Modelling Term Rewriting Systems by Sesqui-Categories

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1 Introduction

The simplest abstract model for a rewriting system is a binary relation on a set; the structures being rewritten are the elements of the set and a pair of elements is related if and only if the first rewrites to the second. In several kinds of rewriting the structures being rewritten form not merely a set, but support a composition so that they are the morphisms of a category. This leads to models of rewriting in which the rewrites appear as 2-cells between morphisms.

The binary relation model has been applied [Hue80] to term rewriting systems in order to separate those parts of confluence results which depend essentially on the rewritten structures being terms, from those aspects of such results which are true of an arbitrary binary relation. One aspect of confluence results which cannot be handled by the binary relation model of a term rewriting system is that of critical pairs [KB70]. The significance of the compositional structure in term rewriting is that this structure, together with coproducts, is sufficient to account for critical pairs. This is demonstrated in section 5 below.

It has been shown [RS87] that a term rewriting system gives rise to a 2-category in which the objects are finite sets of variables, the morphisms are substitutions, and the 2-cells are rewrites. In this paper the question of whether a 2-category really is the most appropriate structure to model a term rewriting system is considered, and a more general structure, called a sesqui-category is proposed as an alternative.

There are two advantages in using the weaker structure of a sesqui-category, rather than a 2-category. The first advantage is conceptual. A sesqui-category may be described informally as having all the structure of a 2-category except that there is no general horizontal composition $\circ$ for combining a 2-cell with a 2-cell. There are instead two compositions $\circ_L$ and $\circ_R$ for combining a 2-cell with a morphism and vice versa respectively. The operations $\circ_L$ and $\circ_R$ each has a natural interpretation from the viewpoint of term rewriting, whereas the operation $\circ$ of the 2-category has no such straightforward interpretation. The second advantage is that the hom-categories of the sesqui-category are ℓ-categories, in the sense of [Mit72]; this means that we have a notion of length for 2-cells. This notion of length allows a distinction between confluence and local confluence, and hence a treatment of critical pairs.

2 Term Rewriting Systems

For a more detailed introduction to term rewriting the survey [DJ90] will be found helpful.

A signature Ω gives rise to an endofunctor T_Ω on **Set**, XT_Ω being the set of terms with variables in X. It is usual to fix a countably infinite set $\mathcal{V}$ of variable symbols, and to call a function from $\mathcal{V}$ to $\mathcal{V}T_\Omega$ which is the identity except for finitely many variables, a **substitution**. The functor T_Ω is part of a monad, and **Fin** will denote the full subcategory of the Kleisli category for this monad determined by the finite subsets of $\mathcal{V}$.

The objects of **Fin** are thus finite sets of variables, and a morphism $f : X \to Y$ is a function $f : X \to YT_\Omega$. If $g : Y \to Z$ in **Fin**, then the composite fg is defined by setting, for each $x \in X$, the term xfg to be the term obtained by substituting yg for every variable y in xf. The morphisms of **Fin** correspond to substitutions, however some care is needed as the correspondence does not preserve composition. For example, the composite in **Fin** of $\{\mathtt{x} \mapsto \mathtt{y}\} : \{\mathtt{x}\} \to \{\mathtt{y}\}$ and $\{\mathtt{y} \mapsto \mathtt{z}\} : \{\mathtt{y}\} \to \{\mathtt{z}\}$ is $\{\mathtt{x} \mapsto \mathtt{z}\}$, but the composite of the substitutions $\{\mathtt{x} \mapsto \mathtt{y}\}$ and $\{\mathtt{y} \mapsto \mathtt{z}\}$ is $\{\mathtt{x} \mapsto \mathtt{z}, \mathtt{y} \mapsto \mathtt{z}\}$.

The usual notion of occurrences in a term extends to morphisms of **Fin**. An occurrence of $f : X \to Y$ is a pair $x\mathsf{a}$, where $x \in X$ and a is an occurrence in the term xf. The set of all occurrences of f will be written $\mathsf{occ}\,(f)$. The term at the occurrence $x\mathsf{a}$ in f is denoted $f@x\mathsf{a}$, and the result of replacing this term by a term t is denoted $f[t \backslash x\mathsf{a}]$. The notions of occurrences and replacement are also used with respect to substitutions.

A **term rewriting system**, or TRS, is a set of rules where each rule, $l \Rightarrow r$, is a pair of terms with all variables present in r also appearing in l. A TRS induces a graph on each hom-set of **Fin**. An edge from f_1 to f_2 is a triple of the form $(l \Rightarrow r, \varphi, x\mathsf{a})$, where $l \Rightarrow r$ is a rule, φ is a substitution, and $x\mathsf{a}$ is an occurrence of f_1. These data will satisfy the conditions that $f_1@x\mathsf{a} = l\varphi$, and f_2 will be obtained from f_1 by replacing the subterm at $x\mathsf{a}$ by $r\varphi$.

There are two ways in which rewrites interact with composition in **Fin**. Firstly, there is **instantiation**. If $f_1, f_2 : X \to Y$ and $h : Y \to Z$, and $(l \Rightarrow r, \varphi, x\mathsf{a})$ is a rewrite from f_1 to f_2, then $(l \Rightarrow r, \varphi h, x\mathsf{a})$ is a rewrite from f_1h to f_2h. Secondly, there is **embedding**. If $g : W \to X$ and $f_1, f_2 : X \to Y$, and $(l \Rightarrow r, \varphi, x\mathsf{a})$ is a rewrite from f_1 to f_2, then there will be a set of rewrites between gf_1 and gf_2.

3 Sesqui-Categories

In looking for an appropriate categorical abstraction for term rewriting we are led to consider a category $\mathbf{K}_0$, each hom-set $\mathbf{K}_0(X, Y)$ of which is a category, $\mathbf{K}(X, Y)$ where the composition represents the action of performing one rewrite followed by another. For each triple X, Y, Z of objects of $\mathbf{K}_0$ there are two compositions $\circ_R$ and $\circ_L$ to model embedding and instantiation respectively. If $f : X \to Y$ and α is a 2-cell in $\mathbf{K}(Y, Z)$, then $f \circ_R \alpha$ is a 2-cell in $\mathbf{K}(X, Z)$. If β is a 2-cell in $\mathbf{K}(X, Y)$ and $g : Y \to Z$, then $\beta \circ_L g$ is a 2-cell in $\mathbf{K}(X, Z)$.

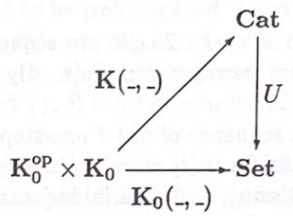
A **sesqui-category** is defined to be data $\mathbf{K}_0$, $\circ_R$ and $\circ_L$ as above, subject to the following equations, which hold whenever the composites are defined.

$$\begin{array}{llcllcl} \text{i.} & 1_X \circ_R \beta & = & \beta & \text{ii.} & (fg) \circ_R \beta & = & f \circ_R (g \circ_R \beta) \\ \text{iii.} & g \circ_R 1_h & = & 1_{gh} & \text{iv.} & g \circ_R (\beta \cdot \delta) & = & (g \circ_R \beta) \cdot (g \circ_R \delta) \\ \text{v.} & \alpha \circ_L 1_X & = & \alpha & \text{vi.} & \alpha \circ_L (hk) & = & (\alpha \circ_L h) \circ_L k \\ \text{vii.} & 1_g \circ_L h & = & 1_{gh} & \text{viii.} & (\alpha \cdot \gamma) \circ_L h & = & (\alpha \circ_L h) \cdot (\gamma \circ_L h) \end{array}$$

$$\text{ix.} \quad (f \circ_R \alpha) \circ_L h = f \circ_R (\alpha \circ_L h)$$

In these equations, 1_X is an identity 1-cell, 1_f an identity 2-cell, and $\cdot$ is the 'vertical' composition in the hom-categories. Sesqui-categories are so called as the prefix 'sesqui' is used to mean 'one and a half' and these structures lie part-way between categories and 2-categories. A sesqui-category satisfying $(\partial_0\alpha \circ_R \beta) \cdot (\alpha \circ_L \partial_1\beta) = (\alpha \circ_L \partial_0\beta) \cdot (\partial_1\alpha \circ_R \beta)$, where ∂_0 and ∂_1 denote domain and codomain, is just a 2-category.

An alternative definition of a sesqui-category is that it is an ordinary category $\mathbf{K}_0$ equipped with a lifting of the hom-functor to **Cat**, as in the commutative diagram at the right, in which $U : \mathbf{Cat} \to \mathbf{Set}$ is the functor which forgets the morphisms.



A third way of describing sesqui-categories is as enriched categories. Besides the usual cartesian closed structure, **Cat** possesses exactly one other symmetric monoidal closed structure [FLK80]. This second structure can be called the unnatural symmetric monoidal closed structure on **Cat**, since the internal hom is the category of functors and transformations which are not required to be natural. If **Cat**′ denotes **Cat** equipped with the unnatural closed structure, then a **Cat**′-category is exactly a sesqui-category. This description of a sesqui-category as a **Cat**′-category leads via the notion of colimits in enriched categories to the appropriate definition of a coproduct in a sesqui-category, viz the **Cat**′-coproduct.

4 Sesqui-Categories Associated to a TRS

In this section we see how a TRS gives rise to a sesqui-category, $\mathcal{F}in$, and a 2-category, $\mathcal{FIN}$, each having **Fin** as its underlying category.

We saw above that a TRS makes each $\mathbf{Fin}(X,Y)$ into a graph. If $f : X \to Y$ in **Fin**, two occurrences $x_1\mathsf{a}_1$ and $x_2\mathsf{a}_2$ of f are said to be **disjoint** if $x_1 \neq x_2$ or neither a_1 nor a_2 is an initial substring of the other. When there are three morphisms in **Fin**, $f_1, f_2, f_3 : X \to Y$, and there are rewrites $\alpha_1 = (l_1 \Rightarrow r_1, \varphi_1, x_1\mathsf{a}_1)$ from f_1 to f_2, and $\alpha_2 = (l_2 \Rightarrow r_2, \varphi_2, x_2\mathsf{a}_2)$ from f_2 to f_3, we shall say the rewrites are **disjoint** if $x_1\mathsf{a}_1$ and $x_2\mathsf{a}_2$ are disjoint occurrences. When this happens, the rewrites may be performed in the other order. That is, if f_2' denotes $f_2[l_1\varphi_1\backslash x_1\mathsf{a}_1,\, r_2\varphi_2\backslash x_2\mathsf{a}_2]$, then α_1 is a rewrite from f_2' to f_3, and α_2 is a rewrite from f_1 to f_2'. To form the category $\mathcal{F}in\,(X,Y)$ from the graph $\mathbf{Fin}\,(X,Y)$, we require that if α_1 and α_2 are disjoint rewrites, such that $\alpha_1 \cdot \alpha_2$ is defined, then $\alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_1$.

The compositions $\circ_R$ and $\circ_L$ can be defined using the ideas of embedding and instantiation of rewrites respectively, and $\mathcal{F}in$ has coproducts which extend those in **Fin**. Each hom-category of $\mathcal{F}in$ has length, that is, for any morphism f there is a natural number $|f|$ such that f is expressible as a composite of $|f|$ non-identity morphisms, but not of more than $|f|$. This notion of length corresponds exactly to the usual notion of length in term rewriting.

A category with a graph on each hom-set is called a **derivation scheme** in [Str95]. The construction of $\mathcal{F}in$, above, used one derivation scheme, but we can also consider the derivation scheme, $\mathbb{D}$, where the edges of the graphs are the rules alone, and not the rewrites derived from these. The following result characterizes $\mathcal{F}in$ by a universal property.

Theorem 4.1 *$\mathcal{F}in$ is the free sesqui-category with coproducts generated by $\mathbb{D}$.* □

The proof of the theorem uses the following normal form lemma, which is also needed for the results in section 5.

Lemma 4.2 *Every length one 2-cell in the free sesqui-category with coproducts generated by $\mathbb{D}$ has a unique expression in the form $f \circ_R (\alpha + X) \circ_L [g, X]$ where f and $\partial_0\alpha$ are epi.* □

In $\mathcal{F}in$ the 2-cells are sequences of one-step rewrites subject to the equivalence that disjoint rewrites commute. By imposing an additional equivalence, we get a 2-category, $\mathcal{FIN}$.

A sequence of $m+1$ one-step rewrites, where $m \geq 0$, is said to be an **outer nesting** if it has the form $(l_1 \Rightarrow r_1, \varphi_1, x\mathsf{a})(l_2 \Rightarrow r_2, \varphi_2, x\mathsf{ab}_1)\cdots(l_2 \Rightarrow r_2, \varphi_2, x\mathsf{ab}_m)$, and there is a substitution ψ, and a variable y occurring in ψ, such that $\varphi_1 = \psi[l_2\varphi_2\backslash y]$, and $\{\mathsf{b}_1, \ldots, \mathsf{b}_m\} = \{\mathsf{b} \in \mathsf{occ}(r_1\psi) \mid (r_1\psi)@\mathsf{b} = y\}$. When the variables of r_1 are a proper subset of those in l_1, we may have $m = 0$. A sequence of $n+1$ one-step rewrites, where $n \geq 1$, is said to be an **inner nesting** if it has the form $(l_2 \Rightarrow r_2, \varphi_2, x\mathsf{ac}_1)\cdots(l_2 \Rightarrow r_2, \varphi_2, x\mathsf{ac}_n)(l_1 \Rightarrow r_1, \varphi_1, x\mathsf{a})$, and there is a substitution ψ, and a variable y occurring in ψ, such that $\varphi_1 = \psi[r_2\varphi_2\backslash y]$, and $\{\mathsf{c}_1, \ldots, \mathsf{c}_n\} = \{\mathsf{c} \in \mathsf{occ}(l_1\psi) \mid (l_1\psi)@\mathsf{c} = y\}$.

To any sequence of rewrites from f to g which form an inner nesting, we can associate a sequence also from f to g which constitutes an outer nesting. By equating each inner nesting with its associated outer nesting, in addition to imposing commutativity of disjoint rewrites, the graph $\mathbf{Fin}\,(X,Y)$ becomes a category $\mathcal{FIN}\,(X,Y)$, which is a quotient of $\mathcal{F}in\,(X,Y)$. The 2-category, $\mathcal{FIN}$, can be characterized by a universal property.

Theorem 4.3 *$\mathcal{FIN}$ is the free 2-category with coproducts generated by $\mathbb{D}$.* □

The most significant distinction between the sesqui-category $\mathcal{F}in$ and the 2-category $\mathcal{FIN}$ is that, in general, the latter lacks a notion of length. For example, with rules `f(x)` $\Rightarrow$ `f(f(x))` and `f(x)` $\Rightarrow$ `a`, the following sequences of one-step rewrites are all identified.

```
ff(x) ⇒ a
ff(x) ⇒ fff(x) ⇒ a
ff(x) ⇒ fff(x) ⇒ ffff(x) ⇒ a.
```

Even when $\mathcal{FIN}$ does have length, this may not correspond to the usual notion of length in term rewriting. For example, with rules `f(x)` $\Rightarrow$ `a` and `a` $\Rightarrow$ `b`, there is no distinction in $\mathcal{FIN}$ between the one-step rewrite `f(a)` $\Rightarrow$ `a` and the two-step rewrite `f(a)` $\Rightarrow$ `f(b)` $\Rightarrow$ `a`.

5 Confluence and Critical Pairs

In this section we see how the critical pairs of a TRS arise from the structure of $\mathcal{F}in$.

By a **span** in a sesqui-category with coproducts and length, **K**, is meant a pair of 2-cells $\langle\alpha,\beta\rangle$, where $\partial_0\alpha = \partial_0\beta$. The span can be **completed** if there exist 2-cells γ, and δ, such that $\partial_1\alpha = \partial_0\gamma$, and $\partial_1\beta = \partial_0\delta$, and $\partial_1\gamma = \partial_1\delta$. A span $\langle\alpha,\beta\rangle$ is said to be **embedded** in a span $\langle\lambda,\mu\rangle$ if there exist 2-cells γ and δ such that $\lambda = \alpha\cdot\gamma$ and $\mu = \beta\cdot\delta$. A span where $|\alpha| = 1 = |\beta|$ is a **local span.** **K** is said to be **confluent** if every span can be completed. If every local span can be completed, **K** is said to be **locally confluent.**

Certain spans owe their existence to the algebraic operations present in **K**. If the 1-cells $\partial_0\alpha$ and $\partial_0\beta$ are composable, then there will be a span $\langle\partial_0\alpha \circ_R \beta,\ \alpha \circ_L \partial_0\beta\rangle$. Such a span is **created** by $\circ_R$ and $\circ_L$. In a similar fashion, two 2-cells α and β give rise to a span $\langle\partial_0\alpha + \beta,\ \alpha + \partial_0\beta\rangle$. Spans of these two forms can always be completed, and will be called **explicable spans.** A span which cannot be embedded in an explicable span will be called **non-trivial.**

Given a span $\langle\alpha,\beta\rangle$, and suitable morphisms f, g, and k, then there must also be spans $\langle f \circ_R \alpha \circ_L g,\ f \circ_R \beta \circ_L g\rangle$, and $\langle\alpha + k,\ \beta + k\rangle$. A span $\langle\alpha,\beta\rangle$ is said to be **propagated from** a span $\langle\gamma,\delta\rangle$ if there is a sequence of spans $\langle\alpha,\beta\rangle = \langle\alpha_0,\beta_0\rangle, \langle\alpha_1,\beta_1\rangle, \cdots, \langle\alpha_n,\beta_n\rangle = \langle\gamma,\delta\rangle$, where, for $0 \le i < n$, either $\langle\alpha_i,\beta_i\rangle = \langle f \circ_R \alpha_{i+1} \circ_L g,\ f \circ_R \beta_{i+1} \circ_L g\rangle$ for some f and g, or $\langle\alpha_i,\beta_i\rangle = \langle\alpha_{i+1} + k,\ \beta_{i+1} + k\rangle$ for some k. If this happens, and $\langle\gamma,\delta\rangle$ can be completed, then so can $\langle\alpha,\beta\rangle$.

A **span basis** for **K** is defined as a set, B, of local spans of **K** such that every non-trivial local span of **K** is propagated from some member of B, and no member of B is propagated from any other. If **K** possesses a span basis, B, and all spans in B can be completed, then **K** will be locally confluent.

The fact that the sesqui-category derived from a TRS does possess a span basis was essentially demonstrated in [KB70], the elements of the span basis being the critical pairs. The usual definition of critical pair depends on notions of subterm and unification, but it is possible to characterize critical pairs solely in terms of the sesqui-category $\mathcal{F}in$.

The notion of propagation provides two relations on local spans in $\mathcal{F}in$. Write $\langle\alpha,\beta\rangle \prec_\circ \langle\alpha',\beta'\rangle$, for local spans $\langle\alpha,\beta\rangle$ and $\langle\alpha',\beta'\rangle$, if $\langle\alpha',\beta'\rangle = \langle f \circ_R \alpha \circ_L g, f \circ_R \beta \circ_L g\rangle$ for some f and g and $\langle\alpha',\beta'\rangle$ is not isomorphic to $\langle\alpha,\beta\rangle$. Similarly, write $\langle\alpha,\beta\rangle \prec_+ \langle\alpha',\beta'\rangle$ for local spans $\langle\alpha,\beta\rangle$ and $\langle\alpha',\beta'\rangle$, if $\langle\alpha',\beta'\rangle = \langle\alpha + k, \beta + k\rangle$ for some k, and $\langle\alpha,\beta\rangle$ is not isomorphic to $\langle\alpha',\beta'\rangle$. The relation $\prec$ on local spans is defined to be the lexicographic product of (firstly) $\prec_+$, and (secondly) $\prec_\circ$. A local span $\langle\alpha,\beta\rangle$ in $\mathcal{F}in$ is **irreducible** if there is no local span $\langle\alpha',\beta'\rangle$ for which $\langle\alpha',\beta'\rangle \prec \langle\alpha,\beta\rangle$. The ordering $\prec$ is well founded, so the non-trivial irreducible spans form a span basis for $\mathcal{F}in$.

Theorem 5.1 *The critical pairs of a TRS are exactly the non-trivial irreducible spans.* □

Recall that a TRS is **left-linear** if no left hand side of a rule contains any repeated variables, and is **orthogonal** if it is left-linear and has no critical pairs. The critical pairs in an orthogonal TRS admit a simpler characterization than those in the general case.

Theorem 5.2 *In a left-linear TRS, the critical pairs are exactly the irreducible spans which are not explicable.* □

Corollary 5.3 *A TRS is orthogonal iff all irreducible spans in $\mathcal{F}in$ are explicable.* □

6 Further Work

It has been proposed [Buc85] that a general framework be found to account for various phenomena similar to critical pairs. Work towards such a framework includes [Sto92] and [Rei91], however these do not use sesqui-categories, and it seems that further development of the above treatment of confluence in sesqui-categories may be relevant.

It is a basic result of term rewriting theory that orthogonal systems are confluent. It is not clear whether this result can be explained in terms of the categorical properties of $\mathcal{F}in$. It seems that other sesqui-categories which can be derived from a TRS may be useful here, in particular one formed by using parallel rewrites as length one 2-cells.

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