The months in a year, A215940, A217626, and A101301

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Abstract

It will be briefly exposed how the application of an alternative treatment for generating a particular sequence, might be "understood" in terms of known information about a concept frequently and daily used like the modern calendar.

Why?, why a triangular matrix might be used as replacement of an arithmetic division for the generation of $A215940$?.

The question is... given for example,

$$P_{24} = (4, 3, 2, 1) \tag{1}$$

And

$$P_1 = (1, 2, 3, 4) \tag{2}$$

Respectively the 1st and the 24th permutations in lexical order expressed as vectors, and the difference between them:

$$\Delta P_{24} = (3, 1, -1, -3) \tag{3}$$

Why the strictly lower uni-triangular matrix 4×4 operating over ΔP_{24} transposed, yields the 24th term of $A215940$?. This is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 3 \end{pmatrix} \tag{4}$$

And also: Why this same behavior is extensible to $A217626$, given $P_{(k+1)}$ and replacing P_1 with P_k there in the difference?...

Well. In order to get an idea about what is actually happening there behind such kind of calculations, we might guess to apply a similar treatment to known information about another concept familiar to us.

Let us think about how it might be defined a year of thirteen months. We can get it done trivially by saying that the 13th month is defined to have zero days. A year having thirteen months with zero days occupied by its last month seems to be a reasonable definition. If for one of those years February has 28 days, then a vector representing the number of days in each month is:

$$\Upsilon = \begin{pmatrix} 31 & 28 & 31 & 30 & 31 & 30 & 31 & 31 & 30 & 31 & 30 & 31 & 0 \end{pmatrix} \tag{5}$$

And now given additionally the strictly lower uni-triangular matrix 13×13 operating over Υ transposed:

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&0 \end{pmatrix} \begin{pmatrix} 31 \\ 28 \\ 31 \\ 30 \\ 31 \\ 30 \\ 31 \\ 31 \\ 30 \\ 31 \\ 30 \\ 31 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 31 \\ 59 \\ 90 \\ 120 \\ 151 \\ 181 \\ 212 \\ 243 \\ 273 \\ 304 \\ 334 \\ 365 \end{pmatrix} \tag{6}$$

We can realize that such operator transform a column vector Υ giving as result another vector Ψ consisting in the partial sums of the components in Υ starting with zero[1]. For the present context: The count of days elapsed after each month with a shift of (+1) place there in the offsets.

Then an arithmetic division might be replaced with the operation of the mentioned matrix for those cases where the same result to be obtained with both methods is built from partial sums in a similar way like in the given example for the calendar.

[1]Clearly zero since for the first component in Ψ, nothing has been added yet to the partial sum carried among the components for Ψ. By context: Because as a side effect due the shift in the offsets, at the first component of the result Ψ, none month had elapsed yet.

Now according to the previous observation, $A215940$ can be defined as: *The vector built from the partial sums of the components in the difference between some permutation and the first, when the permutations are represented by vectors and listed in lexical order.*

Also $A217626$ can be defined as: *First differences of* $A215940$*: The vector built from the partial sums of the components in the difference between two consecutive permutations when they are represented by vectors and listed in lexical order.*

It is noteworthy that these definitions are base-independent. However if the described vectors are all of them such that each one of their components is an integer number ranging between 0 and 9, those vectors can be written directly as integers in the decimal base. This condition is satisfied at least by the first 720 and 719 terms of $A215940$ and $A217626$ respectively.

The proper base where whole larger sets of those vectors are writable directly as integer numbers can be known by looking for $\lfloor \frac{n^2}{4} \rfloor + 1$ ($A033638$). As an example, precisely for the first 6! or 720 permutations, $A033638\,(6) = 10$, meaning this decimal or base 10. Additional information can be read from the links at $A211869$.

Another way of treating this matter, indeed the approach already used there in the description of $A215940$ and $A217626$ is the interpretation of those vectors described above, as representations for polynomials in x, being $x = 10$ the base mostly used by $OEIS$. So, each one of those vectors would have an infinity number of possible integer representations depending on the choice for x. By being extreme for example, nothing forbids to pick $x = 2$ and write each one of the first 13! permutations as a power series evaluated for such choice of the base. The last one of those permutations would be written as follows[2]:

$$(13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$$

$$\sum_{j=1}^{13} \left(jx^{(j-1)}\right)$$

$$13x^{12} + 12x^{11} + 11x^{10} + 10x^9 + 9x^8 + 8x^7 + 7x^6 + 6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x^1 + 1x^0$$

$$98305$$

[2]Notice here that we can always know immediately which is the last term from the first n! of $A215940$ due the way it was defined.

The first permutation would be:

$$(1,2,3,4,5,6,7,8,9,10,11,12,13)$$
$$\sum_{j=1}^{13}\left([13-j+1]\,x^{(j-1)}\right)$$
$$1x^{12}+2x^{11}+3x^{10}+4x^{9}+5x^{8}+6x^{7}+7x^{6}+8x^{5}+9x^{4}+10x^{3}+11x^{2}+12x^{1}+13x^{0}$$
$$16369$$

The difference between both representations is $(98305-16369)=81936$; Since $(x-1)=1$, none division is necessary and 81936 represents in some sense, the $(13!)$th term of $A215940$. To verify this it will be applied the alternative treatment based on the identified behavior for the triangular matrix operating over the differences as it was shown previously:

$$\begin{aligned}&(13,12,11,10,9,8,7,6,5,4,3,2,1)\\ &(-1)\,(1,2,3,4,5,6,7,8,9,10,11,12,13)\\ &=(12,10,8,6,4,2,0,-2,-4,-6,-8,-10,-12)\end{aligned}$$

$$\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0&0&0&0&0\\1&1&0&0&0&0&0&0&0&0&0&0&0\\1&1&1&0&0&0&0&0&0&0&0&0&0\\1&1&1&1&0&0&0&0&0&0&0&0&0\\1&1&1&1&1&0&0&0&0&0&0&0&0\\1&1&1&1&1&1&0&0&0&0&0&0&0\\1&1&1&1&1&1&1&0&0&0&0&0&0\\1&1&1&1&1&1&1&1&0&0&0&0&0\\1&1&1&1&1&1&1&1&1&0&0&0&0\\1&1&1&1&1&1&1&1&1&1&0&0&0\\1&1&1&1&1&1&1&1&1&1&1&0&0\\1&1&1&1&1&1&1&1&1&1&1&1&0\end{pmatrix}\begin{pmatrix}12\\10\\8\\6\\4\\2\\0\\-2\\-4\\-6\\-8\\-10\\-12\end{pmatrix}=\begin{pmatrix}0\\12\\22\\30\\36\\40\\42\\42\\40\\36\\30\\22\\12\end{pmatrix}$$

$$\overrightarrow{u}=(0,12,22,30,36,40,42,42,40,36,30,22,12)$$

Now by applying $\sum_{j=1}^{13}\left(u_j x^{(j-1)}\right)=81936$ the same result is found. By simple inspection it is observed that such vector cannot be written directly as integer number in decimal[3]. It is necessary at least 42 digits, therefore it can be done from base 43 onwards.

[3]These sequences can be verified with the aid of computer techniques like the MD5 checksum comparisons applied to the output data generated using the current definitions against the output data obtained from the alternative methods. For both $A215940$ and $A217626$ it was found that all the methods described here always generate the same results.

The partial sums of the first prime numbers and $A101301$[4]

The partial sum of the first n prime numbers or $\sum_{j=1}^{n} prime(j)$ minus n might be also interpreted as follows:

Let be $\Omega\,(n)$ the strictly lower uni-triangular matrix $n \times n$. Let be $a\,(n) = A101301\,(n)$, and $P(n)$ a polynomial in x built from the first n primes as coefficients. It is assumed for such combinations that the powers of x are there in descending order while the primes coefficients are in ascending order. Let us call $V(n)$ or simply V to the vector representation of P. There exists for each n some polynomial $Q\,(n)$ divisible by $(x-1)$ such that the difference minus n between P and Q is $a\,(n)$.

Q can be found from V because transposing V and operating it with Ω we obtain an associated vector W such that its polynomial representation multiplied by $(x-1)$ is Q.

In other words, $a\,(n)$ is the difference minus n between the constant term in P and the value that such term should have in order to make P divisible by $(x-1)$. *Examples:*

$$P\,(3) = 2x^2 + 3x + 5$$
$$V\,(3) = (2, 3, 5)$$
$$W\,(3) = (0, 2, 5)$$
$$(2x + 5)\,(x - 1) = 2x^2 + 3x - 5$$
$$+5 - (-5) - 3 = 7$$
$$a\,(3) = 7$$

It was the third therm of $A101301$. Now the 8th,

$$P\,(8) = 2x^7 + 3x^6 + 5x^5 + 7x^4 + 11x^3 + 13x^2 + 17x + 19$$
$$V\,(8) = (2, 3, 5, 7, 11, 13, 17, 19)$$
$$W\,(8) = (0, 2, 5, 10, 17, 28, 41, 58)$$
$$\left(2x^6 + 5x^5 + 10x^4 + 17x^3 + 28x^2 + 41x + 58\right)(x - 1) =$$
$$2x^7 + 3x^6 + 5x^5 + 7x^4 + 11x^3 + 13x^2 + 17x - 58$$
$$+19 - (-58) - 8 = 69$$
$$a\,(8) = 69$$

[4] A PARI script for this section is available at [`http://oeis.org/w/images/8/81/Polyvecpri_A101301.gp.txt`]. *Dear reader please consider this*: Might it be used this interpretation there in the search for prime numbers?.

Additional applications: Alternative method for getting the binomial coefficients.

Let be $\Omega(n)$ the strictly lower uni-triangular matrix $n \times n$.
Let be $I(n)$ the identity matrix $n \times n$.
Let be $\Phi(n) = \Omega(n) + I(n)$ the non-strict lower uni-triangular matrix.
And finally let be $V(n)$ a column vector with n components all of them defined to be the unit.

The first n non-zero binomial coefficients can be computed as:

$$\mathbb{C}(n,k) = [\Phi(n)]^k V(n) \tag{7}$$

Example: Find the first 9 non-zero coefficients of the form $\binom{x}{3}$;

$$\left[\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&0&0&0&0\\ 1&1&1&1&1&1&0&0&0\\ 1&1&1&1&1&1&1&0&0\\ 1&1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&1&1 \end{pmatrix}\right]^{\mathbf{3}} \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\4\\10\\20\\35\\56\\84\\120\\165\end{pmatrix}$$

The components in the result are the 9 binomial coefficients between $\binom{3}{3}$ and $\binom{11}{3}$.

In fact, by using this method it will be found always the n binomial coefficients between $\binom{k}{k}$ and $\binom{n+k-1}{k}$.