)$. The proof of the lemma is completed by verifying that J is a projection of $C(Y)$ onto Q which has norm no greater than three.

In the following corollaries let X denote a compact Hausdorff space.

COROLLARY 6. *Let P be a finite dimensional subspace of $C(X)$ which has a weakly separating quotient. There is a projection of norm one from $C(X)$ onto P if and only if P has a basis $\{p_i\}_{i=1}^n$ such that $\| \sum_{i=1}^n c_i p_i \| = \max | c_i |$.*

COROLLARY 7. *$C(X)$ contains a weakly separating subspace of co-dimension n which has a projection of norm one if and only if X contains n isolated points.*

Proof. To prove the necessity of the condition, let L be a projection of norm one of $C(X)$ onto a weakly separating subspace P of co-dimension n in $C(X)$. Define $Y = \text{cl}\,\{x \text{ in } X\colon x' \circ L = x'\}$. We will show that $X - Y$ contains precisely n points. Since $X - Y$ is open, these points will be isolated. We observe that the range, Q, of $I - L$ has dimension n, and that if q is in Q, then q vanishes on Y. Since the functions in Q take all their nonzero values on $X - Y$, $X - Y$ must contain at least n points. If $X - Y$ contained $n + 1$ points, there would exist $n + 1$ open sets U_i in $X - Y$, and corresponding functions f_i of norm one which vanish off U_i. These functions span an $n + 1$ dimensional subspace of $C(X)$; hence there is a nonzero function f in this span that is also in P. But f vanishes on Y. By Lemma 3, the restriction map is an isometry of P onto $C(Y)$. Hence we arrive at the contradiction that f is the zero function.

If X contains n isolated points, the space of all functions in $C(X)$ which vanish on these n points is a weakly separating subspace of $C(X)$ (sinces this space is an ideal) of co-dimension n in $C(X)$. It is also clear there is a projection of norm one from $C(X)$ onto this subspace. The proof is completed.

REMARK. R. Arens [4] has constructed an example of two compact metric spaces X and Z such that $C(X)$ contains an isometric isomorphic-copy of $C(Z)$ which has a weakly separating quotient, but which is not complemented in $C(X)$. Hence the assumption that P has a lower semi-continuous quotient cannot be simply omitted from the theorem, (Also see Amir [1]).

The preceding theorem and lemma should be compared to Theorem 2.2 in (R. Arens [4]). Using the notation preceding the lemma, Professor Arens proved that under the following conditions there will exist a projection of norm less than or equal to three of $C(Y)$ onto Q:

(i) K is a partition of Y into closed subsets

(ii) Y and Y/K are metrizable

(iii) the quotient map of Y onto Y/K is upper semi-continuous[1]

(iv) if $\{x_i\}$ is a sequence in Y such that each x_i belongs to a distinct plural set in K, then a member of K which contains a limit point of $\{x_i\}$ is a singleton.

Apropos to property (ii), A. H. Stone has proved ([23]) that a metrizable space is paracompact. Property (iv) above implies that K' is lower semi-continuous. In the special case that Y is a complete metric space, the preceding lemma contains the above theorem of Arens. If Y is compact, the previous theorem includes both of these results.

In the following, let Y be a metrizable space, and K a partition of Y satisfying properties (i), (iii), and (iv) above. For each K_i in K let P_i be a complemented subspace of $C(K_i)$ which contains the constants. Let L_i denote a projection of $C(K_i)$ onto P_i. We assume that $m = \sup\{\| L_i \|\} < \infty$. Finally, let Q denote the subspace of $C(Y)$ consisting of all functions q such that the restriction of q to K_i is a function in P_i.

THEOREM 8. *There is a projection of $C(Y)$ onto Q which has norm less than or equal to $2 + m$.*

Proof. For a set Z let $B(Z)$ denote the space of bounded functions on Z. Let $D = \cup\{K_i$ contained in Y: K_i is a plural set in $K\}$. Let R and R_i denote the restriction map of $B(Y)$ onte $B(\text{cl}\, D)$ and of $B(Y)$ onto $B(K_i)$ respectively (K_i in K). Let E denote a linear mapping of $C(\text{cl}\, D)$ into $C(Y)$ such that E has norm one, and $R \circ E$ is the identity mapping on $C(\text{cl}\, D)$. Let H be the linear mapping of $C(Y)$ into $B(\text{cl}\, D)$ such that $R_i \circ H = L_i \circ R_i$ for all K_i in K. Let I denote the identity on $C(Y)$, and let $L = I + E \circ R(H - I)$. The proof consists of establishing that L is the desired projection. The variation of a function f defined on a set Z is $\text{var}\,(f) = \max_{z \text{ in } Z} f(z) - \min_{z \text{ in } Z} f(z)$.

We proceed by proving four assertions, the last of which establishes the theorem.

Assertion 1. If x_i is in K_i, K_i is in K, y is not in D and x_i con-

[1] Professor Arens has communicated that the assumption that the quotient mapping be upper semi-continuous had been inadvertently omitted from the statement of his theorem.

verges to y, then $\operatorname{var}(R_i f)$ converges to zero for each f in $C(Y)$.

Assertion 2. $\| L_i \circ R_i f - R_i f \| \leqq 1/2(1 + m) \operatorname{var}(R_i f)$.

Assertion 3. If f is in $C(Y)$, Hf is in $C(\operatorname{cl} D)$.

Assertion 4. The operator L is a projection from $C(Y)$ onto Q of norm at most $2 + m$.

If Assertion 1 is false it will be possible be find points z_i in K_i and a function f in $C(Y)$ such that for some r greater than zero, $f(x_i) - f(z_i)$ is greater than r. Since f is continuous, we may assume that there is a neighborhood N of y such that z_i does not belong to N. Put $Z = \{z_i\}$. Since the quotient map q of Y onto Y/K is, by hypothesis, closed $q(\operatorname{cl} Z)$ is closed in Y/K. But $q(x_i) = q(z_i)$ is in $q(\operatorname{cl} Z)$, and $q(x_i)$ converges to $q(y)$ by the continuity of q. Thus $q(y) = \{y\}$ is in $q(\operatorname{cl} Z)$, and $\{y\} = q(z)$ for some z in $\operatorname{cl} Z$. But $\operatorname{cl} Z$ is contained in $Y - N$ so $z \neq y$. This contradicts the assumption that y is not in D.

To prove the second assertion, let $c = 1/2 \operatorname{var}(R_i f)$. Since 1 is in P_i, $L_i \circ R_i 1 = 1$. Hence

$$\| L_i \circ R_i f - R_i f \| = \| L_i \circ R_i(f - c) - R_i(f - c) \| \leqq \| L_i - I \| \cdot \| R_i(f - c) \| \leqq (m + 1)(1/2) \operatorname{var}(R_i f) \, .$$

To prove Assertion 3 let y be a point in $\operatorname{cl} D$. We distinguish two cases. Case 1, y is in D. Let y be in the plural set K_i of the partition K. From the assumption of property (iv) it follows that there is an open set U containing K_i which meets no other plural set in K. Now let f be in $C(Y)$ and let N be a neighborhood of $Hf(y)$. Let V be a neighborhood of y such that $(L_i \circ R_i f)(V \cap K_i)$ is contained in N. Put $W = V \cap U$ and let x be an arbitrary point in W intersect $\operatorname{cl} D$. Then x is in U, and x is in the closed set K_i. This shows that $W \cap \operatorname{cl} D$ is contained in $K_i \cap V$. Hence on $W \cap \operatorname{cl} D$, $Hf = L_i \circ R_i f$. Thus $Hf(W \cap \operatorname{cl} D)$ is contained in $L_i \circ R_i f(K_i \cap V)$ which in turn is contained in N.

Case 2, y is not in D. In this case $\{y\}$ is in K, and $Hf(y) = f(y)$, since each P_i contains the constant functions. Let x_i converge to y. Then

$$| Hf(x_i) - Hf(y) | \leqq | Hf(x_i) - f(x_i) | + | f(x_i) - f(y) | \, .$$

It is clear that $f(x_i)$ converges to $f(y)$. For the other term we use Assertions 1 and 2 above to write, with x_i in K_i (and K_i in K),

$$| Hf(x_i) - f(x_i) | \leqq | L_i \circ R_i f(x_i) - R_i f(x_i) | \leqq (1/2)(m+1) \operatorname{var} (R_i f) .$$

Since this last term converges to zero, Hf is continuous at y.

To prove Assertion 4, we first observe that linearity and bound for L are obvious. If f is in $C(Y)$ we must show that Lf is in Q. Indeed,

$$R \circ L = R + R \circ H - R = R \circ H .$$

Hence

$$R_i \circ L = R_i \circ R \circ L = R_i \circ R \circ H = L_i \circ R_i$$

for each plural set K_i in K. Thus $R_i \circ Lf$ is in P_i for each plural set K_i in K. If K_i is a member of K which is not a plural set then, $R_i \circ Lf$ is in P_i trivially since P_i contains the constants.

Now we must show that if f is in Q then $Lf = f$. Since $R_i f$ is in P_i for all K_i in K, $R_i \circ Hf = L_i \circ R_i f = R_i f$. Thus $R \circ Hf = Rf$, and $Lf = f + E(Rf - Rf) = f$. This completes the proof of the theorem

REMARK. The assumption that Y is metrizable was used only to guarantee the existence of the linear mapping E. If we drop the hypothesis that Y is metrizable and assume outright the existence of a bounded linear mapping E from $C(\operatorname{cl} D)$ into $C(Y)$ such that $R \circ E$ is the identity on $C(\operatorname{cl} D)$, then the same proof establishes the existence of a projection from $C(Y)$ onto Q which has norm less than or equal to $1 + (m+1) \| E \|$.

COROLLARY 9. *Let Y, K, K_i, P_i, and Q be as in the theorem. If each P_i has dimension less than n, then there is a projection of norm at most $n+1$ from $C(Y)$ onto Q.*

3. Let X be a locally compact, Hausdorff space. A compactification of X is a compact Hausdorff space that contains X (a homeomorphic image of X) as a dense subspace. The Stone-Čech compactification of X will be denoted by βX, and the one-point compactification will be denoted by pX.

If K is an arbitrary compactification of X, the linear mapping which carries a function in $C(K)$ onto the unique function in $C(\beta X)$ which agrees with it on X, is an isometric isomorphism of $C(K)$ into $C(\beta X)$. We will therefore assume that $C(\beta X)$ contains $C(K)$.

If Y is a closed subset of a compact Hausdorff space K, I_Y will denote the ideal of functions in $C(K)$ which vanish on Y. Let N denote the non-negative integers with the discrete topology. If K is a compactification of X, the remainder of K (with respect to X) is the topological space $K - X$ equipped with the relative topology from K. In accordance with the usual terminology let $(m) = C(\beta N)$, $(c) = C(pN)$, and $(c_0) = I_{pN-N} = I_{\beta N-N}$, where the ideals are interpreted as subspaces of $C(pN)$ and $C(\beta N)$ respectively.

THEOREM 10. *Let K be a compactification of X which has a first countable remainder. If there is a bounded linear mapping of $C(\beta X)$ into $C(K)$ which acts as the identity on $I_{\beta X-X}$, then X is pseudocompact.*

We first will prove the following lemma.

LEMMA 11. *Let M be a compactification of N which has a first countable remainder. There does not exist a bounded linear mapping of (m) onto any subspace of $C(M)$ which contains (c_0).*

Proof of lemma. Since N is both locally compact and the union of a countable family of compact sets, $M - N$ is a compact set which is the intersection of a countable family U of open sets in M. Let x be a point in $M - N$. Let V be a countable family of open sets in M whose intersections with $M - N$ form a basis for the neighborhood system for x in $M - N$. Let W be the countable family of open sets in M of the form u intersect v, where u is in U and v is in V. It is easy to see that the intersection of the members of W is the singleton containing x. A compactness argument shows that W is in fact a basis for the neighborhood system for x in M. Since N is first countable we have established that M is first countable. Hence M is sequentially compact.

There is a sequence of points in N, say J, which converges to some point k in M. Now suppose B is a subspace of $C(M)$ which contains (c_0). The restriction of functions in B to J union $\{k\}$ carries B onto a Banach space which is either isometrically isomorphic to (c) or to (c_0). In the former case since (c_0) is complemented in (c), there is a bounded linear mapping of B onto (c_0). In either case if there is a bounded linear mapping of (m) onto B, there is a bounded linear mapping, L, of (m) onto (c_0). But no such mapping can exist. For since (c_0) is a separable Banach space and βN is extremally disconnected, L must be weakly compact (Grothendieck [14], p. 168, Cor. 1). Now an application of the open mapping theorem implies the false assertion that (c_0) is reflexive. This completes the proof of the lemma.

Proof of theorem. If X is not pseudocompact there is countable family of disjoint open sets V_i in X such that $\mathrm{cl} \cup \{V_i\} = \cup \{\mathrm{cl}\, V_i\}$. For each i let U_i be an open set such that $\mathrm{cl}\, U_i \subseteqq V_i$, let u_i be in U_i, and let f_i be a continuous function which vanishes off U_i and attains its norm of one at u_i. For a bounded sequence $x = (x_i, x_2 \cdots)$ in (m), let Ax be the unique function in $C(\beta X)$ which agrees with $\sum_{i=1}^{\infty} x_i f_i$ on X. The mapping A is an isometric isomorphism of (m) onto the range of A. Let L be the hypothesized mapping of the theorem, and let J carry a function in $C(\beta X)$ onto its restriction to $\text{cl}\{u_i\}$. Since $\text{cl}\{u_i\} - \{u_i\}$ is contained in $K - X$, $\text{cl}\{u_i\}$ is homeomorphic to a compactification M of N which has first countable remainder. Let G be the isometric isomorphism of $C(\text{cl}\{u_i\})$ onto $C(M)$ induced by this homeomorphism. The proof is completed by verifying that $G \circ J \circ L \circ A$ is a bounded linear mapping of (m) onto a subspace of $C(M)$ which contains (c_0).

The case in which K is the one-point compactification of X was first proved by J. Conway ([6]). Examples to show that pseudocompactness of X is not sufficient to guarantee the existence of a projection from $C(\beta X)$ onto $I_{\beta X - X}$ have been constructed by J. Conway ([6]) and by A. Pełczýnski and V. N. Sudakov ([21]).

COROLLARY 12. *Let X be an extremally disconnected, compact, Hausdorff space, and let P be a subspace of $C(X)$ which contains the constants and separates the points of X. If P is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z, then the Šilov boundary of P is an extremally disconnected subset of X which has a pseudo-compact complement.*

Proof. Under the hypothesis of the corollary, the Šilov boundary of P is the set Y of Lemma 3. To show that Y is extremally disconnected, we intend to apply a theorem due to Nachbin (Trans. AMS, 68 (1950), 28–46, 1950), Goodner ([13]), Kelley ([11]) and others. A Banach space B is called injective if every Banach space which contains an isometric isomorphic copy B' of B, admits a projection of norm one onto B'. The theorem we wish to apply states that a Banach space is injective if and only if it is isometrically isomorphic to $C(Z)$, for a compact, extremally disconnected, Hausdorff space Z. Now $C(X)$ is injective and from Lemma 3 there is a projection of norm one from $C(X)$ onto P. From this it can be shown that $C(Y)$ is injective, and hence Y is extremally disconnected.

From Lemma 3 it follows that I_Y is complemented in $C(X)$. Let $G = X - Y$. Since $\text{cl}\, G$ is open in X, $I_{\text{cl}G - G}$ is complemented in $C(\text{cl}\, G)$. Since $\text{cl}\, G$ is extremally disconnected, it is the Stone-Čech compactification of G ([10], p. 69, Prob. 6M2). By the theorem, G is pseudocompact (in this case K is the one-point compactification of G), and the corollary is proved.

COROLLARY 13. *If X is a locally compact space such that βX has a first countable remainder, then X is pseudocompact.*

REMARK. Relevant to the last corollary, we observe that if Z is any compact Hausdorff space, there is a pseudocompact, locally compact space X such that $\beta X - X$ is homeomorphic to Z. For let y be a nonisolated point in βN and let $X = (\beta N - \{y\}) \times Z$. From results in ([11]) and ([10], 6M3) we have that X is pseudocompact, and $\beta X = \beta N \times Z$.

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ON THE CHARACTERIZATION OF MEASURES OF THE CONE DUAL TO A GENERALIZED CONVEXITY CONE

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We consider in this paper the cone $C(u_0, \cdots, u_{n-1})$ of functions which are convex with respect to an Extended Complete Tchebycheffian system $\{u_0(t), u_1(t), \cdots, u_{n-1}(t)\}$. The cone dual to $C(u_0, \cdots, u_{n-1})$ is examined and necessary conditions as well as sufficient conditions for a measure to belong to this cone are developed. The merit of these conditions lies in the fact that they involve only the pattern of sign changes of the measure and related functions, and thus are easily verifiable.

Several applications are given. These include new inequalities for the Euler-Fourier coefficients of functions belonging to given convexity cones. Some new inequalities for the Fourier coefficients of the expansion of a function in a series of orthogonal polynomials are also obtained.

We consider in this paper the cone dual to a generalized convexity cone $C(u_0, \cdots, u_{n-1})$ with respect to an Extended Complete Tchebycheffian system $\{u_0(t), u_1(t), \cdots, u_{n-1}(t)\}$. The substantial role that these cones play in various areas of mathematics, such as moment theory, theory of approximation and interpolation and the theory of differential inequalities is discussed in detail in [5], (see also [4], [11], [6] and [7]). In a recent paper, Cargo [3] obtained independently for the special case when $n = 2$ and $u_0 \equiv 1$, some of the results of [4] and [11].

The dual cone was introduced by S. Karlin and A. Novikoff [4] who found necessary and sufficient conditions for a measure to belong to the dual cone. Applications of the results of [4] to the theory of reliability were later explored by Barlow and Marshall [1]. For the case $n = 2$ and ($u_0(t) \equiv 1, u_1(t) \equiv t$) the conditions were stated earlier by Levin and Steckin [8], and a multidimensional version for this special case was recently obtained by Brunk [2].

The necessary and sufficient conditions involve some integral inequalities and thus are not always easily verifiable. Some simple sufficient conditions in terms of equalities and the pattern of sign changes of the measure under examination were also evolved in [4].

In this paper we intend to elaborate on this type of criteria, i.e., necessary conditions as well as sufficient conditions involving only equalities and the pattern of sign changes of the measure. As a by-product, we obtain the interesting fact that the dual cones are essentially mutually disjoint, e. g. no nontrivial measure can belong both to the dual cone of the cone of convex functions and to the cone dual to the cone of monotone functions. Several applications are given in § 4. These include some inequalities for the Euler-Fourier coefficients with respect to the trigonometric system and also for the Fourier coefficients of the expansion of a function in a series of orthogonal polynomials.

We introduce now the generalized convexity cones and their duals. We will not discuss in any detail properties of these cones which can be found elsewhere. The reader is referred to [5] for a thorough discussion of ECT-systems and for the properties of generalized convexity cones which will be used without proof in the sequel.

Let $\{u_i\}_0^{n-1}$ be an Extended Complete Tchebycheffian system (ECT-system) on $[a, b]$. Assume that the functions $u_i(t)$, $i = 0, 1, \cdots, n-1$, admit of the representation

$$(1)\qquad \begin{cases} u_0(t) = w_0(t) \\ \vdots \qquad\quad \vdots \\ u_{n-1}(t) = w_0(t)\displaystyle\int_a^t w_1(\xi_1)\int_a^{\xi_1}\cdots\int_a^{\xi_{n-2}} w_{n-1}(\xi_{n-1})d\xi_{n-1}\cdots d\xi_1 \end{cases}$$

where $w_0(t), \cdots, w_{n-1}(t)$ are continuous strictly positive functions on $[a, b]$. This additional assumption on the set $\{u_i\}_0^{n-1}$ entails no loss of generality in the subsequent discussion.

DEFINITION 1. A function $\phi(t)$ defined on (a, b) is said to be convex with respect to the ECT-system $\{u_i\}_0^{n-1}$ provided

$$(2)\qquad \begin{vmatrix} u_0(t_1) & \cdots & u_0(t_{n+1}) \\ \vdots & & \vdots \\ u_{n-1}(t_1) & \cdots & u_{n-1}(t_{n+1}) \\ \phi(t_1) & \cdots & \phi(t_{n+1}) \end{vmatrix} \geqq 0\,, \qquad \text{for all } a < t_1 < \cdots < t_{n+1} < b\,.$$

The cone of functions satisfying (2) is referred to as a "generalized convexity cone" and is denoted by $C(u_0, \cdots, u_{n-1})$.

Throughout the paper, let $d\mu$ denote a signed measure of bounded variation on (a, b) such that for each $\phi(t) \in C(u_0, \cdots, u_{n-1})$ the integral $\int_a^b \phi d\mu$ is well defined with infinite values permitted. The dual cone of $C(u_0, \cdots, u_{n-1})$ is the set of all measures $d\mu$ which satisfy

$$(3)\qquad \int_a^b \phi(t)d\mu(t) \geqq 0 \text{ for all } \phi(t) \in C(u_0, \cdots, u_{n-1})\,.$$

This cone is designated by $C^*(u_0, \cdots, u_{n-1})$.

The integral operators I_j, $j = 0, 1, \cdots, n-1$ are defined by

$$(4) \qquad \begin{cases} I_0 d\mu(t) = -\int_a^t w_0(t)d\mu(t) \\ I_j f(t) = -\int_a^t w_j(t)f(t)dt\ , \qquad j = 1, 2, \cdots, n-1\ . \end{cases}$$

The following theorem was proved in [4]:

THEOREM A. *A signed measure $d\mu$ belongs to the dual cone $C^*(u_0, \cdots, u_{n-1})$ if, and only if*

$$(5) \qquad \int_a^b u_i(t)d\mu(t) = 0\ , \qquad i = 0, 1, \cdots, n-1\ ,$$

and

$$(6) \qquad I_{n-1}I_{n-2}\cdots I_0 d\mu(t) \geqq 0\ , \qquad \text{for all } a \leqq t \leqq b\ .$$

Furthermore, it was shown that the "moment conditions" (5) *are equivalent to*

$$(7) \qquad I_j I_{j-1}\cdots I_0\, d\mu(b) = 0\ , \qquad j = 0, 1, \cdots, n-1\ .$$

The necessary and sufficient conditions stated in Theorem A are in general hard to verify, the main difficulty being the inequalities (6). Therefore, it seems advantageous to seek simpler conditions even if they will not always be both necessary *and* sufficient. Very weak, but easily verifiable necessary conditions are the "moment conditions" (5). Some simple sufficient conditions which enable us to ascertain that $d\mu \in C^*(u_0, \cdots, u_{n-1})$ by checking its pattern of sign changes were also found. In order to state them we need first introduce some definitions. We adopt the following convention: a signed measure $d\mu$ will be said to have the sign ε (ε can be $(+)$ or $(-)$) on a set s if $\varepsilon\mu(s) > 0$ and there is no subset s' of s for which $\varepsilon\mu(s') < 0$. A function $f(t)$ will be said to have the sign ε on an interval I if and only if $d\mu = f(t)dt$ has the sign ε on I.

DEFINITION 2. A signed measure $d\mu$ defined on (a, b) is said to *possess a first sign* there, if there exists an interval extending to the end-point a on which $d\mu$ has a constant sign (this sign will be called the *first sign of $d\mu$*). Similarly, $d\mu$ is said to *possess a last sign* on (a, b), if there exists an interval extending to the end point b on which $d\mu$ has a constant sign (this sign will be called the *last sign of $d\mu$*).

DEFINITION 3. A signed measure $d\mu$ is said to have k sign changes on (a, b) if there exists a subdivision of (a, b) into disjoint consecutive sets $T_0, T_1, \cdots, T_k$ such that $d\mu$ is of alternating sign on $T_0, T_1, \cdots, T_k$.

We replaced here the "consecutive intervals" of the corresponding definition employed in [5] by "consecutive sets"—thus allowing a T_i to consist of one point only. We note that if the support of the measure consists of a finite number of points or if it is absolutely continuous, the two definitions coincide.

The following theorem stated in [5] (and, in a slightly weaker form, in [4]) is actually true only when one uses the concept of sign changes in the way it is formulated here. The proof involves only minor modifications of the proof presented in [5]. We will not go into details.

THEOREM B. *If a nontrivial signed measure $d\mu$ satisfies the "moment conditions"* (5) *then it has at least n sign changes. If $d\mu$ has exactly n sign changes and its last sign is* $(+)$, *then*

$$d\mu \in C^*(u_0, \cdots, u_{n-1}) .$$

There exists a wide gap between the necessary "moment conditions" and the strong sufficient conditions stated in Theorem B. The main purpose of this note is to narrow it by obtaining stronger necessary conditions as well as weaker sufficient conditions.

2. Necessary conditions. The first results which we will prove concern the simple cone $C^*(u_0)$.

LEMMA 1. *Let $d\mu$ be a signed measure possessing a first sign and a last sign on (a, b). A necessary condition for $d\mu$ to belong to $C^*(u_0)$ is that its first sign be* $(-)$ *and its last sign be* $(+)$.

Proof. Let $d\mu$ be a measure belonging to $C^*(u_0)$. Then, by applying Theorem A, we have

$$\int_a^b u_0(t)d\mu(t) = 0 . \tag{8}$$

We will first establish that the first sign of $d\mu$ is $(-)$. Indeed, suppose there is an interval $(a, t_1]$ on which $d\mu$ is positive.

Consider the function $\phi(t)$ defined by

$$\phi(t) = \begin{cases} c_1 u_0(t) & a \leqq t \leqq t_1 , \\ c_2 u_0(t) & t_1 < t < b , \end{cases} \qquad 0 < c_1 < c_2 .$$

Clearly, $\phi(t)$ belongs to $C(u_0)$. Compute now

$$\int_a^b \phi(t)d\mu(t) = c_1\int_a^{t_1} u_0(t)d\mu(t) + c_2\int_{t_1}^b u_0(t)d\mu(t)$$
$$< c_2\int_a^b u_0(t)d\mu(t) .$$

Using (8), it follows that

$$\int_a^b \phi(t)d\mu(t) < 0 ,$$

which is impossible since $d\mu \in C^*(u_0)$.

Similarly, we will now show that the last sign of $d\mu$ is $(+)$. Indeed, assume that there exists an interval $[t_2, b)$ on which $d\mu$ is negative. Consider the function $\psi(t) \in C(u_0)$ defined by

$$\psi(t) = \begin{cases} -c_2u_0(t) & a < t < t_2 , \\ -c_1u_0(t) & t_2 \leqq t < b , \end{cases} \qquad 0 < c_1 < c_2 .$$

A computation similar to that performed for $\phi(t)$ yields

$$\int_a^b \psi(t)d\mu(t) < 0 ,$$

contrary to the assumption that $d\mu \in C^*(u_0)$. This completes the proof of the lemma.

Corollary 1. *Let $d\mu$ be a signed measure possessing a finite number of sign changes on (a, b). If $d\mu$ belongs to $C^*(u_0)$ then it has an odd number of sign changes and its first sign is $(-)$.*

Let now the signed measure $d\mu$ have $2k - 1$ sign changes on (a, b) and let $\{T_i\}_0^{2k-1}$ be the subdivision of (a, b) associated with the sign changes of $d\mu$. Set

$$S_i = T_{2i-2} \cup T_{2i-1} , \qquad i = 1, 2, \cdots, k$$

and let the points $t_0, \cdots, t_{2k}$ be defined by

$$t_0 = a , \qquad t_i = \sup\{t\colon t \in T_{i-1}\} , \qquad i = 1, 2, \cdots, 2k .$$

Define the numbers $J_1, \cdots, J_k$ by

$$J_i = \int_{S_i} u_0(t)d\mu(t) , \qquad i = 1, 2, \cdots, k . \tag{9}$$

The measure $d\mu_1$ with the k atomic masses $J_1, \cdots, J_k$ situated, respectively, at the points $1, 2, \cdots, k$ will be referred to in this paper as *the measure induced by $d\mu$.*

LEMMA 2. *Let $d\mu$ have $2k - 1$ sign changes on (a, b) and let its* *first sign be* $(-)$. *Then* $d\mu$ *belongs to* $C^*(u_0)$ *on* (a, b) *if, and only if the measure induced by it belongs to* $C^*(1)$ *on* $(0, k+1)$.

Proof. Let $\phi(t)$ be an arbitrary function belonging to $C(u_0)$; then

$$\begin{aligned}\int_a^b \phi(t)d\mu(t) &= \sum_{i=0}^{2k-1} \int_{T_i} \phi(t)d\mu(t) \\ &= \sum_{i=1}^{k} \left[\int_{T_{2i-2}} \phi(t)du(t) + \int_{T_{2i-1}} \phi(t)d\mu(t)\right] \\ &\geqq \sum_{i=1}^{k} \left[\frac{\phi(t_{2i-1})}{u_0(t_{2i-1})} \int_{T_{2i-2}} u_0(t)d\mu(t) + \frac{\phi(t_{2i-1})}{u_0(t_{2i-1})} \int_{T_{2i-1}} u_0(t)d\mu(t)\right].\end{aligned}$$

The inequality follows from the fact that $\phi(t)/u_0(t)$ is non decreasing on (a, b) while $d\mu(t)$ is negative in the first integral and positive in the second.

Using definition (9) we thus obtain

$$\int_a^b \phi(t)d\mu(t) \geqq \sum_{i=0}^{k-1} \frac{\phi(t_{2i-1})}{u_0(t_{2i-1})} J_i . \tag{10}$$

Suppose now that the induced measure $d\mu$ belongs to $C^*(1)$. Then

$$\sum_{i=1}^{k} a_i J_i \geqq 0, \text{ for each sequence } \{a_i\}_1^k \text{ belonging to } C(1) .$$

Since $\{\phi(t_{2i-1})/u_0(t_{2i-1})\}_{i=1}^k$ is a nondecreasing sequence it belongs to $C(1)$. Hence, the right hand side of (10) is nonnegative and $\int_a^b \phi(t)d\mu(t) \geqq 0$. Since $\phi(t)$ was an arbitrary function of $C(u_0)$, this implies that $d\mu$ belongs to $C^*(u_0)$.

Conversely, suppose that $d\mu \in C^*(u_0)$ and let $\{a_i\}_1^k$ be an arbitrary sequence of $C(1)$. Define the function $\bar{\phi}(t)$ by

$$\bar{\phi}(t) = a_i u_0(t) , \qquad \text{for } t \in S_i, i = 1, 2, \cdots, k ,$$

and note that

$$\sum_{i=1}^{k} a_i J_i = \sum_{i=1}^{k} a_i \int_{S_i} u_0(t)d\mu(t) = \int_a^b \bar{\phi}(t)d\mu(t) \geqq 0 . \tag{11}$$

The inequality is due to the fact that $\bar{\phi}(t)/u_0(t)$ is a nondecreasing function, i.e., that $\bar{\phi}(t)$ belongs to $C(u_0)$.

Since the sequence $\{a_i\}_1^k$ was an arbitrary sequence of $C(1)$, this completes the proof of the lemma.

Appealing to Corollary 1, we can deduce

COROLLARY 2. *Let* $d\mu$ *be a measure of* $C^*(u_0)$ *possessing a finite* *number of sign changes on* (a, b). *Then, either the induced measure* $d\mu_1$ *is the trivial measure or it has an odd number of sign changes and its first sign is* $(-)$.

Observe next that if the induced measure $d\mu_1$ has an odd number of sign changes, the discussion preceding Lemma 2 can be applied to $d\mu_1$ and a measure $d\mu_2$, induced by $d\mu_1$, can be obtained. To this end, we only have to substitute $u_0(t) \equiv 1$ in (9) and replace (a, b) by $(0, k + 1)$. By Corollary 2, $d\mu_2$ is either trivial or it has an odd number of sign changes. Thus, if $d\mu_2$ is nontrivial, we can define a measure $d\mu_3$ induced by $d\mu_2$. This process can be continued as long as the induced measure is nontrivial.

LEMMA 3. *Let* $d\mu$ *be a measure of* $C^*(u_0)$ *possessing a finite number of sign changes on* (a, b). *Then the sequence of nontrivial successively induced measures* $d\mu_1, d\mu_2, \cdots$, *is finite.*

Proof. Observe that the induced measures $d\mu_1, du_2, \cdots$, have finite supports. Note next that the number of points in the support of $d\mu_{i+1}$, $i = 1, 2, \cdots$, is at most half the number of points in the support of $d\mu_i$, $i = 1, 2, \cdots$. Hence, the assertion of the lemma follows.

THEOREM. 1. *Let* $d\mu$ *possess a finite number of sign changes on* (a, b). *Necessary and sufficient conditions for* $d\mu$ *to belong to* $C^*(u_0)$ *are*: (a) *that it satisfy* (8), *and* (b) *that* $d\mu$ *and each measure in the finite sequence of nontrivial successively induced measures* $d\mu_1, d\mu_2, \cdots$, *exhibit the pattern of sign changes specified in Lemma* 1.

Proof. Necessity. The necessity of (a) follows from Theorem A. The necessity of (b) follows by a repeated application of Corollary 2.

Sufficiency. Let $d\mu_N$ be the last nontrivial measure in the sequence, so that $d\mu_{N+1}$ is the trivial measure.

Since $d\mu$ has a finite number of sign changes, each nontrivial measure $d\mu_i$, $i = 1, \cdots, N$, also has a finite number of sign changes. Since, by assumption, the measures exhibit the pattern of sign changes specified in Lemma 1, they satisfy the requirements of Lemma 2.

By Lemma 2, if $d\mu_{i+1}$, $i = 1, \cdots, N$ belongs to $C^*(1)$, then so does $d\mu_i$. Furthermore, if $d\mu_1$ belongs to $C^*(1)$ then $d\mu$ belongs to $C^*(u_0)$. Thus, the fact that $d\mu_{N+1}$, the trivial measure, belongs to $C^*(1)$, implies that $d\mu$ belongs to $C^*(u_0)$ and the theorem is proved.

We next derive necessary conditions for a measure possessing a first sign and a last sign on (a, b) to belong to $C^*(u_0, \cdots, u_{n-1})$.

THEOREM 2. *A necessary condition for a measure $d\mu$ possessing a first sign and a last sign on (a, b) to belong to $C^*(u_0, \cdots, u_{n-1})$ is that its first sign be $(-1)^n$ and its last sign be $(+)$.*

Proof. The proof proceeds by induction on n. For $n = 1$, the assertion is simply a restating of Lemma 1. Assuming that the assertion is valid for $n \leqq k - 1$, we will now prove it for $n = k$.

We introduce the first order differential operators (see [5])

$$D_j f(t) = \frac{d}{dt}\left[\frac{1}{w_j(t)} f(t)\right], \qquad j = 0, 1, \cdots, n-1, \tag{12}$$

where the w_i's are the functions introduced in (1).

Let now $d\mu$ be a measure of $C^*(u_0, \cdots, u_{k-1})$ possessing a first sign and a last sign on (a, b). Using integration by parts and the definitions (4) and (12), we find

$$\begin{aligned} \int_a^b \phi(t) d\mu(t) &= \int_a^b \frac{\phi(t)}{w_0(t)} w_0(t) d\mu(t) \\ &= -\frac{\phi(t)}{w_0(t)} I_0 d\mu(t) \Big|_a^b + \int_a^b [D_0\phi(t)][I_0 d\mu(t)] dt . \end{aligned} \tag{13}$$

The integrated part vanishes, since $I_0 d\mu(b) = 0$ is a necessary condition by Theorem A. It is very easy to see (cf. [11] or [5]) that the set of functions $\{D_0\phi(t) \mid \phi(t) \in C(u_0, \cdots, u_{k-1})\}$ comprises a generalized convexity cone. This cone is called the first "reduced" cone, and is denoted, in terms of its basic ECT-system, by $C(D_0u_1, \cdots, D_0u_{k-1})$. Thus, (13) implies that a necessary condition for $d\mu$ to belong to $C^*(u_0, \cdots, u_{k-1})$ is that $I_0 d\mu(t) dt$ belong to $C^*(D_0u_1, \cdots, D_0u_{k-1})$.

Since $d\mu$ has a first sign and a last sign, so does $I_0 d\mu(t) dt$. Utilizing now the fact that the condition on the pattern of signs formulated in the theorem depends only on the order of the cone, i.e. on the number of functions in its basic ECT-system, we can apply the induction hypothesis. We thus deduce that the first sign of $I_0 d\mu(t)$ is $(-1)^{k-1}$ and its last sign in $(+)$.

Note further that

$$I_0 d\mu(t) = -\int_a^t u_0(t) d\mu(t), \tag{14}$$

and that, using relation (8), which is valid by Theorem A, we also have

$$I_0 d\mu(t) = \int_t^b u_0(t) d\mu(t) . \tag{15}$$

Relations (14) and (15) imply that the first sign of $d\mu(t)$ is opposite to that of $I_0 d\mu(t)$ and that the last sign of $d\mu(t)$ is the same as that of $I_0 d\mu(t)$. This completes the induction step and thereby the theorem is proved.

The set of measures of a dual cone $C^*(u_0, \cdots, u_{n-1})$ which possess a first sign on (a, b) is a subcone. This subcone will be called the *restricted dual cone*. Note that the trivial measure does not belong to the restricted dual cone.

The condition on the pattern of signs proved in Theorem 2 readily yield

COROLLARY 3. *A restricted dual cone of odd order and a restricted dual cone of even order are always mutually disjoint.*

Note that in Corollary 3, the cones may be based on different ECT-systems. For a fixed ECT-system, a more comprehensive result in this direction is true, viz.

THEOREM 3. *Let an* ECT-*system be given. Two dual cones with respect to this system which are of different orders have only the trivial measure in common.*

Proof. Consider $C^*(u_0, \cdots, u_{n-1})$ and $C^*(u_0, \cdots, u_{k-1})$ with $n > k$. Let $d\mu$ be a measure belonging to $C^*(u_0, \cdots, u_{k-1})$. Then the necessary conditions of Theorem A imply that

$$I_{k-1} I_{k-2} \cdots I_0 d\mu(t) \geqq 0 \, , \qquad \text{for } a \leqq t \leqq b \, . \tag{16}$$

Suppose now that $d\mu$ belongs also to $C^*(u_0, \cdots, u_{n-1})$. By repeated integration by parts similar to that performed in (13), we find

$$\int_a^b \phi(t) d\mu(t) = - \frac{\phi(t)}{w_0(t)} I_0 d\mu(t) \Big|_a^b - \sum_{j=1}^{k-1} \frac{D_{j-1} \cdots D_0 \phi(t)}{w_j(t)} I_j \cdots I_0 d\mu(t) \Big|_a^b$$
$$+ \int_a^b [D_{k-1} D_{k-2} \cdots D_0 \phi(t)] \, [I_{k-1} I_{k-2} \cdots I_0 d\mu(t)] dt \, .$$

Tne integrated part vanishes by virtue of the conditions (7) which are necessary conditions for $d\mu$ to belong to $C^*(u_0, \cdots, u_{n-1})$. Hence, as in the proof of Theorem 2, we deduce that a necessary condition for $d\mu$ to belong to $C^*(u_0, \cdots, u_{n-1})$ is that $I_{k-1} I_{k-2} \cdots I_0 d\mu(t) dt$ belong to the dual to the k-th "reduced" cone

$$C^*(D_{k-1} \cdots D_0 u_k, D_{k-1} \cdots D_0 u_{k+1}, \cdots, D_{k-1} \cdots D_0 u_{n-1}) \, .$$

This is a dual cone of order n-k, so that by Theorem B, a necessary condition for this to happen, is that either $I_{k-1} \cdots I_0 d\mu(t)$ have at least n-k sign changes on (a, b), or that $I_{k-1} \cdots I_0 d\mu(t) \equiv 0$. Since

(16) has to be satisfied, we deduce that $I_{k-1} \cdots I_0 d\mu(t) \equiv 0$; this is equivalent to $d\mu$ being the trivial measure, so that the proof is complete.

We have seen that for a fixed ECT-system, the intersection of two dual cones of different order contains only the trivial measure. The question of the structure and properties of unions of such cones will be explored by the author in a future publication.

3. **Sufficient conditions.** We have, in the last section, strengthened the necessary conditions given by Theorem A, by adding that if a signed measure $d\mu$ belongs to $C^*(u_0, \cdots, u_{n-1})$ and possesses a first sign and a last sign, then its first sign must be $(-1)^n$ and its last sign must be $(+)$.

We shall obtain in this section weaker sufficient conditions than those specified in Theorem B.

Let the functions $U_i(\mu; t)$, $i = 0, 1, \cdots, n-1$, be defined by

$$U_i(\mu; t) = \int_a^t u_i(t) d\mu(t) , \qquad i = 0, 1, \cdots, n-1 . \tag{17}$$

These functions are smoother than the measure $d\mu(t)$ and therefore it is sometimes easier to check their respective patterns of signs than to check the pattern of signs of $d\mu$.

THEOREM 4. *Let $d\mu$ satisfy the "moment conditions"* (5) *and let its first sign be $(-1)^m$ and its last sign be $(+)$. If there exists a j, $0 \leqq j \leqq n-1$, such that $U_j(\mu; t)$ has at most $n-1$ sign changes on (a, b), then $d\mu \in C^*(u_0, \cdots, u_{n-1})$.*

Proof. The proof proceeds by induction. Let $(u_0, \cdots, u_{m-1})$, $m \geqq 1$, be an arbitrary ECT-system. (Note that this is a completely arbitrary ECT-system. We have chosen to denote its functions by $(u_0, \cdots, u_{m-1})$ in order to be able to avail ourselves of other theorems of the paper without undue change of notation).

Assume that $d\mu(t)$ satisfies the "moment conditions" (5) (where n is replaced by m), and that its first sign is $(-1)^m$ and its last sign is $(+)$. Assume further that $U_0(\mu; t)$ has at most $m-1$ sign changes on (a, b). We will now show that these assumptions imply that

$$d\mu \in C^*(u_0, \cdots, u_{m-1}) .$$

We note that $U_0(\mu; t) = -I_0 d\mu(t)$, and observe that (13) and (5) imply that

$$\int_a^b \phi(t) d\mu(t) = \int_a^b [D_0\phi(t)]\,[I_0 d\mu(t)] dt . \tag{18}$$

Thus, it will suffice if we show that $I_0 d\mu(t)dt$ belongs to

$$C^*(D_0 u_1, \cdots, D_0 u_{m-1}) .$$

Relations (18) and (5) imply that $I_0 d\mu(t)dt$ satisfies the $m-1$ "moment conditions" with respect to $(D_0 u_1, \cdots, D_0 u_{m-1})$. Hence, by Theorem B, it has at least $m-1$ sign changes. However, by our assumption, $I_0 d\mu(t)$ has at most $m-1$ sign changes, so that it must have exactly $m-1$ sign changes. Furthermore, following the same reasoning as in the proof of Theorem 2, we deduce that the first sign of $I_0 d\mu(t)$ is $(-1)^{m-1}$ and its last sign is $(+)$. Therefore, by Theorem *B*, $I_0 d\mu(t)dt$ belongs to $C^*(D_0 u_1, \cdots, D_0 u_{m-1})$.

We have thus proved that if an ECT-system of order $m, m \geqq 1$, is given and $d\mu$ is a signed measure with first sign $(-1)^m$ and last sign $(+)$ satisfying the corresponding "moment conditions", then the condition that $U_0(\mu; \mathrm{t})$ have at most $m-1$ sign changes on (a, b) implies that $d\mu$ belongs to the corresponding dual cone.

Assume now that we have established that, given *any* ECT-*system* of order m and a signed measure $d\mu$ satisfying the corresponding "moment conditions" and having the appropriate first and last signs, the condition that $U_{r-1}(\mu; t), 1 \leqq r < m$, have at most $m-1$ sign changes on (a, b) implies that $d\mu$ belongs to the corresponding dual cone.

We wish to show that the same conclusion is implied by the condition that $U_r(\mu; t)$ have at most $m-1$ sign changes. This will be the induction step and thereby the validity of the theorem will be established.

Let $d\mu(t)$ be a signed measure whose first sign is $(-1)^m$ and whose last sign is $(+)$ and let it satisfy (5). Furthermore, assume that $U_r(\mu; t)$ has at most $m-1$ sign changes. We wish to show that these assumptions together with the induction hypothesis imply that $d\mu \in C^*(u_0, \cdots, u_{m-1})$. It will suffice, as explained earlier, if we show that $I_0 d\mu(\mathrm{t})dt \in C^*(D_0 u_1, \cdots, D_0 u_{m-1})$.

Consider the ECT-system $(D_0 u_1, \cdots, D_0 u_{m-1})$ and define

$$\text{(19)} \qquad U_i^*(\mu; t) = \int_a^t D_0 u_{i+1}(t) d\mu(t) , \qquad i = 0, 1, \cdots, m-2 .$$

In the case where $d\mu(t) = f(t)dt$, the left hand side of (19) will be written as $U_i^*(f; t)$.

Integration by parts similar to that performed in (13) yields

$$\text{(20)} \quad U_{j-1}^*(I_0 d\mu; t) = \frac{u_j(t)}{w_0(t)} I_0 d\mu(t) + U_j(\mu; t) , \qquad j = 1, 2, \cdots, m-1 .$$

Note that the functions $U_j^*(\mu; t), j = 0, 1, \cdots, m-2$ are defined with respect to the ECT-system of order $m-1$ $(D_0 u_1, \cdots, D_0 u_{m-1})$ in exactly the same way that $U_j(\mu; t), j = 0, 1, \cdots, m-1$ were defined in (17) with respect to $(u_0, \cdots, u_{m-1})$. Note further that our assumptions on $d\mu$ imply that the first sign of $I_0 d\mu(t)$ is $(-1)^{m-1}$ and its last sign is $(+)$ and that $I_0 d\mu(t)$ satisfies the "moment conditions" with respect to $(D_0 u_1, \cdots, D_0 u_{m-1})$. Thus, if we show that $U^*_{r-1}(I_0 d\mu; t)$ has at most $m-2$ sign changes, the induction hypothesis, which is applicable since $r-1 < m-1$, will imply that

$$I_0 d\mu(t) \in C^*(D_0 u_1, \cdots, D_0 u_{m-1}) .$$

We start with an analysis of the patterns of signs of $U^*_{r-1}(I_0 d\mu; t)$ and $U_r(\mu; t)$. Since the first sign of $I_0 d\mu(t)$ is $(-1)^{m-1}$ the same is true for $U^*_{r-1}(I_0 d\mu; t)$. Similarly, since the first sign of $d\mu$ is $(-1)^m$ the same is true for $U_r(\mu; t)$. On the other hand, the last signs of both $d\mu$ and $I_0 d\mu(t)$ are $(+)$ so that the last signs of both $U^*_{r-1}(I_0 d\mu; t)$ and $U_r(\mu; t)$ are $(-)$.

Let ν be the number of sign changes of $U^*_{r-1}(I_0 d\mu; t)$; the above analysis of first and last signs implies that

$$\nu \equiv m \pmod 2 . \tag{21}$$

Suppose now that $U^*_{r-1}(I_0 d\mu; t)$ has more than $m-2$ sign changes. Then, by (21), it must have at least m sign changes. We assert that this is incompatible with the assumption that $U_r(\mu; t)$ has at most $m-1$ sign changes.

We divide the proof of this assertion in two parts.

(a) Let $(T^*_0, \cdots, T^*_\nu)$ be the subdivision of (a, b) associated with the sign changes of $U^*_{r-1}(I_0 d\mu; t)$ and let $\{t^*_i\}^\nu_1$, the points of sign change of $U^*_{r-1}(I_0 d\mu; t)$, be defined by $t^*_i = \sup\{t: t \in T^*_{i-1}\}, i = 1, 2, \cdots, \nu$. Then $U_r(\mu; t)$ changes sign at least once in (a, t^*_1).

Note first that $U^*_{r-1}(I_0 d\mu; t)$ is a continuous function, so that the points $t^*_i, i = 1, 2, \cdots, \nu$, are among its zeros. By considering the pattern of signs of $U^*_{r-1}(I_0 d\mu; t)$ we see that $(-1)^{m-1} U^*_{r-1}(I_0 d\mu; t)$ is positive on $(a, t^*_1]$ and changes its sign to negative at t^*_1. Hence, there must exist a point x, $a < x < t^*_1$, such that

$$(-1)^{m-1} D_0 u_{r-1}(x) I_0 d\mu(x) < 0 .$$

Moreover, since $D_0 u_{r-1}(t)$ is strictly positive on (a, b), we have

$$(-1)^{m-1} I_0 d\mu(x) < 0 .$$

This inequality, taken together with relation (20) and the fact that $u_j(t)$ and $w_0(t)$ are strictly positive on (a, b), implies that

$$(-1)^{m-1} U_r(\mu; x) > 0 .$$

However, we know that the first sign of $(-1)^m U_r(\mu; t)$ is $(+)$.

Hence, a sign change must have occurred for some $\bar{t}$, $a < \bar{t} < x < t_1^*$. This completes the first part.

(b) In each interval $[t_i^*, t_{i+1}^*)$, $i = 1, 2, \cdots, \nu$ (where $t_{\nu+1}^* = b$), the function $U_r(\mu; t)$ has at least one point of sign change.

Indeed, with no loss of generality we may assume that $U_{r-1}^*(I_0 d\mu; t)$ is positive for $t \in [t_i^*, t_{i+1}^*]$. Since there exists a point s, $t_i^* < s < t_{i+1}^*$ such that $U_{r-1}^*(I_0 d\mu; s) > 0$ and we also have $U_{r-1}^*(I_0 d\mu; t_{i+1}^*) = 0$, it follows that there exists a point x_2, $t_i^* < x_2 < t_{i+1}^*$ for which

$$D_0 u_{r-1}(x_2) I_0 d\mu(x_2) < 0 .$$

Since $U_{r-1}^*(I_0 d\mu; x_2) \geqq 0$, relation (20) implies that $U_r(\mu; x_2) > 0$.

On the other hand, t_i^* is a point where $U_{r-1}^*(I_0 d\mu; t)$ changes sign from negative to positive. Hence, for each y, $y < t_i^*$, there exists a point x_1, $y < x_1 < t_i^*$ such that $D_0 u_{r-1}(x_1) I_0 d\mu(x_1) > 0$ and

$$U_{r-1}^*(I_0 d\mu; x_1) \leqq 0 .$$

We deduce from (20) that $U_r(\mu; x_1) < 0$. Hence, $U_r(\mu; t)$ must change sign between x_1 and x_2. Noting that y was an arbitrary point satisfying $y < t_i^*$, we conclude that there exists a point x, $t_i^* \leqq x < x_2 < t_{i+1}^*$, which is a point of sign change for $U_r(\mu; t)$.

Combining parts (a) and (b) we see that $U_r(\mu; t)$ has at least as many sign changes as $U_{r-1}^*(I_0 d\mu; t)$. Thus, if $U_{r-1}^*(I_0 d\mu; t)$ has at least m sign changes, then so does $U_r(\mu; t)$, proving the assertion. This completes the proof of Theorem 4.

Remark. The conditions specified in Theorem 4 are weaker than those specified in Theorem B. Indeed, if $d\mu$ has exactly n sign changes on (a, b) and conditions (5) are satisfied, it follows easily that the functions $U_i(\mu; t)$, $i = 0, 1, \cdots, n-1$ can have at most $n-1$ sign changes. The converse is not true. There exist, in fact, examples such that $d\mu$ possesses in excess of n sign changes, while there exists a j, $0 \leqq j \leqq n-1$, such that $U_j(\mu; t)$ has no more than $n-1$ sign changes.

4. Applications. In this section we discuss several applications of the foregoing analysis to Fourier series [part a)] and to expansions of functions in terms of orthogonal polynomials [part b)]. Some of the results stated here might have been discussed elsewhere, but even in that case, the power of our criteria is exemplified by the simplicity of the derivation of the results. Thanks are due to Prof. B. Schwarz who drew our attention to the fact that a special case of assertion (B) below is discussed in [9, Vol. 2, p. 81]. This is the only case which, to the best of our knowledge, has been discussed in the literature.

The inequalities discussed in this section are necessary conditions for functions to be included in given convexity cones. The following converse problem is suggested:

Determine a set of conditions on the Fourier coefficients of a function which will be sufficient to insure the inclusion of the function in a given convexity cone.

(a) *Fourier series.* Let $f(t)$ denote throughout this subsection a function of $L_2(-\pi, \pi)$ and let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt \tag{22}$$

be the corresponding Fourier series.

We shall present the inequalities for the Euler-Fourier coefficients of functions belonging to convexity cones in the form of a series of assertions.

(A) *Let $f(t)$ be monotone nondecreasing on $(-\pi, \pi)$. Then*

$$(-1)^{n+1} b_n \geqq 0 , \qquad n = 1, 2, \cdots . \tag{23}$$

Proof. The assertion is equivalent to the relation

$$(-1)^{n+1} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \geqq 0 , \qquad \text{for each } f(t) \text{ of } C(1) .$$

Thus, we have to show that $d\mu_A(t) = (-1)^{n-1} \sin nt \, dt$ belongs to $C^*(1)$. We note first that the last sign of $d\mu_A$ is $(+)$ and that $d\mu_A$ is odd. Hence, it has the pattern of signs specified in Lemma 2. The zeros of $d\mu_A$ inside $(-\pi, \pi)$, which are simple zeros and therefore points of sign change for $d\mu_A$, are the points $\{-\pi + k\pi/n, k = 1, 2, \cdots, 2n-1\}$. Thus, we have

$$J_i = \int_{-\pi+(2i\pi/n)}^{-\pi+\{(2i+2)/n\}\pi} (-1)^{n+1} \sin nt \, dt , \qquad i = 0, 1, \cdots, n-1 ,$$

and this expression is zero for each $i, 0 \leqq i \leqq n-1$. Hence, the measure induced by $d\mu_A$ belongs to $C^*(1)$, and by Lemma 2 so does $d\mu_A$.

(B) *Let $f(t)$ be convex on $(-\pi, \pi)$.* Then

$$(-1)^n a_n \geqq 0 , \qquad n = 1, 2, \cdots . \tag{24}$$

Proof. The assertion is equivalent to the relation

$$(-1)^n \int_{-\pi}^{\pi} f(t) \cos nt \, dt \geqq 0 \qquad \text{for all } f(t) \text{ of } C(1, t) .$$

Thus, we have to show that $d\mu_B(t) = (-1)^n \cos nt\, dt$ belongs to $C^*(1, t)$. Observe that

$$I_0 d\mu_B(t) = -\int_{-\pi}^{t} (-1)^n \cos nx\, dx = \frac{(-1)^{n+1} \sin nt}{n} ,$$

so that $I_0 d\mu_B(\pi) = 0$ and $I_0 d\mu_B(t)dt \in C^*(1)$. By the remark following equation (18), these are sufficient conditions for $d\mu_B(t)$ to belong to $C^*(1, t)$.

(C) *Let $f(t)$ be monotone nondecreasing on $(-\pi, \pi)$.* Then

$$\left|\frac{b_{kn}}{n}\right| \leqq |b_k| , \qquad \begin{array}{l} k = 1, 2, \cdots , \\ n = 1, 2, \cdots . \end{array} \tag{25}$$

Proof. In view of (23), we have to show that

$$(-1)^{k+1} b_k \geqq (-1)^{kn+1} b_{kn}/n ,$$

i.e., that $d\mu_C(t) = [(-1)^{k+1} \sin kt - (-1)^{kn+1} (sin\ knt/n)]dt$ belongs to $C^*(1)$. We note that

$$I_0 d\mu_C(\pi) = \int_{-\pi}^{\pi} d\mu_C(\pi) = 0 . \tag{26}$$

From the well known inequality (see e.g. [9])

$$|\sin Nx| \leqq N |\sin x| , \qquad N = 1, 2, \cdots ,$$

it follows that

$$\left|\frac{\sin knt}{n}\right| \leqq |\sin kt| , \qquad \begin{array}{l} k = 1, 2, \cdots , \\ n = 1, 2, \cdots , \end{array}$$

so that the sign of $d\mu_C(t)$ is identical, for each t, with the sign of $(-1)^{k+1} \sin kt$. Thus, the first sign of $d\mu_C$ is $(-)$ and its last sign is $(+)$, so that $d\mu_C$ has the pattern of sign changes specified in Lemma 2. Noting that the points of sign change of $d\mu_C$ inside $(-\pi, \pi)$ are $\{-\pi + (i\pi/k), i = 1, \cdots, 2k-1\}$, we have

$$J_i = \int_{-\pi+(2i\pi/k)}^{-\pi+\{(2i+2)\pi/k\}} \left[(-1)^{k+1} \sin kt - (-1)^{kn+1} \frac{\sin knt}{n}\right] dt ,$$
$$i = 0, 1, \cdots, k-1 .$$

This expression is zero for each $i, i = 0, 1, \cdots, k-1$. Thus, by Lemma 2, $d\mu_C$ belongs to $C^*(1)$.

(D) *Let $f(t)$ be convex on $(-\pi, \pi)$. Then*

$$(27)\qquad |a_k| \geqq |a_{nk}|\,, \qquad \begin{array}{l} k = 1, 2, \cdots\,, \\ n = 1, 2, \cdots\,. \end{array}$$

Proof. In view of (24), we have to show that

$$(-1)^k a_k \geqq (-1)^{nk} a_{nk}\,,$$

i.e., that $d\mu_D(t) = [(-1)^k \cos kt - (-1)^{nk} \cos nkt]dt$ belongs to $C^*(1, t)$. We note that

$$I_0 d\mu_D(t) = -\int_{-\pi}^{t} d\mu_D(t) = \frac{1}{k}\left[(-1)^{k+1} \sin kt - (-1)^{kn+1} \frac{\sin knt}{n}\right].$$

Thus, $I_0 d\mu_D(\pi) = 0$ and, by assertion C), $I_0 d\mu_D(t)$ belongs to $C^*(1)$. These conditions imply that $d\mu_D(t)$ belongs to $C^*(1, t)$.

(E) *Let $f(t)$ be monotone nondecreasing on $(-\pi, \pi)$. Then*

$$(28)\qquad \sum_{k=1}^{n} b_k + \frac{1}{2} b_{n+1} \geqq 0\,, \qquad n = 1, 2, \cdots\,.$$

Proof. We need only observe that

$$d\mu_E(t) = \left[\sum_{k=1}^{n} \sin kt + \frac{1}{2} \sin (n+1)t\right] dt$$

is nonnegative for $0 \leqq t \leqq \pi$ (see [9]) and odd. The "moment condition" $I_0 d\mu_E(\pi) = 0$ is clearly satisfied, and the previous observation implies that there exists precisely one sign change. The assertion follows then by appealing to Theorem B. Note that if n is odd, relations (28) and (23) imply

$$(29)\qquad \sum_{k=1}^{n} b_k \geqq 0\,, \qquad \text{for each odd } n\,.$$

(F) *Let $f(t)$ be convex on $(-\pi, \pi)$. Then*

$$(30)\qquad \sum_{k=1}^{n} k a_k + \frac{(n+1)}{2} a_{n+1} \leqq 0\,, \qquad n = 1, 2, \cdots\,.$$

Proof. Set $d\mu_F(t) = -[\sum_{k=1}^{n} k \cos kt + \{(n+1)/2\} \cos (n+1)t]dt$; it is easily seen that $I_0 d\mu_F(\pi) = 0$ and that $I_0 d\mu_F(t) = d\mu_E(t)$ belongs to $C^*(1)$. These conditions imply that $d\mu_F$ belongs to $C^*(1, t)$. Note also that relations (30) and (24) imply that

$$(31)\qquad \sum_{k=1}^{n} k a_k \leqq 0\,, \qquad \text{for each odd } n\,.$$

(G) *Let $f(t)$ be monotone nondecreasing on $(-\pi, \pi)$. Then*

$$\sum_{k=1}^{n} k(n+1-k)b_k \geqq 0 \,, \qquad \text{for each odd } n \,. \tag{32}$$

Proof. Set $d\mu_G(t) = [\sum_{k=1}^{n} k(n+1-k)\sin kt]dt$. Straight computation yields

$$I_0 d\mu_G(t) = \sum_{k=1}^{n} (n+1-k)\cos kt + C \,.$$

We recall the equation (see [9])

$$\sum_{k=1}^{n} (n+1-k)\cos kt + \frac{n+1}{2} = \frac{1}{2}\left[\frac{\sin(n+1)t/2}{\sin t/2}\right]^2 . \tag{33}$$

The right hand side of (33) differs from $I_0 d\mu_G(t)$ by a constant at most. However, for an odd n the right hand side of (33) vanishes for $t = \pi$ and so does $I_0 d\mu_G(t)$ as is clear from the definition of $d\mu_G$. Therefore we have

$$I_0 d\mu_G(t) = \frac{1}{2}\left[\frac{\sin(n+1)t/2}{\sin t/2}\right]^2$$

so that $I_0 d\mu_G(t)$ is nonnegative on $(-\pi, \pi)$ and vanishes for $t = \pi$. This implies, using Theorem A, that $d\mu_G$ belongs to $C^*(1)$.

(H) *Let $f(t)$ be convex on $(-\pi, \pi)$. Then*

$$\sum_{k=1}^{n} k^2(n+1-k)a_k \leqq 0 \,, \qquad \text{for each odd } n \,. \tag{34}$$

Proof. This assertion follows from assertion (G) in precisely the same way as (F) followed from (E).

(I) *Let $f(t)$ be a function of $C(1, t, t^2)$ on $(-\pi, \pi)$. Then*

$$(-1)^{k+1}b_k \leqq (-1)^{kn+1}nb_{nk} \,, \qquad k = 1, 2, \cdots \,, \quad n = 1, 2, \cdots \,. \tag{35}$$

Proof. Set $d\mu_I(t) = [(-1)^{kn+1}n\sin knt - (-1)^{k+1}\sin kt]dt$. Simple integration yields

$$I_0 d\mu_I(t) = \frac{1}{k}[(-1)^{kn+1}\cos knt - (-1)^{k+1}\cos kt] \,,$$

so that $I_0 d\mu_I(\pi) = 0$. Furthermore, $I_0 d\mu_I(t)$ belongs to $C^*(1, t)$ by assertion (D). These facts imply that $d\mu_I(t)$ belongs to $C^*(1, t, t^2)$.

COROLLARY I. *If* $f(t) \in C(1) \cap C(1, t, t^2)$, *then we have*

$$\left|\frac{b_{nk}}{n}\right| \leqq |b_k| \leqq |nb_{nk}|, \qquad k = 1, 2, \cdots, \quad n = 1, 2, \cdots. \tag{36}$$

(J) *Let* $f(t)$ *be a function of* $C(1, t, t^2, t^3)$ *on* $(-\pi, \pi)$. *Then*

$$(-1)^k a_k \leqq (-1)^{kn} n^2 a_{nk}, \qquad k = 1, 2, \cdots, \quad n = 1, 2, \cdots. \tag{37}$$

Proof. Set $d\mu_J(t) = [(-1)^{k+1}\cos kt - (-1)^{kn+1} n^2 \cos nkt]dt$. The familiar integration yields now

$$I_0 d\mu_J(t) = \frac{1}{k}[(-1)^{kn+1} n \sin kt - (-1)^{k+1} \sin kt],$$

so that $I_0 d\mu_J(\pi) = 0$. Furthermore, $I_0 d\mu_J(t)$ belongs to $C^*(1, t, t^2)$ by assertion 1). These facts imply that $d\mu_J(t)$ belongs to $C^*(1, t, t^2, t^3)$.

COROLLARY J. *If* $f(t) \in C(1, t) \cap C(1, t, t^2, t^3)$, *then we have*

$$\left|\frac{a_{nk}}{n^2}\right| \leqq |a_k| \leqq |n^2 a_{nk}|. \qquad k = 1, 2, \cdots, \quad n = 1, 2, \cdots. \tag{38}$$

Corollaries (I) and (J) imply the following theorem relating any two Euler-Fourier coefficients.

THEOREM 5. *Let* $P(n, m)$ *denote the least common multiple of the natural numbers* m *and* n. *The following inequalities are satisfied*:

$$\left|\frac{b_m}{P(m, n)}\right| \leqq |b_n| \leqq |P(m, n)b_m|, \tag{39}$$

for all $f(t) \in C(1) \cap C(1, t, t^2)$,

and

$$\left|\frac{a_m}{P^2(m, n)}\right| \leqq |a_n| \leqq |P^2(m, n)a_m|, \tag{40}$$

for all $f(t) \in C(1, t) \cap C(1, t, t^2, t^3)$.

(K) *Let* $f(t)$ *be a convex function on* $(-\pi, \pi)$. *Then*

$$\int_{-\pi}^{\pi} tf(t)dt \leqq \frac{\pi^3}{3}(b_1 - a_1). \tag{41}$$

Proof. Consider the measure $d\mu_k(t) = (\sin t - \cos t - 3t/\pi^2)dt$. It is easily verified that both the first sign and the last sign of $d\mu_K(t)$ are (+). A direct computation demonstrates that the "moment conditions" $\int_{-\pi}^{\pi} d\mu_K(t) = 0$ and $\int_{-\pi}^{\pi} t d\mu_K(t) = 0$ are satisfied. Moreover, an examination of the graph of $\sin t$-$\cos t$ versus the graph of $3t/\pi^2$ shows that $d\mu_K(t)$ has precisely two sign changes. Hence, Theorem B implies that $d\mu_K(t)$ belongs to $C^*(1, t)$; this is equivalent to assertion (K).

(L) *Let $f(t)$ be a monotone nondecreasing function on $(-\pi, \pi)$. Then*

$$\int_{-\pi}^{\pi} t^2 f(t)dt \geqq \frac{2\pi^2}{3}\left(\frac{a_0}{2} - b_1 - a_1\right). \tag{42}$$

Proof. Let $d\mu_L(t) = (\cos t + \sin t + 3t^2/2\pi^2 - 1/2)dt$. It is easily verified that $d\mu_L(t) = I_0 d\mu_K(t)dt$. Since $I_0 d\mu_K(\pi) = 0$, we can conclude from assertion (K) and the remark following equation (13) that $d\mu_L(t)$ belongs to $C^*(1)$, i.e., that (42) is indeed valid for all $f(t) \in C(1)$.

Since Theorems 1 and 2 specify necessary conditions for a measure to belong to a dual cone, some results of a negative nature are also to be expected. In fact, the following results can readily be deduced from Theorem 2.

THEOREM 6. *Let $(u_0, \cdots, u_{2n-1})$, $n \geqq 1$, be an ECT-system on $[-\pi, \pi]$. No finite linear inequality involving only b_i's can be valid for all $f(t) \in C(u_0, \cdots, u_{2n-1})$.*

Proof. It suffices to observe that a measure which is a linear combination of $\{\sin kt\}$ is an odd function on $(-\pi, \pi)$ and thus has an odd number of sign changes.

A similar reasoning yields also

THEOREM 7. *Let $(u_0, \cdots, u_{2n})$, $n \geqq 0$, be an ECT-system on $[-\pi, \pi]$. No finite linear inequality involving only a_i's can be valid for all $f(t) \in C(u_0, \cdots, u_{2n})$.*

One might conjecture, on the basis of assertion (D), that $\{|a_n|\}$ is a monotone decreasing sequence whenever $f(t)$ is a convex function. A computation of the corresponding $\{J_i\}$ and reliance on Theorem 1, show, however, that neither $|a_2| \geqq |a_3|$ nor $|a_3| \geqq |a_5|$ are valid for all convex functions.

We conclude with the following

REMARK. An inequality for the Euler-Fourier coefficients which holds for all functions of $C(1, t, \cdots, t^n)$ cannot hold, by Theorem 3, for all functions of $C(1, t, \cdots, t^m)$, $m \neq n$.

(b) *Expansion in series of orthogonal polynomials.* Let $\{P_n(t)\}_{n=0}^{\infty}$ be an orthonormal family of polynomials with respect to a weight function $w(t)$ on (a, b), and let $P_n(t)$ be so normalized that the coefficient of t^n is positive. Let $f(t)$ denote a function of $L_2(w(t); a, b)$ throughout this subsection, and let c_n, $n = 0, 1, \cdots$, denote the Fourier coefficients of $f(t)$ with respect to the system $\{P_n(t)\}$, *i.e.*,

$$c_n = \int_a^b f(t)P_n(t)w(t)dt\,, \qquad n = 0, 1, \cdots\,. \tag{43}$$

Given that $f(t)$ belongs to a convexity cone, certain inequalities have to be satisfied by the coefficients c_n, $n = 0, 1, \cdots$. The derivation of such inequalities is the substance of this subsection.

THEOREM 8. *Let $f(t)$ be a function of $C(1, t, \cdots, t^{n-1})$. Then the following conditions are satisfied:*

$$c_n \geqq 0\,, \tag{44}$$

and

$$\frac{c_n}{P_n(b)} \geqq \frac{c_{n+1}}{P_{n+1}(b)}\,, \qquad \frac{(-1)^{n+1}c_n}{P_n(a)} \leqq \frac{(-1)^{n+1}c_{n+1}}{P_{n+1}(a)}\,. \tag{45}$$

Proof. Set $d\mu_1(t) = P_n(t)w(t)dt$. Then relation (44) will follow if we show that $d\mu_1$ belongs to $C^*(1, t, \cdots, t^{n-1})$. The orthogonality properties of the polynomials $P_n(t)$ imply that $d\mu_1$ satisfies the "moment conditions" (5). We recall now that $P_n(t)$ has n simple zeros, i.e. n sign changes, inside (a, b) (see [10], Th. 3.3.1). Furthermore, since these are all the zeros, the normalization implies that the last sign of $d\mu_1$ on (a, b) is $(+)$. Hence, relation (44) follows by appealing to Theorem B.

Consider next the measure

$$d\mu_2(t) = \left[\frac{P_{n+1}(b)}{P_n(b)}P_n(t) - P_{n+1}(t)\right]w(t)dt\,.$$

The "moment conditions" are clearly satisfied by $d\mu_2$ due to the orthogonality properties. Observe next that the polynomial

$$P_{n+1}(b)P_n(t)/P_n(b) - P_{n+1}(t)$$

has exactly n sign changes inside (a, b) (see [10], Th. 3.3.4). Since the $n+1-$ st zero is at b, the normalization implies that the polynomial must change its sign there from positive to negative. Hence, the last sign of $d\mu_2$ on (a, b) is $(+)$ and the first of relations (45) is established by appealing to Theorem B.

Similarly, the measure

$$d\mu_3(t) = \left[P_{n+1}(t) - \frac{P_{n+1}(a)}{P_n(a)}P_n(t)\right]w(t)dt$$

has n sign changes inside (a, b) and an $n+1$-st sign change at a (see [10], Th. 3.3.4). Its last sign on (a, b) is $(+)$ and the "moment conditions" are satisfied. Thus, Theorem B implies that $d\mu_3$ belongs to $C^*(1, t, \cdots, t^{n-1})$, i.e. that the inequality $c_{n+1} \geqq P_{n+1}(a)c_n/P_n(a)$ is valid for all $f(t) \in C(1, t, \cdots, t^{n-1})$. Using the fact that

$$(-1)^{n+1}P_{n+1}(a) > 0 \ ,$$

we obtain the second relation of (45).

COROLLARY 8.1. *If $f(t)$ is absolutely monotone on (a, b) then $c_n \geqq 0$, $n = 0, 1, \cdots$, and the sequence $\{c_n/P_n(b)\}_{n=0}^{\infty}$ is monotone decreasing. It $f(t)$ is completely monotone on (a, b) then*

$$(-1)^n c_n \geqq 0, n = 0, 1, \cdots ,$$

and the sequence $\{c_n/P_n(a)\}_{n=0}^{\infty}$ is monotone decreasing.

For special classes of orthogonal polynomials, some further results can be obtained. Let (a, b) be a finite interval. Then, with no loss of generality we may assume that $a = -1, b = 1$.

THEOREM 9. *Let the weight function $w(t)$ be an even function and let $f(t)$ be a function of $C(t, t, \cdots, t^{n-1})$. Then in addition to* (44) *and* (45), *we have*

$$\frac{c_n}{P_n(1)} \geqq \frac{c_{n+2}}{P_{n+2}(1)} \ . \tag{46}$$

Proof. Consider the measure

$$d\mu(t) = \left[\frac{P_n(t)}{P_n(1)} - \frac{P_{n+2}(t)}{P_{n+2}(1)}\right]w(t)dt \ .$$

The "moment conditions" (5) are satisfied by $d\mu$ by virtue of the orthogonality properties. Thus, by Theorem B, the polynomial

$$Q(t) \equiv P_n(t)/P_n(1) - P_{n+2}(t)/P_{n+2}(1)$$

has at least n zeros inside $(-1, 1)$. On the other hand, it can have at most n zeros inside $(-1, 1)$ since $Q(1) = 0$ and the symmetry of $w(t)$ implies that $Q(-1) = 0$. Hence, $Q(t)$ has exactly n zeros inside $(-1, 1)$. Noting that $t = 1$ is the largest zero of $Q(t)$, we deduce from the normalization of the polynomials $P_n(t), n = 0, 1, \cdots$, that the last sign of $d\mu$ on $(-1, 1)$ is $(+)$. Relation (46) follows now by appealing again to Theorem B.

Note that the ultraspherical polynomials have a symmetric weight function, so that for them relations (44)—(46) are valid.

Consider now the expansion in terms of Tchebycheff polynomials. As a result of the strong affinity of these polynomials to the trigonometric functions, a general inequality for the coefficients of the expansion can be derived from the sole assumption that $f(x)$ is monotone nondecreasing.

Let $T_n(x), n = 0, 1, \cdots$, denote the n-th order Tchebycheff polynomial, and let the coefficients $a_n, n = 0, 1, \cdots$, be defined by

$$a_n = \int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1 - x^2}} dx, \qquad n = 0, 1, \cdots. \tag{47}$$

THEOREM 10. *Let $f(x)$ be a monotone nondecreasing function on $(-1, 1)$. Then*

$$|a_1| \geqq |a_n|, \qquad n = 2, 3, \cdots. \tag{48}$$

Proof. Note first that since $f(x) \in C(1)$, Theorem 8 implies that $a_1 \geqq 0$. Hence, relation (48) is equivalent to $a_1 \geqq |a_n|$.

We start by proving that $a_1 \geqq a_n$. Consider the measure

$$d\mu_1(x) = \frac{T_1(x) - T_n(x)}{\sqrt{1 - x^2}} dx .$$

We wish to prove that this measure belongs to $C^*(1)$ on $(-1, 1)$. Making the monotone change of variable $x = \cos t, 0 < t < \pi$, we see that our problem reduces to showing that

$$d\mu_2(t) = (\cos nt - \cos t)dt$$

belongs to $C^*(1)$ on $(0, \pi)$. The "moment condition $\int_0^\pi d\mu_2(t) = 0$ is trivially satisfied. Furthermore, $d\mu_2(t)$ is negative on an interval extending to 0 and it is positive on an interval extending to π.

The elementary trigonometric identity

$$\cos nt - \cos t = -2 \sin \frac{(n + 1)t}{2} \sin \frac{(n - 1)t}{2}$$

shows that the zeros, i.e. the points of sign change, of $[\cos nt - \cos t]$ inside $(0, \pi)$ are the points $2k\pi/(n+1)$, $k = 1, 2, \cdots, [n/2]$, and the points $2k\pi/(n-1)$, $k = 1, 2, \cdots, [(n-2)/2]$. Thus, for $n = 2$ or $n = 3$, $d\mu_2$ changes sign only once so that the desired conclusion follows from Theorem B.

Assume now that $n \geqq 4$. Since $r/(n-1) < (r+2)/(n+1)$ for all r, $1 \leqq r < n-1$, the ordered sequence of points of sign change of $d\mu_2$ inside $(0, \pi)$ is

$$\frac{2\pi}{n+1}, \frac{2\pi}{n-1}, \frac{4\pi}{n+1}, \cdots, \frac{2[(n-2)/2]\pi}{n-1}, \frac{2[n/2]\pi}{n+1} .$$

The numbers J_i, $i = 0, 1, \cdots [(n-4)/2]$, defined in (9), are thus given by

$$\begin{aligned} J_i &= \int_{2i\pi/(n-1)}^{(2i+2)\pi/(n-1)} (\cos nt - \cos t)dt \\ &= \frac{1}{n}\left[\sin\frac{(2i+2)n\pi}{n-1} - \sin\frac{2in\pi}{n-1}\right] - \left[\sin\frac{(2i+2)\pi}{n-1} - \sin\frac{2i\pi}{n-1}\right] . \end{aligned}$$

Since $n\pi/(n-1) = \pi + \pi/(n-1)$, the expression for J_i reduces to

$$J_i = \left(\frac{1}{n} - 1\right)\left(\sin\frac{(2i+2)\pi}{n-1} - \sin\frac{2i\pi}{n-1}\right), \qquad i = 0, 1, \cdots, [(n-4)/2] . \tag{49}$$

The last J_i is given by

$$\begin{aligned} J_{[(n-2)/2]} &= \int_{2[n-2)/2]\pi/(n-1)}^{\pi} (\cos nt - \cos t)dt \\ &= -\frac{1}{n}\sin\frac{2[(n-2)/2]\pi}{n-1} + \sin\frac{2[(n-2)/2]\pi}{n-1} \\ &= \left(1 - \frac{1}{n}\right)\sin\frac{2[(n-2)/2]}{n-1} > 0 . \end{aligned}$$

From (49) we can deduce that $J_0 < 0$ and that the sequence

$$\{J_i, i = 0, 1, \cdots, [(n-2)/2]\}$$

has precisely one sign change, which is a change from negative to positive. Hence, by appealing to Lemma 2, we conclude that $d\mu_2(t)$ belongs to $C^*(1)$ on $(0, \pi)$. Thus $d\mu_1(x)$ belongs to $C^*(1)$ on $(-1, 1)$ and the inequality $a_1 \geqq a_n$ is established for all $f(x) \in C(1)$.

For the proof of the inequality $a_1 \geqq -a_n$ we consider the measure $d\mu_3(x) \equiv [T_1(x) + T_n(x)](1 - x^2)^{-1/2}dx$ defined on $(-1, 1)$. This measure belongs to $C^*(1)$ on $(-1, 1)$ if, and only if $d\mu_4(t) \equiv -(\cos nt + \cos t)dt$ belongs to $C^*(1)$ on $(0, \pi)$. The proof that $d\mu_4(t)$ belongs to $C^*(1)$

proceeds in exactly the same way as the proof that $d\mu_2(t) \in C^*(1)$. We will not repeat the details. This completes the proof of the theorem.

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Pacific Journal of Mathematics
Vol. 24, No. 3 July, 1968

Duane W. Bailey, *On symmetry in certain group algebras* ... 413
Lawrence Peter Belluce and Surender Kumar Jain, *Prime rings with a one-sided ideal satisfying a polynomial identity* ... 421
L. Carlitz, *A note on certain biorthogonal polynomials* ... 425
Charles O. Christenson and Richard Paul Osborne, *Pointlike subsets of a manifold* ... 431
Russell James Egbert, *Products and quotients of probabilistic metric spaces* ... 437
Moses Glasner, Richard Emanuel Katz and Mitsuru Nakai, *Bisection into small annuli* ... 457
Karl Edwin Gustafson, *A note on left multiplication of semigroup generators* ... 463
I. Martin (Irving) Isaacs and Donald Steven Passman, *A characterization of groups in terms of the degrees of their characters. II* ... 467
Howard Wilson Lambert and Richard Benjamin Sher, *Point-like* 0-*dimensional decompositions of* S^3 ... 511
Oscar Tivis Nelson, *Subdirect decompositions of lattices of width two* ... 519
Ralph Tyrrell Rockafellar, *Integrals which are convex functionals* ... 525
James McLean Sloss, *Reflection laws of systems of second order elliptic differential equations in two independent variables with constant coefficients* ... 541
Bui An Ton, *Nonlinear elliptic convolution equations of Wiener-Hopf type in a bounded region* ... 577
Daniel Eliot Wulbert, *Some complemented function spaces in* $C(X)$ ... 589
Zvi Ziegler, *On the characterization of measures of the cone dual to a generalized convexity cone* ... 603