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F. E. Masat,
Letter to NJAS,
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"A Note on Prime
Number Sequences"
(unpublished manuscript)

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April 18, 1991

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Dear Dr. Sloane:

The reason I am writing to you is that since you have written about sequences, I thought that the enclosed paper may be of interest to you; it involves the sequence of prime numbers.

The sequence of primes (called a^n in the paper) is not new; the associated sequences (called d^n, s_n, and r_n) are all new, or so I believe.

I hope that the paper may prove to be of use or interest or both, and I'll be pleased to receive any comments.

Cordially yours,

Francis E. Masat
Professor

Enc: "A Note on Prime Number Sequences"

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A Note on Prime Number Sequences

Francis E. Masat

The work presented here considers prime numbers from the view point used by Ulam [GLMU] and others [Br, Ro] in their work on lucky numbers. In [GLMU], the lucky numbers were created by using a sequence of sequences of natural numbers. The approach here is similar and it also is intended to generate a renewed interest in prime twins and their properties and applications.

We will use $N - \{1\}$, the set of natural numbers greater than one. Starting with 2, delete every number greater than 2 that is divisible by 2. The next smallest number remaining is 3, so delete every number greater than 3 that is divisible by 3. Continuing, we obtain the following sequence of sequences. We also have listed the n-th deletion term, p_n, used to create the next sequence.

The n-th Prime Number Sequences	The n-th Deletion Term
$a^1 = \{ 2, 3, 4, 5, \dots \} = N - \{1\}$	$p_1 = 2$
$a^2 = \{ 2, 3, 5, 7, \dots \}$ = the odds, plus 2	$p_2 = 3$
$a^3 = \{ 2,3,5,7,11,13,\dots \}$ = all twins, plus 2	$p_3 = 5$
$a^4 = \{ 2,3,5,7,11,13,17,23,29, \dots \}$.	$p_4 = 7$
$\vdots$	$\vdots$

Since we delete multiples of p_k to form a^{k+1}, we may characterize a^{n+1} as $a^{n+1} = a^n - p_k\{ x \in a^n : x \geq p_k \}$. Moreover, if we view a^n in a double subscript manner, as [GLMU] did, then for each i in N, $\lim_{n \to \infty} a^n_i = p_i$, and in general, $\lim_{n \to \infty} a^n = P$, the set of primes.

RESULT 1. There are an infinite number of primes.

Proof. Since each p_n is prime and there are infinitely many numbers left after the deletion of the multiples of p_n from a^n, then there are infinitely many a^n's and p_n's remaining. ■

Since a^3 is the union of { 2, 3} and twins of the form $6x \pm 1$ for some x in N, then in each subsequent sequence, either a left or right twin is deleted, producing a singleton, or a singleton produced earlier is deleted. Moreover, the process is periodic. To see this, we define a second sequence of sequences: for each a^n, let d^n denote the sequence of differences between the terms of a^n. If we let r_n denote the period of repeated digits, we then have:

The Difference Sequences "d^n"	r_n
$d^1 = \overline{1}$	1
$d^2 = 1, \overline{2}$	1
$d^3 = 1, 2, \overline{2, 4}$	2
$d^4 = 1, 2, 2, \overline{4, 2, 4, 2, 4, 6, 2, 6}$	8
$d^5 = 1, 2, 2, 4, \overline{2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4,}$	
$\overline{2, 6, 4, 6, 8, 4, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 6,}$	
$\overline{2, 6, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2, 10, 2, 10}$	48
⋮	⋮

Intuitively, the period of d^n depends on the period of d_{n-1} and p_n. For example, to find r_5, we use $p_4 = 7$ and, since $r_4 = 8$, a "spanning" set from a^4 of nine numbers such as {7, 11, 13, 17, 19, 23, 29, 31, 37}. The numbers deleted from a^4 to form a^5 then will be 7 times each of these or { 49, ..., 259}. Note that the first span of numbers is contained in the interval [p_4, $p_4 + 30$] and that the second span is contained in [p_4^2, $p_4^2 + 30p_4$]. That is,

$30p_4 = 210$ is the span of numbers needed for a period in a^5. If we use the binomial value $B_k = (1 - 1/2)\cdots(1 - 1/p_k)$ with $k = 4$ to delete multiples of 2, 3, 5, and 7 from [49, 259], then

$$r_5 = 259B_4 - 49B_4 = 210B_4 = 210(1\cdot 2\cdot 4\cdot 6)/(2\cdot 3\cdot 5\cdot 7) = 48.$$

In general, we will let s_k denote the span of numbers needed for a period in a^k. For $k > 1$ we then have by inspection that

$$s_k = \mathrm{lcm}(p_1, \ldots, p_{k-1}) = p_1\cdots p_{k-1} = s_{k-1}p_{k-1}.$$

RESULT 2. If k is in N and greater than 2, then the sequence of differences between the terms of the prime number sequence a^{k+1} has period r_{k+1} given by

$$r_{k+1} = r_k(p_k - 1) = B_k s_{k+1} = \phi(s_{k+1}),$$

where $B_k = (1 - 1/2)\cdots(1 - 1/p_k)$ and ϕ is Euler's phi-function. Also, the sequence of periods is strictly increasing for all $k > 1$.

Proof. Deletions repeat in a^k after s_k numbers are processed, producing a^{k+1}. Applying B_k to $[p_k^2, p_k^2 + p_k s_k]$, we have

$$\begin{aligned} r_{k+1} &= B_k p_k(p_k + s_k) - B_k p_k^2 \\ &= B_k p_k s_k \\ &= B_k p_k(p_1\cdots p_{k-1}) \\ &= (p_1 - 1)\cdots(p_{k-1} - 1)(p_k - 1) \\ &= r_k(p_k - 1). \end{aligned}$$

That $r_{k+1} = B_k s_{k+1} = \phi(s_{k+1})$ follows easily. Lastly, since p_k is odd for $k > 1$, then $r_k = r_{k+1}$ implies that $p_k = 2$, a contradiction. ■

For reference, and to gain insight into the structure of the d^n, we list a few of the values provided by Result 2.

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Table of Values for the Difference Sequences "d^n"

k	p_k	r_k	s_k
1	1	1	1
2	3	1	2
3	5	2	6
4	7	8	30
5	11	48	210
6	13	480	2310
7	17	5760	30030
⋮	⋮	⋮	⋮
n	n-th prime	$(p_{n-1} - 1)r_{n-1}$	$p_1 \cdots p_{n-1}$
⋮	⋮	⋮	⋮

RESULT 3. Using the notation of Result 2, we also have:

a. The periods in a^k can be characterized as

$$(*) \quad \{ p_k + (m - 1)s_k, \ldots, ms_k - p_k, ms_k - 1, ms_k + 1 \}.$$

b. The size of the maximum gap in d^k is $g_k = p_k - 1$.

c. There exists at least one "2" in each period of d_k; i.e., $ms_k \pm 1$ are twins in a^k for $m = 1, 2, \ldots$.

d. The maximum gap in d^k first occurs in positions $r_k + k - 3$ and $r_k + k - 1$ of d^k.

e. There are arbitrarily large gaps in P.

Proof. (a). In a^k, $\{x \in a^k: p_k \le x < p_k + s_k\}$ is the first period. Relative to it, consider $I = [s_k - p_k, \ldots, s_k + p_k]$ in N. Since the next number remaining after pk-1 is p_k, there are no numbers between $s_k + p_{k-1}$ and $s_k + p_k$, and none between $s_k - p_k$ and $s_k - p_{k-1}$. Moreover, one of $p_1, \ldots, p_{k-1}$ divides each of number in I except $s_k \pm 1$ and $s_k \pm p_k$. Thus we may rewrite the first period of a^k as $\{ p_k, \ldots, s_k - p_k, s_k - 1, s_k + 1 \}$. Since the same pattern occurs in each period, we may characterize the m-th period as in Equation *.

(b). For $\{p_k, \ldots, s_k - p_k, s_k - 1, s_k + 1, s_k + p_k\}$, the difference sequence ends with $p_k - 1$, 2, and $p_k - 1$; i.e., the maximum gap in d^k is $g_k = p_k - 1$.

(c). The proof of part Part (b) also shows that there is a "2" in each period of d^k, viz., the one corresponding to the twins $ms_k \pm 1$ $(m = 1, 2, 3, \ldots)$ in a^k.

(d). Since the start of d^k shifts right with each new p_k, g_k will occur first in positions $(r_k - 2) + (k - 1)$ and $r_k + (k - 1)$ of d^k, and the result follows.

(e). Since $g_k = p_k - 1$ and the p_k form an unbounded increasing sequence, then there are arbitrarily large gaps in P. ■

Note that the twins described in Result 3(c) may not be primes.

We next consider twins in a^k; i.e., the 2's in d^k. For example, if $n = 5$, then a bound for the number of twins in one period of a^5 is the bound for the number of twins in a period of a4, multiplied by p_4 (the number of periods of r_4 needed for a repetition in a^5), minus r_4 (the number of deletions from a^4 generated by p_4), plus 2 since $p_4(s_4 - 1)$ and $p_4(s_4 + 1)$ do not account for the deletion of any 2's from a^5. That is, $b_5 \geq b_4 p_4 - r_4 + 2 = 3 \cdot 7 - 8 + 2 = 15$. Generalizing this example we have

RESULT 4. A lower bound for the number of sets of twins in a period of a^n is $b_{n-1}p_{n-1} - r_{n-1} + 2$, where b_{n-1} is a lower bound for the number of twins in a period of a^{n-1}, and p_{n-1} and r_{n-1} are as defined previously.

We note that this result parallels the lower bound result found for lucky twins in [Mas]: $b_{n+1} \geq (b_n t_n - r_n)[r_n, t_n]/r_n t_n + 2$, where t_n is the n-th deletion term (not necessarily prime as here) and $[r_n, t_n]$ denotes the lcm(r_n, t_n). Since $t_n = p_n$ in the work here, then $(r_n, p_n) = 1$ and $[r_n, t_n]/r_n t_n = 1$.

Since the $(r_k - 2)$ deletions which produce a^{k+1} may not yield all singletons, then b_{k+1} is a lower bound for the number of sets of twins in each r_{k+1} elements of a^{k+1}. Moreover, by Result 3(c), $b_k > 0$ for all $k > 1$. There is, however, an important open question here: is $b_n < b_{n+1}$ for all $n > 1$? If so, it may tell us something about the number or distribution of prime twins.

Lastly, we look at the relationship between prime twins and the prime number sequences. An inspection of the prime twins in the a^n suggests that between a prime and its square there is at least one set of prime twins. If this is true, then we would have

RESULT 5. a. If x and $x + 2$ are natural numbers such that $p_k \leq x$ and $x + 2 < p_k^2$ for some k, then they are prime twins.

b. If for each k in N there are natural numbers x and $x + 2$ in the interval $[p_k, p_k^2)$, then the number of prime twins is infinite.

Proof. For (a), suppose that x and $x + 2$ are in $[p_k, p_k^2)$, viz., within a^k. In forming a^{k+1}, the deletions begin with p_k^2, leaving x and $x + 2$ as twins in each subsequent a^n. Thus, since x and $x + 2$ remain, they are prime twins.

The proof of Part (b) follows from Part (a). ■

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April 2, 1991.