SIGGRAPH 2012

The **39th** International **Conference** and **Exhibition** on **Computer Graphics** and **Interactive Techniques**

FEM Simulation of 3D Deformable Solids: A practitioner's guide to theory, discretization and model reduction

Part One : The classical FEM method and discretization methodology

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University of Wisconsin - Madison

Find the latest version of course notes at : **www.femdefo.org**

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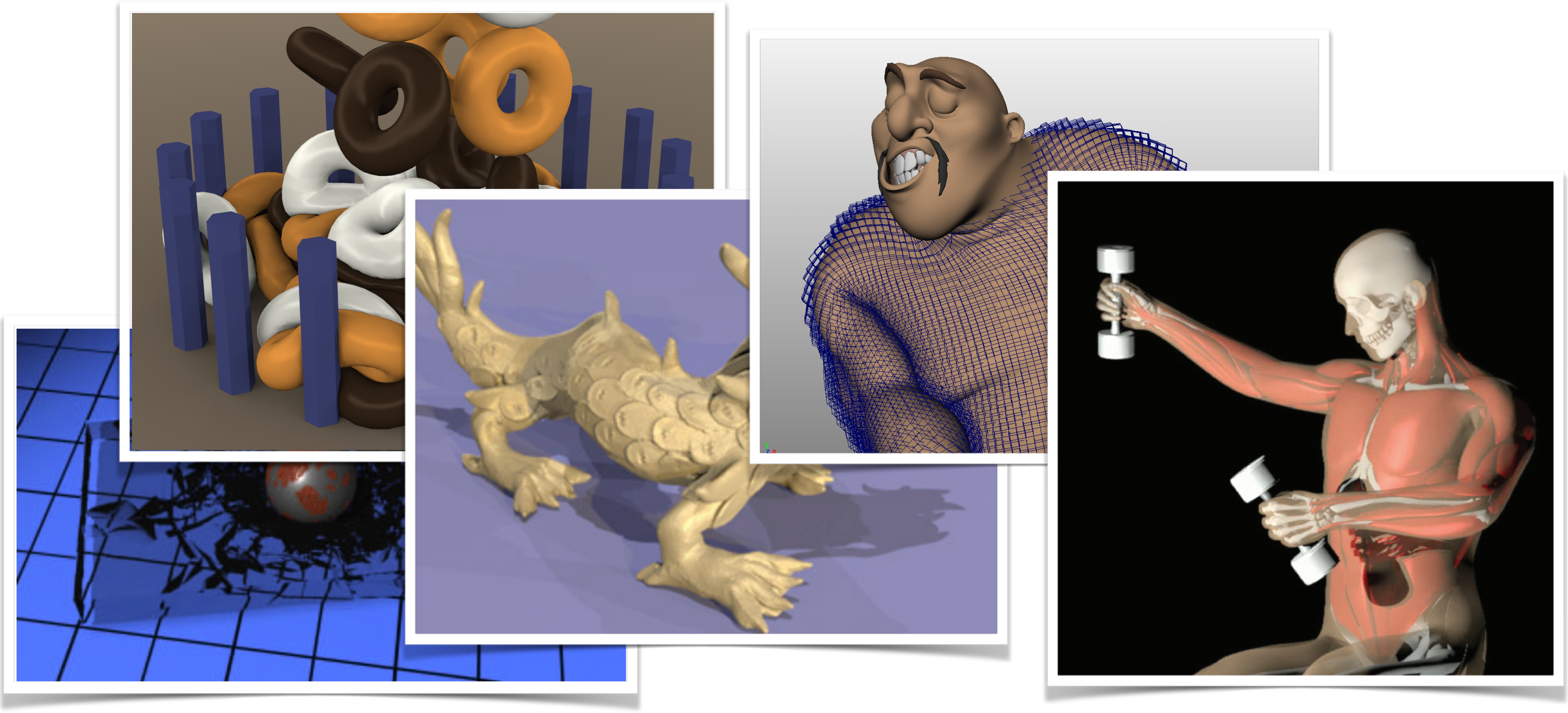
SIGGRAPH2012

Introduction



What does this course aim to do?

- ✓ Give you a brief exposure to the concepts and methods associated with Finite Elements
- ✓ Provide a primer on continuum mechanics
- ✓ Give you enough insight to start implementing
- ✓ Encourage you to study further, and improve your understanding

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Introduction

How do graphics practitioners describe FEM methods?

... a way to model elastic bodies that provides more detail and fidelity than using mass-spring networks ...

... a simulation technique for deformable models represented by tetrahedral (or triangle) meshes ...

... a method for deriving the governing equations of 3D solids, based on the potential energy they store when deformed ...

Introduction

We associate FEM with ....

- The Galerkin-based discretization method (core concept)
- Continuum mechanics concepts (stress, strain, energy, etc.)
- Common material models (corotated, StVK, Neohookean, etc.)

Introduction

FEM: Just *one* possible method for solving partial differential equations (PDEs)

Finite Elements vs. Finite Differences (the executive summary) :

Finite Differences replace the differential equation with an approximate algebraic expression

Finite Elements replace the solution with a parametric approximation, and then compute the best parameter values

FEM vs. Finite Differences

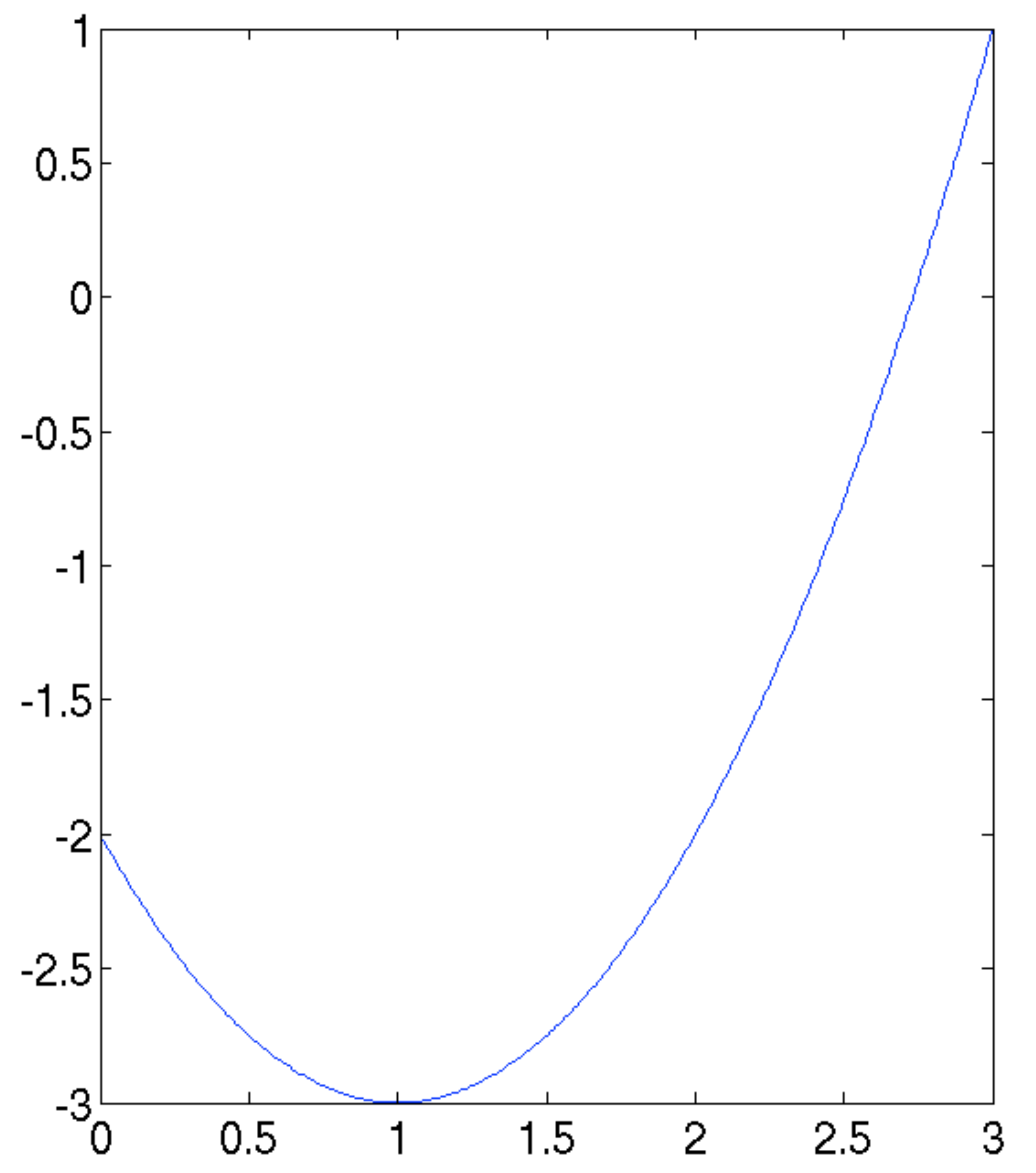
Example : The Poisson equation

Problem statement:

$$\begin{aligned} f''(x) &= 2 \quad x \in (0,3) \\ f(0) &= -2 \\ f(3) &= 1 \end{aligned}$$

Solution:

$$f(x) = x^2 - 2x - 2$$



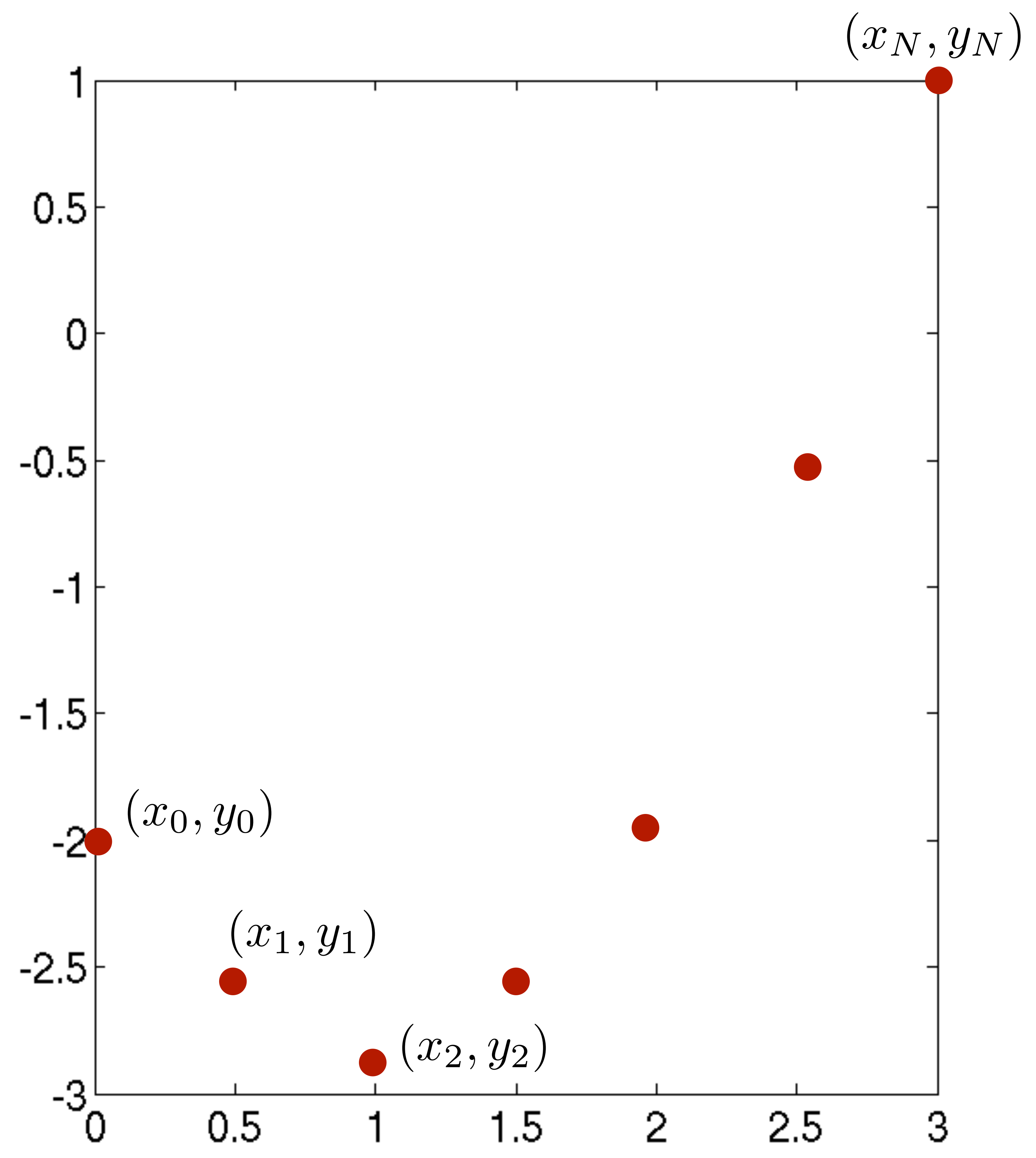
FEM vs. Finite Differences

Example : The Poisson equation

Using Finite Differences:

i. Introduce a number of data points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), \quad x_k := x_0 + kh$$



FEM vs. Finite Differences

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Using Finite Differences:

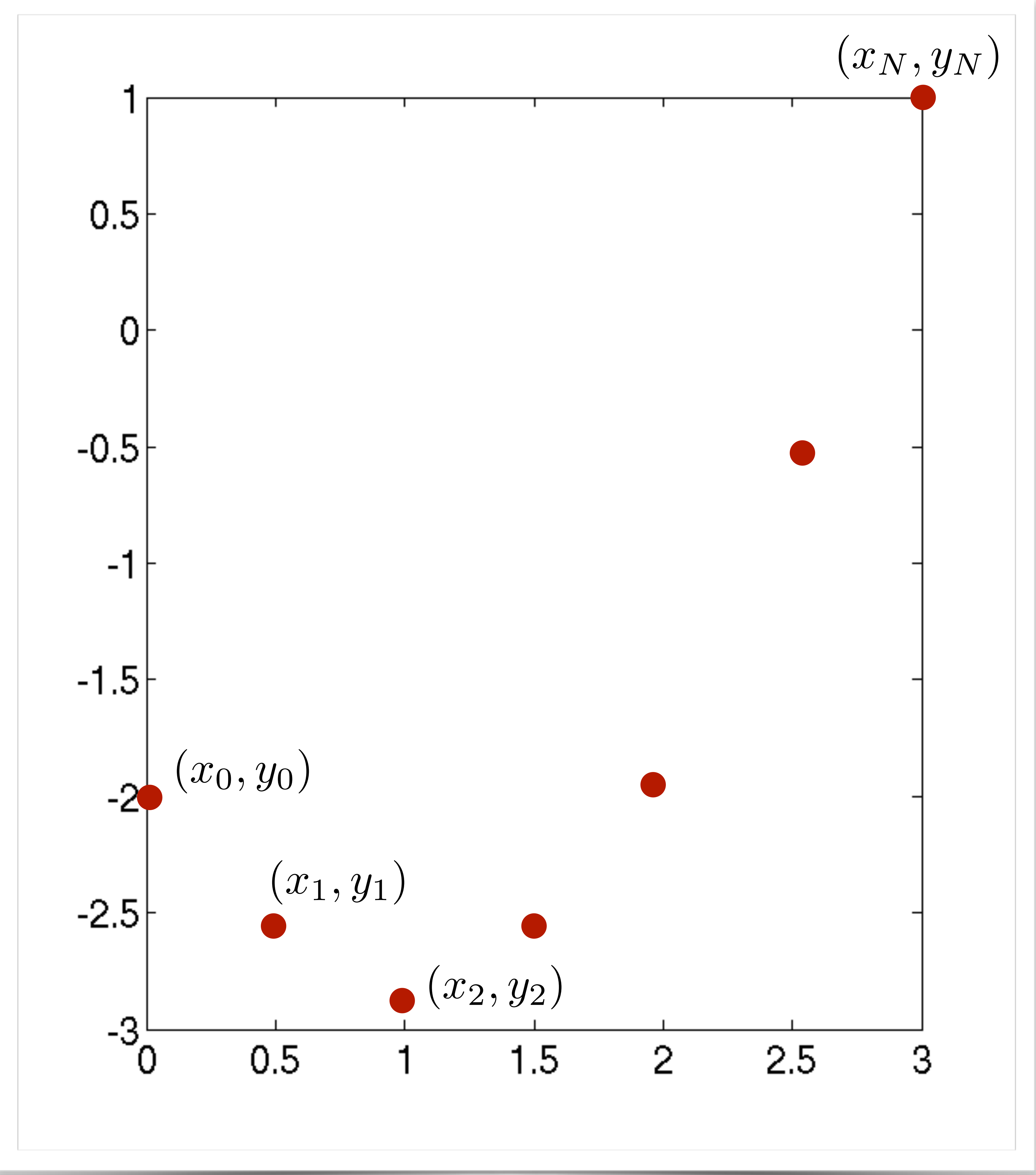
i. Introduce a number of data points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), \quad x_k := x_0 + kh$$

ii. Approximate the PDE with a finite difference formula at each point

$$2 = f''(x_k) \approx \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2}$$

iii. Solve all FD equations as a system



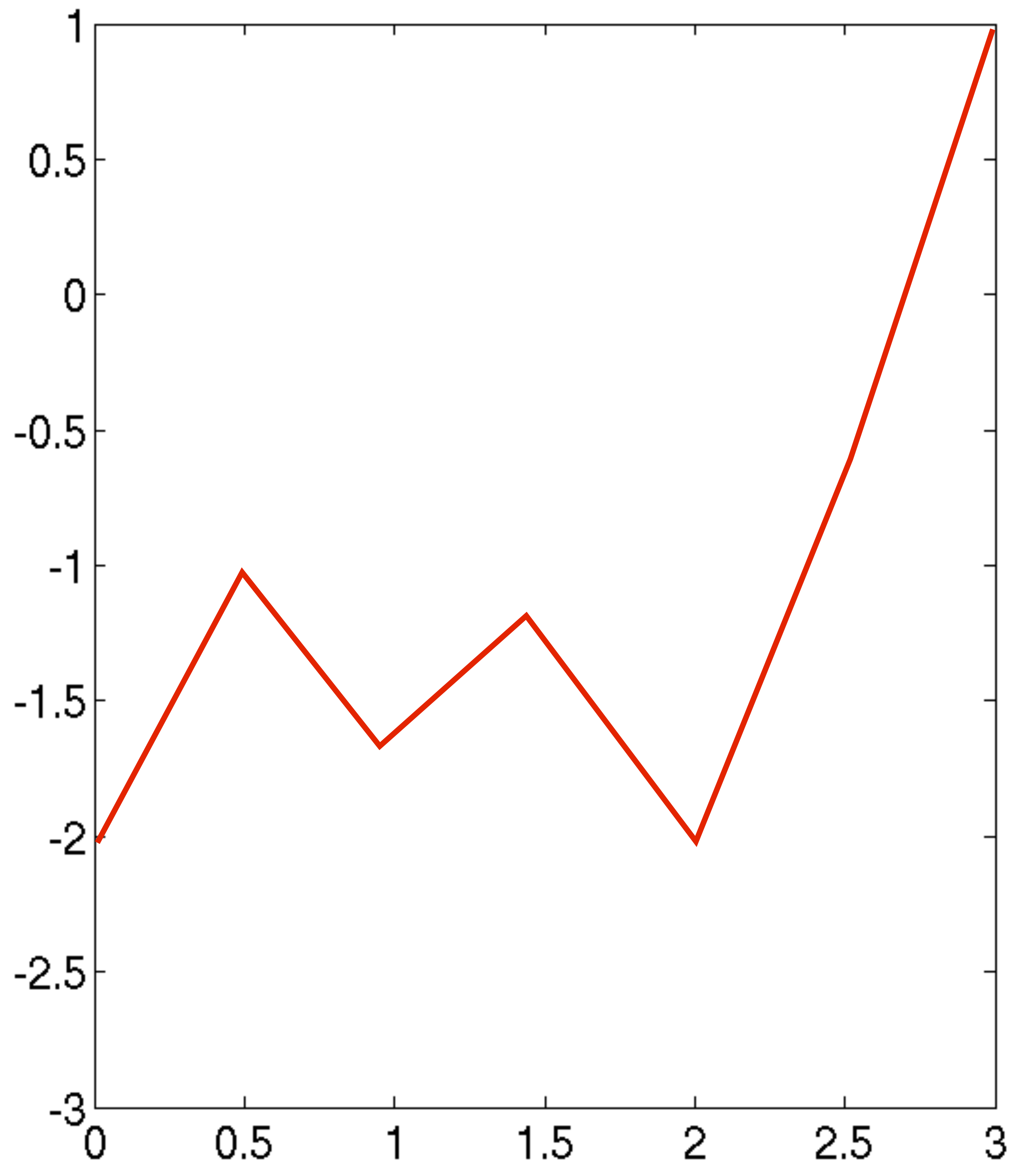
FEM vs. Finite Differences

Example : The Poisson equation

Using Finite Elements:

i. Define a family of candidate functions (which can approximate the solution)

- Piecewise linear polynomials
- Splines
- etc.



FEM vs. Finite Differences

Example : The Poisson equation

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- Splines
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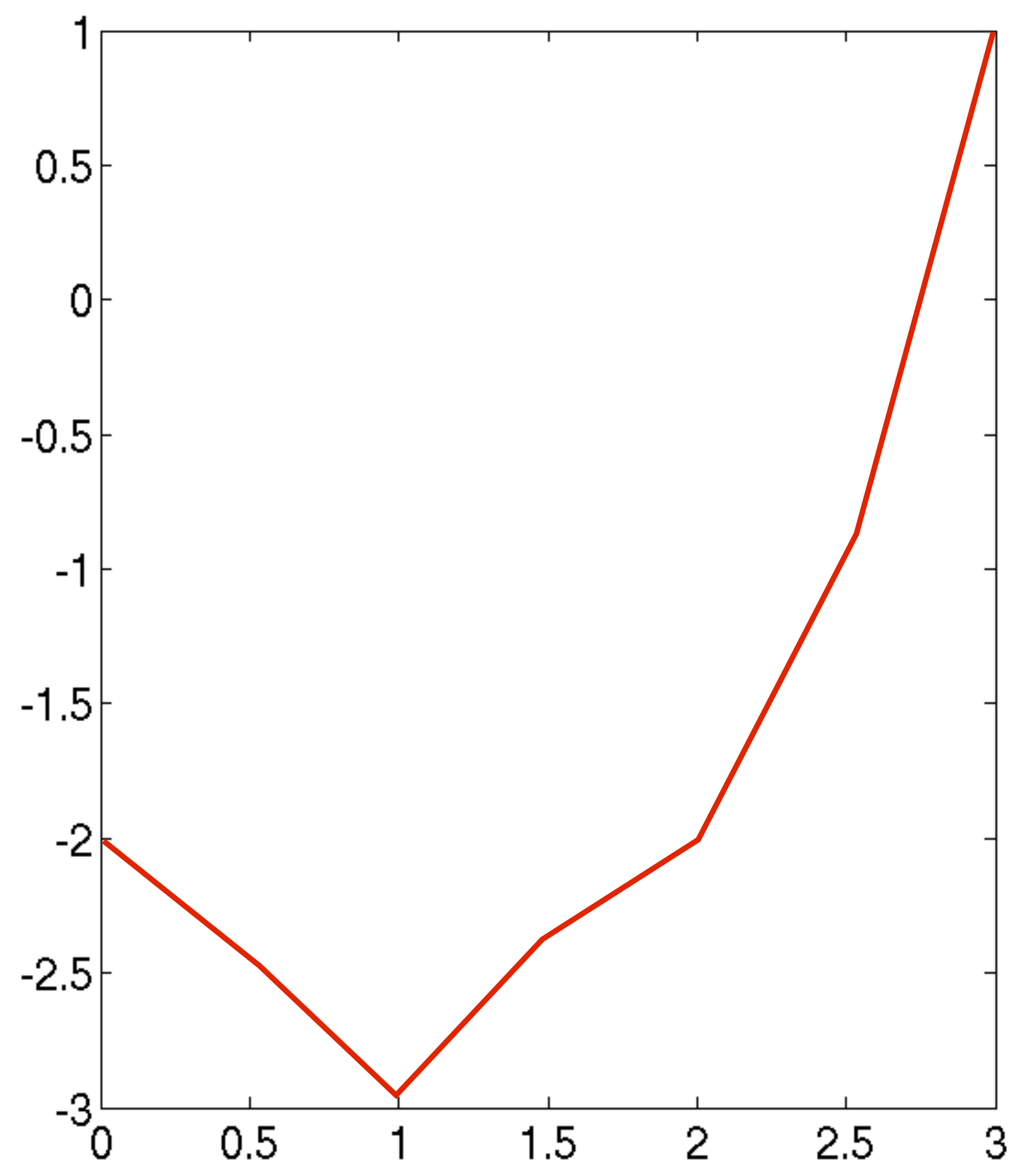
ii. Tune the available parameters to best approximate the solution to the PDE



FEM vs. Finite Differences

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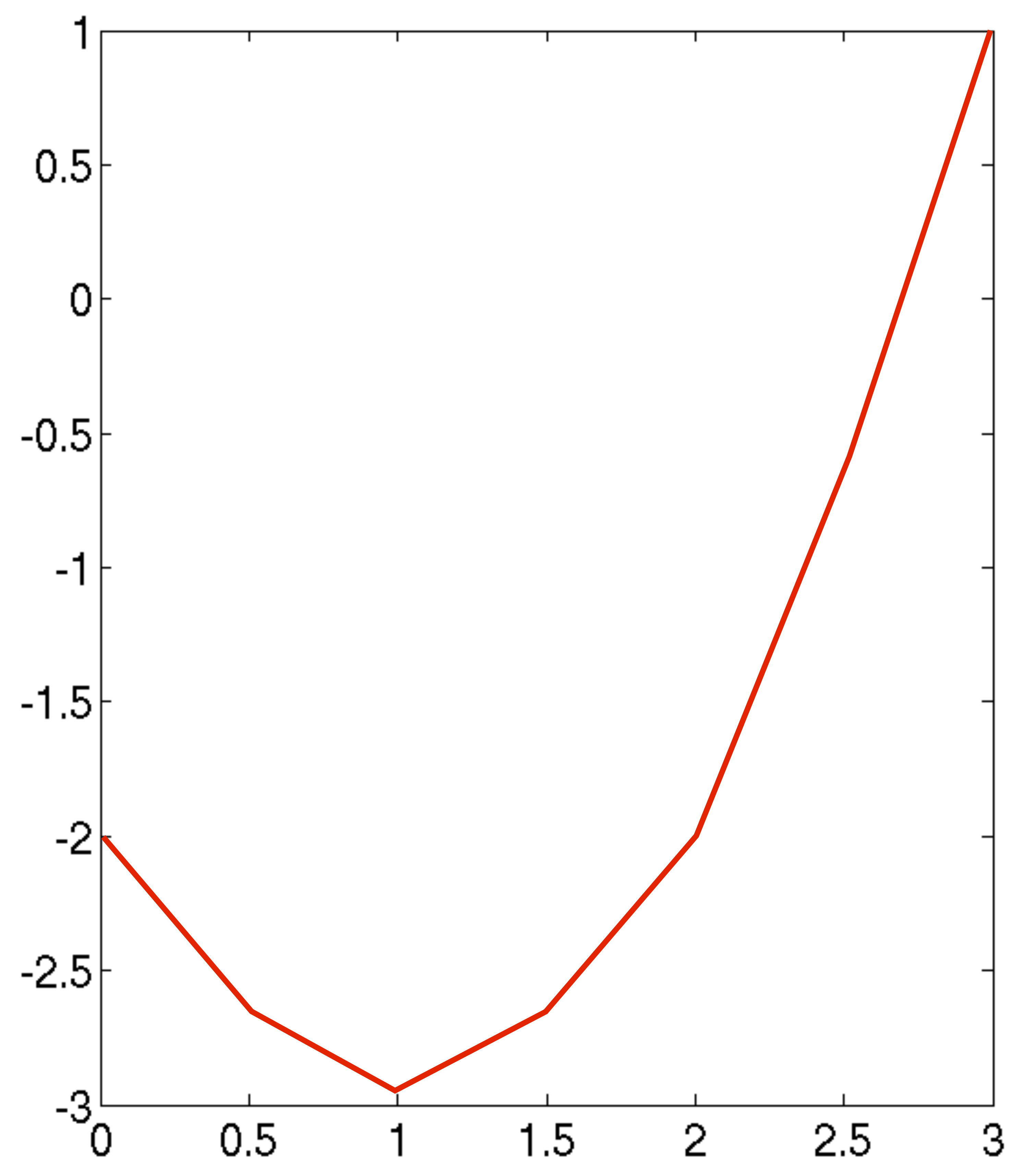
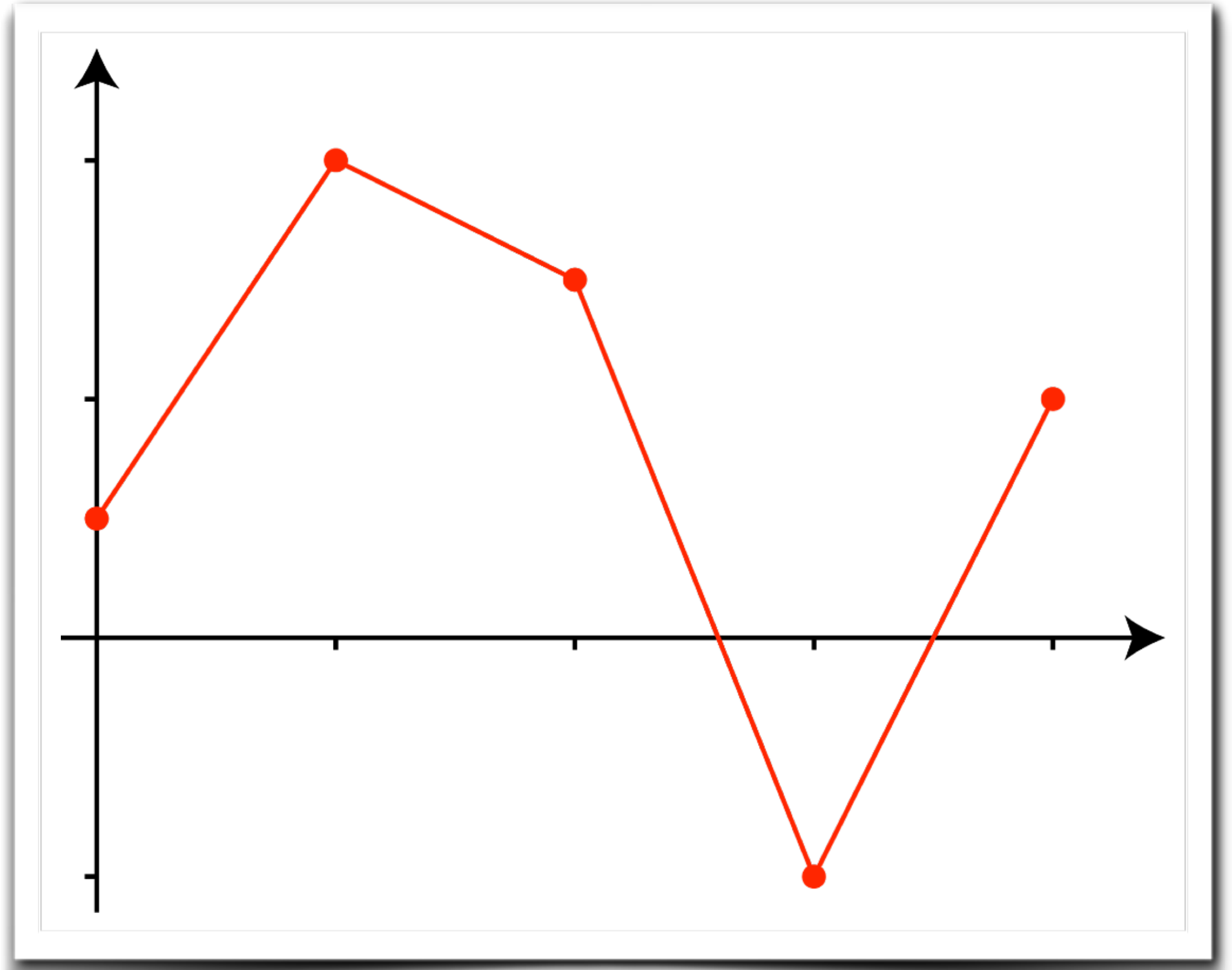
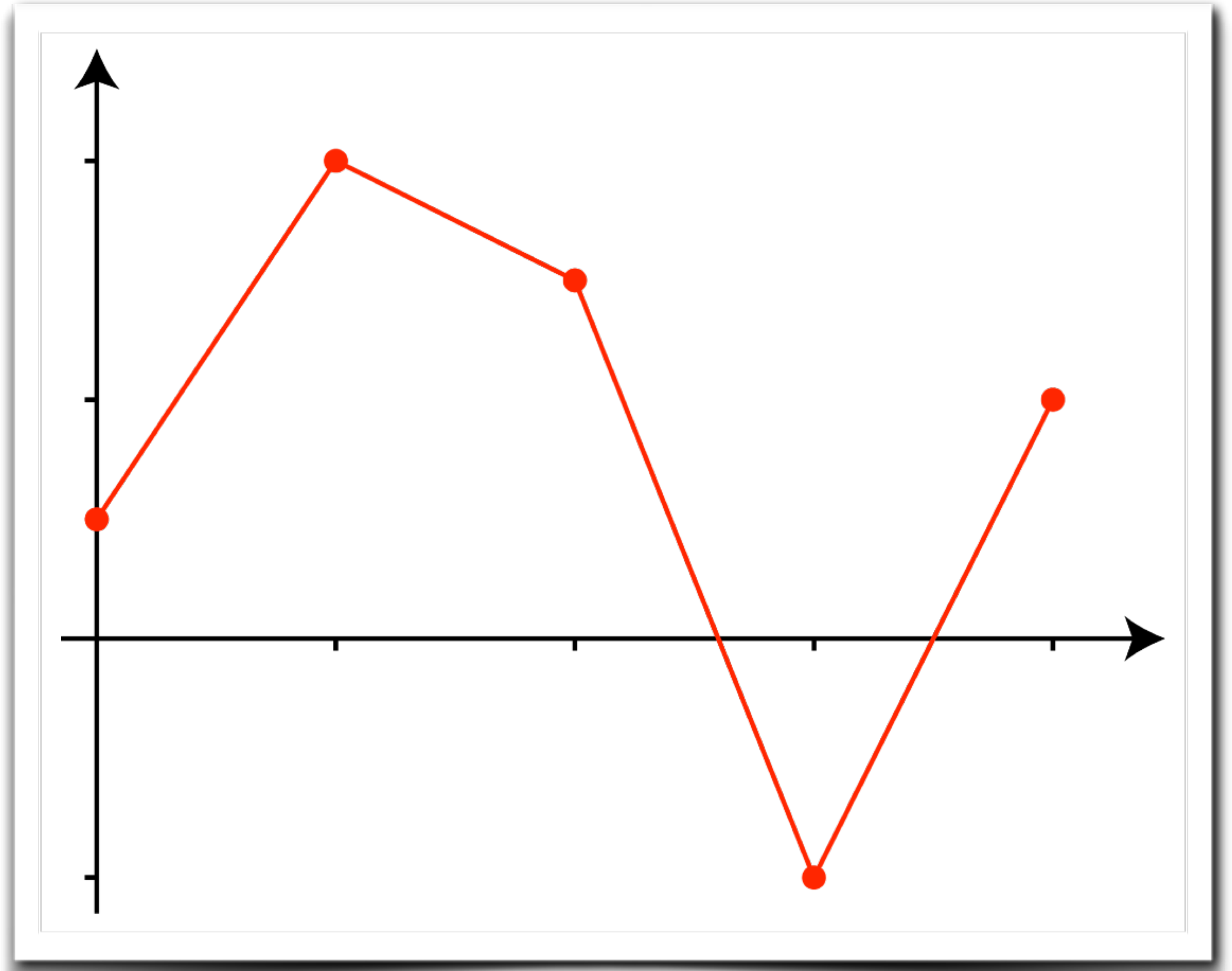


FEM vs. Finite Differences

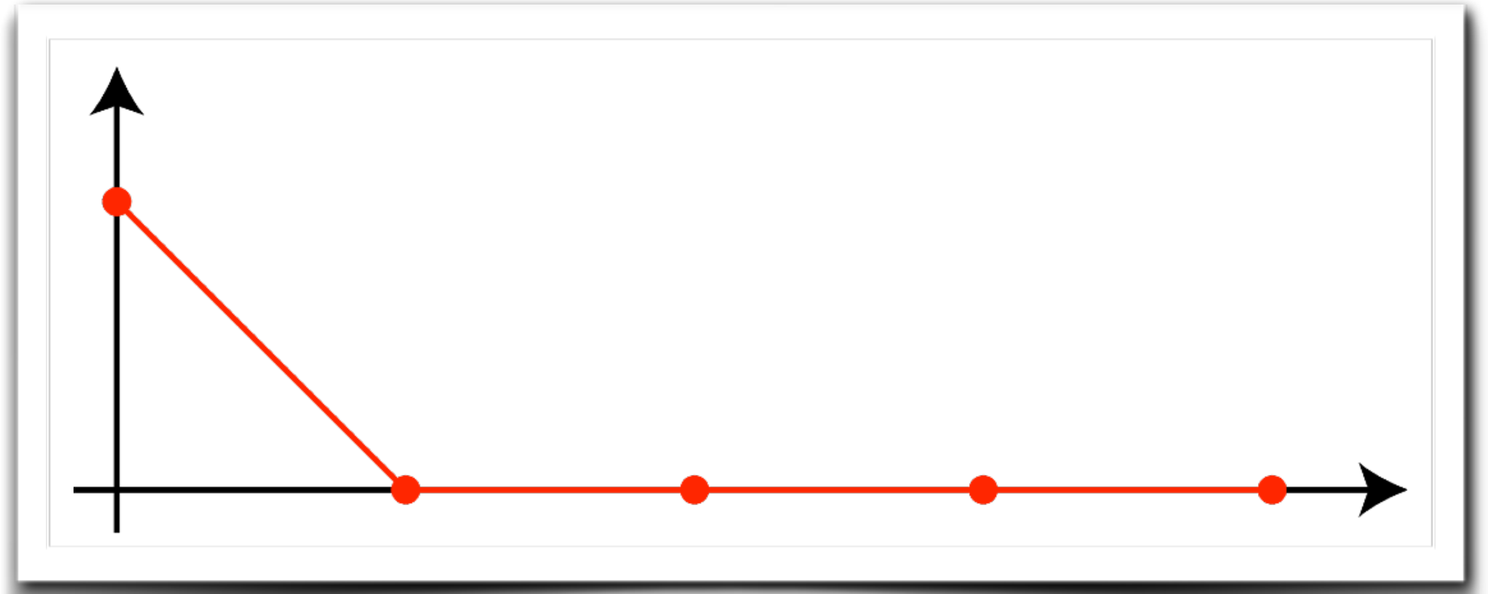


FEM vs. Finite Differences

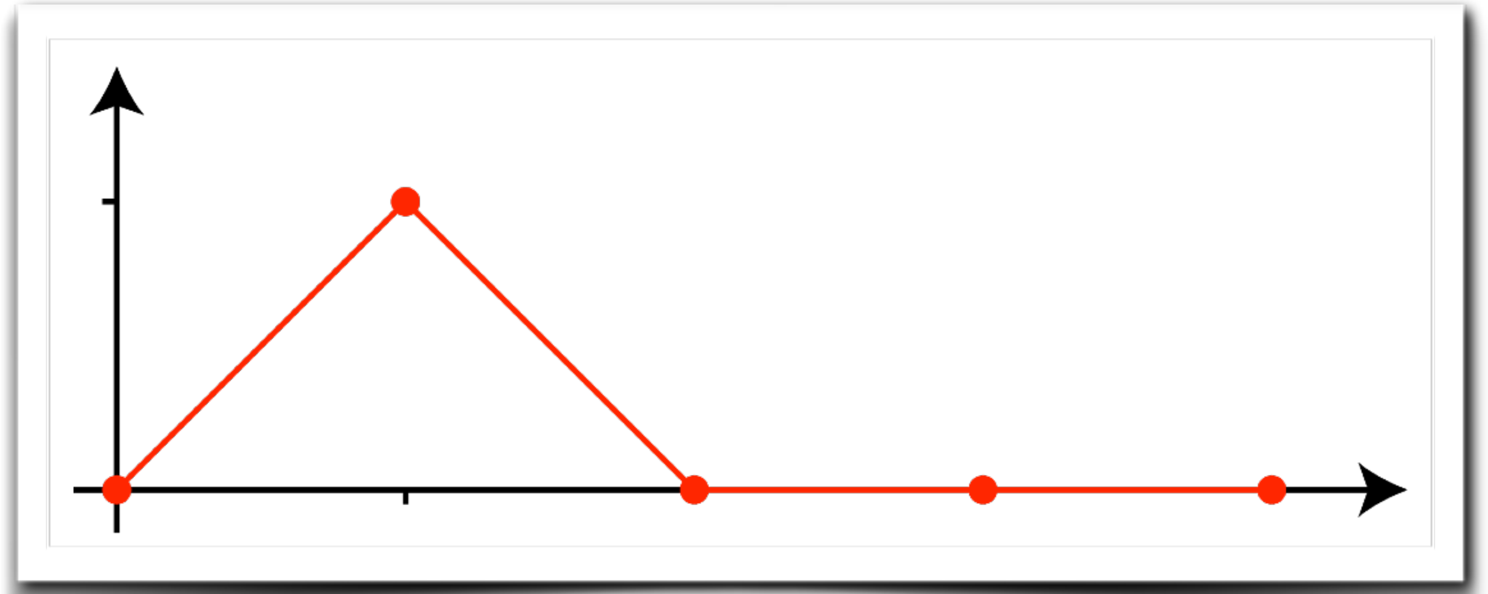
FEM vs. Finite Differences



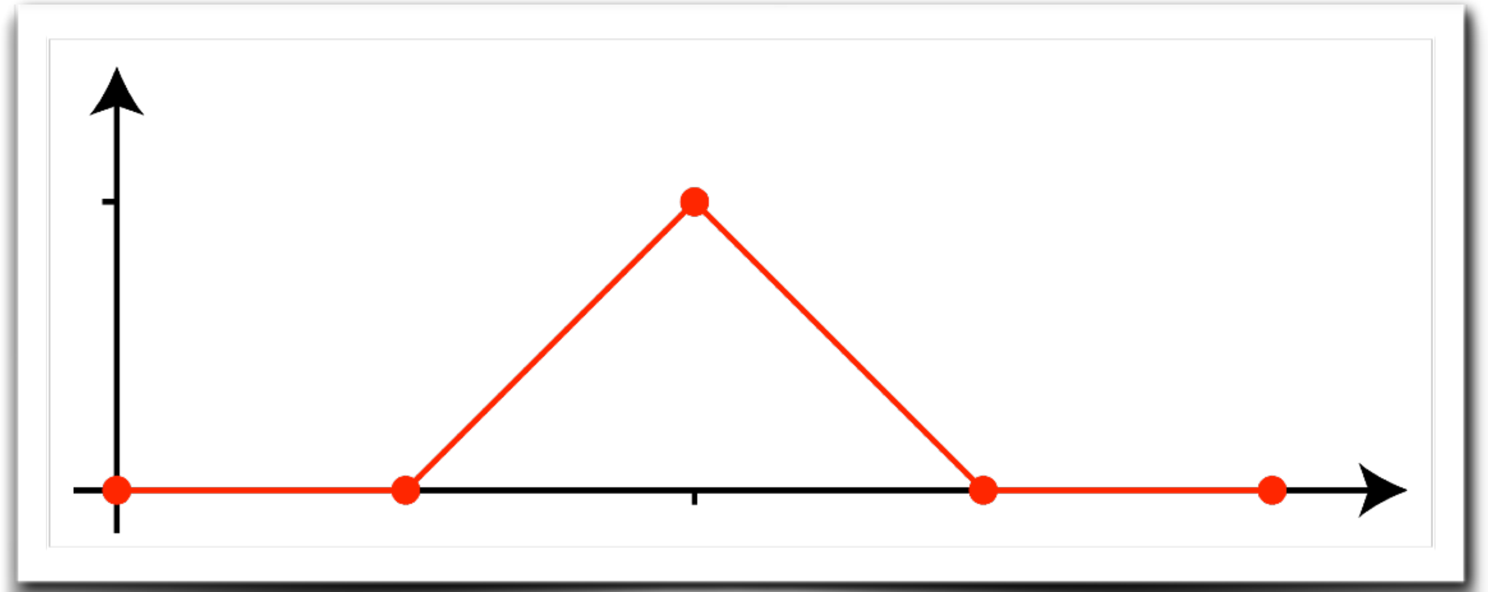
$= \frac{1}{2} \times$



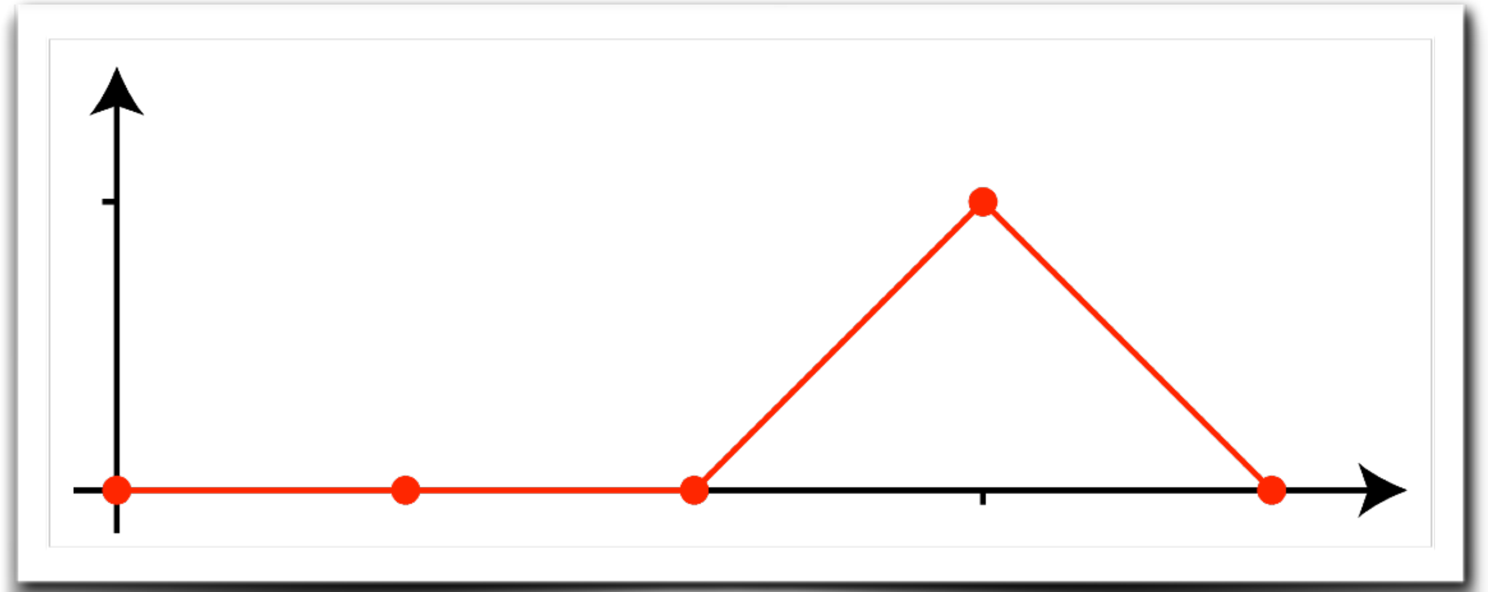
$+2\times$



$+\frac{3}{2} \times$



$-1\times$



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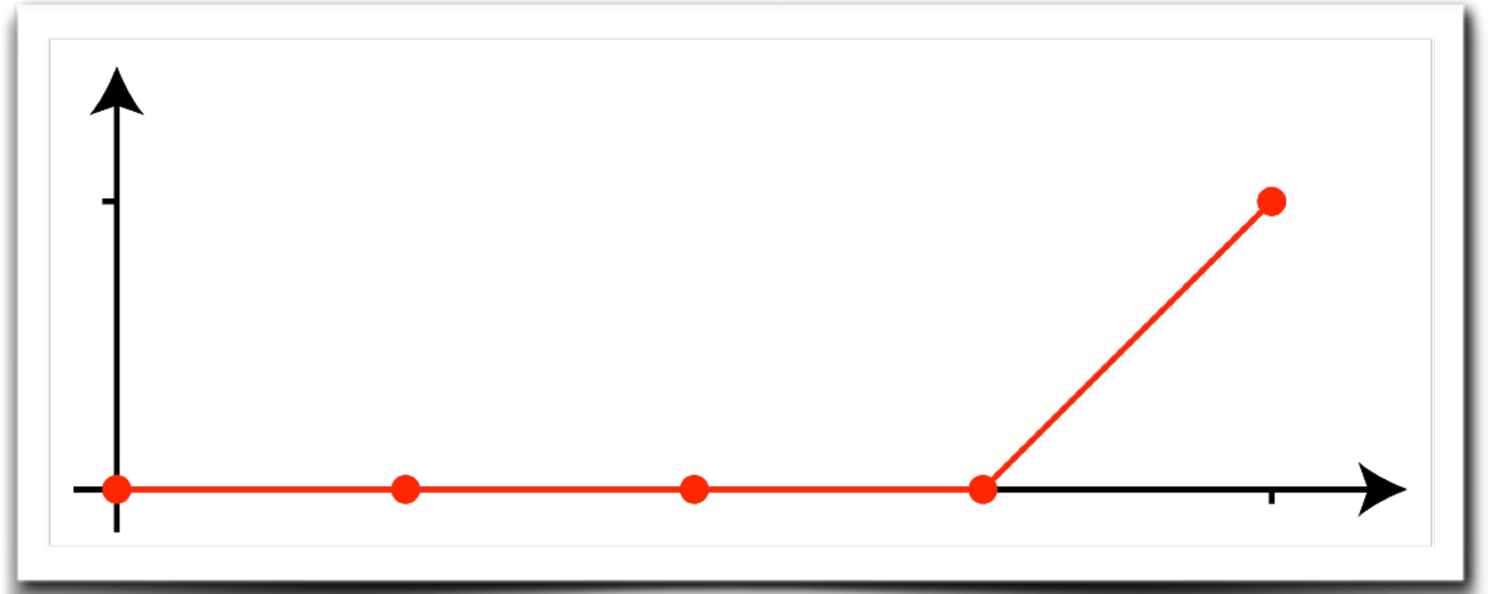
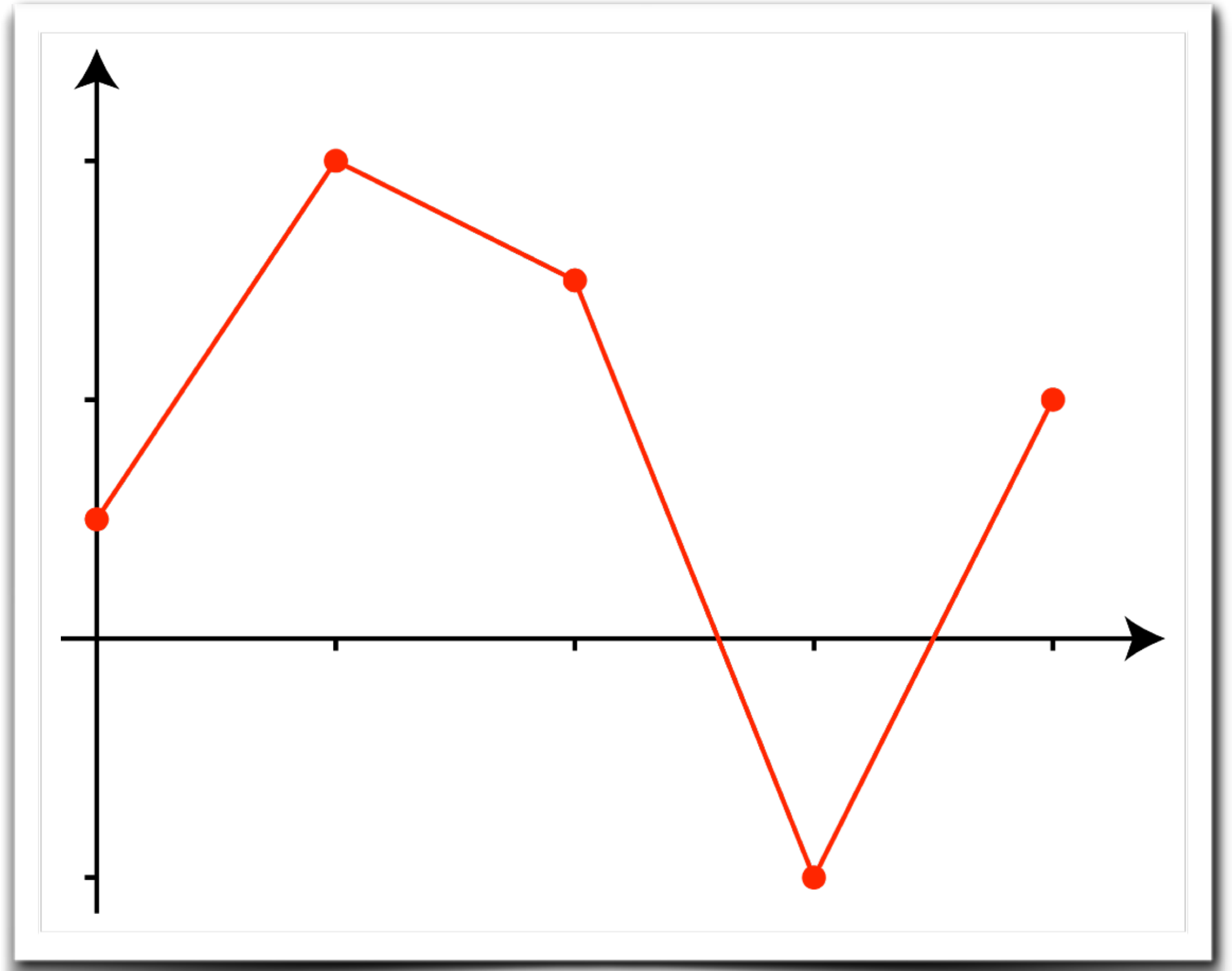


$$y(x) := \sum_k y_k \mathcal{N}_k(x)$$

FEM vs. Finite Differences



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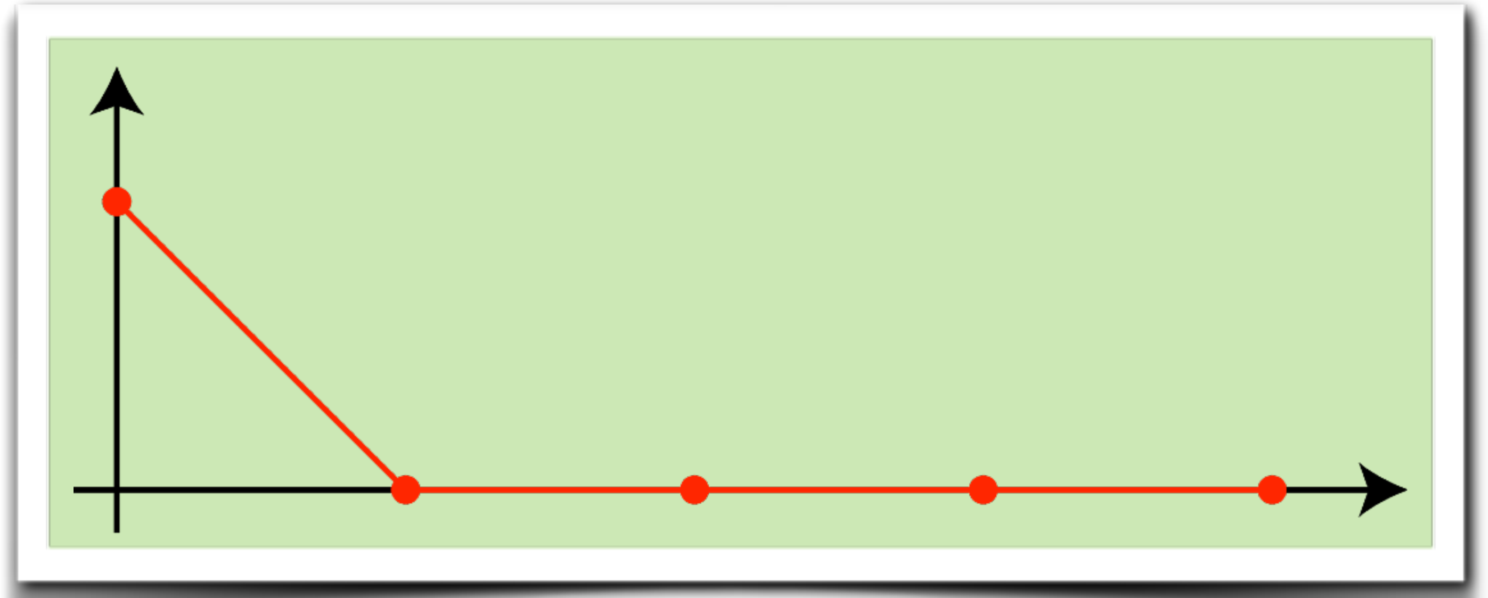
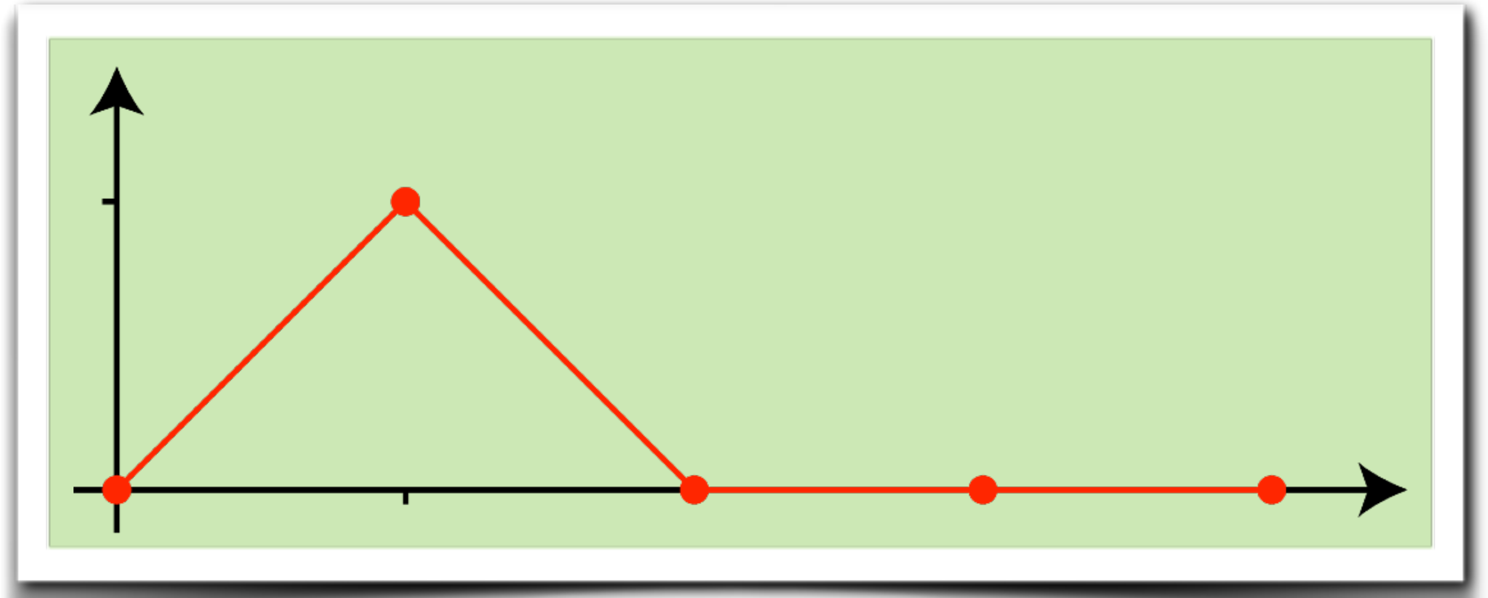
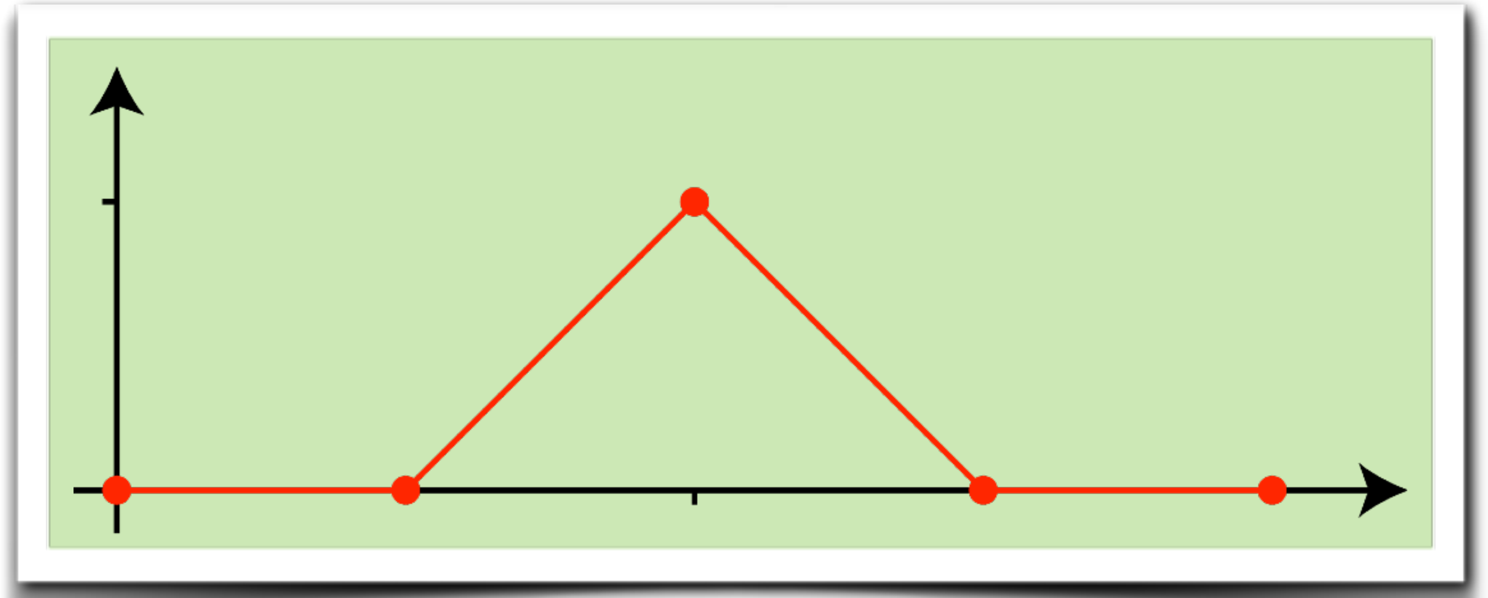


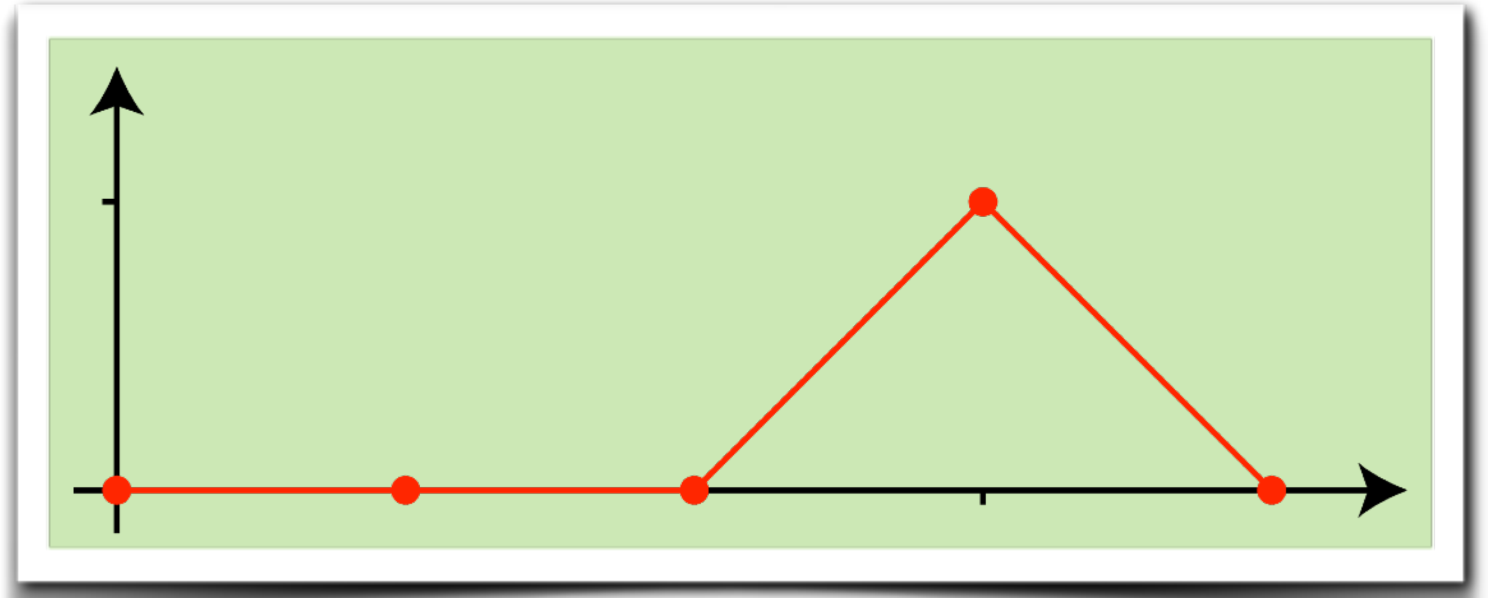
$+2 \times$



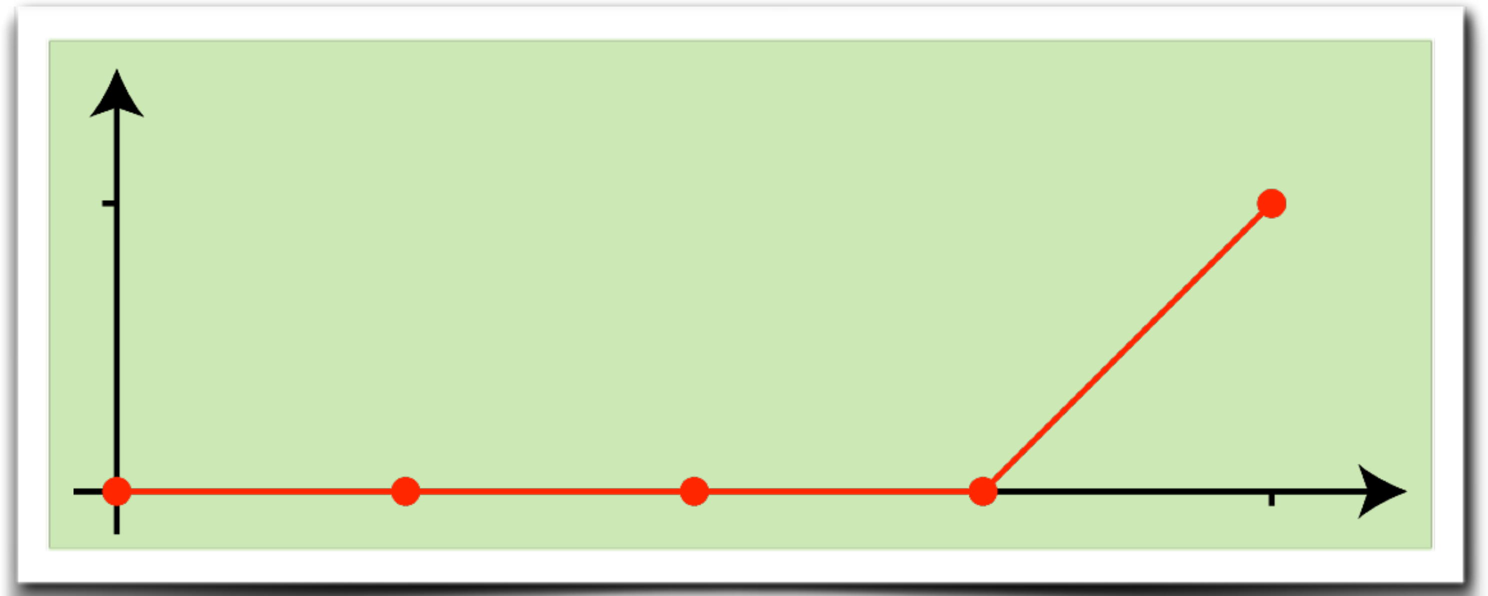
$+\frac{3}{2} \times$



$-1 \times$



$+1 \times$



$$y(x) := \sum_k y_k \boxed{\mathcal{N}_k(x)}$$

FEM vs. Finite Differences

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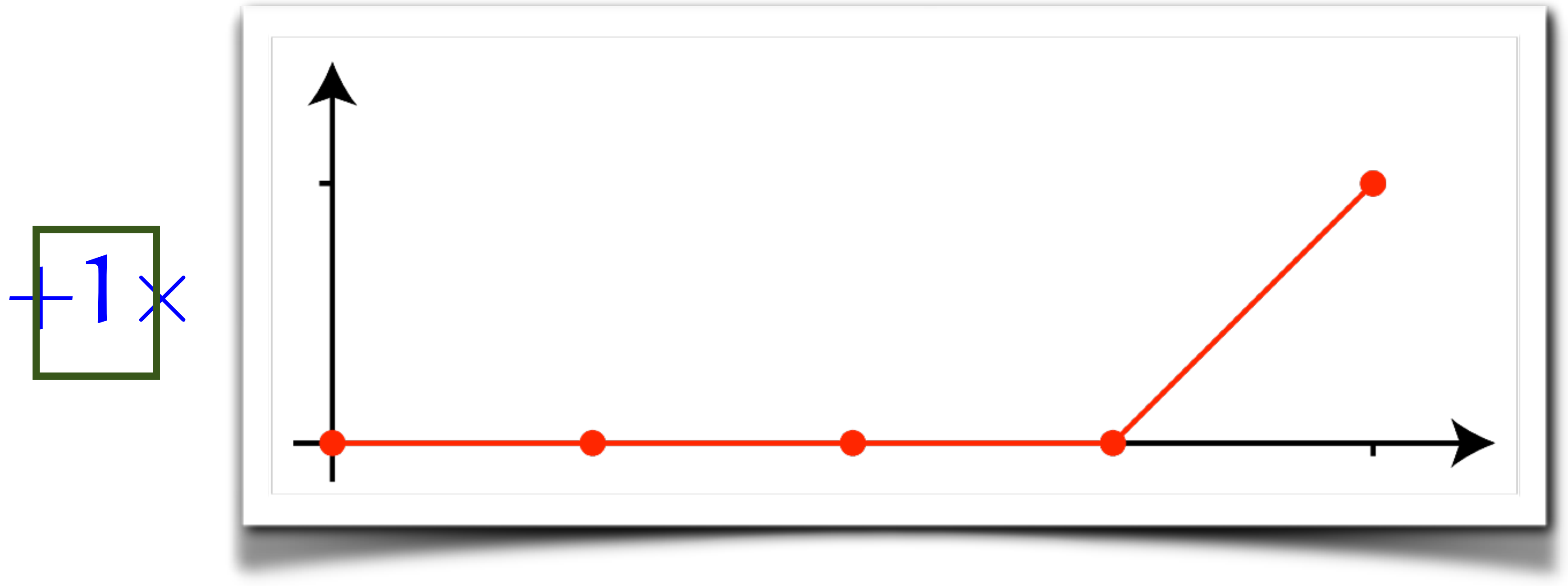

FEM vs. Finite Differences

How do we find the optimal values $y_0, y_1, \ldots, y_n$ *?*

$$y(x) := \sum_k y_k \mathcal{N}_k(x)$$

FEM vs. Finite Differences

How do we find the optimal values $y_0, y_1, \ldots, y_n$?

Can we substitute into the PDE?

$$\begin{aligned} f''(x) &= 2 \qquad x \in (0,3) \\ f(0) &= -2 \\ f(3) &= 1 \end{aligned}$$

$$y(x) := \sum_k y_k \mathcal{N}_k(x)$$

FEM vs. Finite Differences

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Can we substitute into the PDE?

$$\begin{aligned} f''(x) &= 2 \qquad x \in (0,3) \\ f(0) &= -2 \\ f(3) &= 1 \end{aligned}$$

$y(x)$ *is not differentiable enough!*

$$y(x) := \sum_k y_k \mathcal{N}_k(x)$$

FEM vs. Finite Differences

Solve ...

$$y''(x) = 0$$

↔

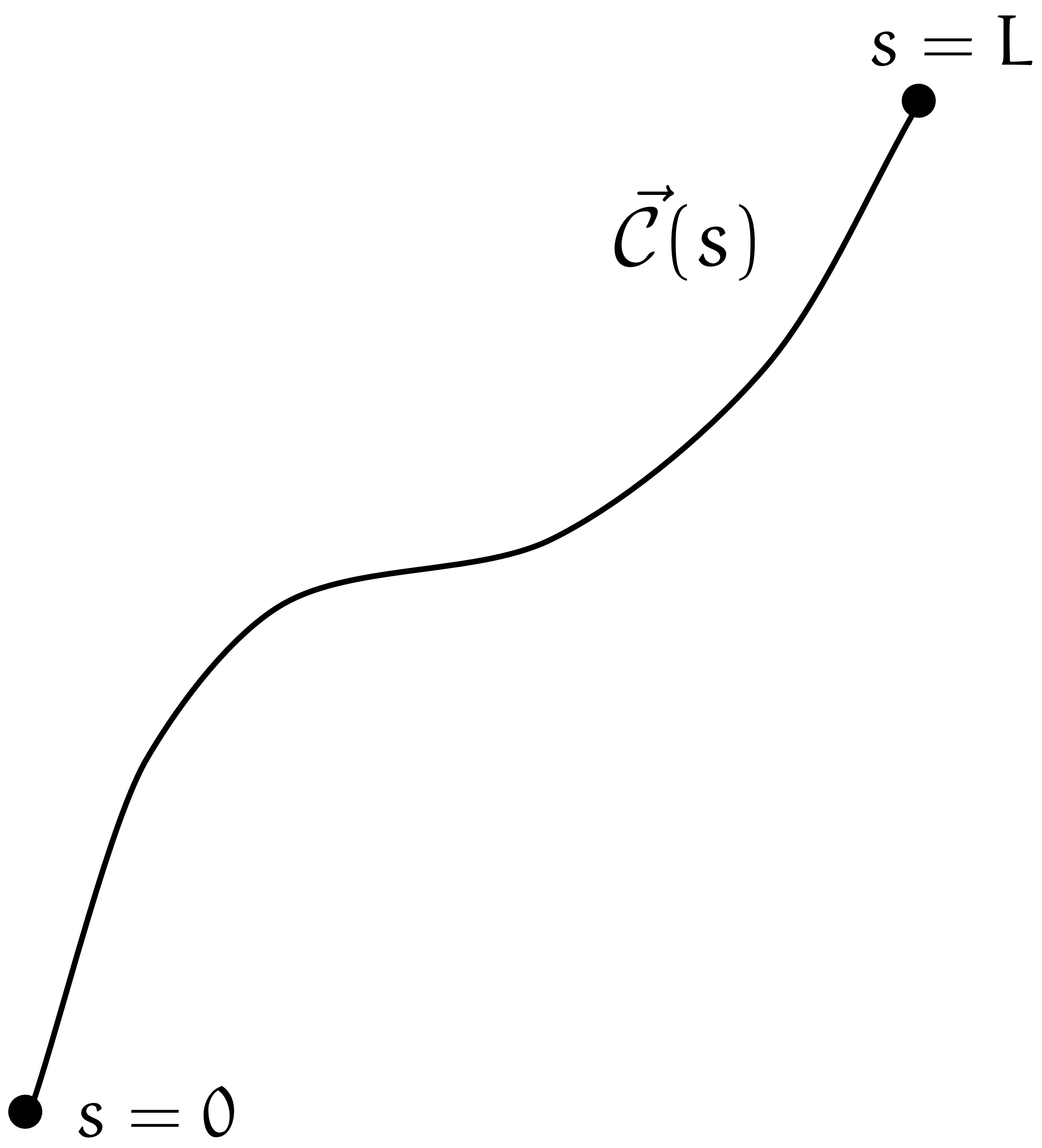
Minimize ...

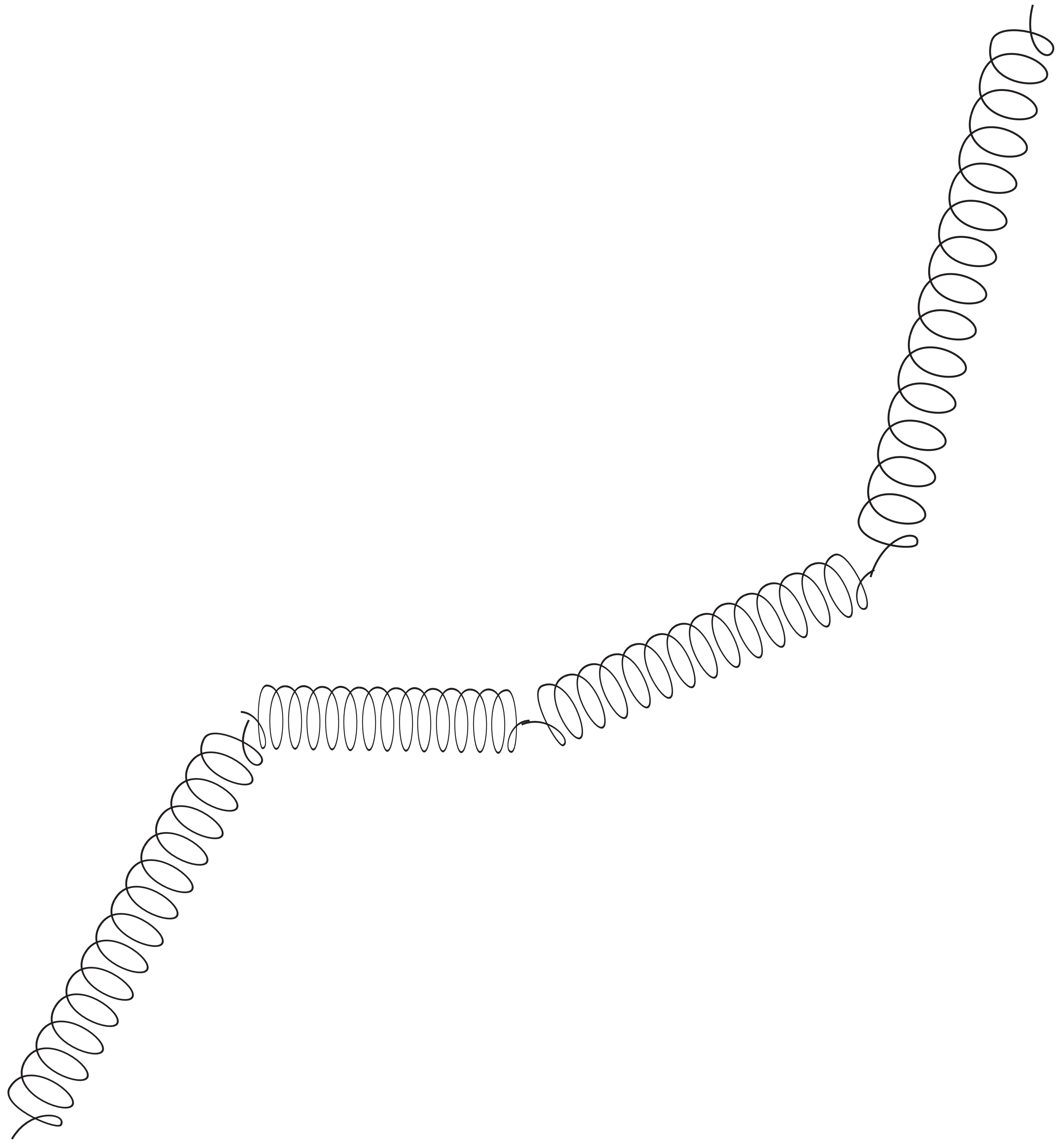
$$E[y] = \int |y'(x)|^2 dx$$

$$y(x) := \sum_k y_k \mathcal{N}_k(x)$$

$$E[y] = E(y_1, y_2, \ldots, y_N)$$

Elasticity on a flexible string





$$\frac{d}{ds}\left(k\frac{\|\vec{C}'(s)\|-1}{\|\vec{C}'(s)\|}\vec{C}'(s)\right)+\vec{f}=0$$

$$E = l_0\frac{k}{2}\left(\frac{l}{l_0}-1\right)^2$$

FEM vs. Finite Differences

Finite Elements

✓Works naturally with mesh-based discretizations

✓Produces numerically nice (sparse, symmetric, definite) discrete systems

✗ Requires attention in choosing proper elements

✗ Discretization is not as sparse as finite differences

Finite Differences

✓Very straightforward to write

✓Generally produces sparse systems (often sparser than FEM)

✗ Accommodating irregular geometries (e.g. meshes) is nontrivial

✗ Need to be very careful to preserve useful numerical properties (e.g. symmetry)

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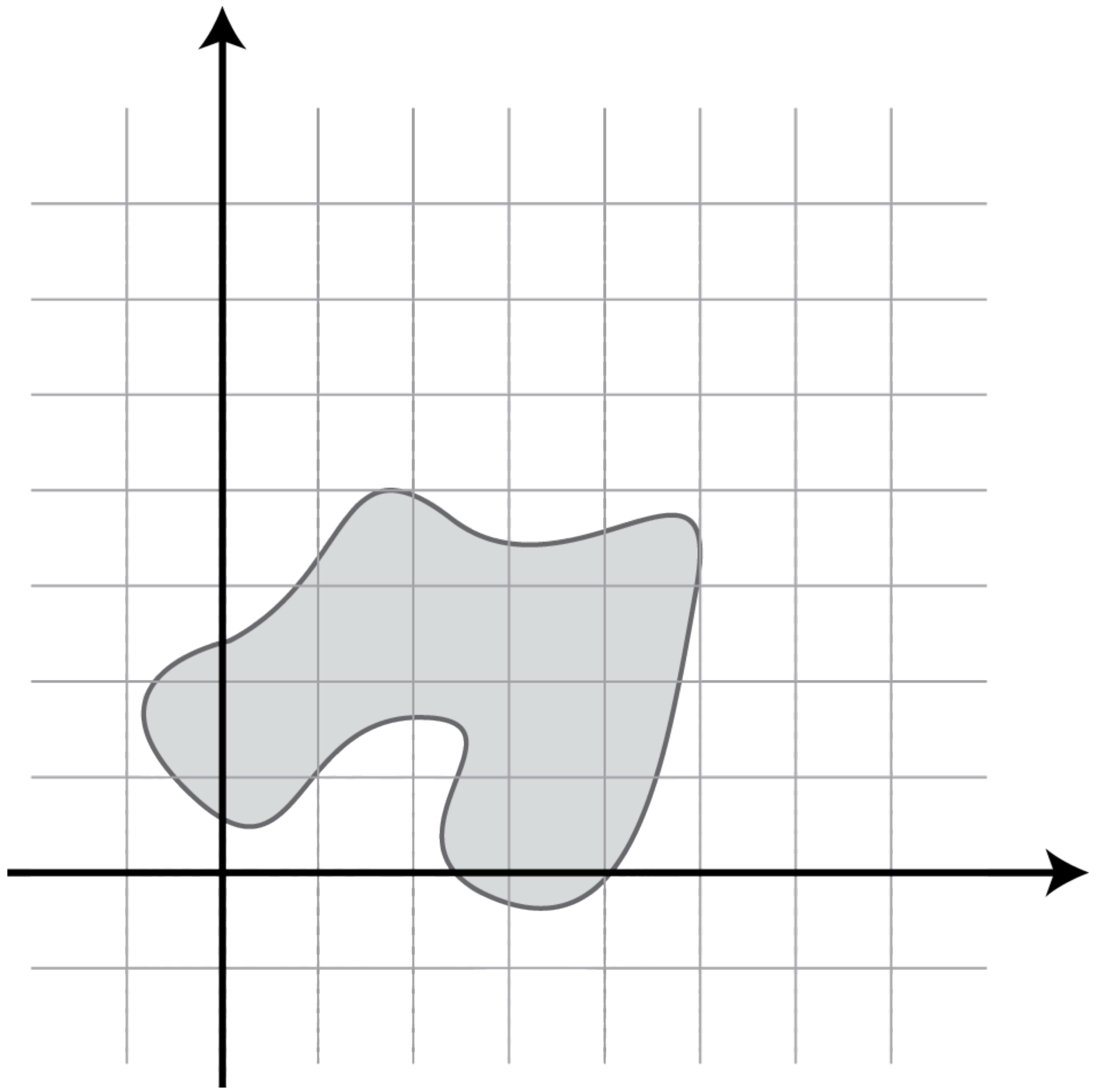
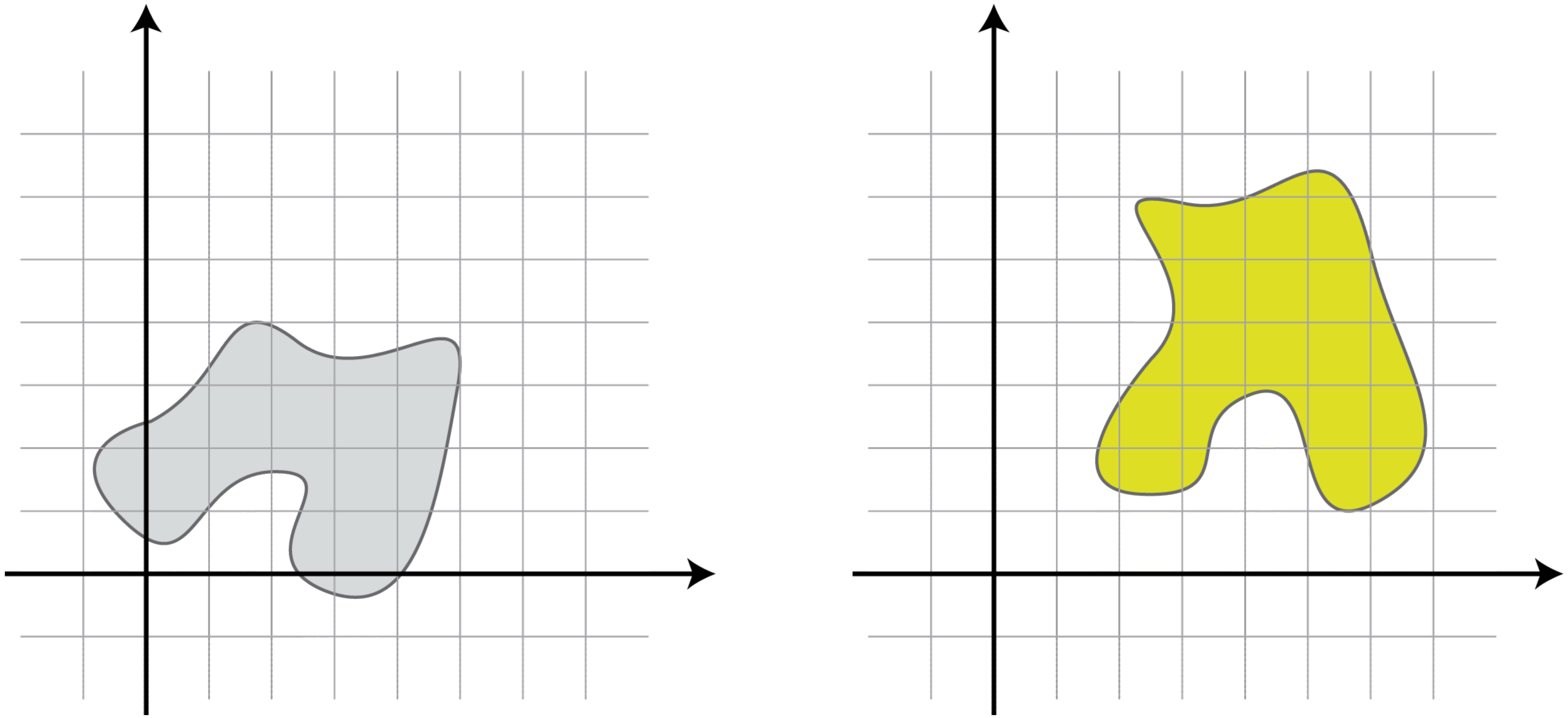
2D/3D Elasticity - The deformation map



2D/3D Elasticity - The deformation map



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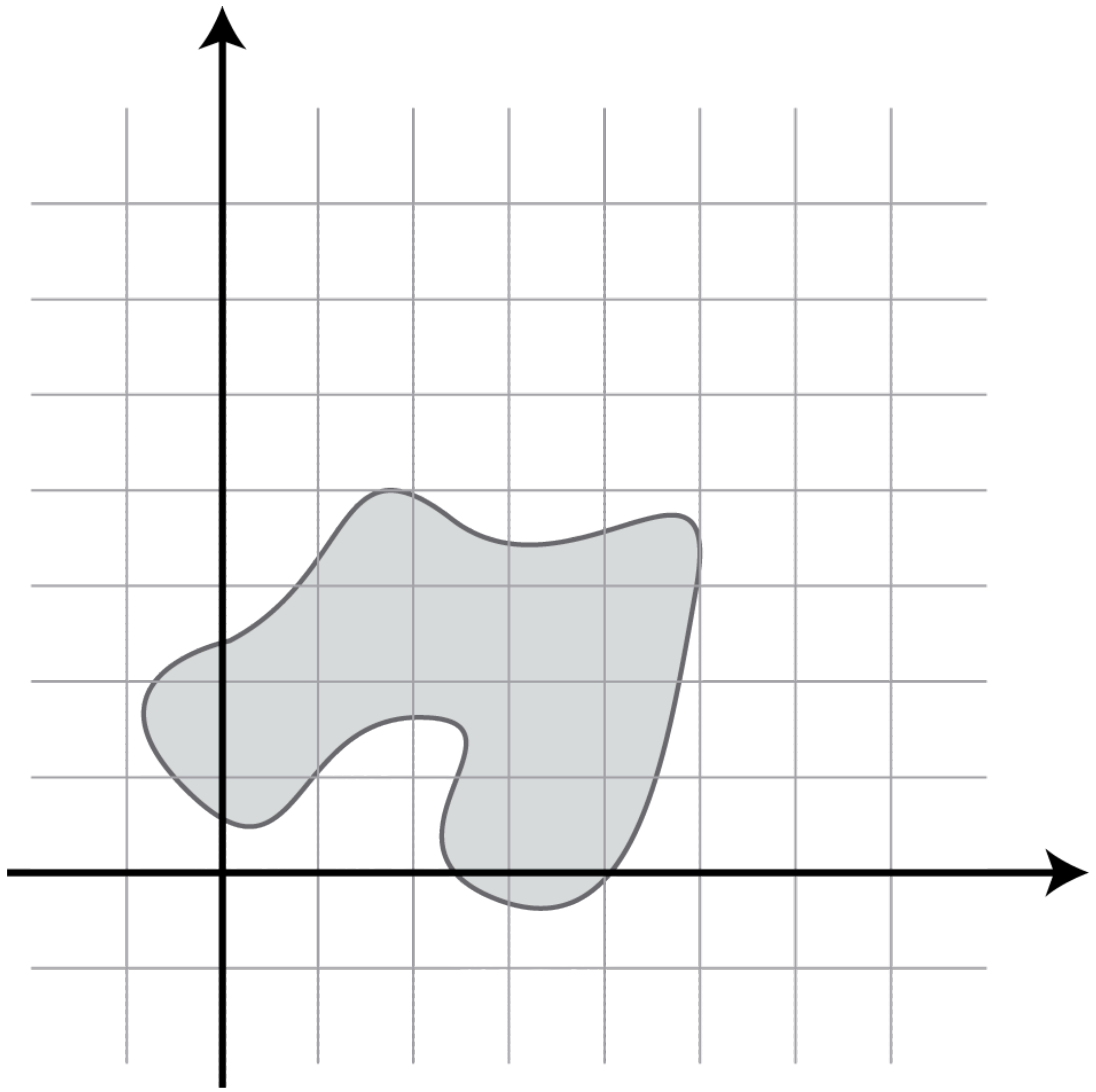
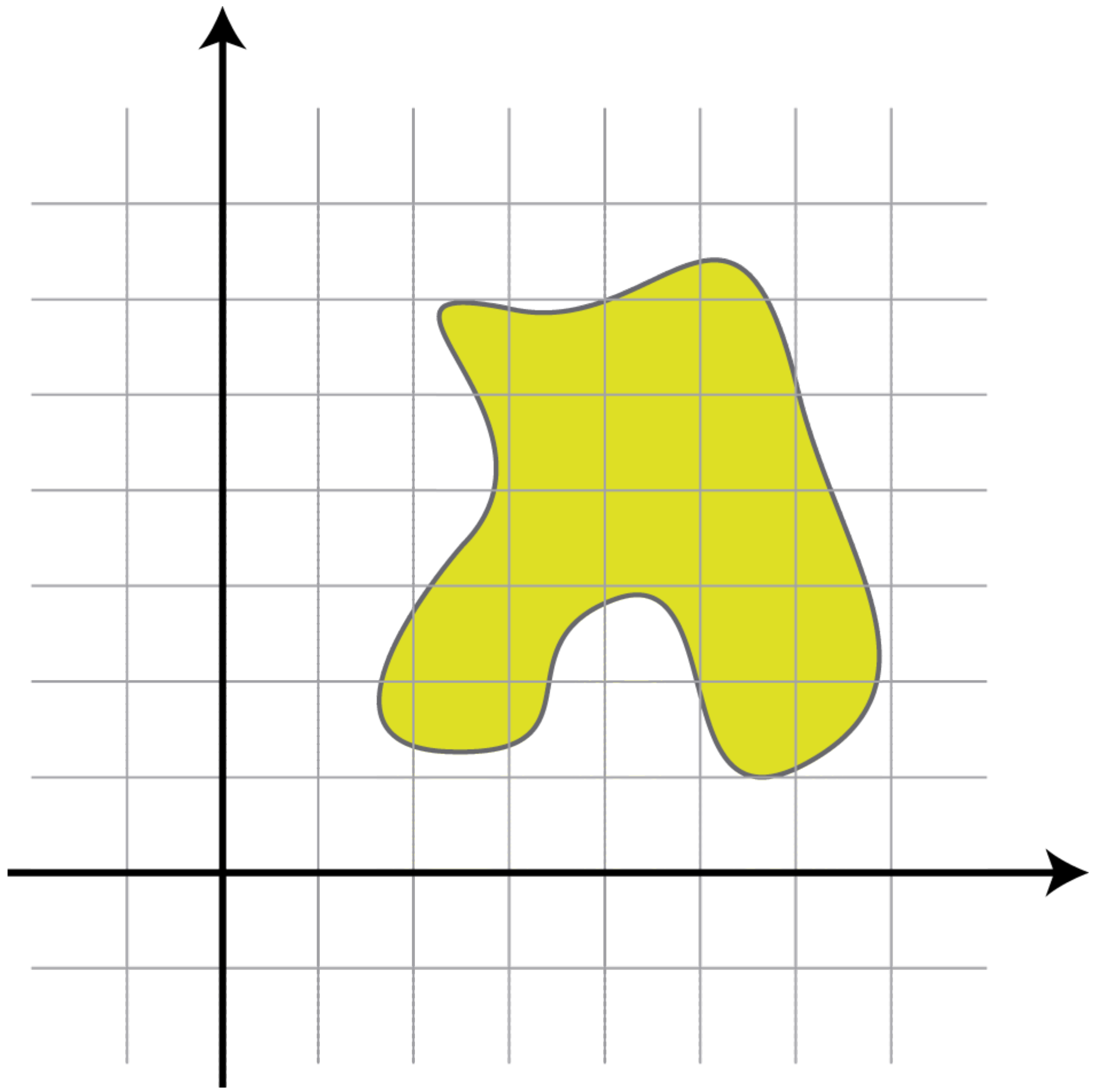
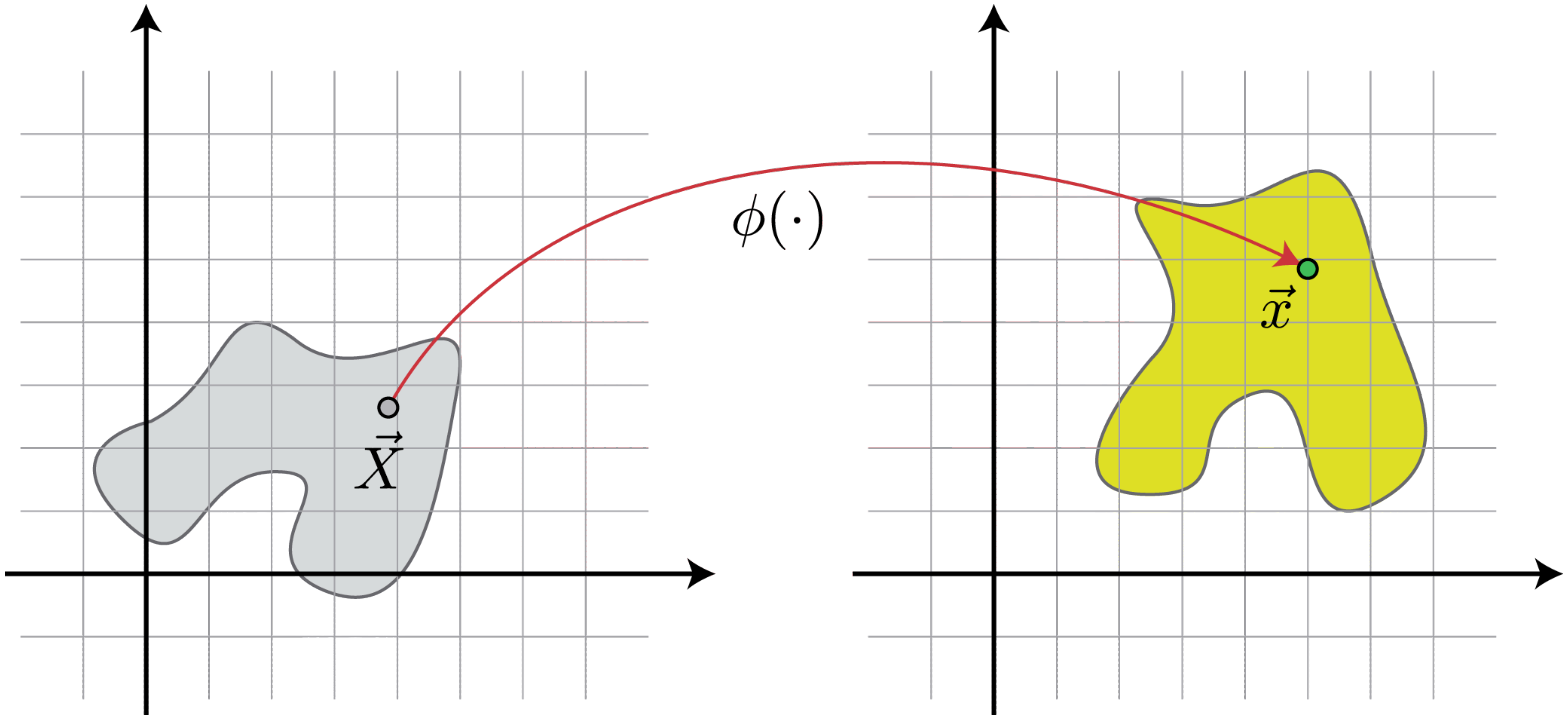


Undeformed configuration
*(**material** coordinates)*



Deformed configuration
*(**spatial** coordinates)*

2D/3D Elasticity - The deformation map



Undeformed configuration
*(**material** coordinates)*

Deformed configuration
*(**spatial** coordinates)*

2D/3D Elasticity - The deformation map

*$\phi(X)$ is a map from **R^3** to **R^3***

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \phi(\vec{X}) = \begin{pmatrix} x(X,Y,Z) \\ y(X,Y,Z) \\ z(X,Y,Z) \end{pmatrix}$$

***Deformation gradient:** the Jacobian of $\phi(X)$*

$$\mathbf{F} := \frac{\partial}{\partial \vec{X}} \phi(\vec{X}) = \begin{pmatrix} \partial x/\partial X & \partial x/\partial Y & \partial x/\partial Z \\ \partial y/\partial X & \partial y/\partial Y & \partial y/\partial Z \\ \partial z/\partial X & \partial z/\partial Y & \partial z/\partial Z \end{pmatrix}$$

2D/3D Elasticity - The deformation map

*$\phi(X)$ is a map from **R^3** to **R^3***

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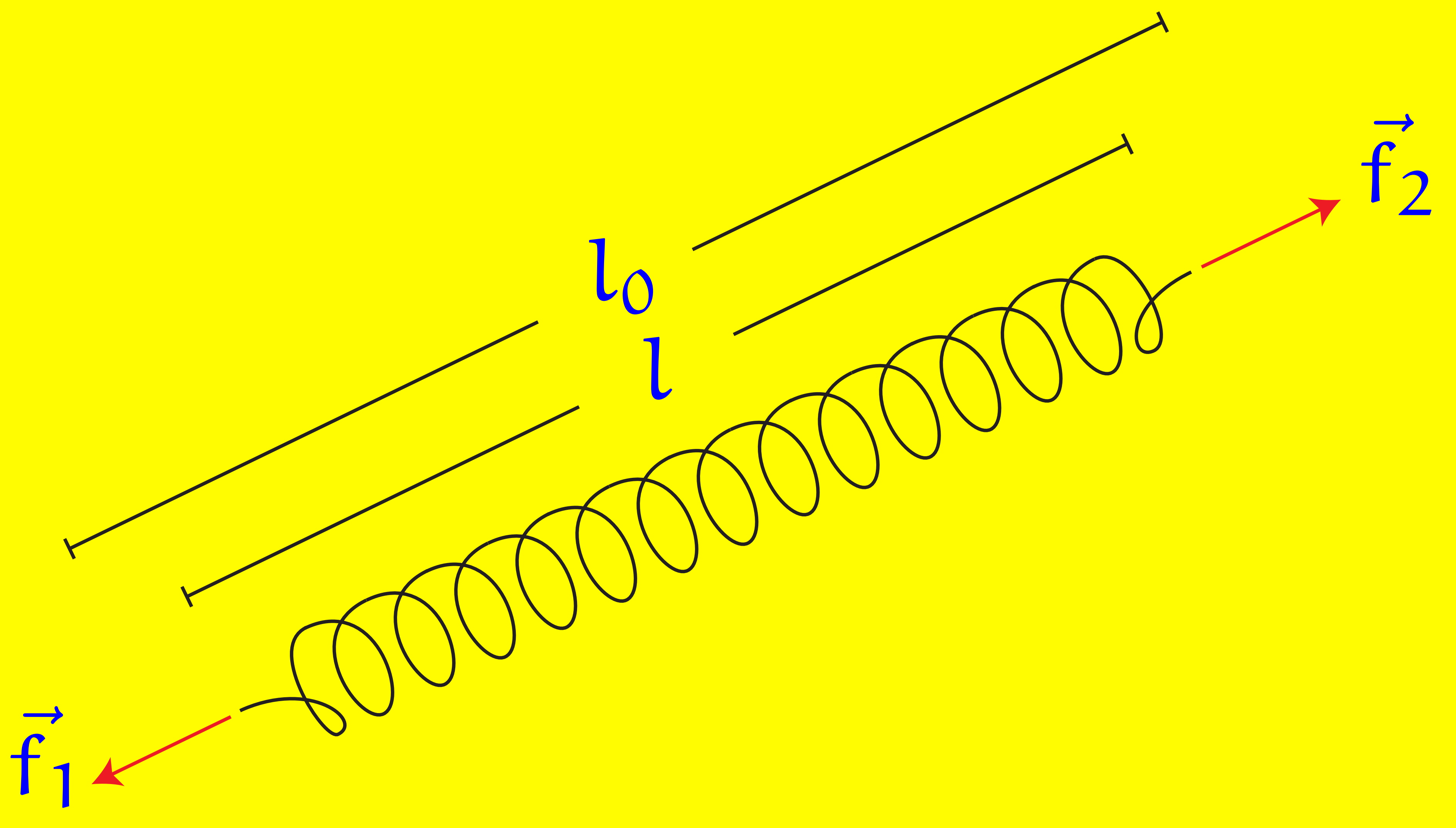
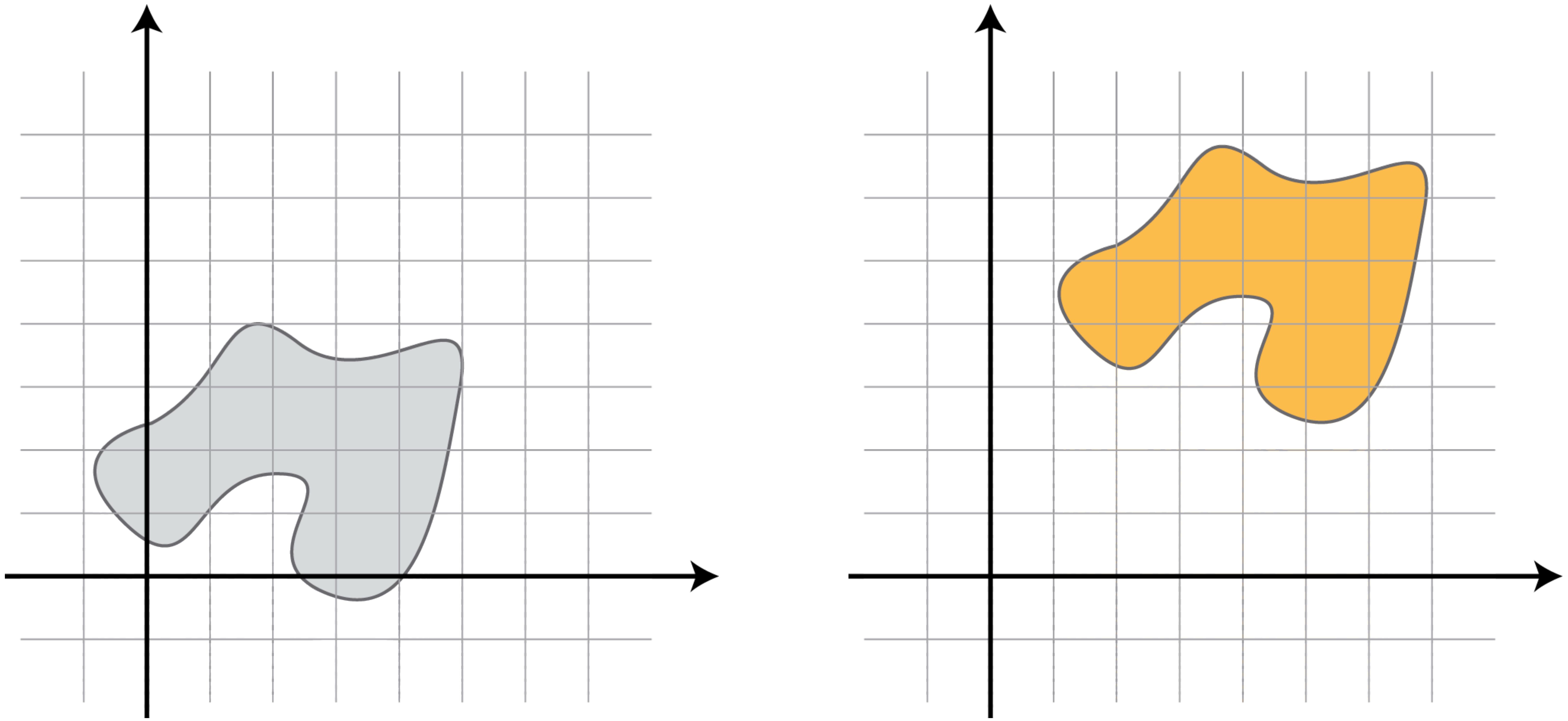
Spring analogue:



$$E = l_0 \frac{k}{2} \left(\boxed{\frac{l}{l_0}} - 1 \right)^2$$

2D/3D Elasticity - Deformation examples

Simple translation

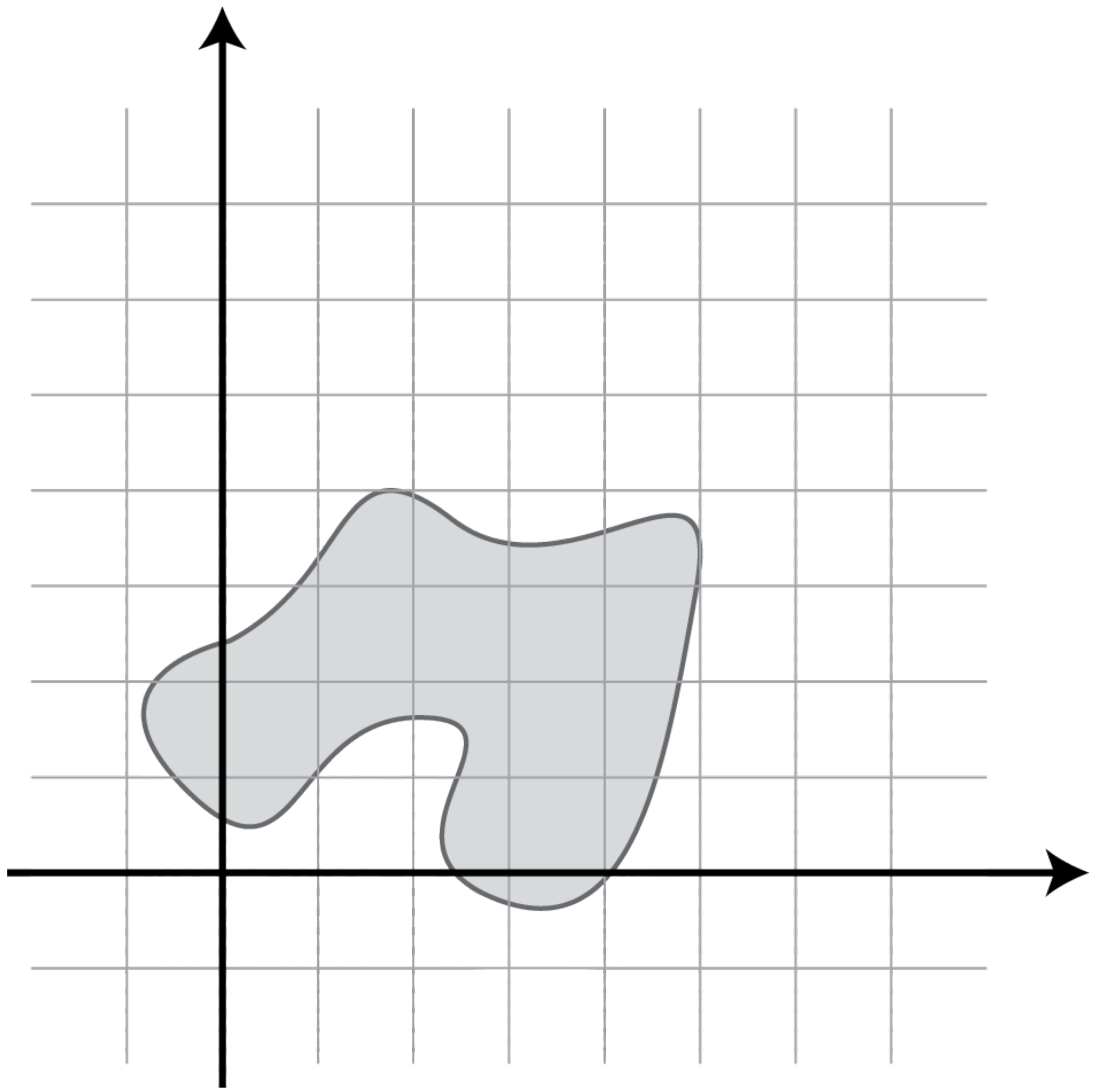


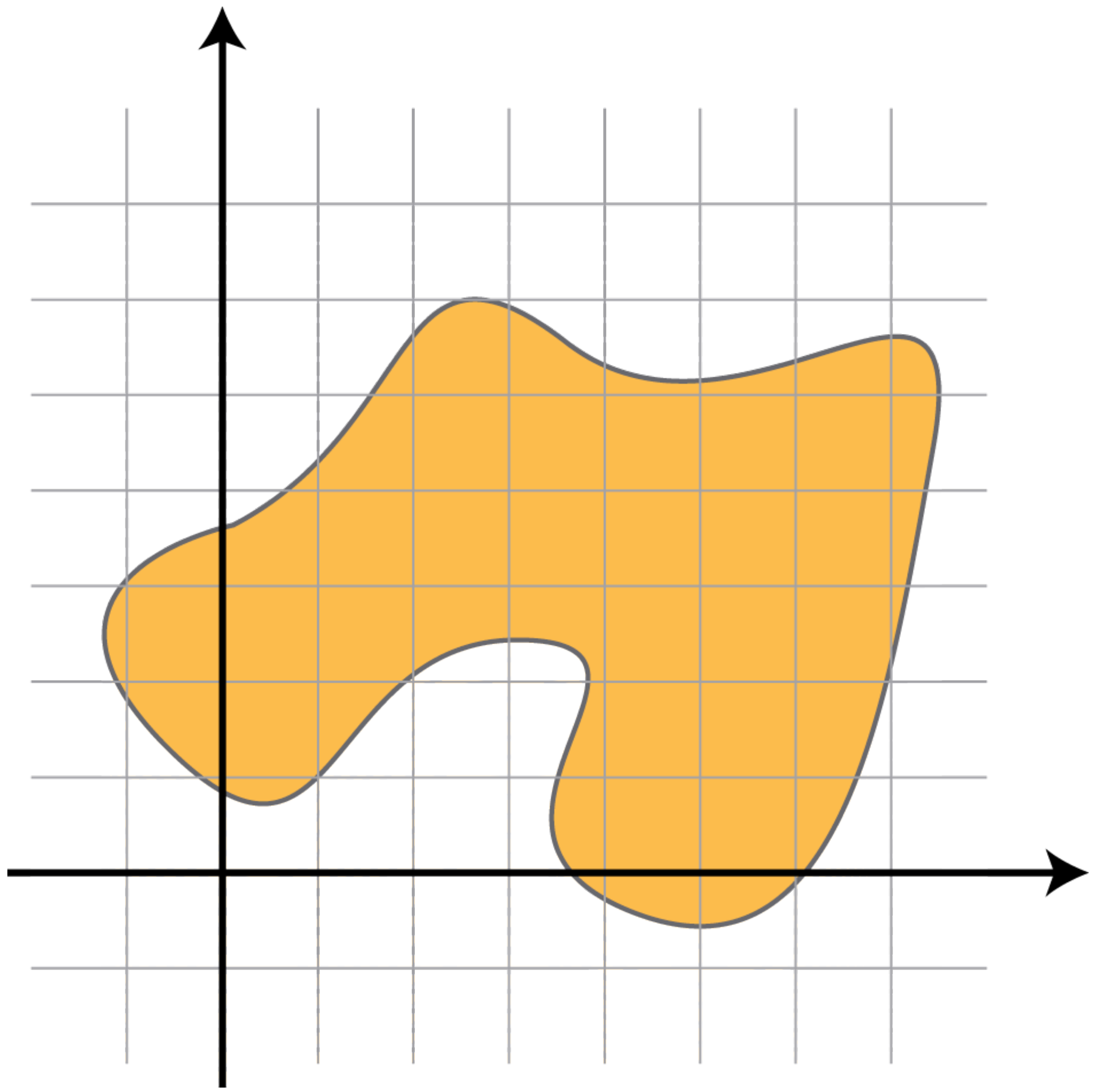
$$\vec{x} = \phi(\vec{X}) = \vec{X} + \vec{t}$$

$$\mathbf{F} = \mathbf{I}$$

2D/3D Elasticity - Deformation examples

Uniform Scaling



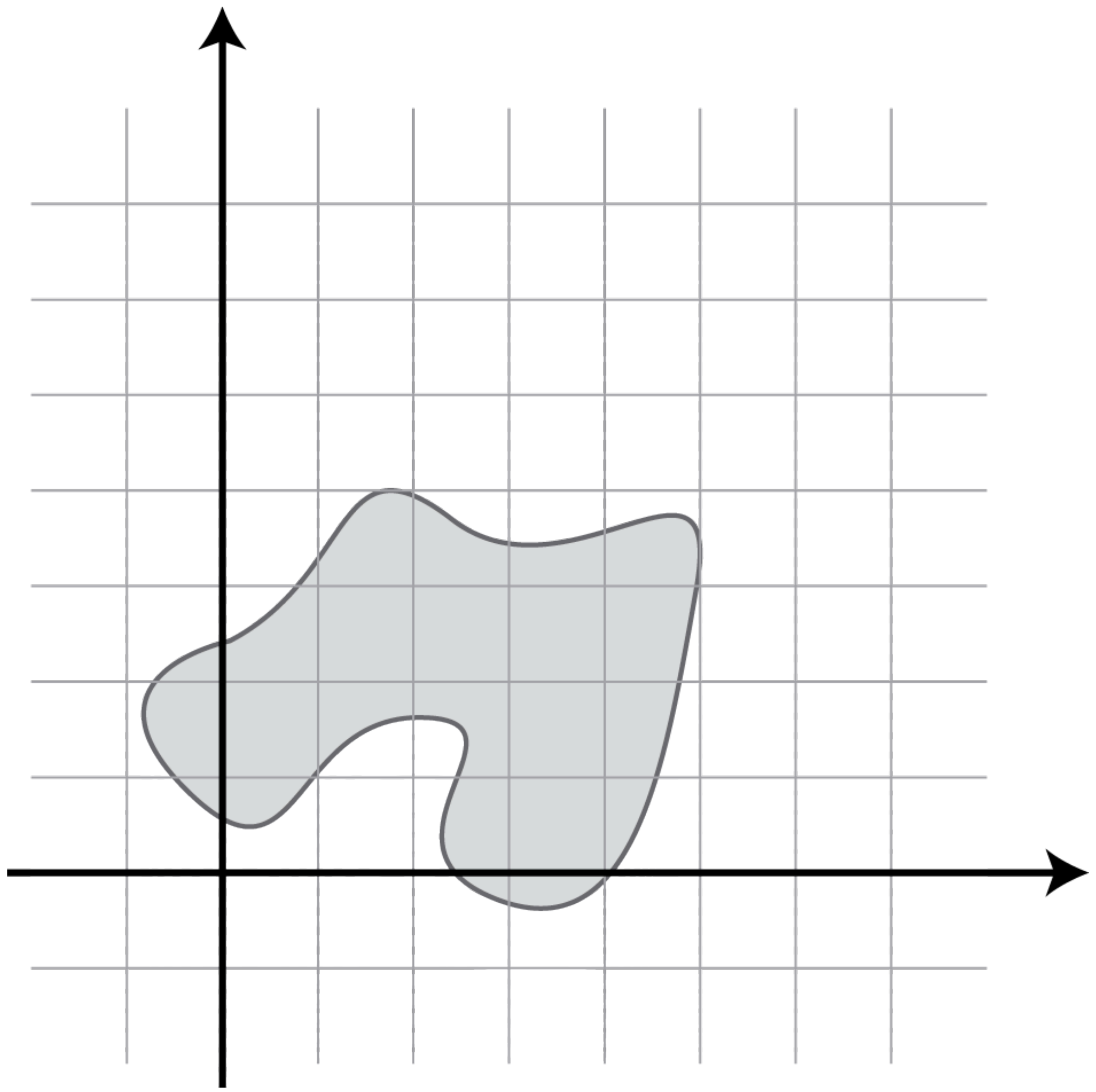


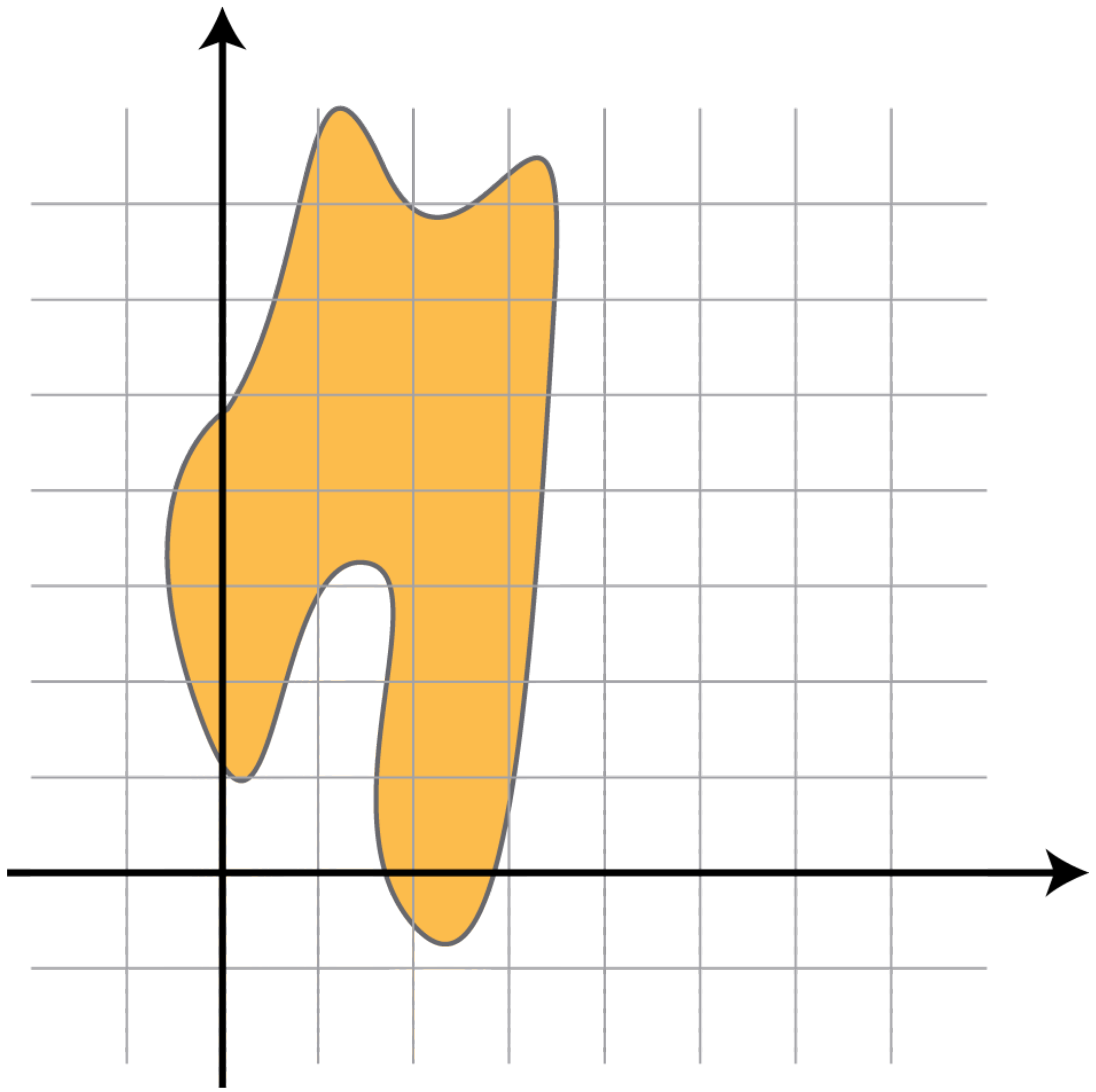
$$\vec{x} = \phi(\vec{X}) = \gamma \vec{X}$$

$$\mathbf{F} = \gamma \mathbf{I}$$

2D/3D Elasticity - Deformation examples

Anisotropic scaling



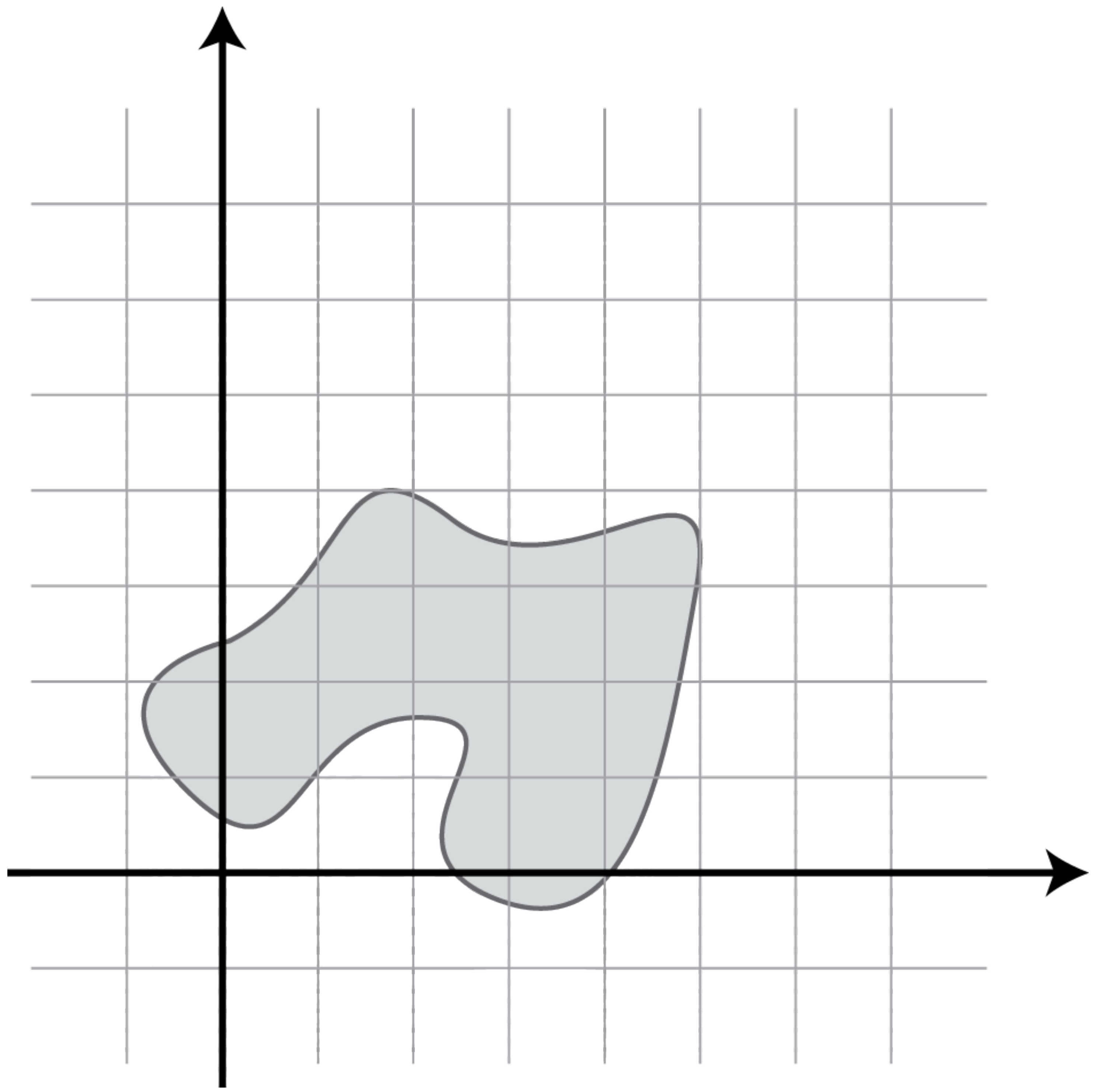


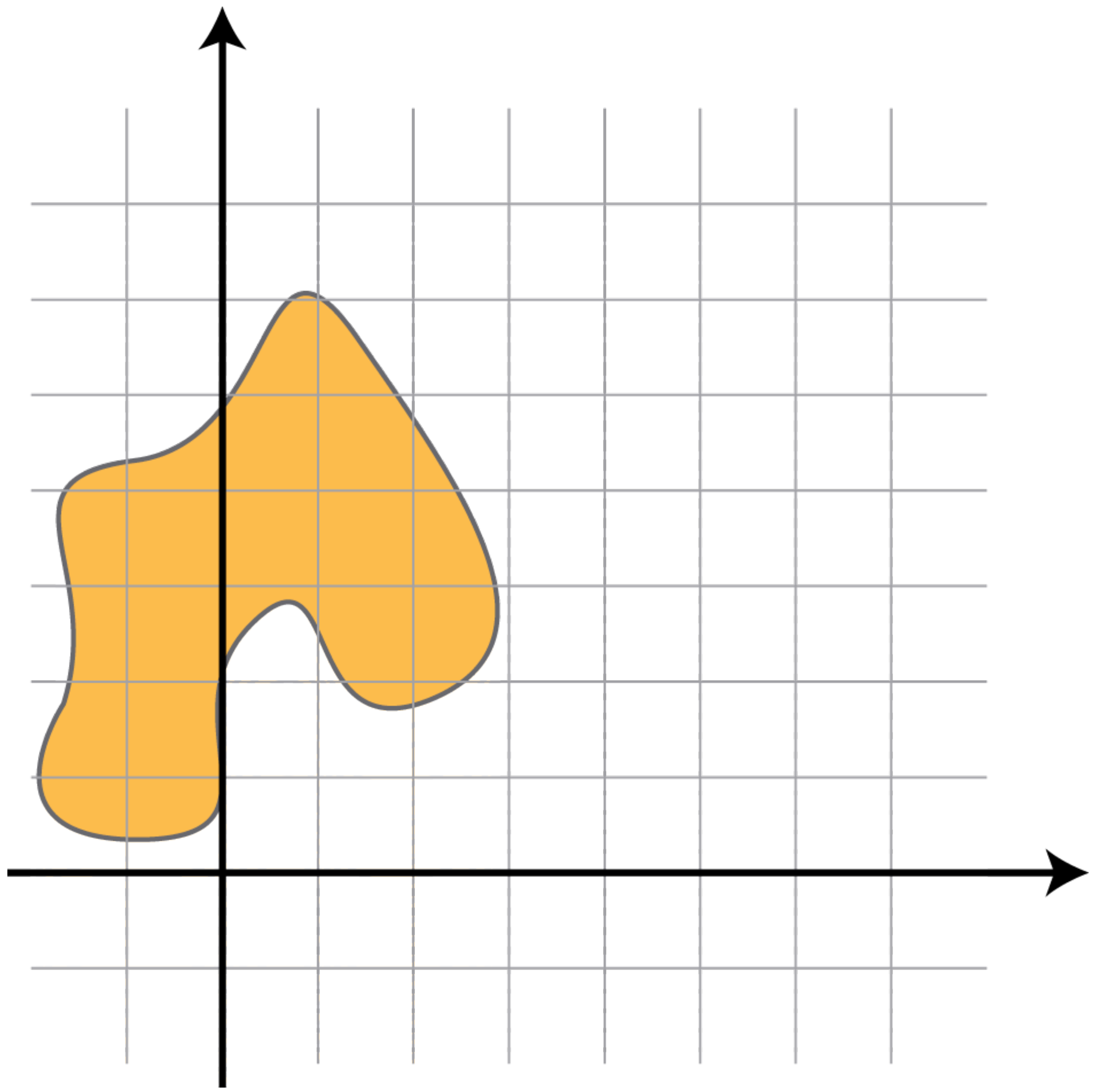
$$\vec{x} = \phi\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0.7X \\ 2Y \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 0.7 & 0 \\ 0 & 2 \end{pmatrix}$$

2D/3D Elasticity - Deformation examples

Rotation only



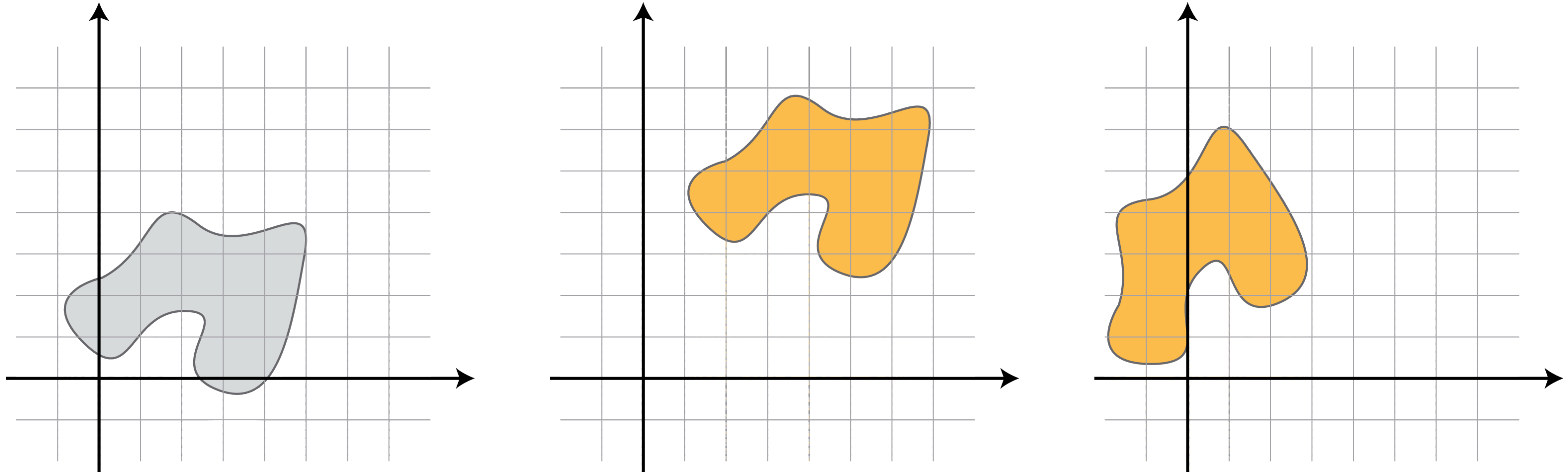


$$\vec{x} = \phi \begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{R}_{45^0} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\mathbf{F} = \mathbf{R}_{45^0}$$

2D/3D Elasticity - Strain measures

How do we quantify shape change? ~~$\mathbf{F}-\mathbf{I}$??~~



Translation
$\mathbf{F}=\mathbf{I}$

Rotation
$\mathbf{F}=\mathbf{R}$

2D/3D Elasticity - Strain measures

Strain measure: *A tensor (matrix) which encodes the severity of shape change*

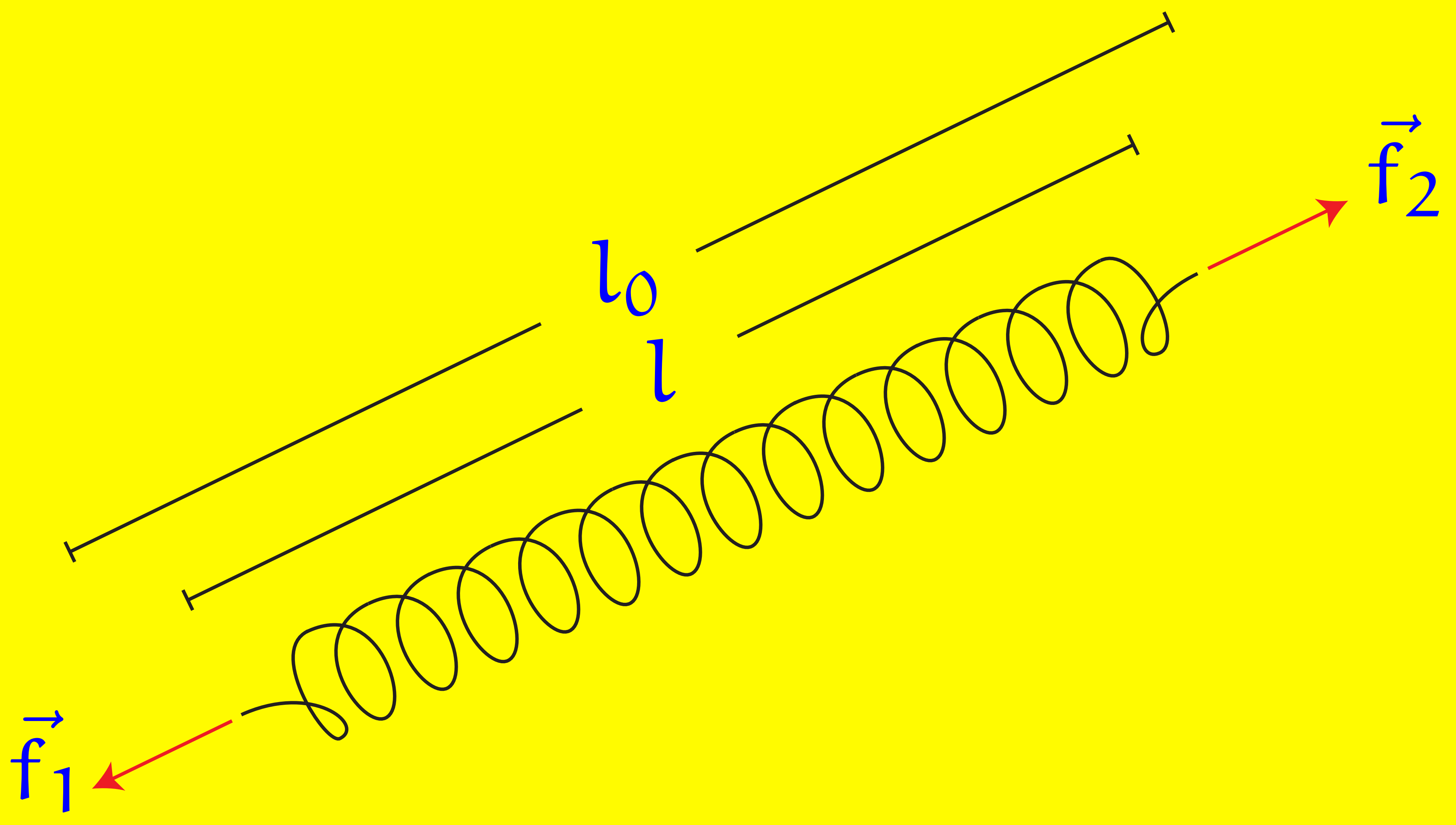
$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^\top\mathbf{F} - \mathbf{I})$$

Green strain

$$\epsilon = \frac{1}{2}(\mathbf{F} + \mathbf{F}^\top) - \mathbf{I}$$

Infinitesimal strain
(small strain tensor)

Spring analogue:



$$E = l_0 \frac{k}{2}\left(\frac{l}{l_0} - 1\right)^2$$

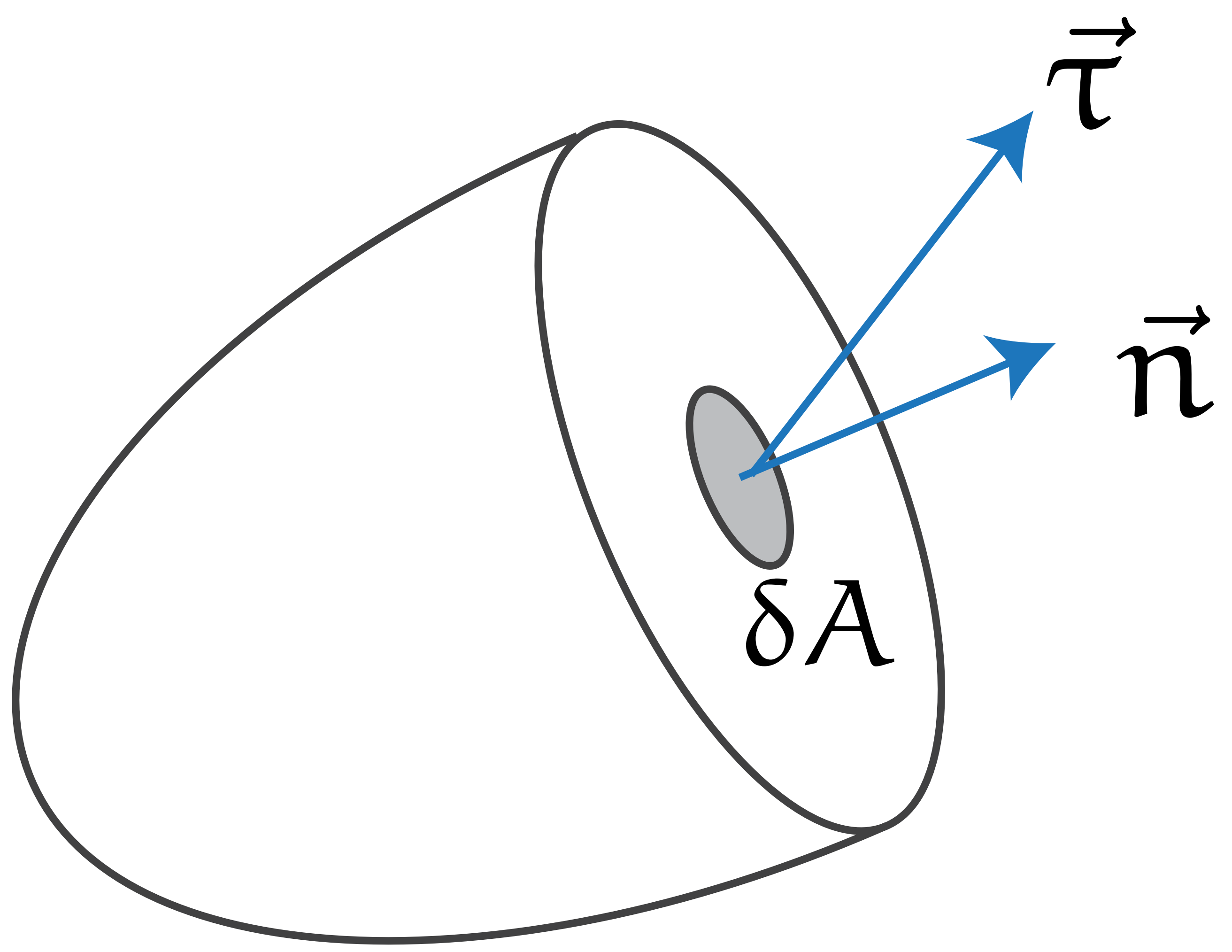
2D/3D Elasticity - Force, traction and stress

Force density (f) **:**

Measures the internal elastic *force per unit (undeformed) volume*

Traction (τ) **:**

Measures the *force per unit area* on a material **cross-section**



2D/3D Elasticity - Force, traction and stress

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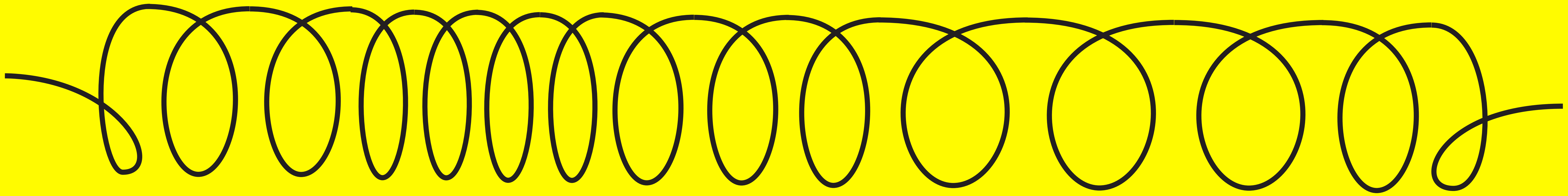
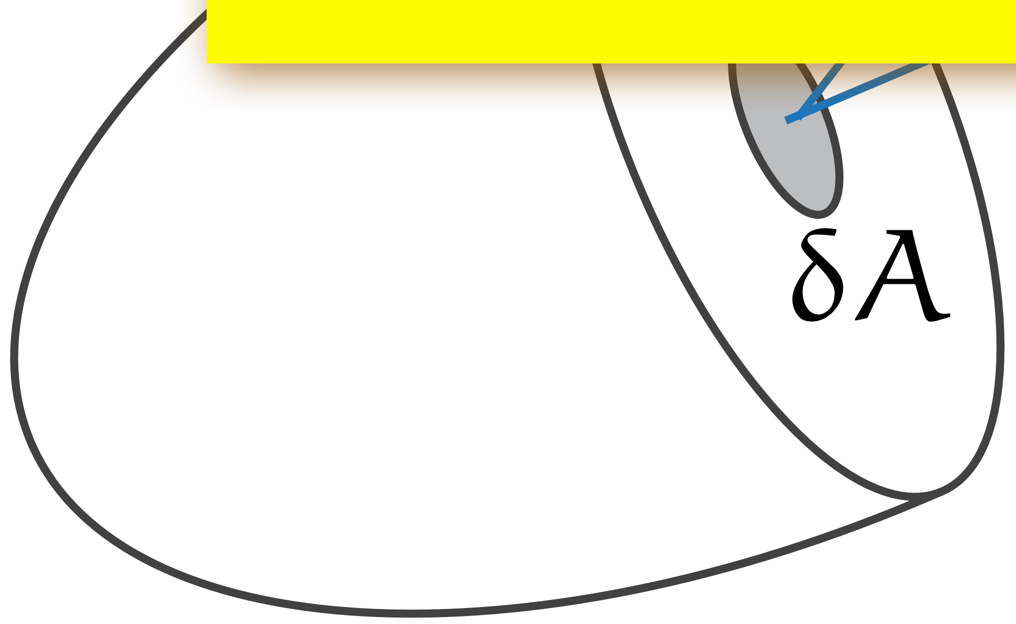
Tractio

Mea ... ion

What is the difference of force and traction?





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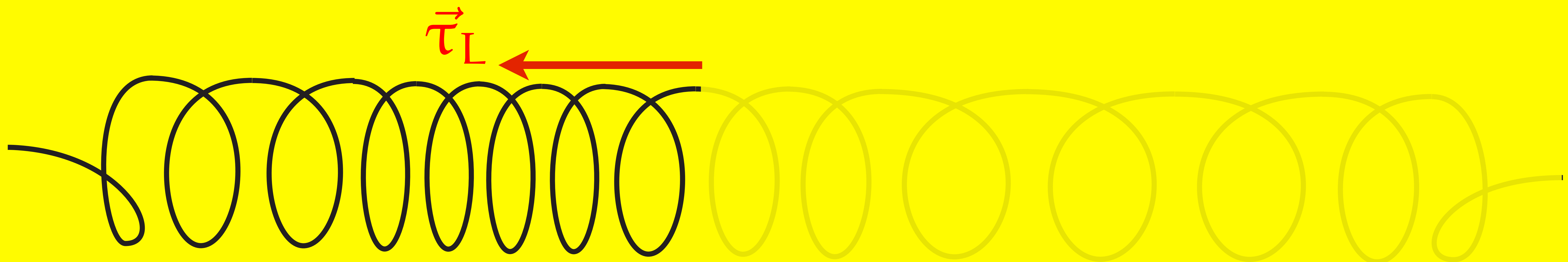

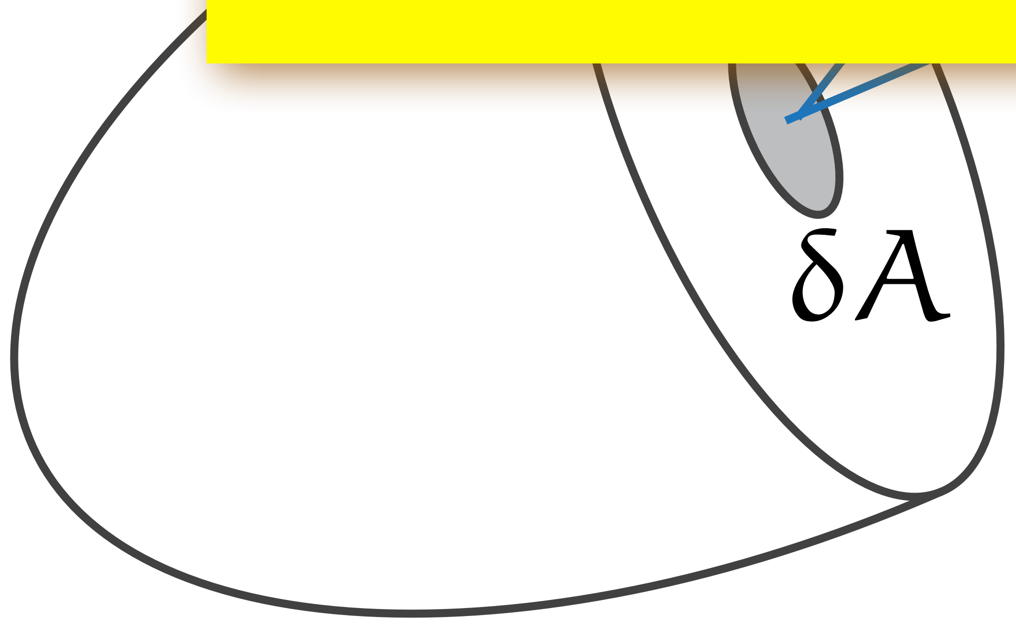

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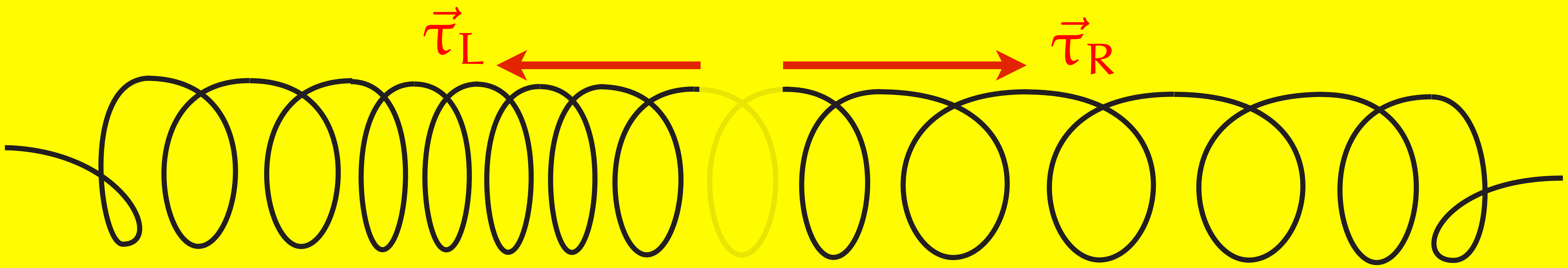
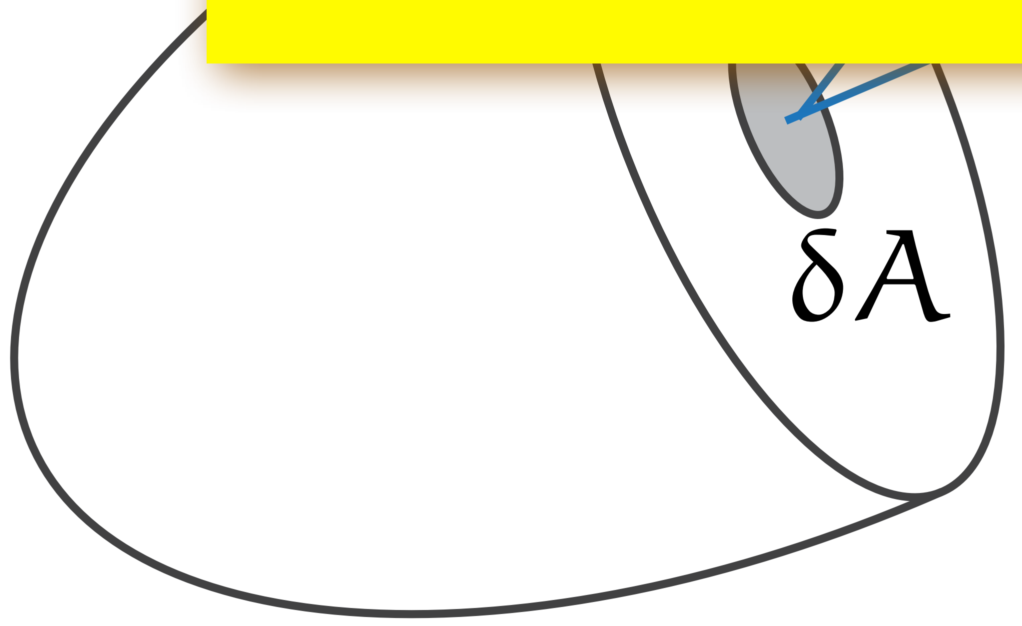




2D/3D Elasticity - Force, traction and stress

Force density (f) **:**

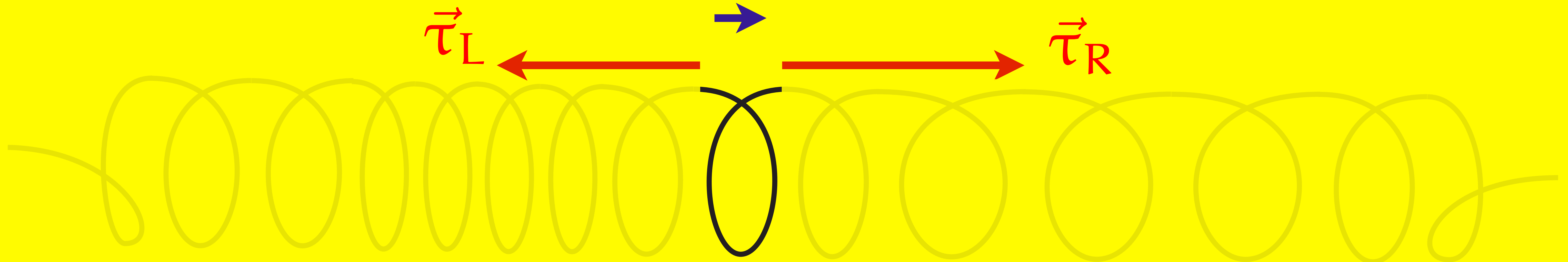
Measures the internal elastic *force per unit (undeformed) volume*

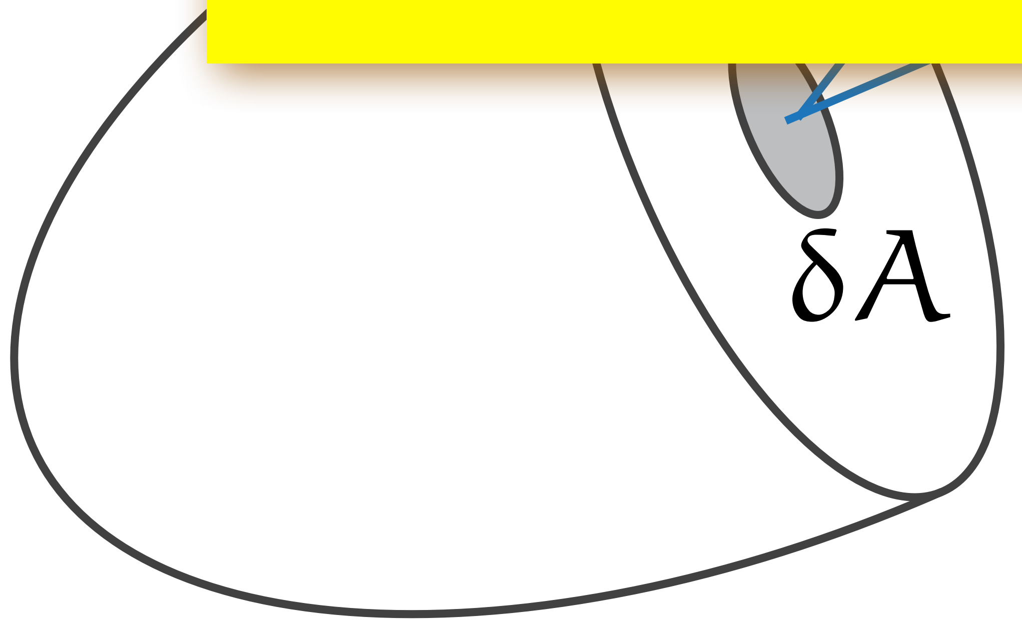
Tractio

Mea ... ion

What is the difference of force and traction?

$$\vec{f} = \vec{\tau}_L + \vec{\tau}_R$$





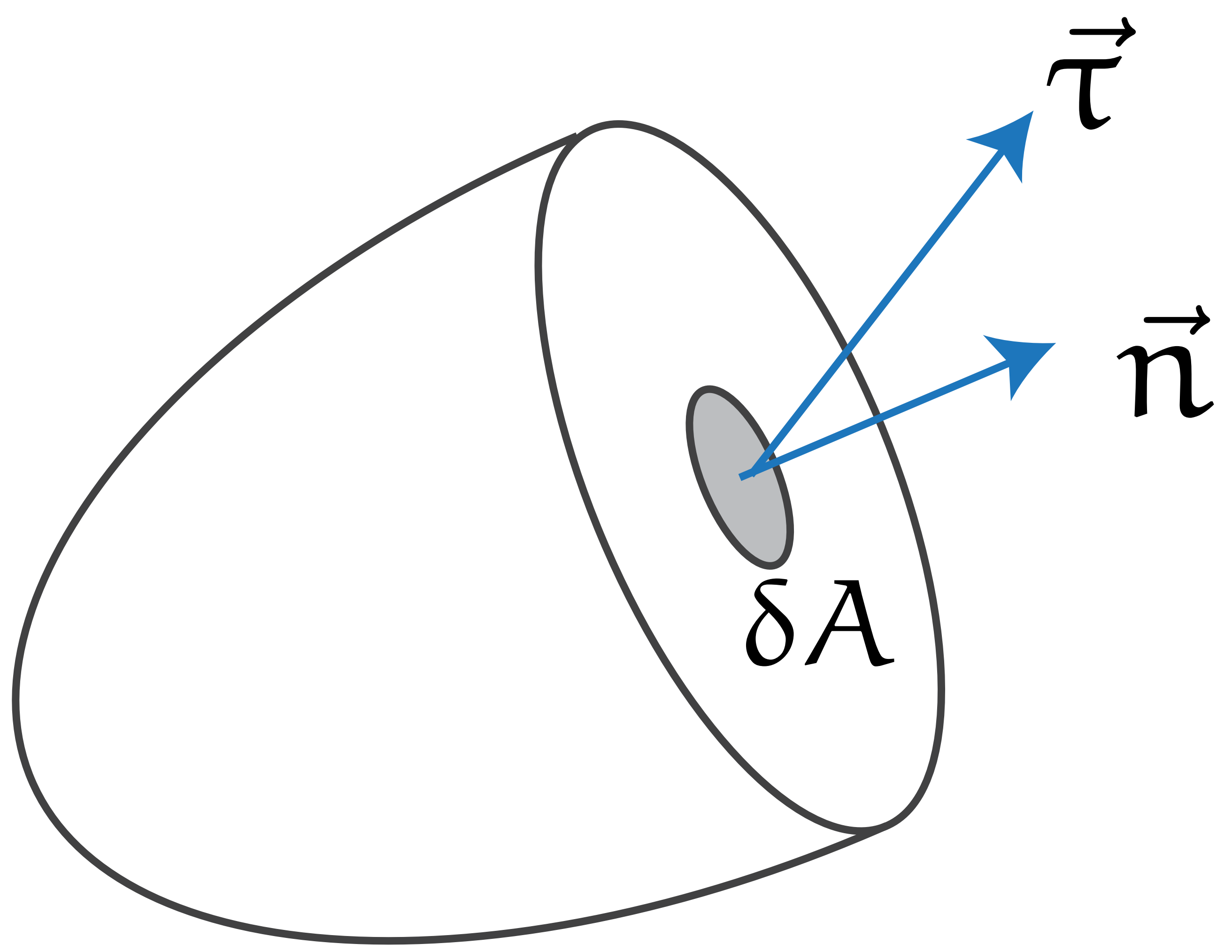
2D/3D Elasticity - Force, traction and stress

Force density (f) **:**

Measures the internal elastic *force per unit (undeformed) volume*

Traction (τ) **:**

Measures the *force per unit area* on a material **cross-section**

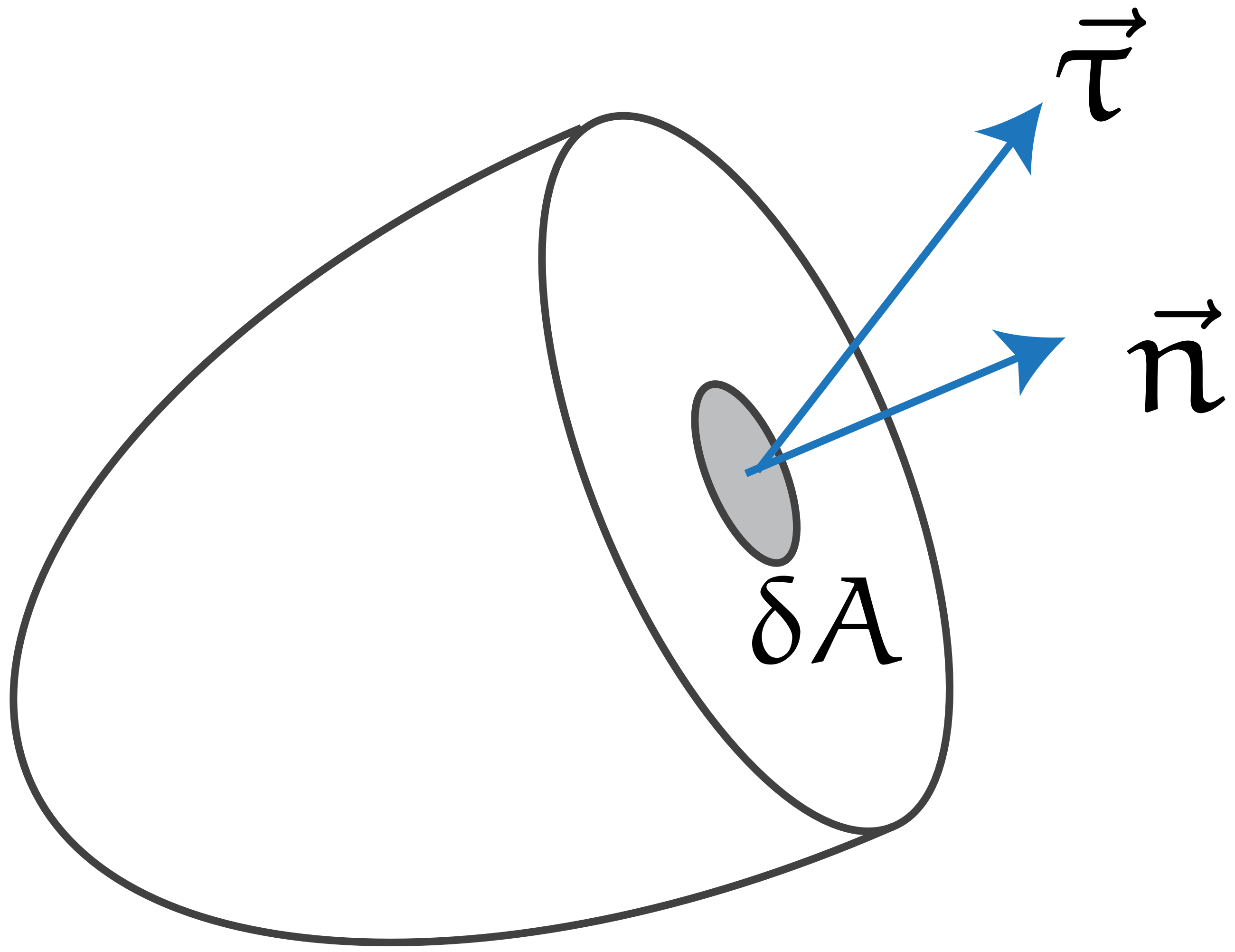


$$\vec{\tau} = \mathbf{P}\vec{n}$$

2D/3D Elasticity - Force, traction and stress

Traction (τ) **:**

Measures the *force per unit area* on a material **cross-section**



$$\vec{\tau} = \mathbf{P}\vec{n}$$

(Piola) Stress tensor (P) **:**

A matrix that describes force response along different orientations

2D/3D Elasticity - Strain energy

Deformation Energy *(E)* [also known as *strain energy*] **:**
Potential energy stored in elastic body, as a result of deformation.

Energy density *(Ψ)* **:**
Ratio of strain energy *per unit (undeformed) volume.*

$$E[\phi] := \int \Psi[\phi] d\vec{X}$$

Total potential energy

$$\Psi[\phi] := \Psi(\mathbf{F})$$

(for typical materials)

2D/3D Elasticity - Strain energy

Deformation Energy *(E)* [also known as
Potential energy stored in elastic body,

Energy density *(Ψ)* **:**
Ratio of strain energy *per unit (undefor*

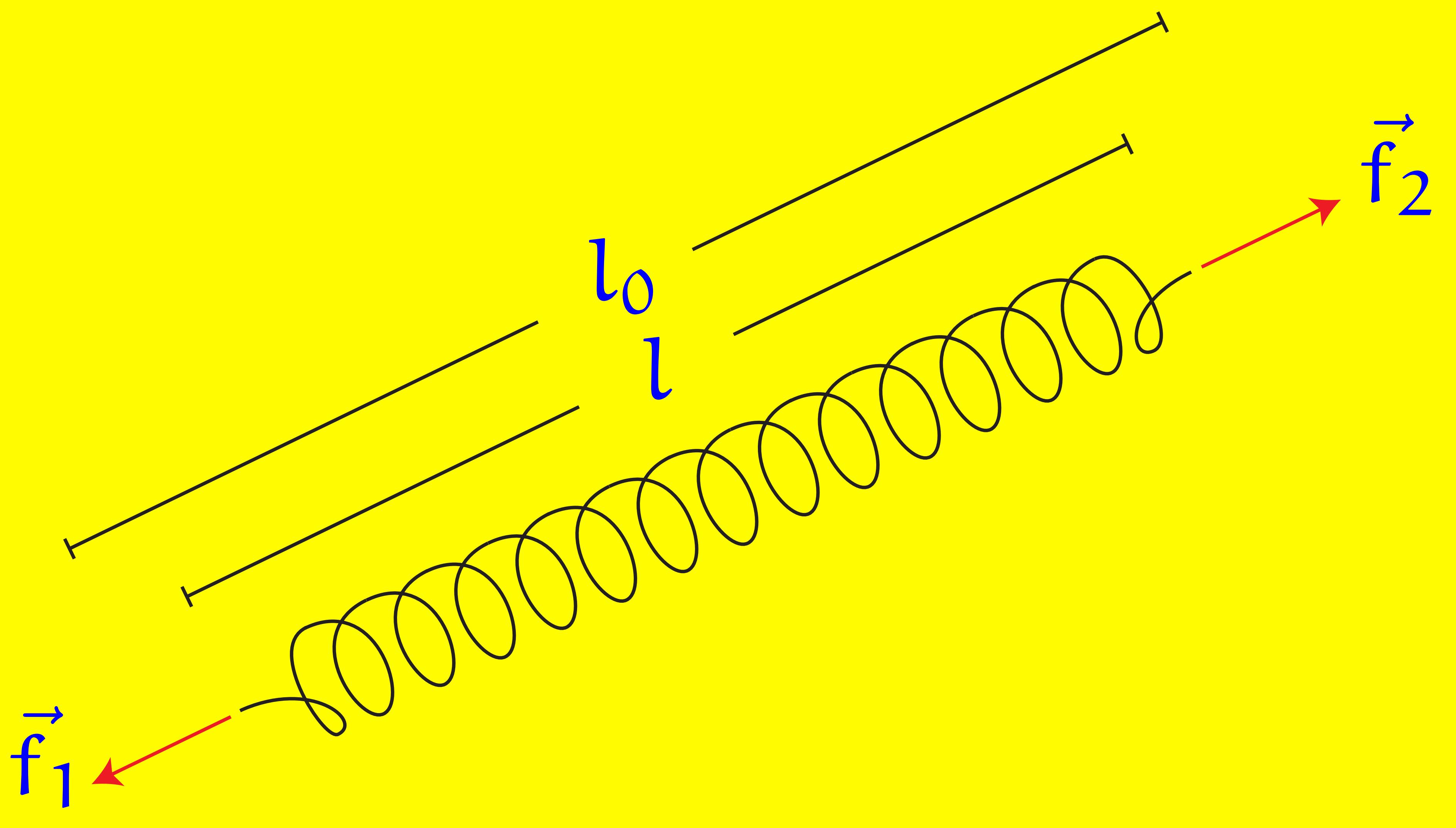
$$E[\phi] := \int \Psi[\mathbf{F}]\, d\vec{X}$$

Total potentia

$$\Psi[\phi] := \Psi(\mathbf{F})$$

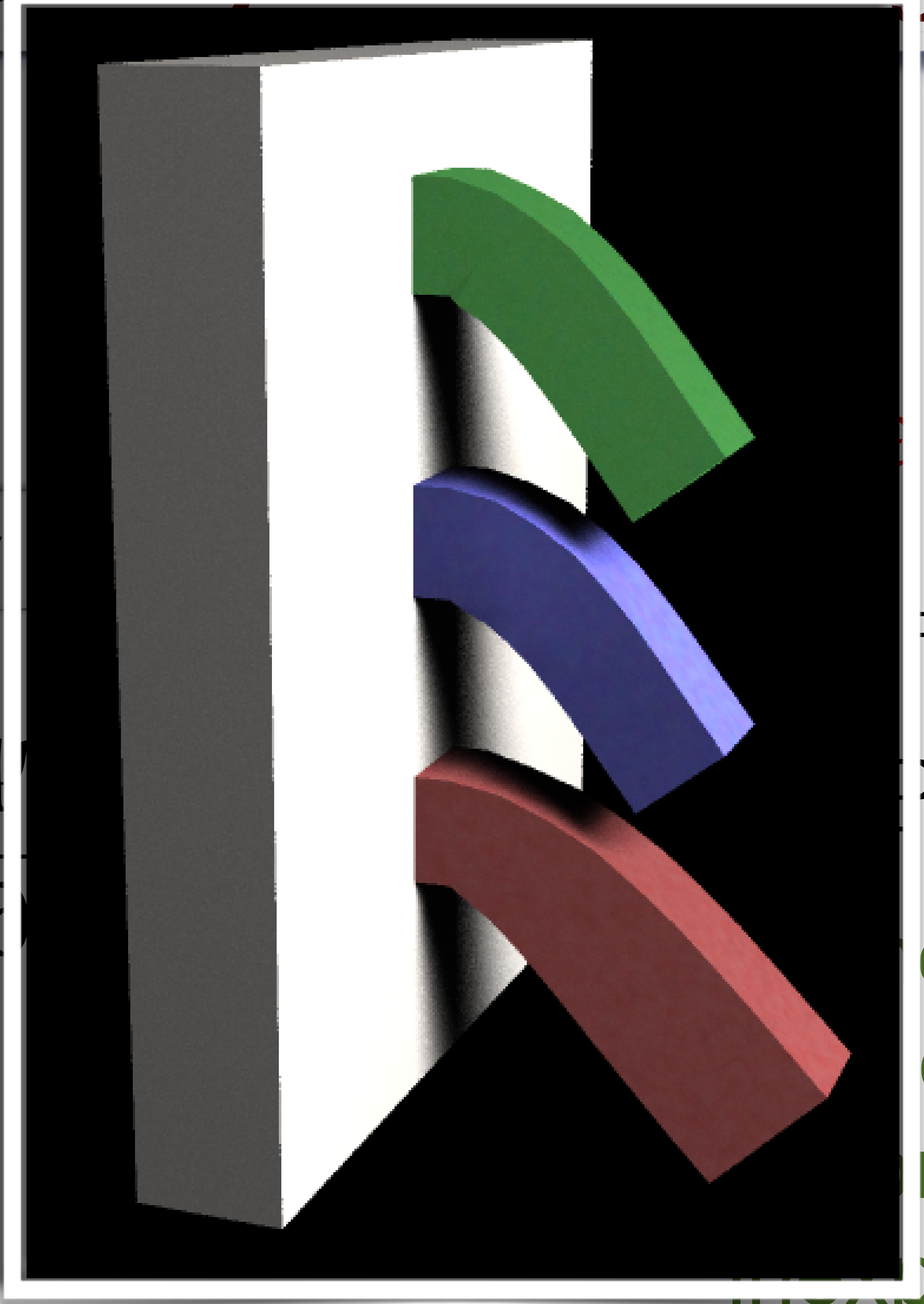
(for typical m

Spring analogue:



$$E = l_0 \underbrace{\frac{k}{2}\left(\frac{l}{l_0} - 1\right)^2}_{\Psi}$$

2D/3D Elasticity - Strain energy



Stress-energy

$$\mathbf{P} := \frac{\partial \Psi}{\partial \ldots}$$

Linear elasticity

$$= \tfrac{1}{2}(\mathbf{F} + \mathbf{F}^{\mathsf{T}}) - \mathbf{I}$$

$$= \mu \|\epsilon\|_F^2 + \tfrac{\lambda}{2}\mathrm{tr}^2(\epsilon)$$

$$= 2\mu\epsilon + \lambda \mathrm{tr}(\epsilon)\mathbf{I}$$

ear force-position
on
mputationally
pensive

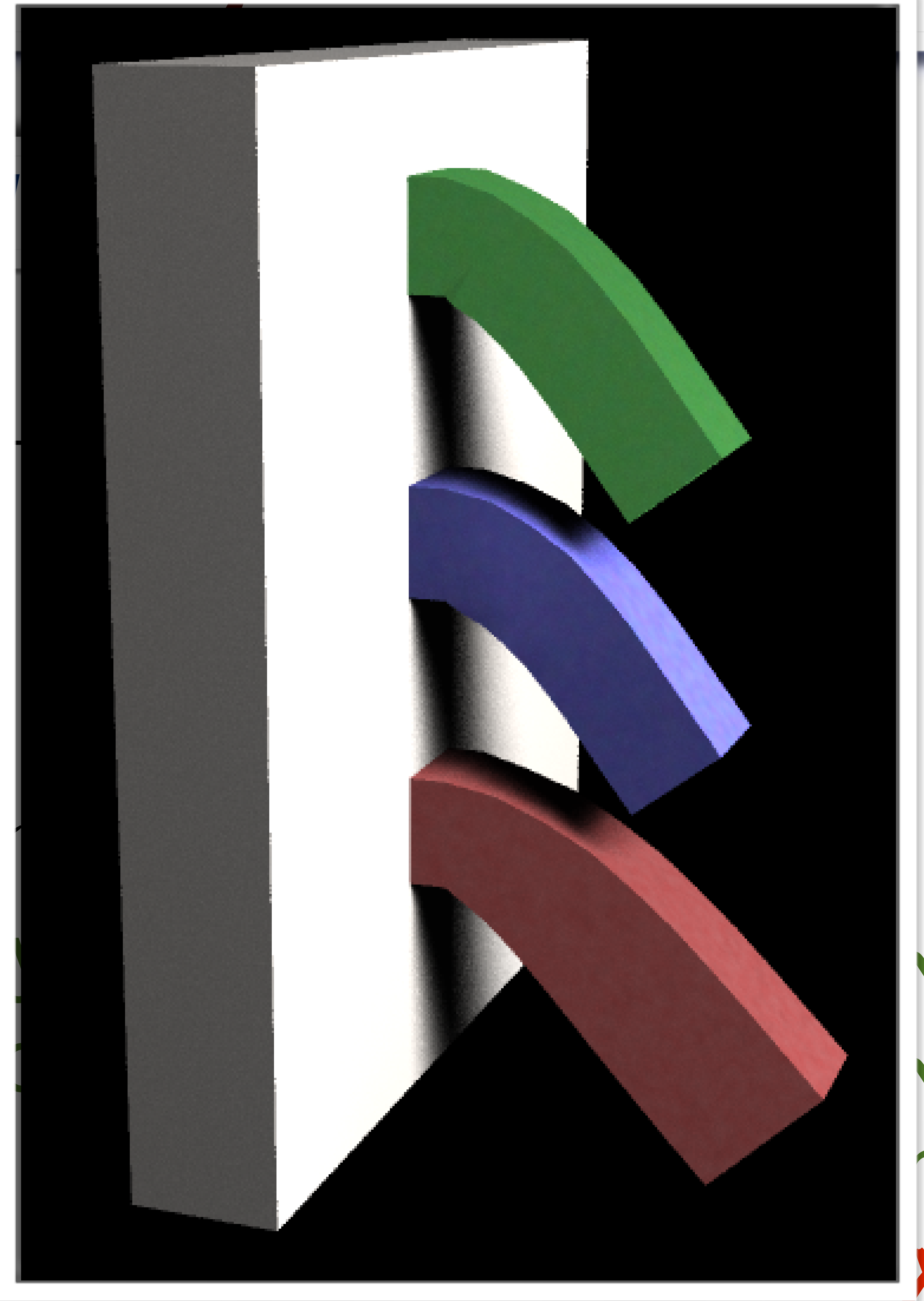
✗ Bad for large deformations

[Source: Müller et al, "Stable real-time deformations", 2002]

2D/3D Elasticity - Material models

Linear elasticity	*Corotated linear elasticity*
$\epsilon = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \mathbf{I}$	$\mathbf{E} = \mathbf{S} - \mathbf{I} \quad [\mathbf{F} = \mathbf{R}\mathbf{S}]$
$\Psi = \mu\|\epsilon\|_F^2 + \frac{\lambda}{2}\mathrm{tr}^2(\epsilon)$	$\Psi = \mu\|\mathbf{E_r}\|_F^2 + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E_r})$
$\mathbf{P} = 2\mu\epsilon + \lambda\mathrm{tr}(\epsilon)\mathbf{I}$	$\mathbf{P} = \mathbf{R}\left[2\mu\mathbf{E_r} + \lambda\mathrm{tr}(\mathbf{E_r})\mathbf{I}\right]$
✓Linear force-position relation	✓Rotationally invariant
✓Computationally inexpensive	✓Survives collapse & inversion
✗ Bad for large deformations	✗ Polar decomposition overhead
	✗ Inaccurate volume

2D/3D Elasticity - Material models



Corotated

$$\mathbf{E} = \mathbf{S} -$$

$$\Psi = \mu\|\mathbf{E}$$

$$\mathbf{P} = \mathbf{R}\,[2\mu$$

✓ Rotational

✓ Survives c

✗ Polar deco
overhead

✗ Inaccurate volume

St. Venant-Kirchhoff material

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

$$\Psi = \mu\|\mathbf{E}\|_F^2 + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E})$$

$$\mathbf{P} = \mathbf{F}\,[2\mu\mathbf{E} + \lambda\mathrm{tr}(\mathbf{E})\mathbf{I}]$$

✓ Rotationally invariant

✓ No polar decomposition needed

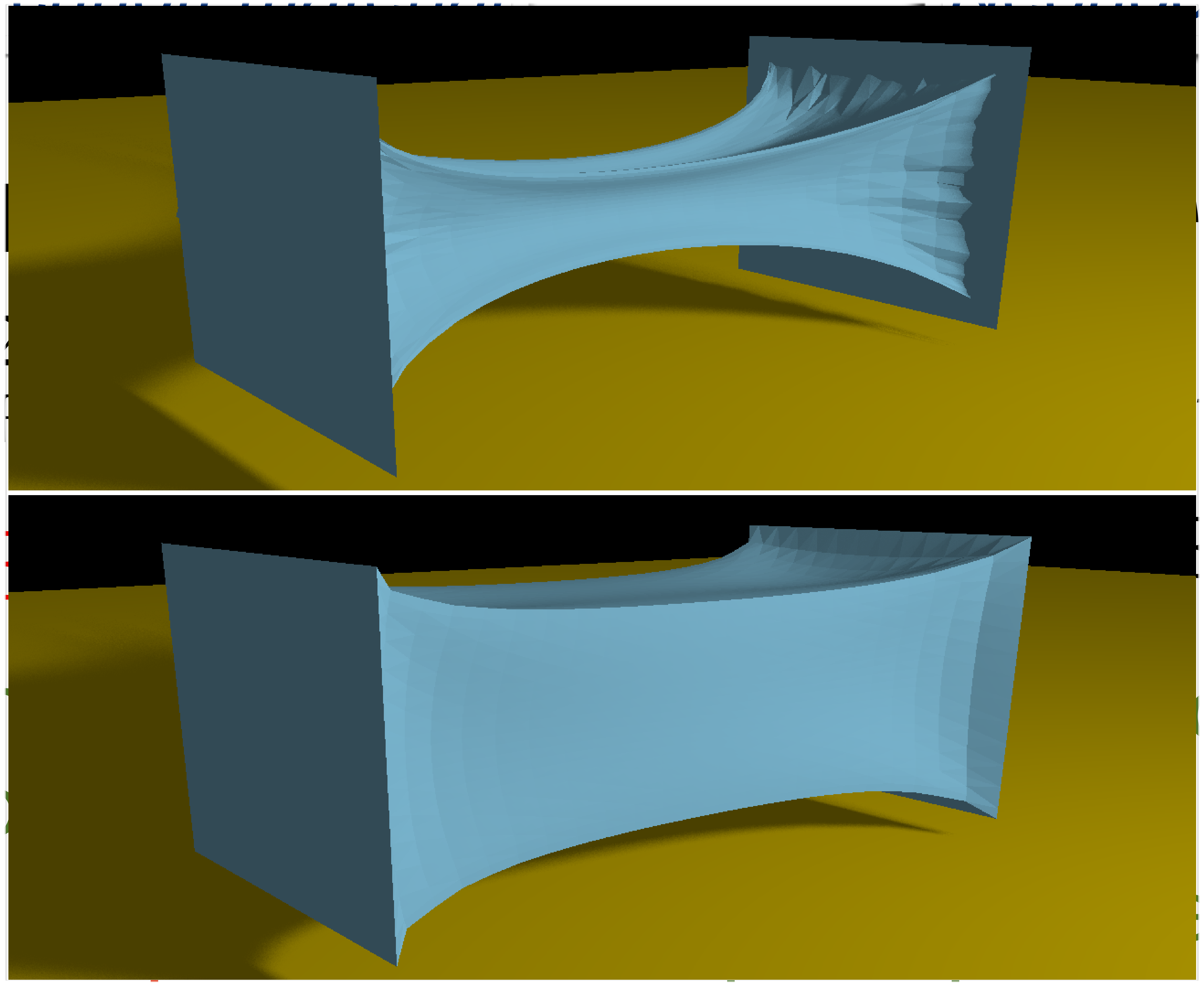
✗ Weak resistance to compression

[Source: Müller et al, "Stable real-time deformations", 2002]

2D/3D Elasticity - Material models

St. Venant-Kirchhoff material

Neohookean elasticity



$\mathbf{E} = \frac{1}{2}($

$\Psi = \mu \|\mathbf{E}\|$

$\mathbf{P} = \mathbf{F}\,[2\mu]$

$\|_F^2, \quad J = \det \mathbf{F}$

$- \mu \log(J) + \frac{\lambda}{2} \log^2(J)$

$^{-T}) + \lambda \log(J)\mathbf{F}^{-T}$

✓ Rotationally i

✓ No polar deco needed

✗ Weak resistance to compression

olume

s collapse/ inversion

✗ Undefined when inverted

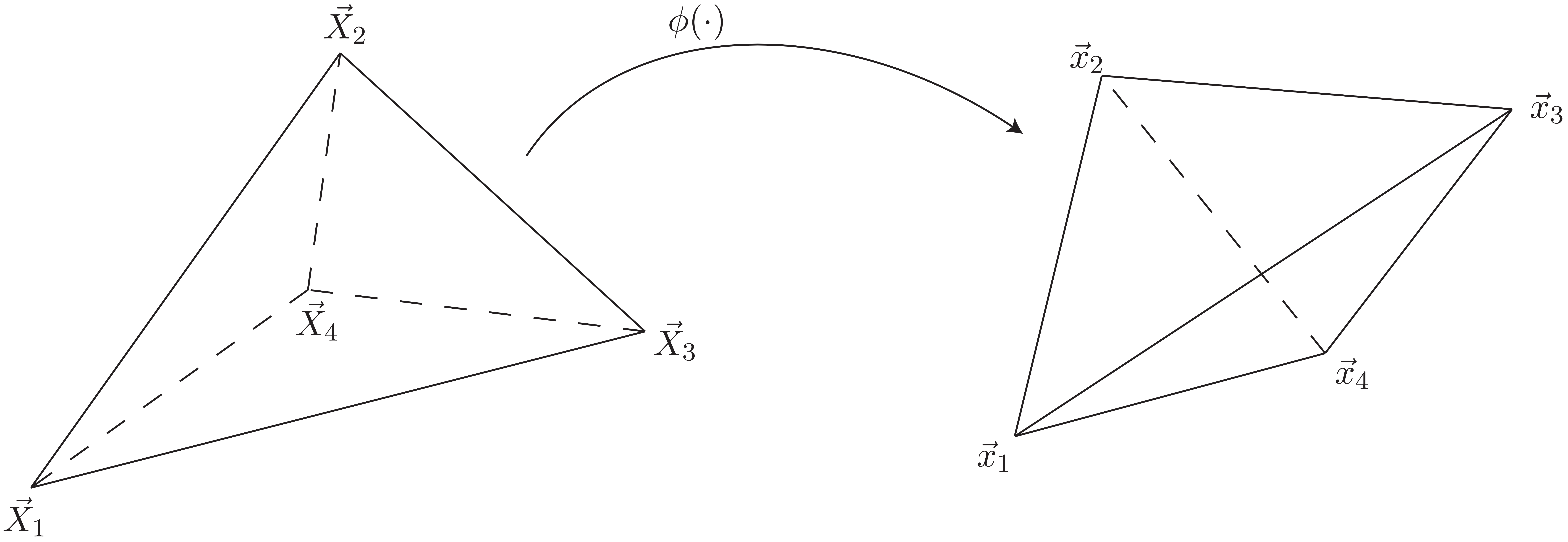
Additional information on course notes

- ✓ Extended discussion of rotational invariance, isotropy and the common isotropic invariants
- ✓ PDE form of elasticity equations
- ✓ Stress formulas for general isotropic materials
- ✓ Benefits and drawbacks of individual material models

SIGGRAPH 2012

The **39th** International **Conference** and **Exhibition**
on **Computer Graphics** and **Interactive Techniques**

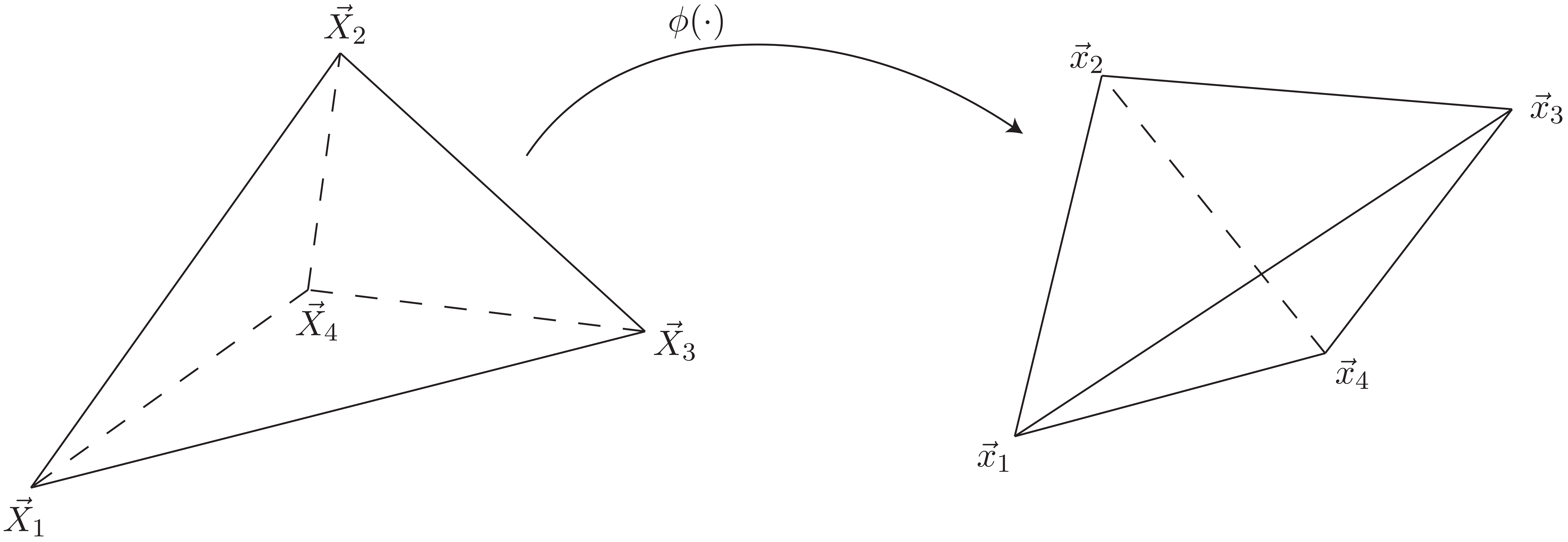
Tetrahedral models - Deformation measures



Undeformed shape

Deformed shape

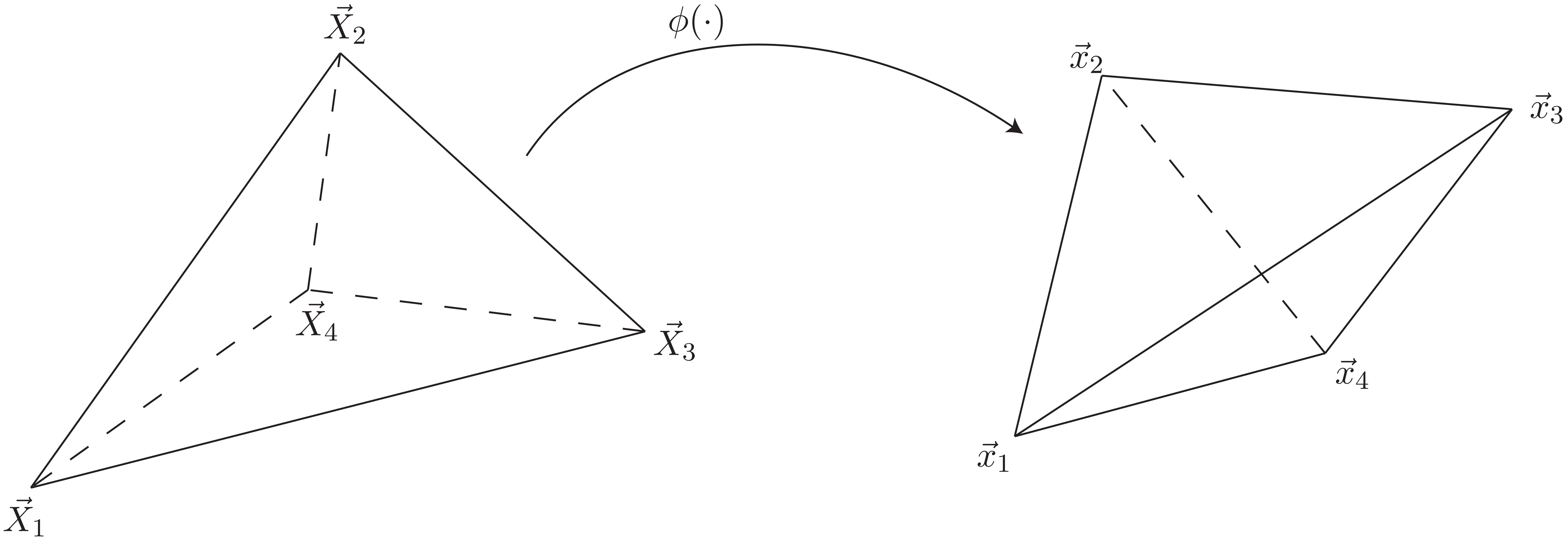
Tetrahedral models - Deformation measures



Hypothesis:
On every tetrahedron
*$\phi(X)$ is **linear!***

$$
\begin{aligned}
x &= a_{11}X + a_{12}Y + a_{13}Z + b_1 \\
y &= a_{21}X + a_{22}Y + a_{23}Z + b_2 \\
z &= a_{31}X + a_{32}Y + a_{33}Z + b_3
\end{aligned}
$$

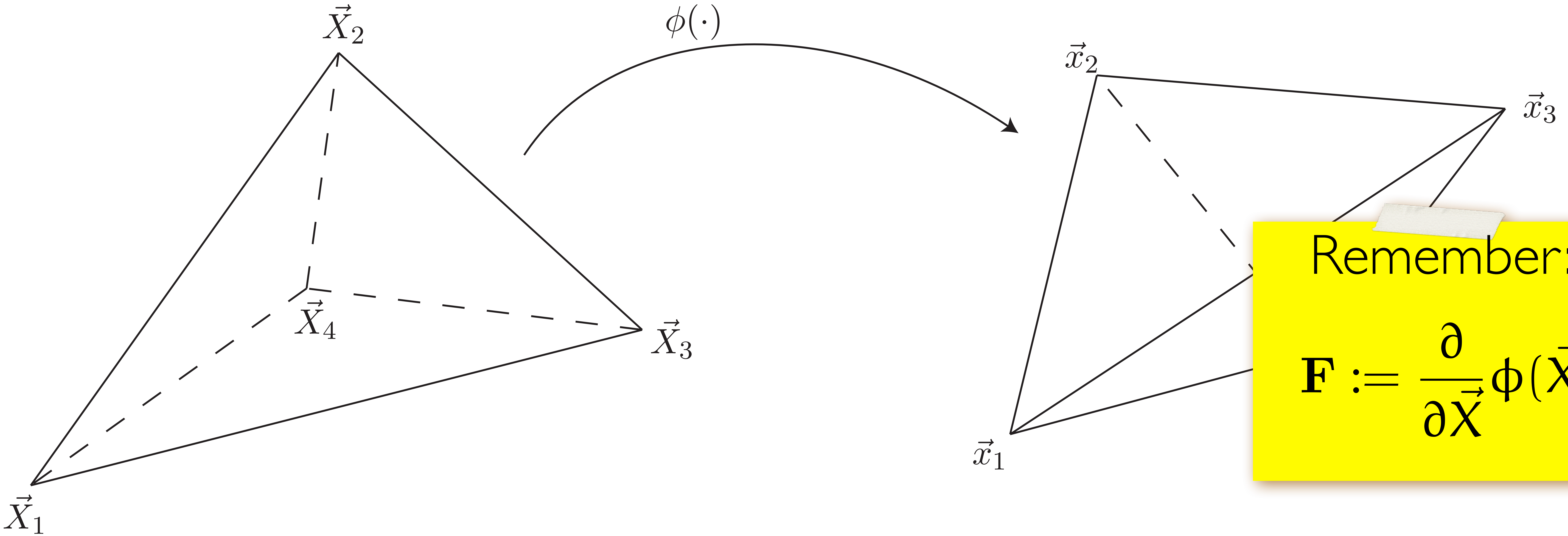
Tetrahedral models - Deformation measures



Hypothesis:
On every tetrahedron
*$\phi(X)$ is **linear!***

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Tetrahedral models - Deformation measures



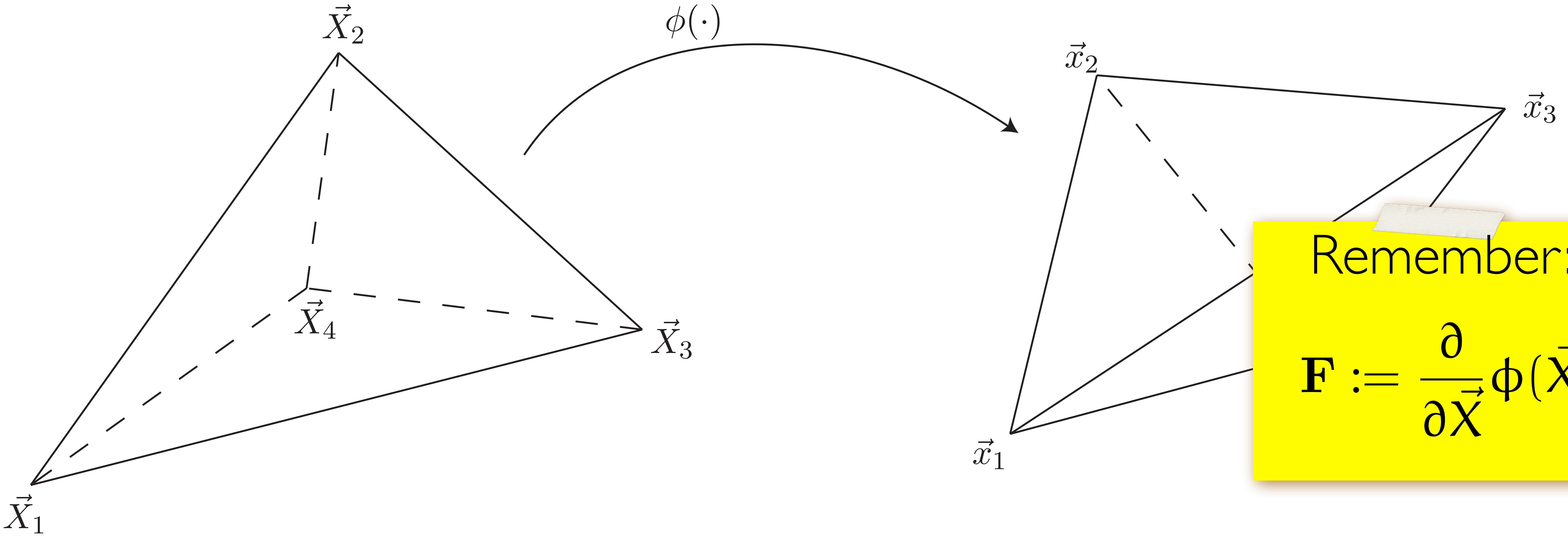
Remember:

$$\mathbf{F} := \frac{\partial}{\partial \vec{X}} \phi(\vec{X})$$

Hypothesis:
On every tetrahedron
*$\phi(X)$ is **linear!***

$$\vec{x} = \phi(\vec{X}) = \mathbf{A}\vec{X} + \vec{t}$$

Tetrahedral models - Deformation measures



Remember:

$$\mathbf{F} := \frac{\partial}{\partial \vec{X}} \phi(\vec{X})$$

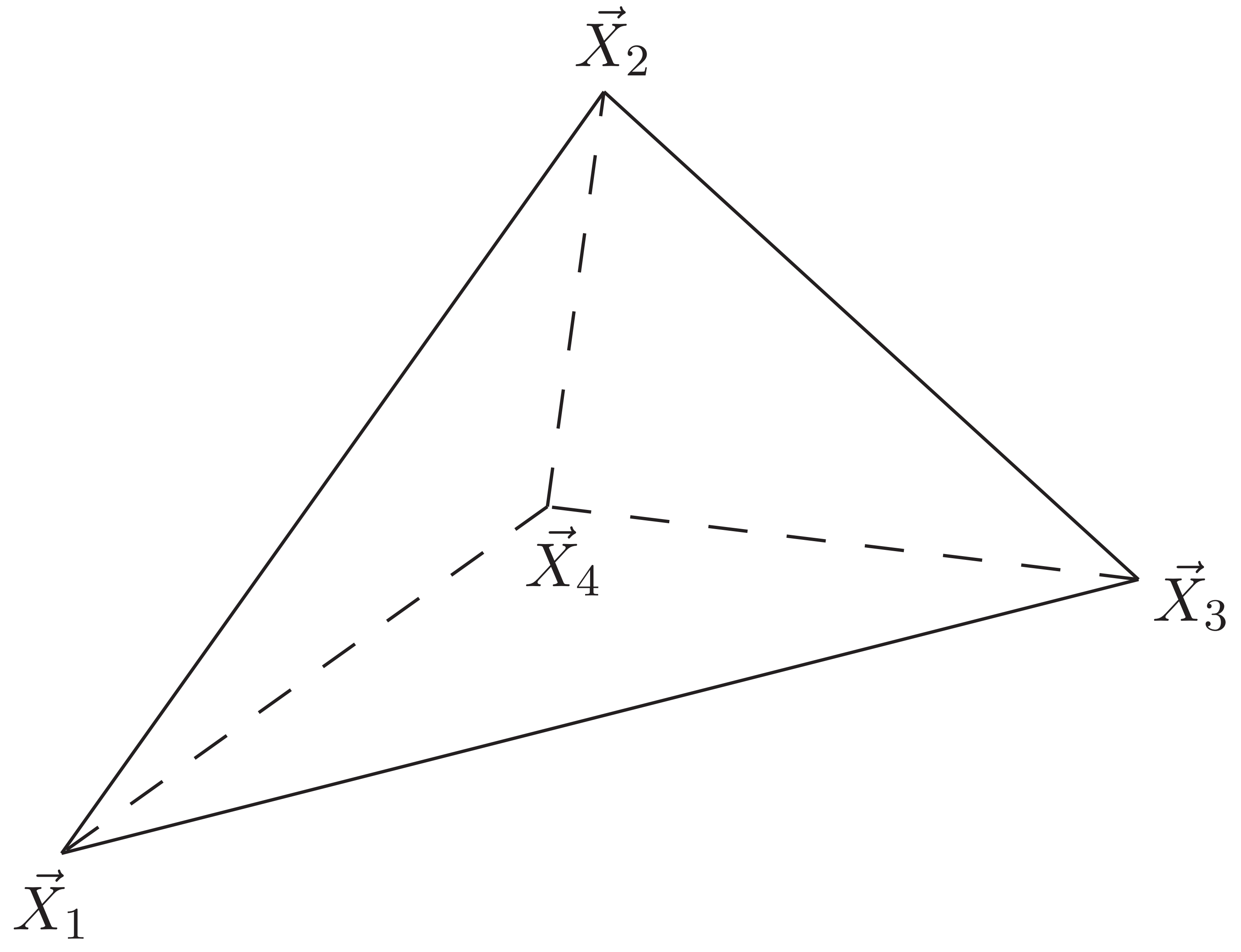
Hypothesis:
On every tetrahedron
*$\phi(X)$ is **linear!***

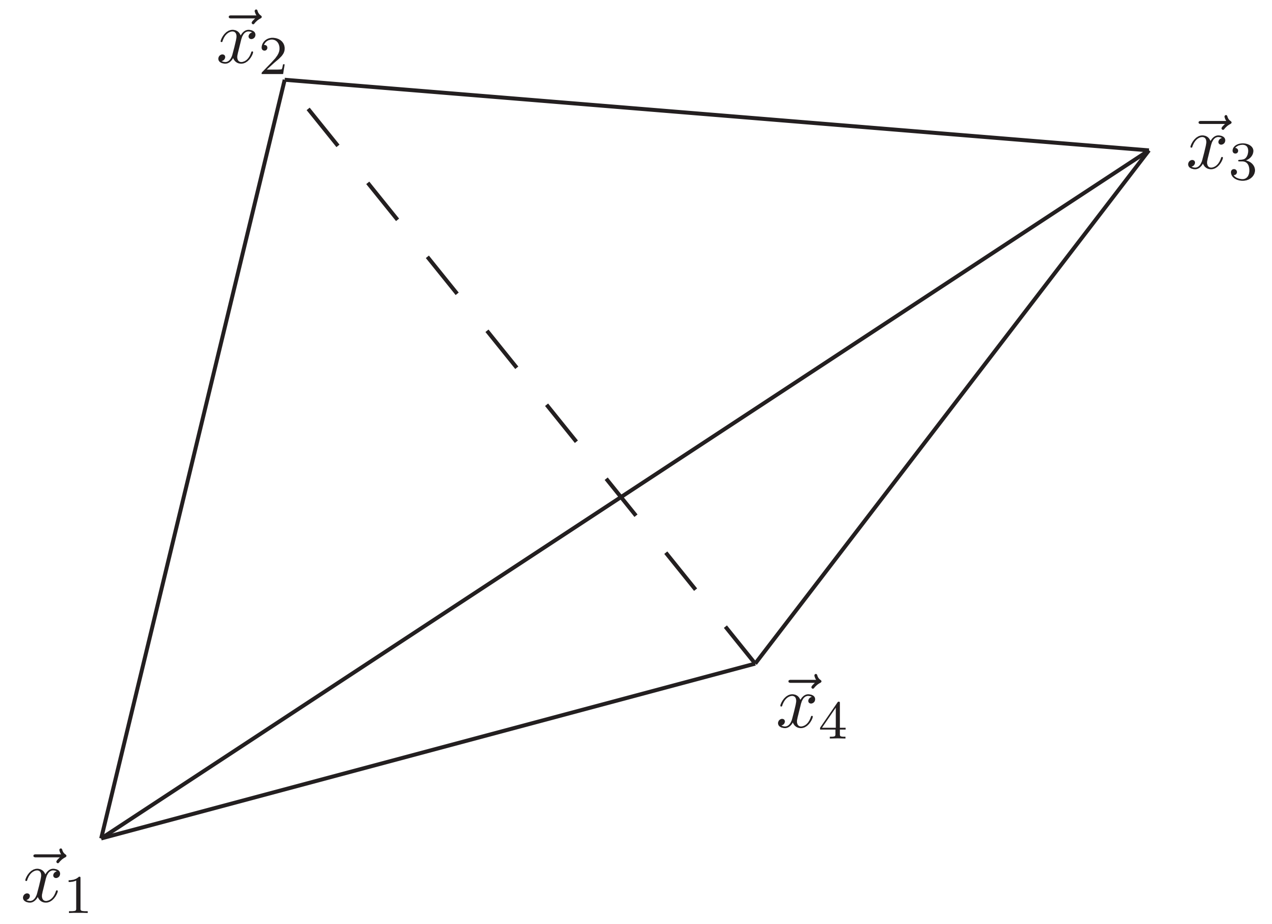
$$\vec{x} = \phi(\vec{X}) = \mathbf{F}\vec{X} + \vec{t}$$

Tetrahedral models - Deformation measures

$$\vec{x} = \mathbf{F}\vec{X} + \vec{t}$$

$$\begin{aligned} \vec{x}_1 &= \mathbf{F}\vec{X}_1 + \vec{t} \\ \vec{x}_2 &= \mathbf{F}\vec{X}_2 + \vec{t} \\ \vec{x}_3 &= \mathbf{F}\vec{X}_3 + \vec{t} \\ \vec{x}_4 &= \mathbf{F}\vec{X}_4 + \vec{t} \end{aligned}$$

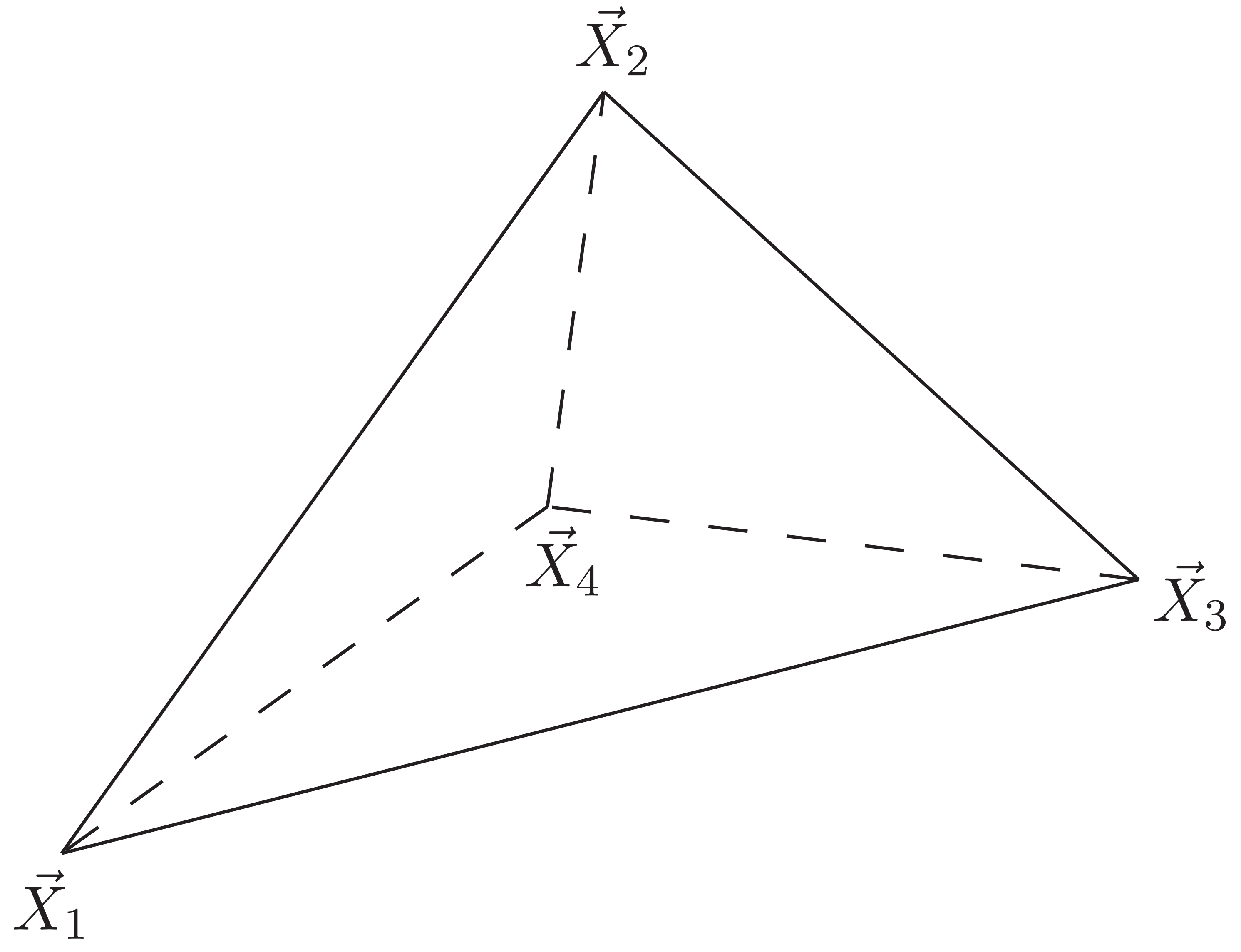


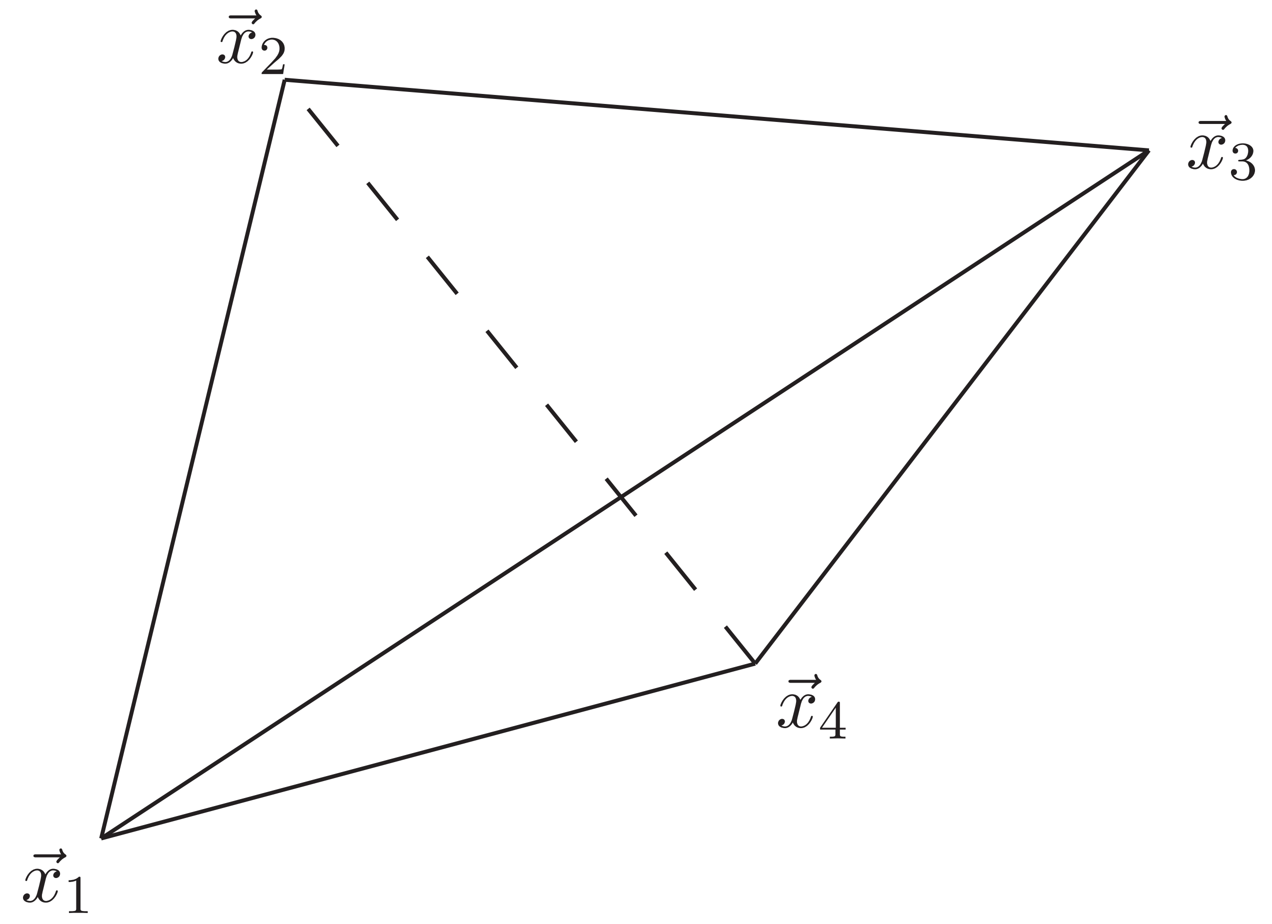


Tetrahedral models - Deformation measures

$$\vec{x} = \mathbf{F}\vec{X} + \vec{t}$$

$$\begin{aligned}
\vec{x}_1 &= \mathbf{F}\vec{X}_1 + \vec{t} \\
\vec{x}_2 &= \mathbf{F}\vec{X}_2 + \vec{t} \\
\vec{x}_3 &= \mathbf{F}\vec{X}_3 + \vec{t} \\
-\,(\vec{x}_4 &= \mathbf{F}\vec{X}_4 + \vec{t}\,)
\end{aligned}$$

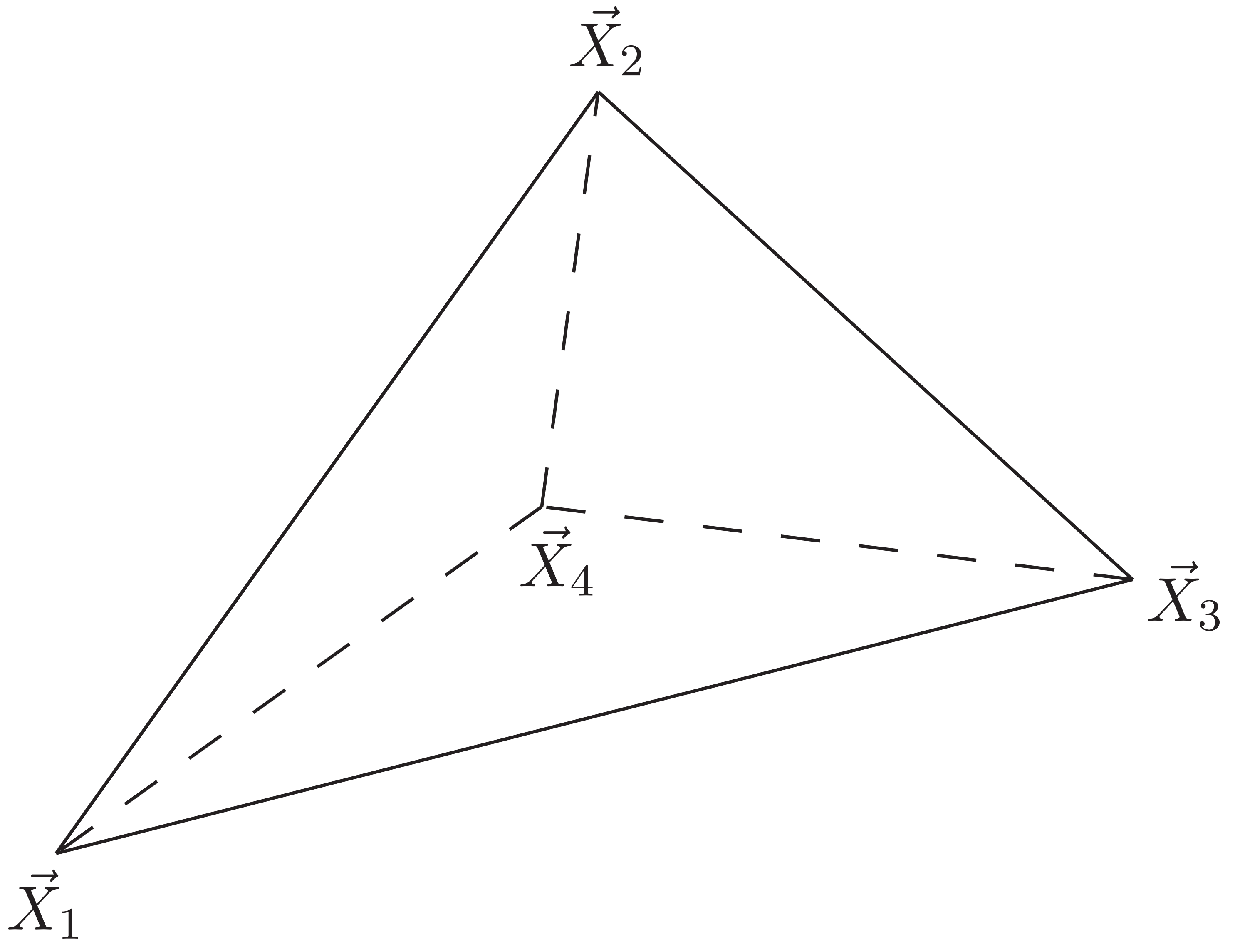


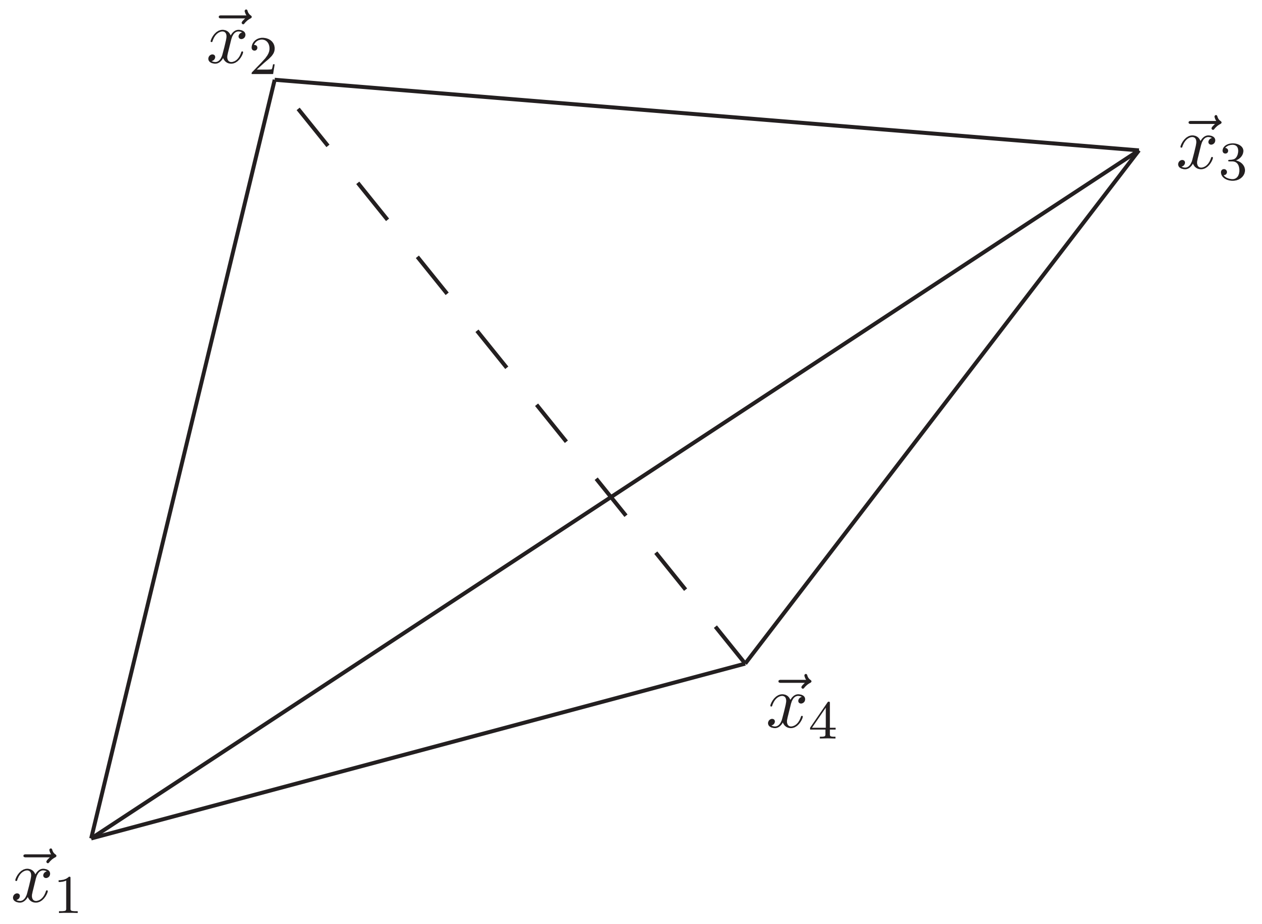


Tetrahedral models - Deformation measures

$$\vec{x} = \mathbf{F}\vec{X} + \vec{t}$$

$$\left.\begin{aligned} \vec{x}_1 &= \mathbf{F}\vec{X}_1 + \vec{t} \\ \vec{x}_2 &= \mathbf{F}\vec{X}_2 + \vec{t} \\ \vec{x}_3 &= \mathbf{F}\vec{X}_3 + \vec{t} \\ -(\vec{x}_4 &= \mathbf{F}\vec{X}_4 + \vec{t}) \end{aligned}\right\} \Rightarrow \begin{aligned} \vec{x}_1 - \vec{x}_4 &= \mathbf{F}(\vec{X}_1 - \vec{X}_4) \\ \vec{x}_2 - \vec{x}_4 &= \mathbf{F}(\vec{X}_2 - \vec{X}_4) \\ \vec{x}_3 - \vec{x}_4 &= \mathbf{F}(\vec{X}_3 - \vec{X}_4) \end{aligned}$$





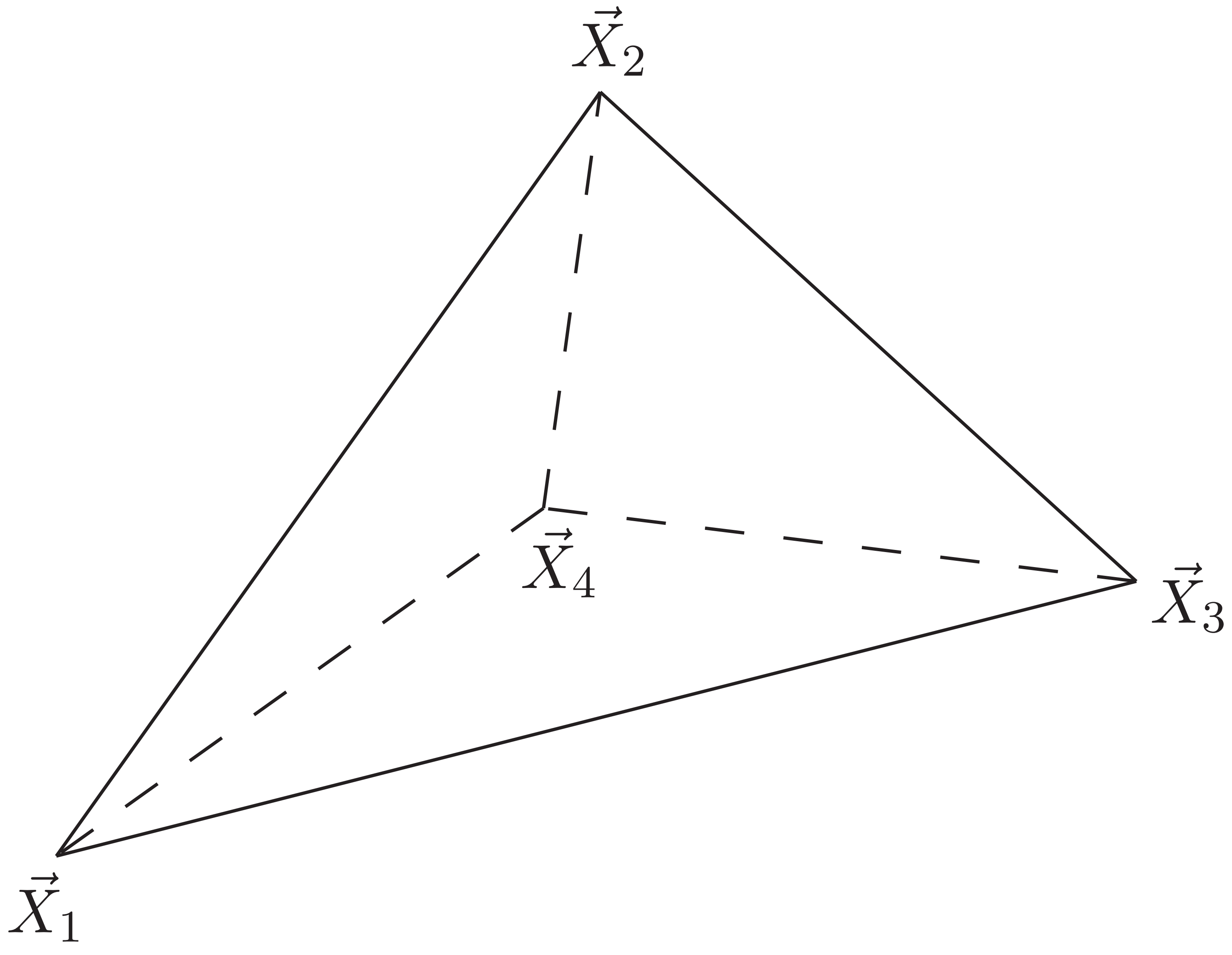
Tetrahedral models - Deformation measures

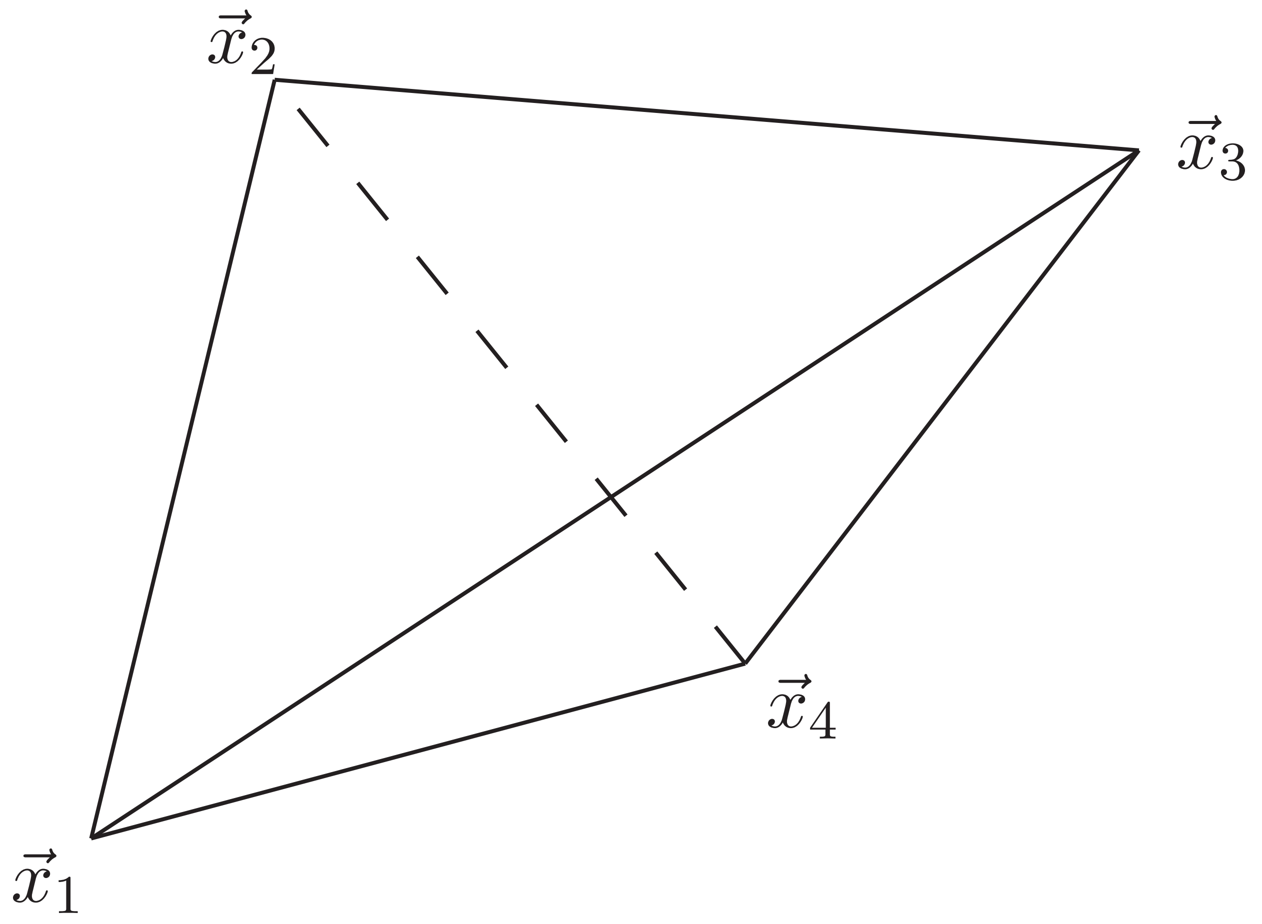
$$\vec{x} = \mathbf{F}\vec{X} + \vec{t}$$

$$\left.\begin{array}{rcl} \vec{x}_1 & = & \mathbf{F}\vec{X}_1 + \vec{t} \\ \vec{x}_2 & = & \mathbf{F}\vec{X}_2 + \vec{t} \\ \vec{x}_3 & = & \mathbf{F}\vec{X}_3 + \vec{t} \\ -\,(\vec{x}_4 & = & \mathbf{F}\vec{X}_4 + \vec{t}\,) \end{array}\right\} \Rightarrow \begin{array}{rcl} \vec{x}_1 - \vec{x}_4 & = & \mathbf{F}(\vec{X}_1 - \vec{X}_4) \\ \vec{x}_2 - \vec{x}_4 & = & \mathbf{F}(\vec{X}_2 - \vec{X}_4) \\ \vec{x}_3 - \vec{x}_4 & = & \mathbf{F}(\vec{X}_3 - \vec{X}_4) \end{array}$$

$$(\vec{x}_1 - \vec{x}_4 \mid \vec{x}_2 - \vec{x}_4 \mid \vec{x}_3 - \vec{x}_4) =$$

$$= \Big(\mathbf{F}(\vec{X}_1 - \vec{X}_4) \mid \mathbf{F}(\vec{X}_2 - \vec{X}_4) \mid \mathbf{F}(\vec{X}_3 - \vec{X}_4)\Big)$$

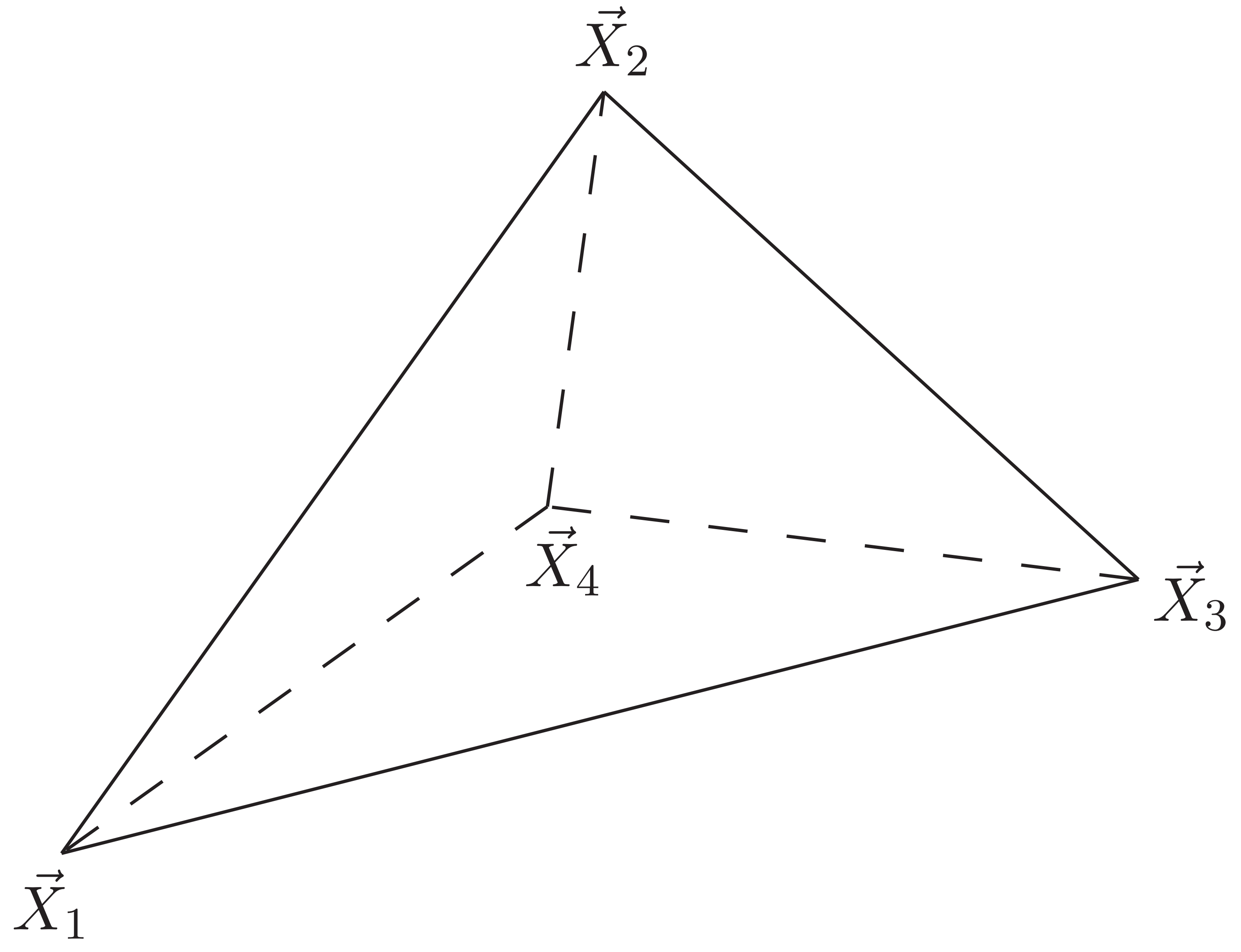




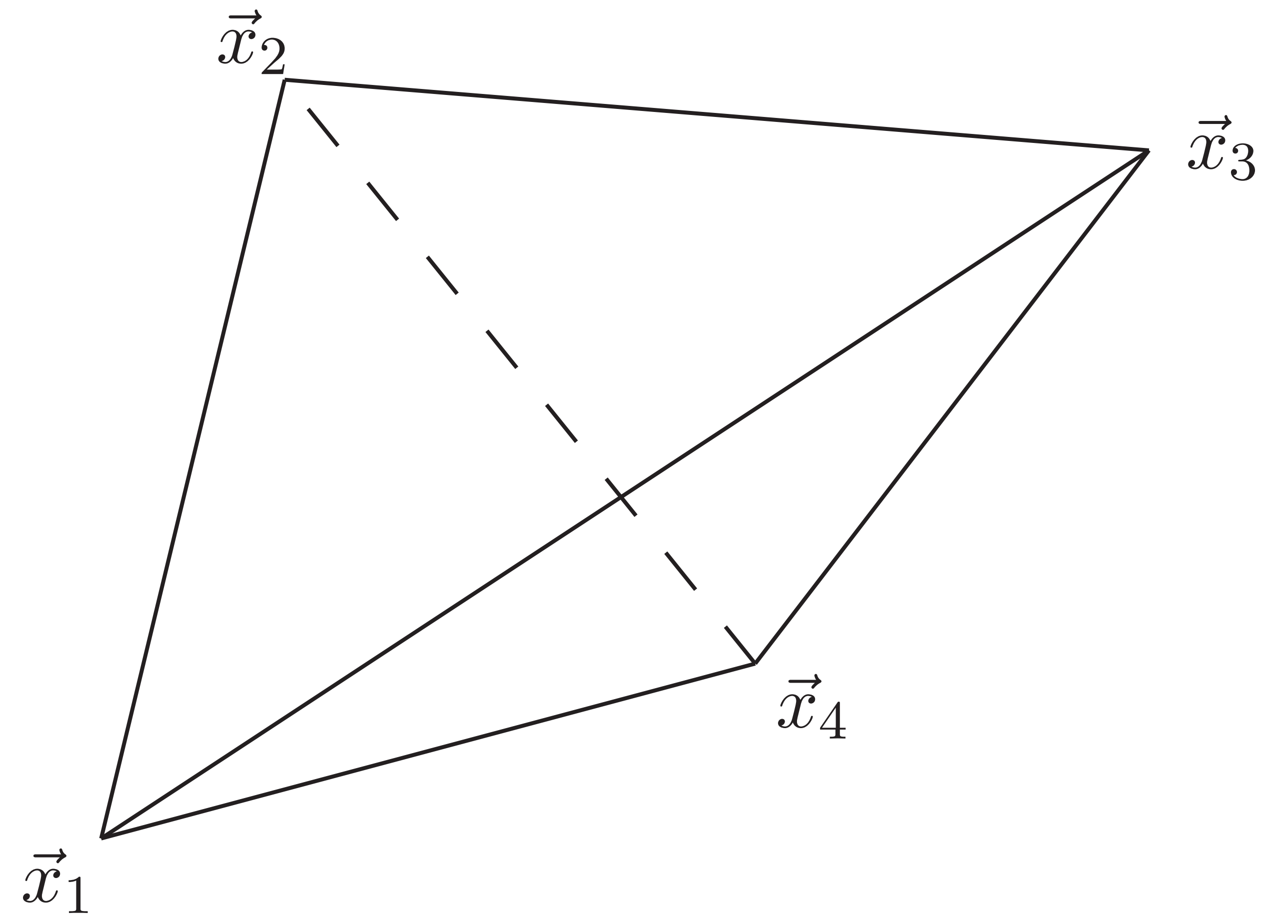
Tetrahedral models - Deformation measures

$$\vec{x} = \mathbf{F}\vec{X} + \vec{t}$$

$$\left.\begin{array}{rcl} \vec{x}_1 & = & \mathbf{F}\vec{X}_1 + \vec{t} \\ \vec{x}_2 & = & \mathbf{F}\vec{X}_2 + \vec{t} \\ \vec{x}_3 & = & \mathbf{F}\vec{X}_3 + \vec{t} \\ -(\vec{x}_4 & = & \mathbf{F}\vec{X}_4 + \vec{t}) \end{array}\right\} \Rightarrow \begin{array}{rcl} \vec{x}_1 - \vec{x}_4 & = & \mathbf{F}(\vec{X}_1 - \vec{X}_4) \\ \vec{x}_2 - \vec{x}_4 & = & \mathbf{F}(\vec{X}_2 - \vec{X}_4) \\ \vec{x}_3 - \vec{x}_4 & = & \mathbf{F}(\vec{X}_3 - \vec{X}_4) \end{array}$$



$$(\vec{x}_1 - \vec{x}_4 \mid \vec{x}_2 - \vec{x}_4 \mid \vec{x}_3 - \vec{x}_4) =$$
$$= \left(\mathbf{F}(\vec{X}_1 - \vec{X}_4) \mid \mathbf{F}(\vec{X}_2 - \vec{X}_4) \mid \mathbf{F}(\vec{X}_3 - \vec{X}_4)\right)$$
$$= \mathbf{F}\left(\vec{X}_1 - \vec{X}_4 \mid \vec{X}_2 - \vec{X}_4 \mid \vec{X}_3 - \vec{X}_4\right)$$

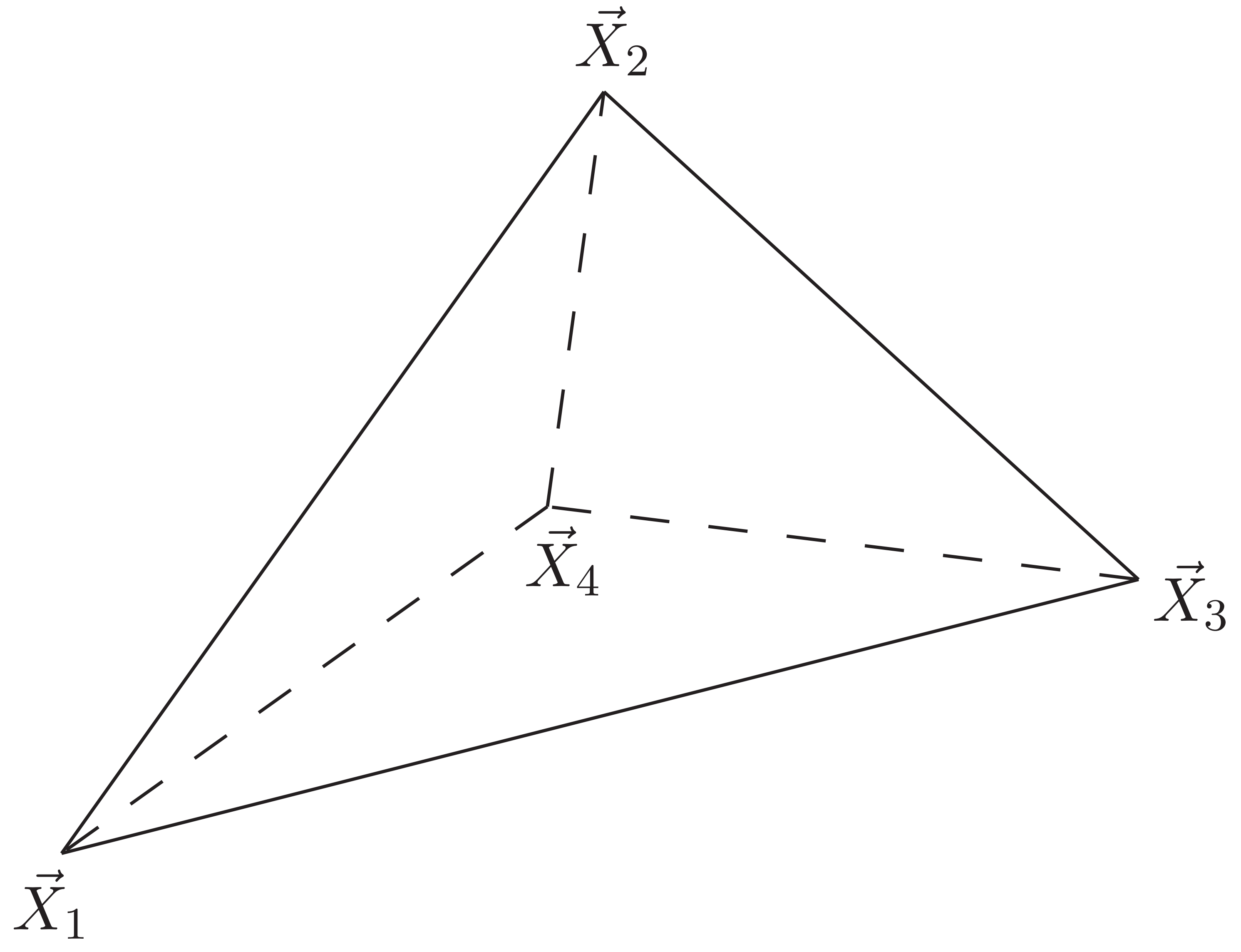


Tetrahedral models - Deformation measures

$$(\vec{x}_1 - \vec{x}_4 \mid \vec{x}_2 - \vec{x}_4 \mid \vec{x}_3 - \vec{x}_4) =$$

$$= \left(\mathbf{F}(\vec{X}_1 - \vec{X}_4) \mid \mathbf{F}(\vec{X}_2 - \vec{X}_4) \mid \mathbf{F}(\vec{X}_3 - \vec{X}_4)\right)$$

$$= \mathbf{F}\left(\vec{X}_1 - \vec{X}_4 \mid \vec{X}_2 - \vec{X}_4 \mid \vec{X}_3 - \vec{X}_4\right)$$



$$\mathbf{D}_s = \mathbf{F}\mathbf{D}_m$$

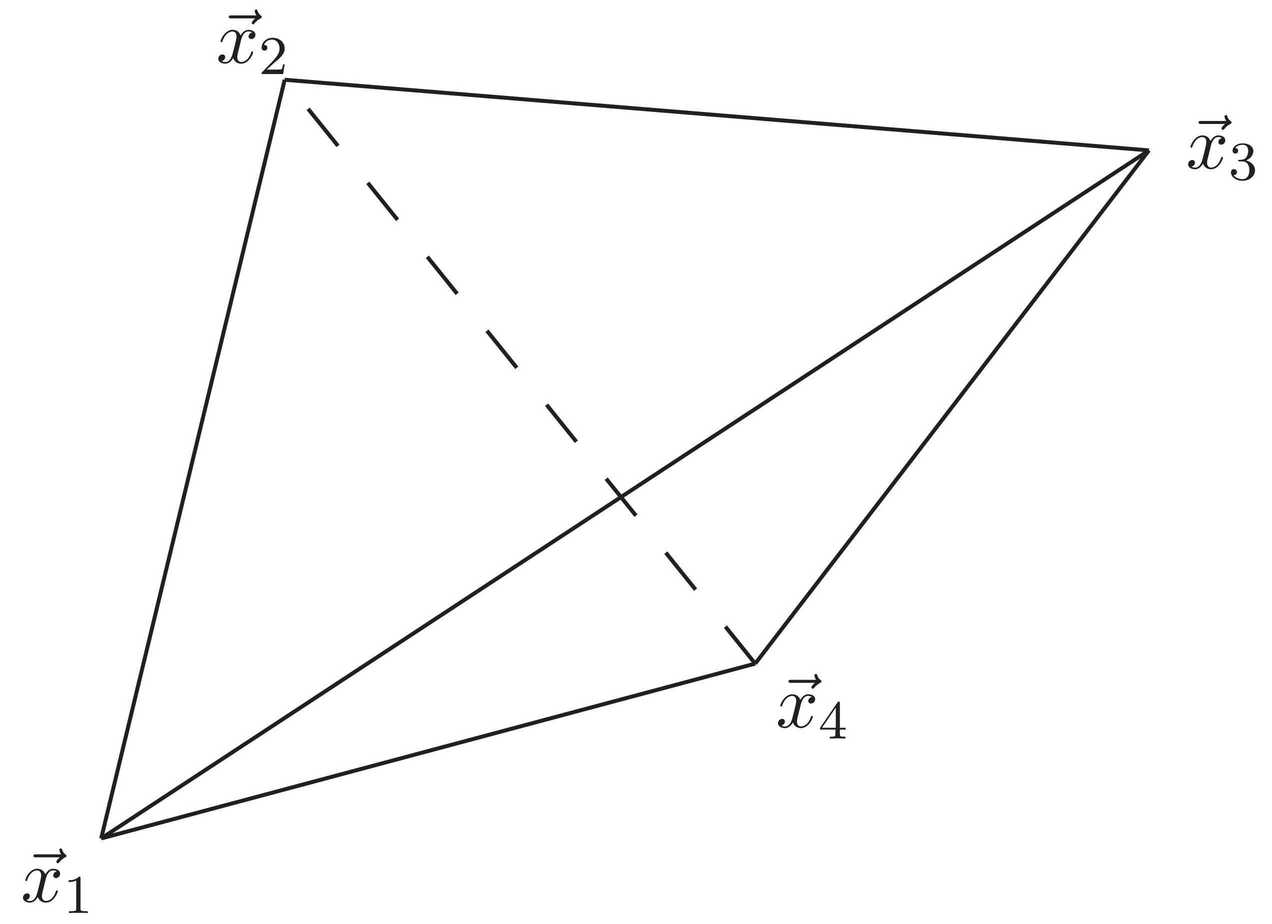
$$\mathbf{F} = \mathbf{D}_s\mathbf{D}_m^{-1}$$

$$\mathbf{D}_s = (\vec{x}_1 - \vec{x}_4 | \vec{x}_2 - \vec{x}_4 | \vec{x}_3 - \vec{x}_4)$$

Spatial shape matrix

$$\mathbf{D}_m = (\vec{X}_1 - \vec{X}_4 | \vec{X}_2 - \vec{X}_4 | \vec{X}_3 - \vec{X}_4)$$

Material shape matrix



Tetrahedral models - Force computation

$$\mathbf{F} = \mathbf{F}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$$

Deformation gradient

$$\Psi(\mathbf{F}) = \Psi(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$$

Energy density

$$E(tet) = \mathrm{Vol}(tet)\Psi(\mathbf{F}) = E(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$$

Total tetrahedron energy

$$\vec{f}_i := -\frac{\partial}{\partial \vec{x}_i} E(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = ?!?!?$$

Compute:

$$\mathbf{D}_s = (\vec{x}_1 - \vec{x}_4 | \vec{x}_2 - \vec{x}_4 | \vec{x}_3 - \vec{x}_4)$$

$$\mathbf{D}_m = (\vec{X}_1 - \vec{X}_4 | \vec{X}_2 - \vec{X}_4 | \vec{X}_3 - \vec{X}_4)$$

$$\mathbf{F} = \mathbf{D}_s \mathbf{D}_m^{-1}$$

Tetrahedral models - Force computation

$$\mathbf{H} = \left(\vec{f}_1 \mid \vec{f}_2 \mid \vec{f}_3 \right)$$

$$= -\mathrm{Vol}(tet)\mathbf{P}(\mathbf{F})\mathbf{D}_m^{-T}$$

$$\vec{f}_4 = -\vec{f}_1 - \vec{f}_2 - \vec{f}_3$$ *(Balance of forces)*

$$\mathbf{P} = \mathbf{P}(\mathbf{F})$$ *(from material model definition)*

Compute:

$$\mathbf{D}_s = (\vec{x}_1 - \vec{x}_4 | \vec{x}_2 - \vec{x}_4 | \vec{x}_3 - \vec{x}_4)$$

$$\mathbf{D}_m = (\vec{X}_1 - \vec{X}_4 | \vec{X}_2 - \vec{X}_4 | \vec{X}_3 - \vec{X}_4)$$

$$\mathbf{F} = \mathbf{D}_s \mathbf{D}_m^{-1}$$

Tetrahedral models - Force computation

Your material definition goes here

Compute:

$$\mathbf{D}_s = (\vec{x}_1 - \vec{x}_4 | \vec{x}_2 - \vec{x}_4 | \vec{x}_3 - \vec{x}_4)$$

$$\mathbf{D}_m = (\vec{X}_1 - \vec{X}_4 | \vec{X}_2 - \vec{X}_4 | \vec{X}_3 - \vec{X}_4)$$

$$\mathbf{F} = \mathbf{D}_s \mathbf{D}_m^{-1}$$

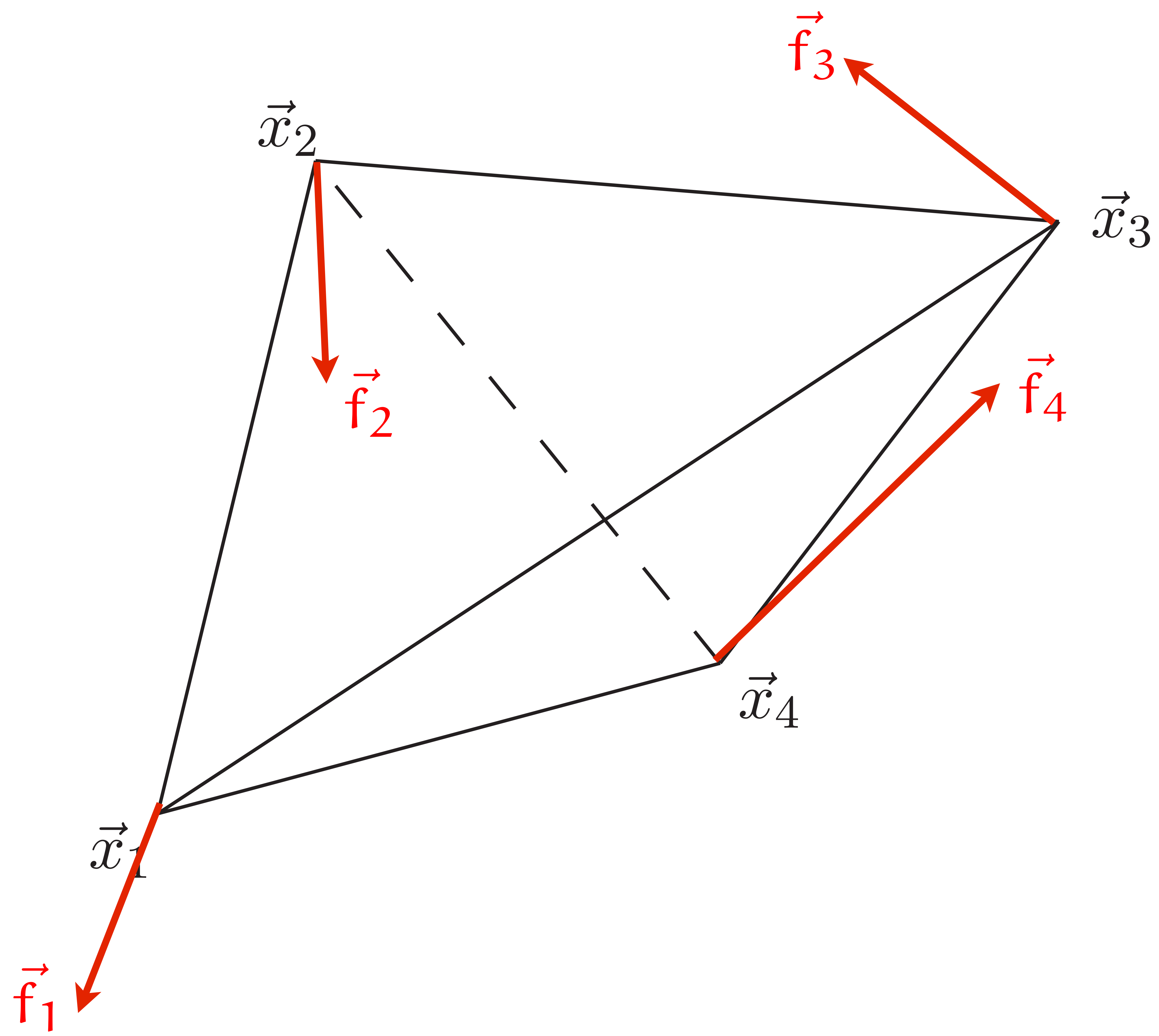
$$\mathbf{P} = \mathbf{P}(\mathbf{F})$$

$$\mathbf{H} = \left(\vec{f}_1 \mid \vec{f}_2 \mid \vec{f}_3 \right)$$

$$= -\mathrm{Vol}(\mathrm{tet}) \mathbf{P}(\mathbf{F}) \mathbf{D}_m^{-T}$$

$$\vec{f}_4 = -\vec{f}_1 - \vec{f}_2 - \vec{f}_3$$

Tetrahedral models - Force computation



Compute:

$$\mathbf{D}_s = (\vec{x}_1 - \vec{x}_4 | \vec{x}_2 - \vec{x}_4 | \vec{x}_3 - \vec{x}_4)$$

$$\mathbf{D}_m = (\vec{X}_1 - \vec{X}_4 | \vec{X}_2 - \vec{X}_4 | \vec{X}_3 - \vec{X}_4)$$

$$\mathbf{F} = \mathbf{D}_s \mathbf{D}_m^{-1}$$

$$\mathbf{P} = \mathbf{P}(\mathbf{F})$$

$$\mathbf{H} = \left(\vec{f}_1 \mid \vec{f}_2 \mid \vec{f}_3 \right)$$

$$= -\mathrm{Vol}(\mathrm{tet}) \mathbf{P}(\mathbf{F}) \mathbf{D}_m^{-T}$$

$$\vec{f}_4 = -\vec{f}_1 - \vec{f}_2 - \vec{f}_3$$

Additional information on course notes

- ✓ Optimizations and precomputation opportunities
- ✓ Newton methods for implicit integration of nonlinear materials
- ✓ Outline of an unconditionally stable, Backward Euler integration scheme
- ✓ Force differentials for matrix-free implementation of implicit solvers

FEM Simulation of 3D Deformable Solids: A practitioner's guide to theory, discretization and model reduction

Part One : The classical FEM method and discretization methodology

Eftychios Sifakis
University of Wisconsin - Madison

Find the latest version of course notes at : **www.femdefo.org**

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