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Minimal Coverability Set for Petri Nets: Karp and Miller Algorithm with Pruning

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Abstract. This paper presents the Monotone-Pruning algorithm (MP) for computing the minimal coverability set of Petri nets. The original Karp and Miller algorithm (K&M) unfolds the reachability graph of a Petri net and uses acceleration on branches to ensure termination. The MP algorithm improves the K&M algorithm by adding pruning between branches of the K&M tree. This idea was first introduced in the Minimal Coverability Tree algorithm (MCT), however it was recently shown to be incomplete. The MP algorithm can be viewed as the MCT algorithm with a slightly more aggressive pruning strategy which ensures completeness. Experimental results show that this algorithm is a strong improvement over the K&M algorithm as it dramatically reduces the exploration tree.

1. Introduction

Petri nets form an important formalism for the description and analysis of concurrent systems. While the state space of a Petri net may be infinite, many verification problems are decidable. The minimal coverability set (MCS) [2] is a finite representation of a well-chosen over-approximation of the set of reachable markings. As proved in [10], it can be used to decide several important problems. Among them we mention the *coverability* problem to which many safety problems can be reduced (is it possible to reach a marking dominating a given one?); the *boundedness* problem (is the set of reachable markings

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finite?); the *place boundedness* problem (given a place p, is it possible to bound the number of tokens in p in any reachable marking?); the *semi-liveness* problem (is there a reachable marking in which a given transition is enabled?). Finally, the *regularity problem* asks whether the set of reachable markings is regular.

Karp and Miller (K&M) introduced an algorithm for computing the MCS [8]. This algorithm builds a finite tree representation of the (potentially infinite) unfolding of the reachability graph of the given Petri net. It uses acceleration techniques to collapse branches of the tree and ensure termination. By taking advantage of the fact that Petri nets are strictly monotonic transition systems, the acceleration essentially computes the limit of repeatedly firing a sequence of transitions. The MCS can be extracted from the K&M tree. The K&M Algorithm thus constitutes a key tool for Petri nets, and has been extended to other classes of well-structured transition systems [1].

However, the K&M Algorithm is not efficient and in real-world examples it often does not terminate in reasonable time. This inefficiency is due to processing comparable but inequal markings. This observation led to the Minimal Coverability Tree (MCT) algorithm [2]. This algorithm introduces clever optimizations for ensuring that all markings in the tree are incomparable. At each step the new node is added to the tree only if its marking is not smaller than the marking of an existing node. Then, the tree is pruned: each node labelled with a marking that is smaller than the marking of the new node is removed together with all its successors. The idea is that a node that is not added or that is removed from the tree should be covered by the new node or one of its successors. It was recently shown that the MCT algorithm is incomplete [9, 7]. The flaw is intricate and, according to [7], difficult to patch. As an illustration, an attempt to resolve this issue has been done in [9]. However, as proved in [6], the algorithm proposed in [9] may not terminate. In [7], an alternative algorithm, the CoverProc algorithm, is proposed for the computation of the MCS of a Petri net. This algorithm follows a different approach and is not based on the K&M Algorithm.

We propose here the Monotone-Pruning algorithm (MP), an improved K&M algorithm with pruning. This algorithm can be viewed as the MCT Algorithm with a slightly more aggressive pruning strategy which ensures completeness. The MP algorithm constitutes a simple modification of the K&M algorithm, and is thus easily amenable to implementation and to extensions to other classes of systems [1, 3, 4]. Moreover, as the K&M Algorithm, and unlike the algorithm proposed in [7], any strategy of exploration of the Petri net is correct: depth first, breadth first, random. It is thus possible to develop heuristics for subclasses of Petri nets. Finally experimental results show that our algorithm is a strong improvement over the K&M Algorithm, it indeed dramatically reduces the exploration tree. In addition, optimizations based on symbolic computations (as those proposed in [5] for MCT) can be integrated in the MP Algorithm.

While the algorithm in itself is simple and includes the elegant ideas of the original MCT Algorithm, the proof of its correctness is long and technical. The main difficulty is to prove the completeness of the algorithm, i.e. to show that the set returned by the algorithm covers every reachable marking. To overcome this difficulty, we reduce the problem to the correctness of the algorithm for a particular class of finite state systems, which we call widened Petri nets (WPN). These are Petri nets whose semantics is widened w.r.t. a given marking m: as soon as the number of tokens in a place p is greater than $m(p)$, this value is replaced by ω. Widened Petri nets generate finite state systems for which the proof of correctness of the Monotone-Pruning algorithm is easier as accelerations can be expressed as finite sequences of transitions.

Definitions of Petri nets and widened Petri nets are given in Section 2, together with the notions of

minimal coverability set and reachability tree. In Section 3, we recall the K&M Algorithm and present the Monotone-Pruning Algorithm. We prove its termination and correctness under the assumption that it is correct on WPN. In Section 4, we develop the proof of correctness of MP Algorithm on widened Petri nets. In Section 5, we give a comparison of MP and MCT algorithms, and illustrate them on the Petri net proposed in [7] to prove the incompleteness of MCT. Finally, implementation and experimental results are discussed in Section 6.

2. Preliminaries

$\mathbb{N}$ denotes the set of natural numbers. To denote that the union of two sets X and Y is disjoint, we write $X \biguplus Y$. A quasi order $\leq$ on a set S is a reflexive and transitive relation on S. Given a quasi order $\leq$ on S, a state $s \in S$ and a subset X of S, we write $s \leq X$ iff there exists an element $s' \in X$ such that $s \leq s'$.

Given a finite alphabet Σ, we denote by Σ^* the set of words on Σ, and by ε the empty word. We denote by $\prec$ the (strict) prefix relation on Σ^*: given $u, v \in \Sigma^*$ we have $u \prec v$ iff there exists $w \in \Sigma^*$ such that $uw = v$ and $w \neq \varepsilon$. We denote by $\preceq$ the relation obtained as $\prec \cup =$.

2.1. Markings, ω-markings and labelled trees

Given a finite set P, a *marking on* P is an element of the set $\mathsf{Mark}(P) = \mathbb{N}^P$. The set $\mathsf{Mark}(P)$ is naturally equipped with a partial order denoted $\leq$.

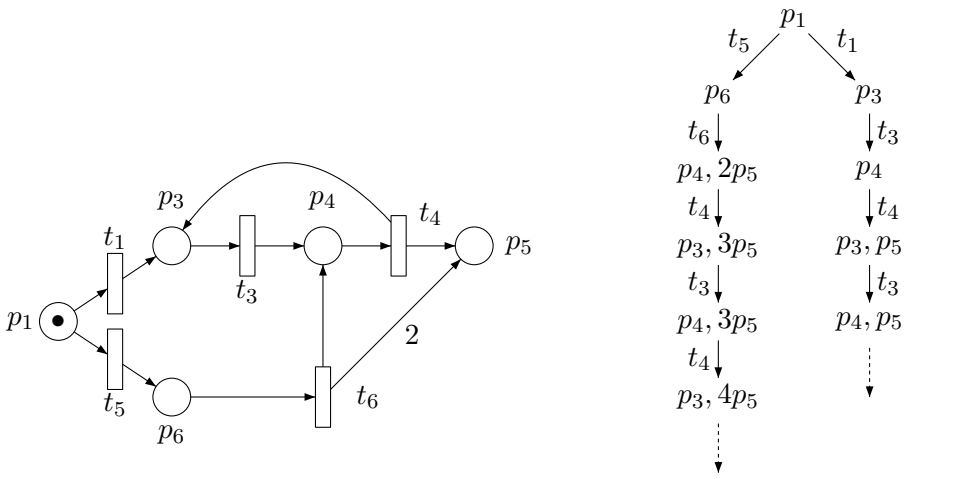
Given a marking $m \in \mathsf{Mark}(P)$, we represent it by giving only the positive components. For instance, $(1, 0, 0, 2)$ on $P = (p_1, p_2, p_3, p_4)$ is represented by the multiset $\{p_1, 2p_4\}$. An *ω-marking on* P is an element of the set $\mathsf{Mark}^\omega(P) = (\mathbb{N} \cup \{\omega\})^P$. The order $\leq$ on $\mathsf{Mark}(P)$ is naturally extended to this set by letting $n < \omega$ for any $n \in \mathbb{N}$, and $\omega \leq \omega$. Addition and subtraction on $\mathsf{Mark}^\omega(P)$ are obtained using the rules $\omega + n = \omega - n = \omega$ for any $n \in \mathbb{N}$. The ω-marking $(\omega, 0, 0, 2)$ on $P = (p_1, p_2, p_3, p_4)$ is represented by the multiset $\{\omega p_1, 2p_4\}$.

Given two sets Σ_1 and Σ_2, a labelled tree is a tuple $\mathcal{T} = (N, n_0, E, \Lambda)$ where N is the set of nodes, $n_0 \in N$ is the root, $E \subseteq N \times \Sigma_2 \times N$ is the set of edges labelled with elements of Σ_2, and $\Lambda : N \to \Sigma_1$ labels nodes with elements of Σ_1. We extend the mapping Λ to sets of nodes: for $S \subseteq N$, $\Lambda(S) = \{\Lambda(n) \mid n \in S\}$. Given a node $n \in N$, we denote by $\mathsf{Ancestor}_{\mathcal{T}}(n)$ the set of ancestors of n in $\mathcal{T}$ (n included). If n is not the root of $\mathcal{T}$, we denote by $\mathsf{parent}_{\mathcal{T}}(n)$ its first ancestor in $\mathcal{T}$. Finally, given two nodes x and y such that $x \in \mathsf{Ancestor}_{\mathcal{T}}(y)$, we denote by $\mathsf{path}_{\mathcal{T}}(x, y) \in E^*$ the sequence of edges leading from x to y in $\mathcal{T}$. We also denote by $\mathsf{pathlabel}_{\mathcal{T}}(x, y) \in \Sigma_2^*$ the label of this path.

2.2. Petri nets

Definition 2.1. (Petri net (PN))
A Petri net $\mathcal{N}$ is a tuple (P, T, I, O, m_0) where P is a finite set of *places*, T is a finite set of *transitions* with $P \cap T = \emptyset$, $I : T \to \mathsf{Mark}(P)$ is the backward incidence mapping, representing the *input* tokens, $O : T \to \mathsf{Mark}(P)$ is the forward incidence mapping, representing *output* tokens, and $m_0 \in \mathsf{Mark}(P)$ is the initial marking.

The semantics of a PN is usually defined on markings, but can easily be extended to ω-markings. We define the semantics of $\mathcal{N} = (P, T, I, O, m_0)$ by its associated labelled transition system $(\mathsf{Mark}^\omega(P), m_0, \Rightarrow)$ where $\Rightarrow \subseteq \mathsf{Mark}^\omega(P) \times \mathsf{Mark}^\omega(P)$ is the transition relation defined by $m \Rightarrow m'$ iff $\exists t \in T$ s.t. $m \geq$



(a) A Petri net $\mathcal{N}$. (b) Prefix of its reachability tree.

Figure 1. A Petri net with its reachability tree.

$I(t) \wedge m' = m - I(t) + O(t)$. For convenience we will write, for $t \in T$, $m \overset{t}{\Rightarrow} m'$ if $m \geq I(t)$ and $m' = m - I(t) + O(t)$. In addition, we also write $m' = \mathsf{Post}(m, t)$, this defines the operator Post which computes the successor of an ω-marking by a transition. We naturally extend this operator to sequences of transitions. Given an ω-marking m and a transition t, we write $m \overset{t}{\Rightarrow} \cdot$ iff there exists $m' \in \mathsf{Mark}^{\omega}(P)$ such that $m \overset{t}{\Rightarrow} m'$. The relation $\Rightarrow^*$ represents the reflexive and transitive closure of $\Rightarrow$. We say that a marking m is *reachable in* $\mathcal{N}$ iff $m_0 \Rightarrow^* m$. We say that a Petri net is bounded if the set of reachable markings is finite.

Example 2.1. We consider the Petri net $\mathcal{N}$ depicted on Figure 1(a), which is a simplification of the counter-example proposed in [7], but is sufficient to present our definitions. The initial marking is $\{p_1\}$, depicted by the token in the place p_1. For any integer n, we have $\mathsf{Post}(\{p_1\}, t_1(t_3t_4)^n) = \{p_3, np_5\}$. In particular, this net is not bounded as place p_5 can contain arbitrarily many tokens. ⌟

2.3. Minimal Coverability Set of Petri Nets

We recall the definition of minimal coverability set introduced in [2].

Definition 2.2. A coverability set of a Petri net $\mathcal{N} = (P, T, I, O, m_0)$ is a finite subset C of $\mathsf{Mark}^{\omega}(P)$ such that the two following conditions hold:

1) for every reachable marking m of $\mathcal{N}$, there exists $m' \in C$ such that $m \leq m'$,

2) for every $m' \in C$, either m' is reachable in $\mathcal{N}$ or there exists an infinite strictly increasing sequence of reachable markings $(m_n)_{n \in \mathbb{N}}$ converging to m'.

A coverability set is minimal iff no proper subset is a coverability set.

One can prove (see [2]) that a PN $\mathcal{N}$ admits a unique minimal coverability set, which we denote by by $\mathrm{MCS}(\mathcal{N})$.

Note that every two elements of a minimal coverability set are incomparable. Computing the minimal coverability set from a coverability set is easy. Note also that if the PN is bounded, then the set of reachable markings is finite, and thus the notion of reachable maximal marking is well-defined. In this case, a set of markings is a coverability set iff it contains all maximal reachable markings.

Example 2.2. (Example 2.1 continued)
The MCS of the Petri net $\mathcal{N}$ is composed of the following ω-markings: $\{p_1\}$, $\{p_6\}$, $\{p_3, \omega p_5\}$, and $\{p_4, \omega p_5\}$.

2.4. Reachability tree of Petri nets

We recall the notion of a reachability tree for a PN. This definition corresponds to the execution of the PN as a labelled tree. We require it to be coherent with the semantics of PN (soundness), to be complete w.r.t. the fireable transitions, and to contain no repetitions. Naturally, if the PN has an infinite execution, then this reachability tree is infinite.

Definition 2.3. (Reachability tree of a PN)
The reachability tree of a PN $\mathcal{N} = (P, T, I, O, m_0)$ is (up to isomorphism) a labelled tree $\mathcal{R} = (N, n_0, E, \Lambda)$, with $E \subseteq N \times T \times N$ and $\Lambda : N \to \mathsf{Mark}(P)$, such that:

Root: $\Lambda(n_0) = m_0$

Soundness: $\forall (n, t, n') \in E, \Lambda(n) \stackrel{t}{\Rightarrow} \Lambda(n')$

Completeness: $\forall n{\in}N, \forall t{\in}T, (\Lambda(n) \stackrel{t}{\Rightarrow} \cdot)$ implies $(\exists n'{\in}N \mid (n, t, n'){\in}E)$

Uniqueness: $\forall n, n', n'' \in N, \forall t \in T, (n, t, n') \in E \wedge (n, t, n'') \in E$ implies $n' = n''$

Using notations introduced for labelled trees, the following property holds:

Lemma 2.1. $\forall x, y \in N, x \in \mathsf{Ancestor}_{\mathcal{R}}(y)$ implies $\Lambda(y){=}\mathsf{Post}(\Lambda(x), \mathsf{pathlabel}_{\mathcal{R}}(x, y))$.

Example 2.3. (Example 2.1 continued)
A prefix of the reachability tree of $\mathcal{N}$ is depicted on Figure 1(b). Each node is represented by its label (a marking). ⌟

2.5. Widened Petri nets

We present an operation which, given a (potentially unbounded) Petri net, turns it to a finite state system. Let P be a finite set, and $\varphi \in \mathsf{Mark}(P)$ be a marking. We consider the *finite* set of ω-markings whose finite components (*i.e.* values different from ω) are less or equal than φ. Formally, we define $\mathsf{Mark}^{\omega}_{\varphi}(P) = \{m \in \mathsf{Mark}^{\omega}(P) \mid \forall p \in P, m(p) \leq \varphi(p) \vee m(p) = \omega\}$. The widening operator Widen_{φ} maps an ω-marking into an element of $\mathsf{Mark}^{\omega}_{\varphi}(P)$: $\forall m \in \mathsf{Mark}^{\omega}(P), \forall p \in P$,

$$\mathsf{Widen}_{\varphi}(m)(p) = \begin{cases} m(p) & \text{if } m(p) \leq \varphi(p) \\ \omega & \text{otherwise.} \end{cases}$$

Note that this operator trivially satisfies $m \leq \mathsf{Widen}_{\varphi}(m)$.

Definition 2.4. (Widened Petri net)
A widened Petri net (WPN for short) is a pair $(\mathcal{N}, \varphi)$ composed of a PN $\mathcal{N} = (P, T, I, O, m_0)$ and of a marking $\varphi \in \mathsf{Mark}(P)$ such that $m_0 \leq \varphi$.

The semantics of $(\mathcal{N}, \varphi)$ is given by its associated labelled transition system $(\mathsf{Mark}^{\omega}_{\varphi}(P), m_0, \Rightarrow_{\varphi})$ where for $m, m' \in \mathsf{Mark}^{\omega}_{\varphi}(P)$, and $t \in T$, we have $m \stackrel{t}{\Rightarrow}_{\varphi} m'$ iff $m' = \mathsf{Widen}_{\varphi}(\mathsf{Post}(m, t))$. We carry over from PN to WPN the relevant notions, such as reachable marking and reachability tree. We define the operator Post_{φ} by $\mathsf{Post}_{\varphi}(m, t) = \mathsf{Widen}_{\varphi}(\mathsf{Post}(m, t))$. Subscript φ may be omitted when it is clear from the context. Finally, the minimal coverability set of a widened Petri net $(\mathcal{N}, \varphi)$ is simply the set of its maximal reachable states as its reachability set is finite. It is denoted $\mathrm{MCS}(\mathcal{N}, \varphi)$.

We state the following result, whose proof easily follows by induction.

Proposition 2.1. Let $(\mathcal{N}, \varphi)$ be a WPN, and m be a reachable marking of $\mathcal{N}$. Then there exists an ω-marking m' reachable in $(\mathcal{N}, \varphi)$ such that $m \leq m'$.

Example 2.4. (Example 2.1 continued)
Consider the mapping φ associating 1 to places p_1, p_3, p_4 and p_6, and 3 to place p_5, and the widened Petri net $(\mathcal{N}, \varphi)$. Then for instance from marking $\{p_4, 3p_5\}$, the firing of t_4 results in the marking $\{p_3, \omega p_5\}$, instead of the marking $\{p_3, 4p_5\}$ in the standard semantics. Similarly, consider the prefix of the reachability tree of $\mathcal{N}$ depicted on Figure 1(b). For $(\mathcal{N}, \varphi)$, the prefix of the reachability tree is obtained by substituting the marking $\{p_3, 4p_5\}$ with the ω-marking $\{p_3, \omega p_5\}$, as we have $\varphi(p_5) = 3$. One can compute the MCS of this WPN. Due to the choice of φ, it coincides with the MCS of $\mathcal{N}$. ⌟

3. Monotone-Pruning Algorithm

3.1. Karp and Miller Algorithm.

The K&M Algorithm [8] is a well known solution to compute a coverability set of a PN. It is represented as Algorithm 1 (with a slight modification as in [8], the algorithm computes simultaneously all the successors of a marking). The K&M algorithm uses an external acceleration function $\mathsf{Acc} : 2^{\mathsf{Mark}^{\omega}(P)} \times \mathsf{Mark}^{\omega}(P) \to \mathsf{Mark}^{\omega}(P)$ which is defined as follows:

$$\forall p \in P, \mathsf{Acc}(M, m)(p) = \begin{cases} \omega & \text{if } \exists m' \in M \mid m' < m \wedge m'(p) < m(p) < \omega \\ m(p) & \text{otherwise.} \end{cases}$$

The K&M Algorithm builds a tree in which nodes are labelled by ω-markings and edges by transitions of the Petri net. Roughly, it consists in exploring the reachability tree of the PN, and in applying the acceleration function Acc on branches of this tree. Note that the acceleration may compute ω-markings that are not reachable. The correctness of this procedure relies on the strict monotonicity of the firing rule of PN and on the fact that the order $\leq$ on ω-markings is well-founded.

Theorem 3.1. ([8])
Let $\mathcal{N}$ be a PN. The K&M algorithm terminates and computes a coverability set of $\mathcal{N}$.

Algorithm 1 The K&M Algorithm

Require: A Petri net $\mathcal{N} = (P, T, I, O, m_0)$.
Ensure: A labelled tree $\mathcal{C} = (X, x_0, B, \Lambda)$ such that $\Lambda(X)$ is a coverability set of $\mathcal{N}$.
1: Let x_0 be a new node such that $\Lambda(x_0) = m_0$.
2: $X := \{x_0\}$; $\mathsf{Wait} := \{(x_0, t) \mid \Lambda(x_0) \stackrel{t}{\Rightarrow} \cdot\}$; $B := \emptyset$;
3: **while** $\mathsf{Wait} \neq \emptyset$ **do**
4: Pop (n', t) from Wait. $m := \mathsf{Post}(\Lambda(n'), t)$;
5: **if** $\nexists y \in \mathsf{Ancestor}_{\mathcal{C}}(n') \mid \Lambda(y) = m$ **then**
6: Let n be a new node s.t. $\Lambda(n) = \mathsf{Acc}(\Lambda(\mathsf{Ancestor}_{\mathcal{C}}(n')), m)$;
7: $X := X \cup \{n\}$; $B = B \cup \{(n', t, n)\}$; $\mathsf{Wait} := \mathsf{Wait} \cup \{(n, u) \mid \Lambda(n) \stackrel{u}{\Rightarrow} \cdot\}$;
8: **end if**
9: **end while**
10: Return $\mathcal{C} = (X, x_0, B, \Lambda)$.

3.2. Definition of the algorithm

The K&M Algorithm uses comparisons along the same branch to stop exploration (test of Line 5), that we call *vertical pruning*. We present in this section our algorithm which we call Monotone-Pruning Algorithm as it includes a kind of *horizontal pruning* in addition to the vertical one. We denote this algorithm by MP. It involves the acceleration function Acc used in the Karp and Miller algorithm. However, it is applied in a slightly different manner.

Algorithm 2 Monotone Pruning Algorithm for Petri Nets.

Require: A Petri net $\mathcal{N} = (P, T, I, O, m_0)$.
Ensure: A labelled tree $\mathcal{C} = (X, x_0, B, \Lambda)$ and a set $\mathsf{Act} \subseteq X$ such that $\Lambda(\mathsf{Act}) = \mathrm{MCS}(\mathcal{N})$.
1: Let x_0 be a new node such that $\Lambda(x_0) = m_0$;
2: $X := \{x_0\}$; $\mathsf{Act} := X$; $\mathsf{Wait} := \{(x_0, t) \mid \Lambda(x_0) \stackrel{t}{\Rightarrow} \cdot\}$; $B := \emptyset$;
3: **while** $\mathsf{Wait} \neq \emptyset$ **do**
4: Pop (n', t) from Wait.
5: **if** $n' \in \mathsf{Act}$ **then**
6: $m := \mathsf{Post}(\Lambda(n'), t)$;
7: Let n be a new node such that $\Lambda(n) = \mathsf{Acc}(\Lambda(\mathsf{Ancestor}_{\mathcal{C}}(n') \cap \mathsf{Act}), m)$;
8: $X := X \cup \{n\}$; $B := B \cup \{(n', t, n)\}$;
9: **if** $\Lambda(n) \not\leq \Lambda(\mathsf{Act})$ **then**
10: $\mathsf{Act} := \mathsf{Act} \setminus \{x \mid \exists y \in \mathsf{Ancestor}_{\mathcal{C}}(x)$ s.t. $\Lambda(y) \leq \Lambda(n) \wedge (y \in \mathsf{Act} \vee y \notin \mathsf{Ancestor}_{\mathcal{C}}(n))\}$;
11: $\mathsf{Act} := \mathsf{Act} \cup \{n\}$; $\mathsf{Wait} := \mathsf{Wait} \cup \{(n, u) \mid \Lambda(n) \stackrel{u}{\Rightarrow} \cdot\}$;
12: **end if**
13: **end if**
14: **end while**
15: Return $\mathcal{C} = (X, x_0, B, \Lambda)$ and Act.

As Karp and Miller Algorithm, the MP Algorithm builds a tree $\mathcal{C}$ in which nodes are labelled by ω-markings and edges by transitions of the Petri net. Therefore it proceeds in an exploration of the

reachability tree of the Petri net, and uses acceleration along branches to reach the "limit" markings. In addition, it can prune branches that are covered by nodes on other branches (what we call *horizontal* pruning). Therefore, nodes of the tree are partitioned into two subsets: active nodes, and inactive ones. Intuitively, active nodes will form the minimal coverability set of the Petri net, while inactive ones are kept to ensure completeness of the algorithm.

Given a pair (n', t) popped from Wait, the introduction in $\mathcal{C}$ of the new node obtained from (n', t) proceeds in the following steps:

1. node n' should be active (test of Line 5) ;
2. the "regular" successor marking is computed: $m = \mathsf{Post}(\Lambda(n'), t)$ (Line 6) ;
3. this marking is accelerated w.r.t. the *active ancestors* of node n', and a new node n is created with this marking: $\Lambda(n) = \mathsf{Acc}(\Lambda(\mathsf{Ancestor}_{\mathcal{C}}(n') \cap \mathsf{Act}), m)$ (Lines 7 and 8) ;
4. the new node n is declared as active if, and only if, it is not covered by an existing active node (test of Line 9 and Line 11) ;
5. update of Act: some nodes are "deactivated", i.e. removed from set Act (Line 10).

We detail the update of the set Act. Intuitively, one wants to deactivate nodes (and their descendants) that are covered by the new node n. This would lead to deactivate a node x iff it has an ancestor y dominated by n, i.e. such that $\Lambda(y) \leq \Lambda(n)$. This condition has to be refined to ensure the termination of the algorithm (see Remark 3.1). In MP Algorithm (see Line 10), node x is deactivated iff its ancestor y is either active ($y \in \mathsf{Act}$), or is not itself an ancestor of n ($y \notin \mathsf{Ancestor}_{\mathcal{C}}(n)$). In this case, we say that x *is deactivated by* n. This subtle condition constitutes the main difference between MP and MCT Algorithms (see Section 5).

Consider the introduction of a new node n obtained from $(n', t) \in \mathsf{Wait}$, and a node y such that $\Lambda(y) \leq \Lambda(n)$, y can be used to deactivate nodes in two ways:

- if $y \notin \mathsf{Ancestor}_{\mathcal{C}}(n)$, then no matter whether y is active or not, all its descendants are deactivated (represented in gray on Figure 2(a)),
- if $y \in \mathsf{Ancestor}_{\mathcal{C}}(n)$, then y must be active ($y \in \mathsf{Act}$), and in that case all its descendants are deactivated, except node n itself as it is added to Act at Line 11 (see Figure 2(b)).

Remark 3.1. Note that if one considers the simple condition $\Lambda(y) \leq \Lambda(n)$ to deactivate nodes, i.e. one considers all active and inactive nodes to discard nodes, then one looses the termination of the algorithm. Consider Example 3.1. With this condition, node n_9 covers node n_7 and thus deactivates node n_8 (this does not happen in MP as n_7 is an inactive ancestor of n_9). But then, node n_{10} covers n_8 and deactivates n_{10}, and so on.

The main result of the paper is that MP Algorithm terminates and is correct:

Theorem 3.2. Let $\mathcal{N}$ be a PN. The MP algorithm terminates and computes the minimal coverability set of $\mathcal{N}$.

(a) $y \notin \mathsf{Ancestor}_{\mathcal{C}}(n)$ (b) $y \in \mathsf{Ancestor}_{\mathcal{C}}(n) \cap \mathsf{Act}$

Figure 2. Deactivations of MP Algorithm.

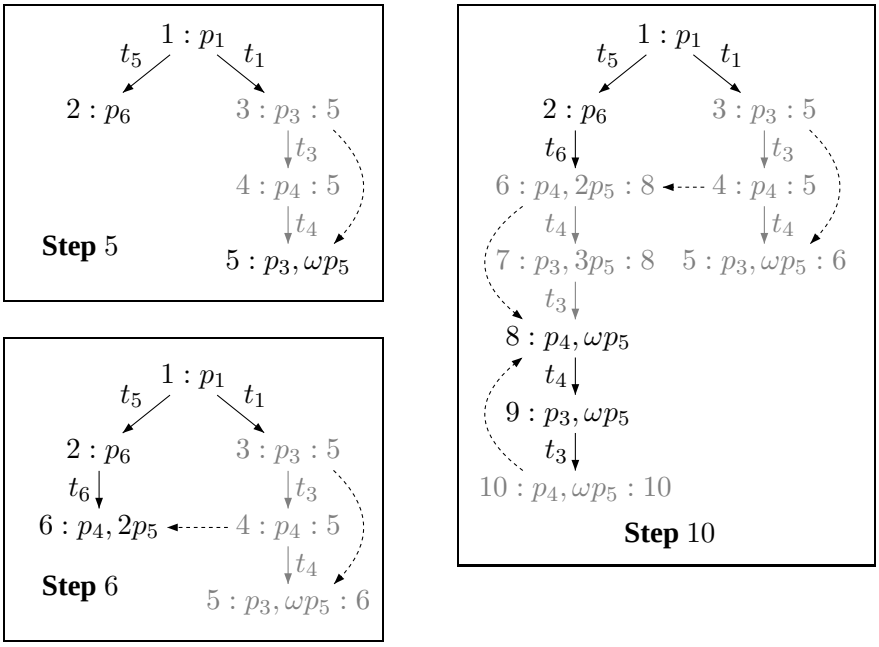


Figure 3. Snapshots of the execution of MP Algorithm on PN $\mathcal{N}$.

Example 3.1. (Example 2.1 continued)
We consider the execution of the MP Algorithm on the PN $\mathcal{N}$. Three intermediary steps (5, 6 and 10) are represented on Figure 3. The numbers written on the left (before the separator ":") of nodes indicate the order in which nodes are created. Nodes that are deactivated are represented in light gray, and dashed arrows indicate how nodes are deactivated. In addition, the step of the algorithm at which the node is deactivated is represented at the right (after the separator ":"). In the following explanations, node n_i denotes the node that has been created at step i:

- At step 5, the new node n_5 ($\{p_3, p_5\}$) covers node n_3 ($\{p_3\}$), which is thus deactivated, together with its descendants, except node n_5 that is just added.
- At step 6, the new node n_6 ($\{p_4, 2p_5\}$) covers node n_4 ($\{p_4\}$). This node was already deactivated but as it lies on another branch, it can be used to discard its descendants. As a consequence node n_5 is deactivated.
- At step 10, the new node n_{10} is covered by node n_8, which is still active. Thus n_{10} is immediately declared as inactive.

After step 10, MP terminates and the active nodes give the MCS of $\mathcal{N}$. ⌟

Remark 3.2. (MP Algorithm for widened Petri nets.)
In the sequel, we will consider the application of the MP Algorithm on widened Petri nets. Let $(\mathcal{N}, \varphi)$ be a WPN. The only difference is that the operator Post (resp. $\Rightarrow$) must be replaced by the operator Post_φ (resp. $\Rightarrow_\varphi$). For WPN, the MP Algorithm satisfies an additional property: we prove in Lemma 4.1 that all markings computed by MP are reachable in $(\mathcal{N}, \varphi)$. Indeed, the acceleration is consistent with the semantics of $(\mathcal{N}, \varphi)$, i.e. all markings computed by Acc belong to $\mathsf{Mark}^\omega_\varphi(P)$ (where P denotes the set of places of $\mathcal{N}$), provided the arguments of Acc do.

Example 3.2. (Example 2.4 continued)
Consider the WPN $(\mathcal{N}, \varphi)$ introduced in Example 2.4. Running MP on this WPN also yields the trees depicted on Figure 3.

3.3. Termination of the MP Algorithm

The proof of the termination of the MP Algorithm relies on two facts. First, only a finite number of accelerations can occur on each branch of the exploration tree built by MP. Second, a branch with no acceleration is finite. This last assertion is a consequence of the fact that $\mathsf{Mark}^\omega(P)$ equipped with partial order $\leq$ is a well-founded quasi-order, yielding the contradiction.

Theorem 3.3. The MP Algorithm terminates.

Proof:
We proceed by contradiction, and assume that the algorithm does not terminate. Let $\mathcal{C} = (X, x_0, B, \Lambda)$ and $\mathsf{Act} \subseteq X$ be the labelled tree and the set computed by MP. As $\mathcal{C}$ is of finite branching (bounded by $|T|$), there exists by König's lemma an infinite branch in this tree. We fix such an infinite branch, and write it $b = x_0 \xrightarrow{t_0} x_1 \xrightarrow{t_1} x_2 \ldots$, with $(x_i, t_i, x_{i+1}) \in B$, for all i.

Let us show that only a finite number of accelerations may occur on this branch b. By definition of the acceleration function Acc, two cases may occur when a new node n is built: either one of the active ancestors of n is strictly dominated, in that case at least one new ω will appear in the resulting marking ($\exists p \mid \Lambda(n)(p) = \omega > m(p)$), or no active ancestor is strictly dominated, and then the mapping Acc has no effect on marking m ($\Lambda(n) = m$). We say that in the first case, there is an "effective acceleration". By definition of the semantics of a Petri net on ω-markings, once a marking has value ω on a place p, so will all its successors. Thus, as there are finitely many places, a finite number of effective accelerations can occur on branch b.

We consider now the largest suffix of the branch b containing no effective accelerations: let i be the smallest positive integer such that for any $j \geq i$, we have $\Lambda(x_{j+1}) = \mathsf{Post}(\Lambda(x_j), t_j)$. We will prove that the set $S = \{x_j \mid j \geq i\}$ is an infinite set of active nodes with pairwise incomparable markings, which is impossible as the set $\mathsf{Mark}^\omega(P)$ equipped with partial order $\leq$ is a well-founded quasi-order, yielding the contradiction.

First note that if a node n deactivates a node x_j of the branch b then n is a node of b. Indeed, all the descendants of x_j are also deactivated except n if it is a descendant of x_j (line 10 of Algorithm 2). If n does not belong to b, this implies that for any $k \geq j$, x_k is deactivated. This is impossible because branch b is infinite and the algorithm only computes successors of active nodes (test of Line 5).

Let us show that all nodes $x_j, j \geq i$ are active (never deactivated). Indeed if x_j is deactivated by n then n is a node of b as shown in the previous paragraph. This implies that n dominates an active ancestor of x_j and therefore that an acceleration occurs. This is a contradiction with the definition of i.

Finally, the markings of nodes of S are all pairwise incomparable. Indeed, let x_j and x_k with $i \leq j < k$. If $\Lambda(x_j) < \Lambda(x_k)$ an acceleration occurs which is impossible by definition of i. Otherwise, if $\Lambda(x_j) \geq \Lambda(x_k)$, then the branch is stopped, which is also a contradiction. □

3.4. Correctness of the MP Algorithm

We reduce the correctness of the MP Algorithm for Petri nets to the correctness of this algorithm for widened Petri nets, which are finite state systems. This latter result is technical, and proved in the next section:

Theorem 3.4. The MP Algorithm for WPN terminates and computes the MCS.

We use this theorem to prove:

Theorem 3.5. The MP Algorithm for Petri nets is correct.

Proof:
Let $\mathcal{N} = (P, T, I, O, m_0)$ be a PN, $\mathcal{C} = (X, x_0, B, \Lambda)$ and $\mathsf{Act} \subseteq X$ be computed by the MP Algorithm on $\mathcal{N}$. As the MP Algorithm terminates, all these objects are finite. We will prove that $\Lambda(\mathsf{Act})$ is the minimal coverability set of $\mathcal{N}$.

First note that elements of $\Lambda(\mathsf{Act})$ are pairwise incomparable: this is a simple consequence of Lines 9, 10 and 11. Thus, we only have to prove that it is a coverability set.

The soundness of the construction, i.e. the fact that elements of $\Lambda(\mathsf{Act})$ satisfy item 2 of Definition 2.2, follows from the correctness of the acceleration function. To prove the completeness, i.e. item 1 of Definition 2.2, we use the correctness of the MP Algorithm on widened Petri nets. We can consider,

for each place $p \in P$, the largest value appearing in a marking during the computation. This defines a marking $\varphi \in \mathsf{Mark}(P)$.

We consider now the widened Petri net $(\mathcal{N}, \varphi)$ and the execution of the MP Algorithm on it (see Remark 3.2). We claim that there exists an execution of this algorithm which builds the same labelled tree $\mathcal{C}$ and the same partition. This execution is obtained by picking the same elements in the list Wait. This property can be proven by induction on the length of the execution of the algorithm. Indeed, by definition of marking φ, operators Post and Post_φ are equivalent on the markings computed by the algorithm. Thus, both algorithms perform exactly the same accelerations and compute the same ω-markings.

By correctness of the MP Algorithm on WPN (see Theorem 3.4), we obtain $\Lambda(\mathsf{Act}) = \mathrm{MCS}(\mathcal{N}, \varphi)$. By Proposition 2.1, any marking reachable in $\mathcal{N}$ is covered by a reachable marking of $(\mathcal{N}, \varphi)$, and thus by $\mathrm{MCS}(\mathcal{N}, \varphi) = \Lambda(\mathsf{Act})$. □

4. MP Algorithm for WPN

We devote this section to the proof that the MP Algorithm is correct on WPN.

4.1. Outline

The main difficulty is to prove the completeness of the set Act returned by MP. We have to show that any reachable marking is covered by an element of Act. In this section, we illustrate our approach using the WPN introduced in Example 2.4 and the execution of the MP algorithm on this WPN described in Example 3.2 and illustrated in Figure 3.

Given a reachable marking m, we consider a sequence of transitions ρ such that $m_0 \stackrel{\rho}{\Rightarrow} m$. For instance, we consider marking $\{p_3, p_5\}$ reachable from $\{p_1\}$ with the fireable sequence $\rho = t_1 t_3 t_4$. We want to find an active node n that covers m. We will try to follow this sequence of transitions ρ in the labelled tree built by MP Algorithm, and reach an active node. To describe this walk in the labelled tree, we consider a sequence of pairs (x_i, ρ_i) composed of an active node x_i of the tree, and a sequence of transitions ρ_i fireable from the node's label $\Lambda(x_i)$. The invariant of this walk is that ρ_i is fireable from $\Lambda(x_i)$ and $\mathsf{Post}(\Lambda(x_i), \rho_i)$ covers the marking m considered initially. This sequence of pairs is called the covering path and formally defined in Definition 4.5.

The first element of the covering path is the pair (n_1, ρ) composed of the root, and the sequence of transitions that we want to follow. In our running example, this first element is the pair $(n_1, t_1 t_3 t_4)$. Obviously, following this sequence may lead us to an inactive node. For instance, the successor of node n_1 by the transition t_1 is the inactive node n_3. We would thus reach the pair $(n_3, t_3 t_4)$, in which n_3 is inactive. To reach an active node, we use the explanation of the deactivation of node n_3. In this example, node n_3 has been covered by node n_5. We thus consider that we have to fire the remaining sequence $t_3 t_4$ from node n_5, we thus obtain the pair $(n_5, t_3 t_4)$. Again, node n_5 has been deactivated, because its ancestor n_4 has been covered by node n_6. We will thus "jump" to node n_6. However, n_6 does not cover n_5, but an ancestor of node n_5. Roughly, we thus have to include in the sequence of transitions that remain to be fired the path from n_4 to n_5. It may however not be sufficient to consider the sequence of transitions labeling this path, as it may hide some accelerations. This is precisely why our proof relies on the model of widened Petri nets: as ω corresponds to a finite value, we can represent the

accelerations explicitly by finite sequences of transitions. In this example, the new pair we consider is the pair $(n_6, t_4(t_3t_4)^4(t_3t_4))$. Again, as n_6 is inactive, it will be replaced by node n_8, and so on. Finally, the core of our proof consists in showing that the walks we define this way in the labelled tree are finite, i.e. end up with a pair of the form (n, ε). The result then trivially follows from the above invariant.

The rest of this section formalizes the above intuition. We introduce in Subsection 4.2 a notion of *exploration of a* WPN which corresponds to a tree built on the reachability tree of the WPN, with some additional properties. This structure allows to make explicit the effect of accelerations. We prove in Subsection 4.3 that MP indeed builds an exploration. In fact, other algorithms like K&M or MCT also do build explorations. Then, we prove that the exploration built by MP is complete in Subsection 4.4: we define the notion of covering path and prove that all covering paths are finite.

4.2. Exploration of a WPN

To build a coverability set the different algorithms we consider (K&M, MCT and MP) proceed in a similar way. Roughly, the algorithm starts with the root of the reachability tree and picks a fireable transition t. Then it picks a descendant that may either be the direct child by t (no acceleration) or a descendant obtained after skipping a few nodes (acceleration), this descendant must be strictly greater than the direct child (by t). Then if this node is not covered by the previously selected (and active) nodes, it can be used to prune other branches (not in the K&M algorithm): some active nodes are deactivated, intuitively because the subtree rooted at the new node should cover those nodes. The process continues with active nodes.

This process can be viewed as an exploration of the reachability tree $\mathcal{R} = (N, n_0, E, \Lambda)$ in the following sense. We define below an exploration as a tuple $\mathcal{E} = (X, B, \alpha, \beta)$, where X is the subset of N explored by the algorithm, B is an edge relation on X, such that $(x, t, x') \in B$ if x' is the node built by the algorithm when processing the transition t fireable from x. The function α gives the order in which nodes of X are explored by the algorithm. The function β gives the position at which a node is deactivated, *i.e.* $\beta(n) = i$ if n is deactivated (pruned) when the i-th node appears.

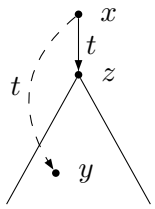


Figure 4. Condition $(v).(a)$ of Def. 4.1.

Definition 4.1. (Exploration)
Given a WPN $(\mathcal{N}, \varphi)$ and its reachability tree $\mathcal{R} = (N, n_0, E, \Lambda)$, an exploration of $\mathcal{R}$ is a tuple $\mathcal{E} = (X, B, \alpha, \beta)$ such that

- X is a finite subset of N,
- $B \subseteq X \times T \times X$,
- $n_0 \in X$,
- $(X, n_0, B, \Lambda_{|X})$ is a labelled tree,
- α is a bijection from X to $\{1, \ldots, |X|\}$, and
- β is a mapping from X to $\{1, \ldots, |X|\} \cup \{+\infty\}$.

For any $1 \leq i \leq |X|$, we define the sets $X_i = \{x \in X \mid \alpha(x) \leq i\}$, $\mathsf{Inact}_i = \{x \in X \mid \beta(x) \leq i\}$, and $\mathsf{Act}_i = X_i \setminus \mathsf{Inact}_i$. We let $\mathsf{Act} = \mathsf{Act}_{|X|}$ and $\mathsf{Inact} = \mathsf{Inact}_{|X|}$.

In addition, we require the following conditions:

(i) $\forall x \in X, \alpha(x) \leq \beta(x)$,

(ii) $\forall x, y \in X$, $x \in \mathsf{Ancestor}_{\mathcal{R}}(y)$ implies $\alpha(x) \leq \alpha(y)$,

(iii) $\forall (x,t,y) \in B, \alpha(y) \leq \beta(x)$,

(iv) T**-completeness:** $\forall x \in \mathsf{Act}, \forall t \in T$ s.t. $\Lambda(x) \stackrel{t}{\Rightarrow}_{\varphi} \cdot, \exists y \in X \mid (x,t,y) \in B$,

(v) $\forall (x,t,y) \in B$, there exists $z \in N$ such that:

(a) $(x,t,z) \in E$, and $z \in \mathsf{Ancestor}_{\mathcal{R}}(y)$,

(b) $\mathsf{Post}_{\varphi}(\Lambda(x),t) = \Lambda(z) \leq \Lambda(y)$.

The first condition states that nodes cannot be deactivated strictly before being selected. The second condition states that nodes are selected downward: one cannot select a node that has a descendant already selected. Condition (iii) states that the algorithm explores subtrees of active nodes only. Condition (iv) enforces that all fireable transitions of active nodes are explored. The last condition (see Figure 4, where the cone below node z denotes the descendants of z in $\mathcal{R}$) requires that the selected descendant is either the direct child by the selected transition t or a descendant of this child whose marking is greater than the marking of the child (acceleration). In the sequel, we denote by $\mathsf{Ancestor}_{\mathcal{E}}(\cdot)$ the ancestor relation considered in the labelled tree $(X, n_0, B, \Lambda_{|X})$. By definition, we have the following simple property: $\forall x \in X, \mathsf{Ancestor}_{\mathcal{E}}(x) = \mathsf{Ancestor}_{\mathcal{R}}(x) \cap X$.

It is easy to verify that sets Act_i and Inact_i form a partition of X_i ($X_i = \mathsf{Act}_i \uplus \mathsf{Inact}_i$) and that sets Inact_i are increasing ($\mathsf{Inact}_i \subseteq \mathsf{Inact}_{i+1}, \forall i < |X|$).

Remark 4.1. A trivial case of exploration is obtained when relation B coincides with the restriction of relation E to the set X. This case in fact corresponds to the exploration obtained by an algorithm that would perform no acceleration.

Remark 4.2. It can be proven that K&M and MCT applied on WPN do build explorations. Consider the K&M Algorithm. As it deactivates no node, it yields $\beta(n) = +\infty$ for any node n. However, it uses some accelerations and therefore some nodes are skipped but it respects condition (v).

4.3. MP-exploration of a WPN

Let $(\mathcal{N}, \varphi)$ be a WPN with $\mathcal{N} = (P, T, I, O, m_0)$, and $\mathcal{C} = (X, x_0, B, \Lambda)$, $\mathsf{Act} \subseteq X$ be the labelled tree and the set returned by the MP Algorithm. We define here the two mappings α and β that allow to show that the labelled tree $\mathcal{C}$ can be interpreted as an exploration in the sense of Definition 4.1.

Mapping α. It is simply defined as the order in which elements of X are built by the MP Algorithm.

Mapping β. Initially, the set Act contains the node x_0. Any new node n can be added only once in set Act, immediately when it is added in X (Line 11) (and can thus be removed from Act at most once). We define mapping β as follows:

- if a node x never enters set Act, then we let $\beta(x) = \alpha(x)$.
- if a node x enters set Act and is never removed, then we let $\beta(x) = +\infty$.
- finally, in the last case, let x be a node which enters set Act and is removed from it during the execution of the algorithm. Nodes can only be removed from set Act at Line 10. Then let n be the node added to X at Line 8 during this execution of the **while** loop, we define $\beta(x) = \alpha(n)$.

Remark 4.3. Using these definitions of mappings α and β, one can verify that intermediary values of sets X and Act computed by the algorithm coincide with sets X_i and Act_i defined in Definition 4.1.

Example 4.1. (Example 3.2 continued)
On Figure 3, numbers indicated on the left and on the right of nodes correspond to values of mappings α and β. When no number is written on the right, this means that the node is active, and then the value of β is $+\infty$.

Embedding of $\mathcal{C} = (X, x_0, B, \Lambda)$ in the reachability tree. In the labelled tree $\mathcal{C}$ built by the algorithm, the label of the new node n obtained from the pair (n', t) is computed by function Acc. To prove that $\mathcal{C}$ can be embedded in the reachability tree of $(\mathcal{N}, \varphi)$, we define a mapping called the concretization function which expresses the marking resulting from the acceleration as a marking reachable in $(\mathcal{N}, \varphi)$ from marking $\Lambda(n')$. Intuitively, an acceleration represents the repetition of some sequences of transitions until the upper bound is reached. As the system is finite (we consider widened Petri nets), we can exhibit a particular sequence of transitions which allows to reach this upper bound.

Definition 4.2. (Concretization function)
The concretization function is a mapping γ from B^* to T^*. Given a sequence of adjacent edges $b_1 \dots b_k \in B$, we define $\gamma(b_1 \dots b_k) = \gamma(b_1) \dots \gamma(b_k)$. We let $M = \max\{\varphi(p) \mid p \in P\} + 1$.

Let $b = (n', t, n) \in B$. The definition of γ proceeds by induction on $\alpha(n')$: we assume γ is defined on all edges $(x, u, y) \in B$ such that $\alpha(x) < \alpha(n')$.

Let $m = \mathsf{Post}_\varphi(\Lambda(n'), t)$, then there are two cases, either :

1. $\Lambda(n) = m$ (t is not accelerated), then we define $\gamma(b) = t$, or
2. $\Lambda(n) > m$. Let $X' = \{x_1, \dots, x_k\}$ (x_i's are ordered w.r.t. α) defined by:
   $X' = \{x \in \mathsf{Ancestor}_\mathcal{C}(n') \cap \mathsf{Act}_{\alpha(n)-1} \mid \Lambda(x) \leq m \wedge \exists p. \Lambda(x)(p) < m(p) < \omega\}$.
   For each $j \in \{1, \dots, k\}$, let $w_j = \mathsf{path}_\mathcal{C}(x_j, n') \in B^*$. Then we define:
   $\gamma(b) = t.(\gamma(w_1).t)^M \dots (\gamma(w_k).t)^M$.

The following Lemma states the expected property of the concretization function:

Lemma 4.1. Let $x, y \in X$ such that $x \in \mathsf{Ancestor}_\mathcal{C}(y)$, and let $w = \mathsf{path}_\mathcal{C}(x, y)$. Then we have $\mathsf{Post}_\varphi(\Lambda(x), \gamma(w)) = \Lambda(y)$.

Proof:
We prove the result for the case where w is a single edge, that is $w = (x, t, n) \in B$. The general result follows easily. We let $m = \mathsf{Post}_\varphi(\Lambda(x), t)$.

We distinguish two cases :
If $\Lambda(n) = m$**:** then by definition we have $\gamma(w) = t$ and the result is trivial.
If $\Lambda(n) > m$**:** then an acceleration has been applied. We prove the property by induction on $\alpha(x)$.

- if $\alpha(x) = 1$, then the acceleration is necessarily applied w.r.t. node x, that is $\mathsf{Post}_\varphi(\Lambda(x), t) > \Lambda(x)$. Let $add \in \mathsf{Mark}^\omega_\varphi(P)$ be defined by $add(p) = \mathsf{Post}_\varphi(\Lambda(x), t)(p) - \Lambda(x)(p)$ for any $p \in P$. Naturally, the vector is positive exactly for places p which have strictly increased, and which will thus be accelerated. For these places, after M iterations of t, the value in these places has exceeded the maximum value, *i.e.* value $\varphi(p)$, and thus is equal to ω. In the other places, the value is let unchanged. As a consequence, we exactly obtain $\mathsf{Post}_\varphi(\Lambda(x), t^M) = \Lambda(n)$.

- otherwise, we have $\alpha(x) > 1$. Following the definition of the concretization function, let $x_1, \ldots, x_k$ denote the nodes used to compute the concretized path $\gamma(w)$, and $w_1, \ldots, w_k$ the paths associated. Let $i \in \{1, \ldots, k\}$. By definition, $w_i = \mathsf{path}_\mathcal{C}(x_i, x) \in B^*$. In particular, any edge $b = (z, u, z') \in B$ composing this path is such that $\alpha(z) < \alpha(x)$. We can thus apply the induction hypothesis on path w_i, for any $i = 1 \ldots k$. Therefore we have $\mathsf{Post}_\varphi(\Lambda(x_i), \gamma(w_i)) = \Lambda(x)$ for any $i = 1 \ldots k$. By definition, we have $\gamma(w) = t.(\gamma(w_1).t)^M \ldots (\gamma(w_k).t)^M$. For $i \in \{0, \ldots, k\}$, let m_i^{acc} denote the marking reached from $\Lambda(x)$ by the sequence $t.(\gamma(w_1).t)^M \ldots (\gamma(w_i).t)^M$, *i.e.* such that $m_i^{acc} = \mathsf{Post}_\varphi(\Lambda(x), t.(\gamma(w_1).t)^M \ldots (\gamma(w_i).t)^M)$. We prove, by induction on $i \in \{0, \ldots, k\}$, the following property:

$$\forall p \in P, m_i^{acc}(p) = \begin{cases} \omega & \text{if } \exists 1 \leq j \leq i \text{ s.t. } \Lambda(x_j)(p) < m(p) < \omega \\ m(p) & \text{otherwise.} \end{cases}$$

  - For $i = 0$, the property is trivial by definition of m.
  - Let $i < k$, assume property holds for i and let prove it for $i + 1$. To prove the result we split the set of places P into three parts and successively prove that for each case the property is satisfied:
    - (i) P_1: $\exists 1 \leq j \leq i \mid \Lambda(x_j)(p) < m(p) < \omega$. Intuitively, P_1 represents places accelerated by one of the nodes $x_1, \ldots, x_i$. By the induction hypothesis, we have $m_i^{acc}(p) = \omega$, and we thus we will still have $m_{i+1}^{acc}(p) = \omega$, as expected.
    - (ii) P_2: $p \notin P_1 \wedge \Lambda(x_{i+1})(p) < m(p) < \omega$. Intuitively, P_2 denotes places not accelerated by one of $x_1, \ldots, x_i$, but that should be accelerated by x_{i+1}. By the induction hypothesis of the external induction, we have $\Lambda(x) = \mathsf{Post}_\varphi(\Lambda(x_{i+1}), \gamma(w_{i+1}))$. Thus we obtain $m = \mathsf{Post}_\varphi(\Lambda(x_{i+1}), \gamma(w_{i+1}).t)$. By definition of x_{i+1}, we have $\Lambda(x_{i+1}) \leq m$ and by induction hypothesis, we have $m \leq m_i^{acc}$. Thus sequence $\gamma(w_{i+1}).t$ can be iterated from marking m_i^{acc}. Let $p \in P_2$. By definition of P_2, we have $\Lambda(x_{i+1})(p) < m(p)$. This entails that the firing of sequence $\gamma(w_{i+1}).t$ adds tokens in place p. By the choice of M, the value $\varphi(p)$ will be exceeded and we obtain $m_{i+1}^{acc}(p) = \omega$ as expected.
    - (iii) P_3: $p \notin P_1 \wedge p \notin P_2$. This last case concerns places that should not be accelerated by any of the x_j's, with $j \leq i + 1$. Thus induction hypothesis entails that $m_i^{acc}(p) = m(p)$

and we have to prove that $m^{acc}_{i+1}(p) = m(p)$. As in the previous case, we have $m = \mathsf{Post}_{\varphi}(\Lambda(x_{i+1}), \gamma(w_{i+1}).t)$. Let $p \in P_3$, then we have $\Lambda(x_{i+1})(p) = m(p) = m^{acc}_i(p)$. Here, the iteration of the sequence $\gamma(w_{i+1}).t$ will let the value of place p unchanged. We thus obtain $m^{acc}_{i+1}(p) = m(p)$ as expected.

It is then trivial to verify that the application of this property for $i = k$ leads to the result.

This concludes the proof. □

This result allows to prove by induction that the labelled tree built by MP is, up to an isomorphism, included in the reachability tree of the WPN, and is thus an exploration:

Proposition 4.1. (MP-exploration)
The execution of the MP Algorithm on a WPN $(\mathcal{N}, \varphi)$ defines an exploration $\mathcal{E}$ of $(\mathcal{N}, \varphi)$. We call this exploration an MP-*exploration of* $(\mathcal{N}, \varphi)$.

Proof:
To embed $\mathcal{C}$ in $\mathcal{R}$, we define the mapping η from X to N which maps a node $x \in X$ to a node $n \in N$ that is labelled with the same marking. η is recursively defined by:

- $\eta(x_0) = n_0$,
- let $b = (x, t, y) \in B$, $n = \eta(x)$, and $\varrho = \gamma(b)$. By Lemma 4.1, we have $\Lambda(n) \stackrel{\varrho}{\Rightarrow} \cdot$. Thus there exists a unique node $n' \in N$ such that $n \in \mathsf{Ancestor}_{\mathcal{R}}(n')$ and $\mathsf{pathlabel}_{\mathcal{R}}(n, n') = \varrho$. We define $\eta(y) = n'$.

One can verify that using this definition, the $\mathcal{C}$-labelling of a node $x \in X$ coincides with the $\mathcal{R}$-labelling of the node $\eta(x) \in N$. As a consequence, we identify in the sequel node $x \in X$ with node $\eta(x) \in N$.

It is then routine to verify that properties (i) to (v) hold. □

4.4. Proof of Theorem 3.4

We prove in this section the:

Theorem 3.4. The MP Algorithm for WPN terminates and computes the MCS.

Termination of MP for WPN can be proved as in Theorem 3.3. As a consequence of Lemma 4.1, MP algorithm only computes markings that are reachable in the WPN, therefore the algorithm is sound. We devote the rest of this section to the proof of its completeness.

Fix a WPN $(\mathcal{N}, \varphi)$, with $\mathcal{N} = (P, T, I, O, m_0)$, and let $\mathcal{E} = (X, B, \alpha, \beta)$ be an MP-exploration of $(\mathcal{N}, \varphi)$. We will use notations X, Act and Inact of Definition 4.1.

4.4.1. Preliminary properties.

Given a node $n \in X$, we define the predicate $\mathsf{disc}(n)$ as $\beta(n) = \alpha(n)$. When this holds, we say that n is discarded as it is immediately added to the set Inact. In that case, no other node is deactivated.

Given two nodes $n, x \in X$ such that $\alpha(n) \leq \alpha(x)$ and $n \in \mathsf{Inact}$, we define the predicate $\mathsf{prune}(n, x)$ as $\exists y \in \mathsf{Ancestor}_{\mathcal{E}}(n).\Lambda(y) \leq \Lambda(x) \wedge (y \in \mathsf{Act}_{\beta(n)-1} \vee y \notin \mathsf{Ancestor}_{\mathcal{E}}(x))$.

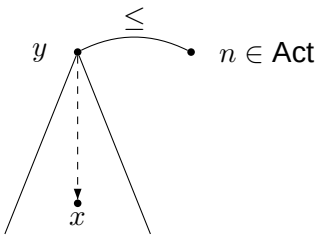


Figure 5. Illustration of Proposition 4.3.

One can check that the MP-exploration $\mathcal{E}$ satisfies the following properties. Arbitrary explorations do not satisfy them.

Proposition 4.2. Let $n \in \mathsf{Inact}$, then:

(i) $\mathsf{disc}(n) \iff \Lambda(n) \leq \mathsf{Act}_{\alpha(n)-1}$.

(ii) $\neg\mathsf{disc}(n)$ implies $\mathsf{prune}(n, x)$, where $x = \alpha^{-1}(\beta(n))$.

(iii) $\forall x \in X$ s.t. $\alpha(n) \leq \alpha(x)$, if $\mathsf{prune}(n, x) \wedge \neg\mathsf{disc}(x)$, then $\beta(n) \leq \alpha(x)$

Proposition 4.3. Let $\mathcal{E}$ an MP-exploration, and $i \in \{1, \ldots, |X|\}$. Let three distinct nodes $x, y, n \in X$ such that $n \in \mathsf{Act}$, $\Lambda(y) \leq \Lambda(n)$, $y \in \mathsf{Ancestor}_{\mathcal{E}}(x)$ and $y \notin \mathsf{Ancestor}_{\mathcal{E}}(n)$. Then we have $\beta(x) \leq \alpha(n)$.

Proof:
The property is illustrated on Figure 5. First note that $\alpha(n) < \alpha(y)$ cannot hold. Otherwise, as $n \in \mathsf{Act}$ and $\Lambda(y) \leq \Lambda(n)$, node y would have been immediately deactivated by node n (predicate $\mathsf{disc}(y)$ would hold). This is impossible as node y owns a descendant, node x. Thus, we have $\alpha(n) > \alpha(y)$ (the equality is impossible as y and n are distinct). Then, one can verify that the introduction of n will deactivate node x, if it has not yet been deactivated: node y is an ancestor of x, covered by n, and y is not an ancestor of n. We obtain $\beta(x) \leq \alpha(n)$. □

4.4.2. Covering Function.

We introduce a function Temp-Cover which explicits why nodes are deactivated. Intuitively, for a node $n \in \mathsf{Inact}$, if we have $\mathsf{Temp\text{-}Cover}(n) = (x, \varrho) \in X \times T^*$, this means that node x is in charge of deactivation of n, and that the firing of the sequence ϱ from $\Lambda(x)$ leads to a state dominating $\Lambda(n)$. Note that to identify the sequence in T^*, we use the path between nodes *in the reachability tree*. This is possible as by definition, the exploration is embedded in the reachability tree.

Definition 4.3. (Temp-Cover)
The mapping Temp-Cover is defined from Inact to $X \times T^*$ as follows. Let $n \in \mathsf{Inact}$, and $i = \beta(n)$. We distinguish two cases: [1]

[1] In the following definitions, any choice of x and y is correct.

Discarded: If $\mathsf{disc}(n)$, then by Proposition 4.2.(i), there exists a node $x \in \mathsf{Act}_{i-1}$ such that $\Lambda(n) \leq \Lambda(x)$. We fix such a node x and define $\mathsf{Temp\text{-}Cover}(n) = (x, \varepsilon)$.

Not discarded: Otherwise, $\neg\mathsf{disc}(n)$ holds. By Proposition 4.2.(ii), $\mathsf{prune}(n, x)$ holds, where $x = \alpha^{-1}(i)$. We fix a witness y of property $\mathsf{prune}(n, x)$, and let $\varrho = \mathsf{pathlabel}_{\mathcal{R}}(y, n) \in T^*$. We define $\mathsf{Temp\text{-}Cover}(n) = (x, \varrho)$.

The following property easily follows from Definition 4.3 and Lemma 2.1:

Lemma 4.2. Let $n \in \mathsf{Inact}$, $\mathsf{Temp\text{-}Cover}(n) = (x, \varrho)$. Then $\Lambda(n) \leq \mathsf{Post}_{\varphi}(\Lambda(x), \varrho)$.

The next proposition follows from the strategy of exploration of MP Algorithm.

Lemma 4.3. Let $n \in \mathsf{Inact}$, $\mathsf{Temp\text{-}Cover}(n) = (x, \varrho)$ and c such that $\mathsf{pathlabel}_{\mathcal{R}}(c, n) = \varrho$. Then we have:

(i) $\beta(n) < \beta(x)$;

(ii) $\varrho \neq \varepsilon$ implies $\beta(n) = \alpha(x)$;

(iii) if $\varrho \neq \varepsilon$, then $\forall y \in X, c \in \mathsf{Ancestor}_{\mathcal{E}}(y) \wedge x \notin \mathsf{Ancestor}_{\mathcal{E}}(y)$ implies $\beta(y) \leq \alpha(x)$.

Proof:
Property (i) is a consequence of the following property: we have $x \in \mathsf{Act}_{\beta(n)}$. Intuitively, this means that x is active when it deactivates node n.

Property (ii): by definition of $\mathsf{Temp\text{-}Cover}(n)$, $\varrho \neq \varepsilon$ implies that property $\neg\mathsf{disc}(n)$ holds. Then we obtain $x = \alpha^{-1}(\beta(n))$, as expected.

Last, consider property (iii). As for the previous property, by definition of $\mathsf{Temp\text{-}Cover}(n)$, $\varrho \neq \varepsilon$ implies that $\mathsf{prune}(n, x)$ holds and that c is a witness of the property $\mathsf{prune}(n, x)$. Then, by definition of the pruning of MP Algorithm, the whole subtree rooted in c is deactivated by node x, except node x itself if it belongs to the subtree. Let $y \in X$ such that $c \in \mathsf{Ancestor}_{\mathcal{E}}(y) \wedge x \notin \mathsf{Ancestor}_{\mathcal{E}}(y)$. Then y belongs to the subtree rooted in c, but not to the subtree rooted in x. As a consequence, it is deactivated by node x, and we obtain $\beta(y) \leq \alpha(x)$. □

The previous definition is temporary, in the sense that it describes how a node is deactivated. However, active nodes may be deactivated, and thus nodes referenced by mapping $\mathsf{Temp\text{-}Cover}$ may not belong to set Act. In order to recover an active node from which a dominating node can be obtained, we define a mapping which records for each inactivate node the successive covering informations:

Definition 4.4. (Covering function)
The covering function Cover is a mapping from X to $(X \times T^*)^*$. It is recursively defined as follows. Let $n \in X$.

1. if $n \in \mathsf{Act}$, then $\mathsf{Cover}(n) = \varepsilon$;
2. otherwise, let $\mathsf{Temp\text{-}Cover}(n) = (x, \varrho)$. We define $\mathsf{Cover}(n) = (x, \varrho) \cdot \mathsf{Cover}(x)$.

Example 4.2. (Example 3.2 continued)
We illustrate the definition of the covering function on Example 3.2. The MP Algorithm terminates at step 10. Consider node n_3, deactivated at step 5. We have $\mathsf{Temp\text{-}Cover}(n_3) = (n_5, \varepsilon)$. Indeed, it is directly covered by node n_5. Node n_5 is deactivated at step 6 by node n_6 through node n_4, which is its ancestor by transition t_4. Looking at the definition of the concretization function with respect to mapping φ, one can observe that we have $\mathsf{Temp\text{-}Cover}(n_5) = (n_6, t_4.(t_3t_4)^4)$. We indeed have that the maximal value of φ is 3. Node n_6 is deactivated at step 8 because it is directly covered by node n_8, thus we have $\mathsf{Temp\text{-}Cover}(n_6) = (n_8, \varepsilon)$. We finally obtain $\mathsf{Cover}(n_3) = (n_5, \varepsilon) \cdot (n_6, t_4.(t_3t_4)^4) \cdot (n_8, \varepsilon)$. One can verify that $\Lambda(n_3) \leq \mathsf{Post}_\varphi(\Lambda(n_8), t_4.(t_3t_4)^4)$. ⌟

We state the next property which follows from Lemma 4.2 by induction:

Lemma 4.4. Let $n \in \mathsf{Inact}$ be such that $\mathsf{Cover}(n) = (x_1, \varrho_1) \cdots (x_k, \varrho_k)$. Then we have $\Lambda(n) \leq \mathsf{Post}_\varphi(\Lambda(x_k), \varrho_k\varrho_{k-1} \dots \varrho_1)$.

We now state a core property of mapping Cover, holding for MP-explorations. It is fundamental to prove the absence of cycles, and thus the fact that the exploration yields a minimal coverability set. Roughly, it states that intermediary markings skipped by accelerations would not modify activations/deactivations:

Proposition 4.4. Let $x \in \mathsf{Inact}$ be such that $\mathsf{Cover}(x) = (x_1, \varrho_1) \cdots (x_k, \varrho_k)$. Define $\varrho = \varrho_k\varrho_{k-1} \dots \varrho_1$, and let $n \in \mathsf{Act}$ and $\varrho' \in T^*$. Then we have:

$$(\varrho' \prec \varrho \wedge \Lambda(n) \geq \mathsf{Post}_\varphi(\Lambda(x_k), \varrho')) \text{ implies } \beta(x) \leq \alpha(n)$$

Proof:
As $\varrho' \prec \varrho$, there exists a unique index j such that $1 \leq j \leq k$ and $\varrho' = \varrho_k\varrho_{k-1} \dots \varrho_{j+1}\varrho''$ with $\varepsilon \preceq \varrho'' \prec \varrho_j$. In particular, this yields that $\varrho_j \neq \varepsilon$.

By definition of Cover, we have that for any $\ell \in \{1, \dots, k\}$, $(x_\ell, \varrho_\ell) = \mathsf{Temp\text{-}Cover}(x_{\ell-1})$ (where we let $x_0 = x$). By Lemma 4.3. (i), this implies $\beta(x_{\ell-1}) < \beta(x_\ell)$. We thus obtain the inequality $\beta(x) \leq \beta(x_{j-1})$, as $x_0 = x$ (the inequality is non strict as we may have $j = 1$).

Consider the peculiar case $x_j \in \mathsf{Ancestor}_\mathcal{E}(n)$. This implies $\alpha(x_j) \leq \alpha(n)$. As $\varrho_j \neq \varepsilon$, we have by Lemma 4.3. (ii) the equality $\beta(x_{j-1}) = \alpha(x_j)$, which yields the result. In the sequel we thus assume $x_j \notin \mathsf{Ancestor}_\mathcal{E}(n)$.

Node x_{j-1} is deactivated by node x_j, and with sequence ϱ_j. This means the ancestor of node x_{j-1} by the sequence ϱ_j in the reachability tree, which we denote by c, belongs to X and is covered by x_j. As $\varrho'' \prec \varrho_j$, we can consider the successor of c by the sequence ϱ'' (in the reachability tree), and denote this node by y, which is thus a (strict) ancestor of $x_{j-1} \in X$. We now distinguish two cases: either $y \in X$ or $y \notin X$. Consider the second case: node y lies in between nodes c and x_{j-1} and as it does not belong to X, it is "skipped" by an acceleration. We denote by $(y_1, t, y_2) \in B$ the edge of the exploration that skips the node y. By the minimal-completeness property (see Appendix A) of the exploration $\mathcal{E}$ applied on edge (y_1, t, y_2) and node y, there exist two nodes z and y' in X verifying the following properties:

(i) $z \in \mathsf{Ancestor}_\mathcal{E}(y_1)$ and $\beta(z) = \alpha(y_2)$,

(ii) $y' \in \mathsf{Ancestor}_\mathcal{R}(y) \cap X$, $\Lambda(y') < \Lambda(y)$ and $z \in \mathsf{Ancestor}_\mathcal{E}(y')$.

We will prove that in the first case ($y \in X$), and respectively in the second one ($y \notin X$), we can apply Proposition 4.3 to nodes x_{j-1}, y and n (resp. x_{j-1}, y' and n in the second case). Therefore, we prove each of the hypotheses:

- First, properties $n \in \mathsf{Act}$ and $y \in \mathsf{Ancestor}_{\mathcal{E}}(x_{j-1})$ (resp. $y' \in \mathsf{Ancestor}_{\mathcal{E}}(x_{j-1})$) are trivial. In addition, we obviously have $x_{j-1} \neq y$ (resp. $x_{j-1} \neq y'$) as y is a strict ancestor of x_{j-1} (and y' is itself a strict ancestor of y). We also have $x_{j-1} \neq n$ as $n \in \mathsf{Act}$ while x_{j-1} is deactivated by x_j.

- Second, we prove that $\Lambda(n) \geq \Lambda(y)$ (resp. $\Lambda(n) \geq \Lambda(y')$ in the second case). Indeed, we can prove using Lemma 4.2 that $\mathsf{Post}_{\varphi}(\Lambda(x_k), \varrho_k \varrho_{k-1} \dots \varrho_{j+1}) \geq \Lambda(x_j)$. By definition of c, we have $\Lambda(x_j) \geq \Lambda(c)$. By definition of y, this yields $\mathsf{Post}_{\varphi}(\Lambda(x_k), \varrho) \geq \Lambda(y)$, and thus $\Lambda(n) \geq \Lambda(y)$. In the second case, the property follows from $\Lambda(y) > \Lambda(y')$.

- Third, we prove that $y \notin \mathsf{Ancestor}_{\mathcal{E}}(n)$ (resp. $y' \notin \mathsf{Ancestor}_{\mathcal{E}}(n)$), which also entails $y \neq n$ (resp. $y' \neq n$). Consider the first case and proceed by contradiction: assume that y is an ancestor of node n. This implies that $c \in \mathsf{Ancestor}_{\mathcal{E}}(n)$, and then by Lemma 4.3.$(iii)$, as $x_j \notin \mathsf{Ancestor}_{\mathcal{E}}(n)$ and $\varrho_j \neq \varepsilon$, we obtain $\beta(n) \leq \alpha(x_j)$ which is impossible as $n \in \mathsf{Act}$.

  Consider now the second case and proceed by contradiction: assume that $y' \in \mathsf{Ancestor}_{\mathcal{E}}(n)$. Then y_2 is necessarily an ancestor of n, otherwise n is deactivated at step $\alpha(y_2)$ (see Lemma 4.3.(iii)). But then we can apply a reasoning similar to that of the first case and prove that n is deactivated by node x_j, what yields a contradiction.

Finally, we obtain by Proposition 4.3 the inequality $\beta(x_{j-1}) \leq \alpha(n)$. Combined with a previous inequality, this entails $\beta(x) \leq \alpha(n)$ as expected. □

4.4.3. Covering Path.

Before turning to the proof of Theorem 3.4, we introduce an additional definition. Our aim is to prove that any reachable state s is covered by some active node. Therefore we define a notion of covering path, which is intuitively a path through active nodes in which each node is labelled with a sequence (a stack) of transitions that remain to be fired to reach a state s' dominating the desired state s. Formally, a covering path is defined as follows:

Definition 4.5. (Covering Path)
A *covering path* is a sequence $(n_i, \varrho_i)_{i \geq 1} \in (\mathsf{Act} \times T^*)^{\mathbb{N}}$ such that $\Lambda(n_1) \overset{\varrho_1}{\Rightarrow}_{\varphi} \cdot$ and for any $i \geq 1$, we have either

(i) $\varrho_i = \varepsilon$, and then it has no successor, or

(ii) $\varrho_i = t_i \varrho_i'$, then let n be such that $(n_i, t_i, n) \in B$ (possible as $\mathcal{E}$ is T-complete). If $n \in \mathsf{Act}$ then $(n_{i+1}, \varrho_{i+1}) = (n, \varrho_i')$. Otherwise, let $\mathsf{Cover}(n) = (x_1, \eta_1) \cdots (x_k, \eta_k)$, we define $(n_{i+1}, \varrho_{i+1}) = (x_k, \eta_k \dots \eta_1 \cdot \varrho_i')$.

Note that given a node $n \in \mathsf{Act}$ and $\varrho \in T^*$ such that $\Lambda(n) \overset{\varrho}{\Rightarrow}_{\varphi} \cdot$, there exists a unique covering path $(n_i, \varrho_i)_{i \geq 1}$ such that $(n_1, \varrho_1) = (n, \varrho)$. We say that this path is *associated with the pair* (n, ϱ).

Example 4.3. (Example 4.2 continued)
We illustrate the definition of covering path on Example 3.2. Consider the covering path associated with the pair $(n_1, t_1t_3t_4)$. Successor of node n_1 by transition t_1 is the inactive node n_3. We have already shown in Example 4.2 that $\mathsf{Cover}(n_3) = (n_5, \varepsilon) \cdot (n_6, t_4.(t_3t_4)^4) \cdot (n_8, \varepsilon)$. In addition, successor of node n_8 by transition t_4 is the active node n_9, and as the successor of node n_9, *i.e.* node n_{10}, is covered by node n_8, firing t_3 in n_9 leads to node n_8. Finally, one can verify that the covering path is of the following form: $(n_1, t_1t_3t_4), (n_8, t_4(t_3t_4)^4t_3t_4), (n_9, (t_3t_4)^4t_3t_4), (n_8, t_4(t_3t_4)^3t_3t_4) \dots (n_9, \varepsilon)$. Note that the marking $\{p_3, p_5\}$ reached from node n_1 by the sequence $t_1t_3t_4$ is covered by the marking $\{p_3, \omega p_5\}$ of node n_9. ⌟

Lemma 4.5. Let $(n_i, \varrho_i)_{i\geq 1}$ be a covering path. Then we have $\mathsf{Post}_\varphi(\Lambda(n_1), \varrho_1) \leq \mathsf{Post}_\varphi(\Lambda(n_i), \varrho_i)$ for all i. In particular, if for some i we have $\varrho_i = \varepsilon$, we obtain $\mathsf{Post}_\varphi(\Lambda(n_1), \varrho_1) \leq \Lambda(n_i)$.

Proof:
We prove that for any i, we have $\mathsf{Post}_\varphi(\Lambda(n_i), \varrho_i) \leq \mathsf{Post}_\varphi(\Lambda(n_{i+1}), \varrho_{i+1})$. In the definition of covering path, we extend the path only in case (ii). Two cases can occur, in the first one, the property is trivial. In the second one, the property follows from Lemma 4.4. □

As a consequence of this lemma, to prove the completeness result, it is sufficient to show that for any reachable marking, there exists a finite covering path that covers it. We introduce a notion of cycle for covering paths:

Definition 4.6. Let $(n, t) \in \mathsf{Act} \times T$ such that $\Lambda(n) \stackrel{t}{\Rightarrow}_\varphi \cdot$, and $(n_i, \varrho_i)_{i\geq 1}$ be the covering path associated with (n, t). The pair (n, t) is said to be *singular* if there exists $i > 1$ such that $(n_i, \varrho_i) = (n, t\varrho)$, with $\varrho \in T^*$.

The following lemma states that all infinite covering paths are periodic:

Lemma 4.6. Let $(n, \varrho) \in \mathsf{Act} \times T^*$ such that $\Lambda(n) \stackrel{\varrho}{\Rightarrow} \cdot$, and $p = (n_i, \varrho_i)_{i\geq 1}$ be the covering path s.t. $(n_1, t_1) = (n, \varrho)$. If p is infinite, then there exists a position $i \geq 1$ such that the pair (n_i, t_i) is singular, where $\varrho_i = t_i\varrho'_i$.

Proof:
We distinguish two cases:

- If there exists a bound $k \in \mathbb{N}^*$ such that infinitely often, the length of the sequence ϱ_i is smaller than k. Then, as the number of sequences of length bounded by k and the number of active nodes are finite, there exist two positions $1 \leq j < l$ such that $(n_j, \varrho_j) = (n_l, \varrho_l)$. Let i be an index in the interval $[j, l]$ such that the length of the sequence ϱ_i is minimal. This implies that the construction of the coverability path from (n_i, ϱ_i) only depends on the first transition t_i of ϱ_i: the pair (n_i, t_i) is singular.

- Otherwise, for any bound k, there exists a position after which all sequences of transitions have a length larger than k. For each k, we note $l(k)$ the first position verifying this property: $\forall l \geq l(k), |\varrho_l| \geq k$. Note that the sequence starting at $l(k)$ only depends on the node $n_{l(k)}$ and the transition $t_{l(k)}$ such that $\varrho_{l(k)} = t_{l(k)}\varrho'_{l(k)}$. As the number of active nodes and the set of transitions

are finite, there exist $k < k'$ such that $l(k) < l(k')$, $n_{l(k)} = n_{l(k')}$ and $t_{l(k)} = t_{l(k)}$. Then the pair $(n_{l(k)}, t_{l(k)})$ is singular.

□

4.4.4. Proof of Theorem 3.4.

We will prove that any reachable marking of $(\mathcal{N}, \varphi)$ is covered by some active node. Let $m \in \mathsf{Mark}^{\omega}_{\varphi}(P)$ be a reachable marking. There exists $\varrho \in T^*$ such that $m_0 \overset{\varrho}{\Rightarrow}_{\varphi} m$. One can prove that there exists a node $n'_0 \in \mathsf{Act}$ such that $\Lambda(n_0) \leq \Lambda(n'_0)$ (n'_0 covers the root):

Lemma 4.7. Let $\mathcal{E}$ be an MP-exploration with root n_0. Then there exists a node $n'_0 \in \mathsf{Act}$ such that $\Lambda(n_0) \leq \Lambda(n'_0)$.

Proof:
We prove that the property holds for the set Act_i all $i \leq |X|$. For all i such that $n_0 \in \mathsf{Act}_i$, there is nothing to prove. Consider, if it exists, the smallest i such that $n_0 \in \mathsf{Act}_i$ and $n_0 \notin \mathsf{Act}_{i+1}$. Let $n = \alpha^{-1}(i+1)$. Following definition of Act_{i+1}, as n_0 is the root of the tree, we must have $\Lambda(n_0) < \Lambda(n)$. As i has been chosen to be minimal, n can not be covered by another node, and thus $n \in \mathsf{Act}_{i+1}$. As a consequence, the property is true at step $i+1$. Moreover, note that as n_0 is the root of the tree, n is necessarily a descendant of n_0. As a consequence, the definition of active and inactive nodes yields that the only remaining active node after step $i+1$ is the node n. We thus have $\mathsf{Act}_{i+1} = \{n\}$. In other terms, this means that from this step, the exploration will start from a new root, and thus by the all new (active or node) nodes are descendant of n. Then, we can inductively apply the same reasoning to node n, and conclude by the transitivity of relation $\leq$. □

As a consequence, there exists $m' \in \mathsf{Mark}^{\omega}_{\varphi}(P)$ such that $\Lambda(n'_0) \overset{\varrho}{\Rightarrow}_{\varphi} m'$ and $m \leq m'$. We can then consider the covering path associated with the pair (n'_0, ϱ).

We now prove that all covering paths are finite. This will conclude the proof by Lemma 4.5. By Lemma 4.6, if a covering path is infinite, then it contains a singular pair. Therefore we prove that the MP-exploration $\mathcal{E}$ cannot admit a singular pair. Consider a singular pair $(n, t) \in \mathsf{Act} \times T$, and denote by $(n_i, \sigma_i)_{i \geq 1}$ its infinite covering path. As it is singular, there exists $k > 1$ such that $(n_k, \sigma_k) = (n, t\sigma'_k)$. For any $1 \leq i \leq k$, we write $\sigma_i = t_i \sigma'_i$ (this is possible as the path is infinite and thus never contains empty stacks).

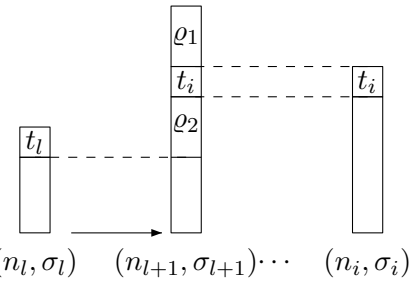


Figure 6. Stacks of a singular pair.

For each $1 < i \leq k$, we define the position $\mathsf{prod}(i) = \max\{1 \leq j < i \mid |\sigma_j| \leq |\sigma_i|\}$. This definition is correct as $|\sigma_1| = |t| = 1$, and for any $1 < i \leq k$, we have $|\sigma_i| \geq 1$ as $\sigma_i \neq \varepsilon$. Intuitively, the value $\mathsf{prod}(i)$ gives the position which is responsible of the addition in the stack of transition t_i. Indeed, let $1 < i \leq k$ and $l = \mathsf{prod}(i)$. As for any position j such that $l < j < i$, we have $|\sigma_j| > |\sigma_i|$, the transition t_i is present in σ_j "at the same height".

Consider now the position $1 < i \leq k$ such that $\alpha(n_i)$ is minimal among $\{\alpha(n_j) \mid 1 \leq j \leq k\}$ (recall that $n_1 = n_k$), and let $l = \mathsf{prod}(i)$. By the T-completeness of $\mathcal{E}$, there exists a node $x \in X$ such that edge (n_l, t_l, x) belongs to B. As we have $|\sigma_l| \leq |\sigma_{l+1}|$, this implies that $x \in \mathsf{Inact}$.

We write the covering function associated with node x as follows: $\mathsf{Cover}(x) = (x_1, \eta_1) \dots (x_k, \eta_k)$. Following the definition of a covering path, we obtain $x_k = n_{l+1}$. In addition, following the above mentioned property of $l = \mathsf{prod}(i)$, there exist two sequences $\varrho_1, \varrho_2 \in T^*$ such that $\eta_k \dots \eta_1 = \varrho_1 t_i \varrho_2$, and verifying:

$$\sigma_{l+1} = \varrho_1 t_i \varrho_2 \sigma'_l \text{ and } \sigma_i = t_i \varrho_2 \sigma'_l$$

This means that the head of the stack σ_l, *i.e.* transition t_l, has been replaced by the sequence $\varrho_1 t_i \varrho_2$, and that between positions $l+1$ and i, transition t_i (which is the head of the stack σ_i), is never consumed. This situation is depicted on Figure 6. In particular, this implies the following property:

$$\Lambda(n_i) \geq \mathsf{Post}_\varphi(\Lambda(n_{l+1}), \varrho_1)$$

Indeed, there are two cases, either $i = l+1$, and then we necessarily have $\varrho_1 = \varepsilon$ and the property is trivial, or $l+1 < i$, and then we have that the covering path starting in pair (n_{l+1}, ϱ_1) ends in pair (n_i, ε). The result then follows from Lemma 4.5.

To conclude, we use the key Proposition 4.4. Indeed, one can verify that the proposition can be applied on nodes x and n_i using sequences $\varrho = \varrho_1 t_i \varrho_2$ and $\varrho' = \varrho_1$. This result yields the following inequality: $\beta(x) \leq \alpha(n_i)$. As x is the successor of n_l by transition t_l, property (ii) of an exploration implies $\alpha(n_l) < \alpha(x)$. As we always have $\alpha(x) \leq \beta(x)$, we finally obtain $\alpha(n_l) < \alpha(n_i)$, which is a contradiction with our choice of i.

5. Comparison with the MCT Algorithm

The MP Algorithm has been proposed as a modification of the MCT Algorithm to obtain completeness, using a slightly different pruning strategy. Apart from minor structural differences in the presentation of the algorithm, the main difference comes from the deactivation of nodes. In MP, inactive nodes can be used to deactivate nodes. In MCT, only active nodes are used to deactivate nodes. More precisely, the pruning strategy of the MCT Algorithm is obtained by replacing Line 10 by the following line :

$$10' : \qquad \mathsf{Act} := \mathsf{Act} \setminus \{x \mid \exists y \in \mathsf{Ancestor}_{\mathcal{C}}(x) \text{ s.t. } \Lambda(y) \leq \Lambda(n) \wedge y \in \mathsf{Act}\};$$

Thus, condition $(y \in \mathsf{Act} \vee y \notin \mathsf{Ancestor}_{\mathcal{C}}(n))$ in MP is replaced by the stronger condition $y \in \mathsf{Act}$ to obtain MCT. In particular, this shows that MP pruning strategy is more aggressive than MCT one, i.e. for a given partial exploration and a given new node, MP always deactivates nodes whenever MCT does, but the reverse is not always true. This last observation does not imply that the exploration tree of MP is always smaller than MCT's. Indeed, one can build an example where a node is deactivated in MP but not in MCT, this node is then used in MCT to deactivate a new node m (successors of m are not explored in MCT), while successors of m are explored in MP.

While the exploration trees of MP and of MCT are in general incomparable (none is included into the other), we proved that the MP Algorithm is complete. The incompleteness of the MCT Algorithm is illustrated with the Petri net first published in [7] and depicted in Figure 7. Let us show that MCT is already incomplete for WPN. Consider the WPN $(\mathcal{N}_{cex}, \varphi)$ where $\varphi = \{p_1, p_2, p_3, p_4, 3p_5, p_6, p_7\}$ (see Definition 2.4). We represent in Figure 8 an execution of the MCT Algorithm on this WPN [2]. The set

[2] Actually running MCT on the Petri net $\mathcal{N}_{cex}$ yields the exact same tree.

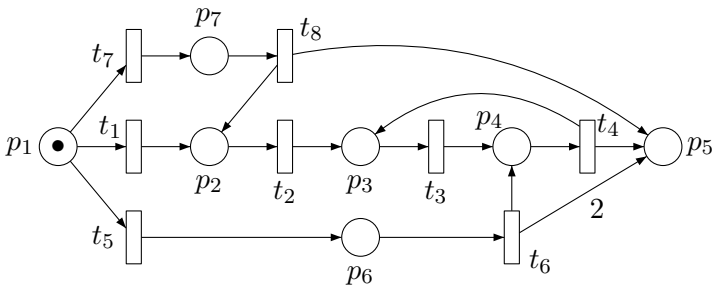


Figure 7. The counter-example of [7]: Petri net $\mathcal{N}_{cex}$.

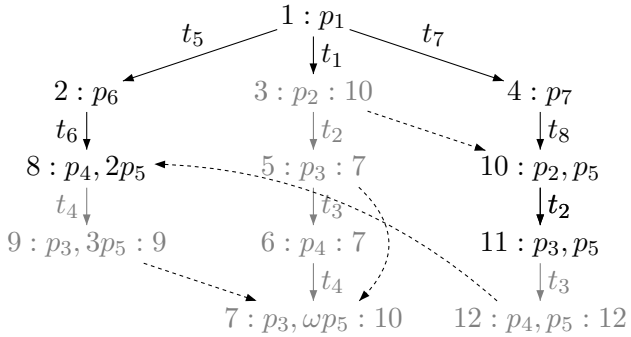


Figure 8. An execution of the MCT Algorithm on $\mathcal{N}_{cex}$.

returned by MCT is $\{\{p_1\}, \{p_7\}, \{p_2, p_5\}, \{p_3, p_5\}, \{p_6\}, \{p_4, 2p_5\}\}$, this set is incomplete as it should include $\{p_3, \omega p_5\}$ and $\{p_4, \omega p_5\}$. The problem arises from the following fact: successors of node n_8 (node created at step 8) are not explored on the hypothesis that they will be covered by node n_7 and its successors; node n_7 and its successors are not explored on the hypothesis that they will be covered by node n_{10} and its successors; but node n_{12}, a successor of node n_{10}, is not explored on the hypothesis that it will be covered by node n_8 and its successors. This cycle in the hypothesis leads to the incompleteness.

The marking $\{p_3, 3p_5\}$ is reachable for instance by the sequence $t_5t_6t_4$, it is not covered by the set returned by MCT in Figure 8. The covering path associated with the root node n_1 and the sequence of transitions $t_5t_6t_4$ is:

$$(n_1, t_5t_6t_4) \cdot (n_2, t_6t_4) \cdot (n_8, t_4) \cdot (n_{10}, t_2(t_3t_4)^4) \cdot (n_{11}, (t_3t_4)^4) \cdot (n_8, t_4(t_3t_4)^3) \cdots$$

note that $(n_{10}, t_2(t_3t_4)^4)$ comes from the "unfolding" of the acceleration in the WPN (Definition 4.2), this covering path is infinite (a loop starts with (n_8, t_4)). One can check that the pair (n_8, t_4) is singular (Definition 4.6). The crux of our proof of the completeness of MP for WPN is to show that no singular pair occurs in the exploration trees built by MP.

Figure 9 illustrates an execution of the MP Algorithm on $\mathcal{N}_{cex}$. The difference with the execution of the MCT Algorithm occurs at step 8: node n_7 is deactivated because node n_6 is covered by node n_8. This deactivation does not happen in MCT because node n_6 is inactive. This exactly corresponds to the difference between the two algorithms.

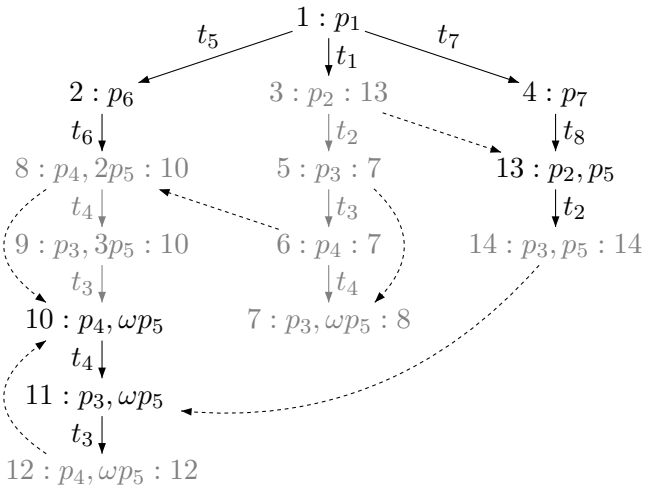


Figure 9. An execution of the MP Algorithm on $\mathcal{N}_{cex}$.

6. Implementation and Experiments

We implemented the MP algorithm in Python [3]. In order to minimize the overhead of the inactive nodes, our implementation takes advantage of the following observation. One of the following two cases occurs when a node x, different from the new node n, is deactivated:

- if the new node n is not a descendant of x, then x and its subtree are completely removed (indeed the algorithm will not need them anymore);
- if the new node n is a descendant of x an acceleration has occurred between n and an ancestor y of x; the node x could be used later to deactivate one of the descendants of node n; in our implementation x is removed and the set of markings of the deactivated nodes lying between y and n (including x) is associated with n; nevertheless, note that we only need to keep the minimal elements of this set.

Therefore our implementation maintains a tree data structure containing only the active nodes and for each active node n it maintains the set of the minimal markings of the inactive ancestors of n (up to the first active ancestor y of n).

We compare the MP Algorithm with the K&M Algorithm (also available in our implementation) and with the procedure CoverProc introduced in [7]. This latter procedure is an alternative for the computation of the MCS. Instead of using a tree structure as in the K&M Algorithm, it computes a set of pairs of markings, with the meaning that the second marking can be reached from the first one. This is sufficient to apply acceleration. To improve the efficiency, only maximal pairs are stored.

Experimental results, obtained on a 3 Ghz Xeon computer, are presented in Table 1. The test set is the one from [7]. We recall in the last column the values obtained for the CoverProc [7] algorithm. Note that the implementation of [7] also was in Python, and the tests were run on the same computer. We

[3] Our prototype can be downloaded at `http://www.lif.univ-mrs.fr/~preynier/coverability/`.

report for each test the number of places and transitions of the net and the size of its MCS, the time the K&M and MP Algorithms took and the number of elements passed in the waiting list.

As expected the MP Algorithm is a lot faster than the K&M Algorithm and the tree it constructs is, in some instances, dramatically smaller. The K&M Algorithm could not compute the MCS for the last five tests (we time out after 20 minutes), while the MP Algorithm took less than 20 seconds for all five tests. Note that the time reported for the K&M Algorithm is the time to build the K&M tree, from this tree one has to extract the minimal coverability set which maybe costly if the set is big (see K&M results in [7]). The MP Algorithm directly computes the minimal coverability set (Act), *i.e.* no additional computation is needed.

We also compare two versions of the MP Algorithm, the Depth-First (DFS) and Breadth-First (BFS) ones. To be more precise, we give the number of nodes of the tree built by the algorithm (# X), and among them how many are deactivated (# Inact). Note that we recover that the number of nodes of the tree that remain active coincides with the size of the MCS. In all the instances we considered, the BFS is slightly faster than the BFS. This illustrates the interest of having an algorithm correct for any exploration strategy.

Regarding CoverProc, the procedure is significantly slower than MP. This can be explained by the fact that considering pairs of markings may increase the number of elements to be stored. Moreover, the MP Algorithm has, in our view, another important advantage over CoverProc. In MP, the order of exploration is totally free (any exploration strategy yields the MCS) while, in the CoverProc procedure, each time an acceleration is applied, the successors of the resulting node must be explored in a depth-first search. This is particularly relevant as we have seen that the MP Algorithm is more efficient in BFS than in DFS on the tests set we consider.

Finally, we have also implemented the (potentially incomplete) MCT procedure. For all instances we considered, the time needed by the MCT procedure is similar to the ones of the MP Algorithm, the difference being less than 1%. Also note that, as already mentioned in [7], the error in MCT is rare, in our experiments MCT always computed the correct MCS.

7. Conclusion

We have proposed in this paper the Monotone-Pruning algorithm, an improved K&M algorithm with pruning. This algorithm is a correction of the MCT Algorithm, based on a slightly more aggressive pruning strategy which ensures completeness. The MP algorithm constitutes a simple modification of the K&M algorithm, and is thus easily amenable to implementation and to extensions to other classes of systems. Moreover, as the K&M Algorithm, and unlike the algorithm proposed in [7], any strategy of exploration of the Petri net is correct: depth first, breadth first, random. The empirical results presented in Section 6 show that our algorithm drastically outperforms K&M and CoverProc algorithms.

A natural continuation of this work is to develop a more efficient prototype based on symbolic data structures. Another issue is the complexity of the current proof of correctness. It would be worthwhile to look for a simpler proof based on an invariant representing the pruning strategy of the MP algorithm.

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Test				K&M		MP (DFS)				MP (BFS)				CoverProc[7]
name	#P	#T	# MCS	# Wait	time (s)	# Wait	# X	# Inact	time (s)	# Wait	# X	# Inact	time (s)	time (s)
BasicME	5	4	3	5	< 0.01	5	5	2	< 0.01	5	5	2	< 0.01	0.12
Kanban	16	16	1	72226	9.1	114	59	58	< 0.01	111	100	99	< 0.01	0.19
Lamport	11	9	14	83	0.02	24	24	10	< 0.01	24	24	10	< 0.01	0.17
Manufacturing	13	6	1	81	0.01	30	20	19	< 0.01	30	23	22	< 0.01	0.14
Peterson	14	12	20	609	0.2	35	35	15	0.02	35	35	15	0.02	0.25
Read-write	13	9	41	11139	6.33	76	76	35	.07	76	76	35	.07	1.75
Mesh2x2	32	32	256	x	x	6241	4998	4742	15.1	5401	3876	3620	10.5	330
Multipool	18	21	220	x	x	2004	1908	1688	5.2	1854	1828	1608	4.7	365
pncsacover	31	36	80	x	x	1615	1177	1097	1.5	1462	1040	960	1.5	113
csm	14	13	16	x	x	102	93	77	.03	122	105	89	.03	0.34
fms	22	20	24	x	x	809	623	599	0.24	867	577	553	0.20	2.1

Table 1. The K&M, MP and CoverProc algorithms comparison. #P, #T, # MCS : number of places and transitions and size of the MCS of the Petri net. # Wait: number of elements passed in the waiting list. # X : number of nodes in the tree constructed by the algorithm. # Inact: number of nodes deactivated by the algorithm. For the MP Algorithm, the size of the MCS is recovered as (# X - # Inact).

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A. Complements for the proof of Proposition 4.4

We prove an additional property of MP Algorithm related to its accelerations. Intuitively, some nodes may be hidden when an acceleration is performed. The property states that if a node y is hidden, then it owns an ancestor y' which has been explored ($y' \in X$), and whose label is strictly less than the label of y. We call this property the minimal completeness of the exploration.

Lemma A.1. (Minimal Completeness)
For any edge $b = (x, t, x') \in B$ corresponding to an acceleration (*i.e.* such that $\Lambda(x') > \mathsf{Post}_\varphi(\Lambda(x), t)$), there exists a node z such that:

(i) $z \in \mathsf{Ancestor}_\mathcal{E}(x)$ and $\beta(z) = \alpha(x')$,

(ii) for any node $y \in \mathsf{Ancestor}_\mathcal{E}(x') \setminus \mathsf{Ancestor}_\mathcal{E}(x)$, there exists a node $y' \in \mathsf{Ancestor}_\mathcal{R}(y) \cap X$ such that $\Lambda(y') < \Lambda(y)$ and $z \in \mathsf{Ancestor}_\mathcal{E}(y')$.

Proof:
The proof proceeds by induction on $\alpha(x)$. If $\alpha(x) = 1$ (x is the root), then one can easily verify that one can choose $z = x$, and for any node y that is skipped by the acceleration, $y' = x$ is a correct candidate.

We now consider x such that $\alpha(x) > 1$, and consider an edge $b = (x, t, x') \in B$. We consider the notations introduced in the definition of the concretization function, and let $\gamma(b) = t(\gamma(w_1)t)^M \dots (\gamma(w_k)t)^M$,

where w_i is the path in B^* associated with node x_i. We assume that nodes x_i's are ordered w.r.t. α, and thus x_1 is an ancestor of all x_i's. We let $z = x_1$, it verifies property (i).

Let us prove property (ii). Let $y \in N$ be a node of the reachability tree such that there exists a word $\varepsilon \neq \varrho \preceq \gamma(b)$ verifying $\Lambda(y) = \mathsf{Post}_\varphi(\Lambda(x), \varrho)$. As $\varrho \preceq \gamma(b)$, there exists a unique i such that $t(\gamma(w_1)t)^M \dots (\gamma(w_i)t)^M \prec \varrho$ and $\varrho \preceq t(\gamma(w_1)t)^M \dots (\gamma(w_{i+1})t)^M$.

We can thus decompose ϱ as $\varrho = t(\gamma(w_1)t)^M \dots (\gamma(w_i)t)^M(\gamma(w_{i+1})t)^l\eta$ where $0 \leq l < M$ and $\varepsilon \prec \eta \preceq \gamma(w_{i+1})t$.

Consider the node y' in the reachability tree defined as follows: it is the successor of node x_{i+1} by the sequence η.

We first prove that $\Lambda(y') \leq \Lambda(y)$. Following notations introduced in the proof of Lemma 4.1, we have that $\Lambda(y)$ can be reached from marking m_i^{acc} by the sequence $(\gamma(w_{i+1})t)^l\eta$. According to previous properties, we obtain $m \leq m_i^{acc} \leq \mathsf{Post}_\varphi(m_i^{acc}, (\gamma(w_{i+1})t)^l)$ and thus $\Lambda(y') = \mathsf{Post}_\varphi(\Lambda(x_{i+1}), \eta) \leq \mathsf{Post}_\varphi(m, \eta) \leq \Lambda(y)$.

Now, we prove that the inequality is strict. By contradiction, if $\Lambda(y') = \Lambda(y)$, according to previous inequalities, we obtain $\mathsf{Post}_\varphi(\Lambda(x_{i+1}), \eta) = \mathsf{Post}_\varphi(m, \eta)$. By completing η to obtain the sequence $\gamma(w_{i+1})t$, we obtain $\mathsf{Post}_\varphi(\Lambda(x_{i+1}), \gamma(w_{i+1})t) = \mathsf{Post}_\varphi(m, \gamma(w_{i+1})t)$. By Lemma 4.1, we have $\mathsf{Post}_\varphi(\Lambda(x_{i+1}), \gamma(w_{i+1})) = \Lambda(x)$. By definition of m, we obtain $m = \mathsf{Post}_\varphi(m, \gamma(w_{i+1})t)$. This is a contradiction with our choice of x_{i+1}! Indeed, in Definition 4.2, we require the following property: $\exists p.\Lambda(x_{i+1})(p) < m(p) < \omega$. One can prove that this implies the following strict inequality: $m(p) < \mathsf{Post}_\varphi(m, \gamma(w_{i+1})t)(p)$, yielding the contradiction

Then, we distinguish two cases:

- if $y' \in X$, then we are done ($z = x_1$ is an ancestor y').
- otherwise ($y' \notin X$), this implies that y' is skipped by an acceleration on the path between x_{i+1} and x, related to an edge $b' = (n, u, n')$. But then we can apply the induction hypothesis on this edge as we have $\alpha(n) < \alpha(x)$, and obtain two nodes z' and y'' verifying properties (i) and (ii). By transitivity, we trivially obtain $y'' \in \mathsf{Ancestor}_\mathcal{R}(y) \cap X$ and $\Lambda(y'') < \Lambda(y)$. It remains to prove that $z \in \mathsf{Ancestor}_\mathcal{E}(y'')$. As x_{i+1} is an ancestor of node n, active at step $\alpha(x)$, it can not be deactivated by the acceleration related to edge b', what implies that z' must be "below" x_{i+1}, *i.e.* $x_{i+1} \in \mathsf{Ancestor}_\mathcal{E}(z')$. This yields the result.

This concludes the proof. □