Learning Splines for Sparse Tomographic Reconstruction

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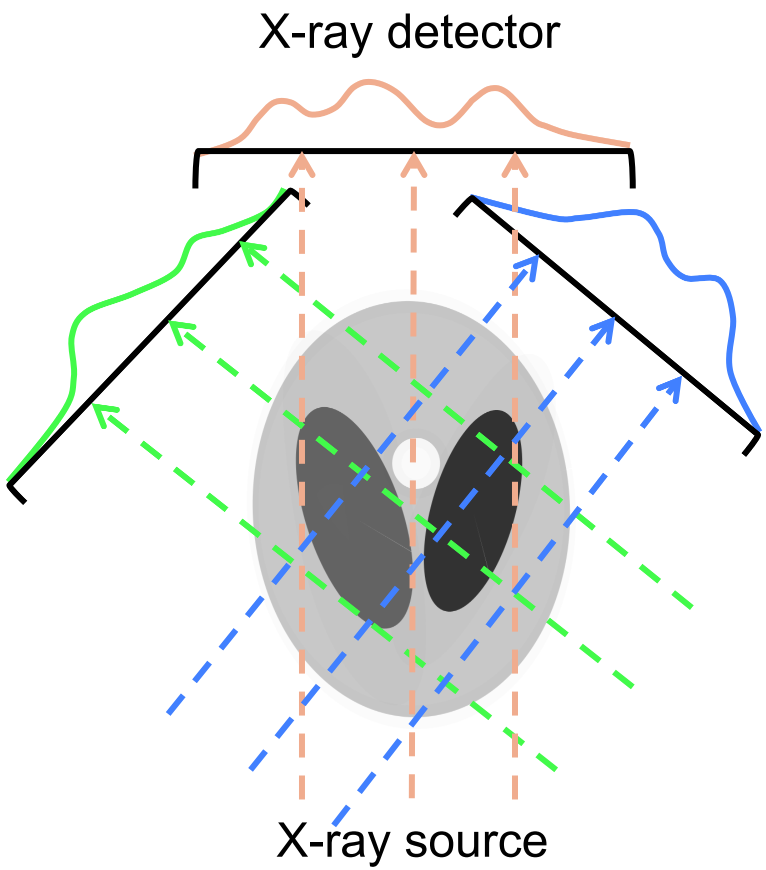
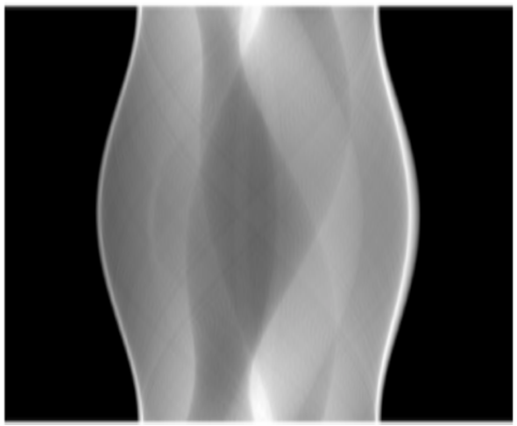
Tomographic Reconstruction

- **Recover the image given X-ray measurements**



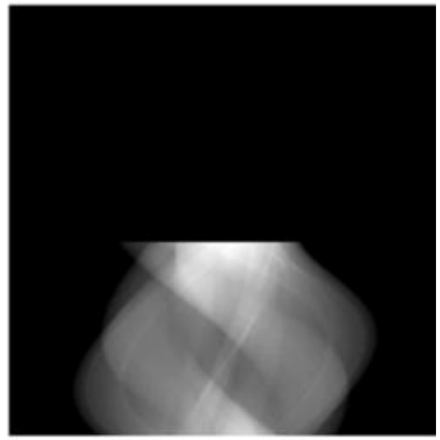


Sinogram

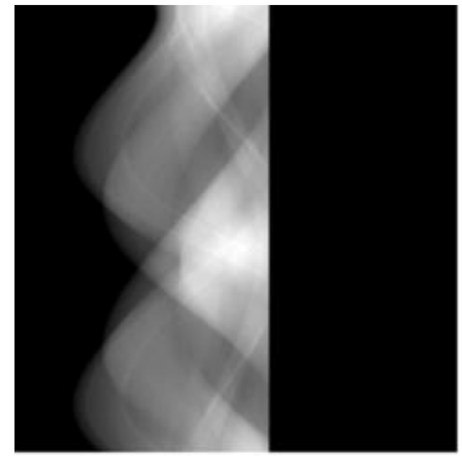
Motivation

- **X-ray Exposure Reduction**

Few-View



Limited-Angle



Half-Detector

Images courtesy of Pan et.al [2]

- **ill-posed problem**

$$\begin{pmatrix} & A & \end{pmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

Sparse CT

$$\begin{pmatrix} \quad \mathbf{A} \quad \end{pmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \end{bmatrix}$$

tomographic system matrix

intensity image

sinogram data

- **Least-squares solution:**

$$\hat{\mathbf{x}} = \min_{\mathbf{x}} \quad ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2$$

- **Regularize the solution:**

$$\hat{\mathbf{x}} = \min_{\mathbf{x}} \quad ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2 + \lambda R(\mathbf{x})$$

- **R(x) can be sparsity promoting regularizer**

Related Work (Sparsity)

- **TV minimization:**
  - Very promising for piece-wise constant images
  - ASD-POCS [Pan & Sidky 2009]
- **Besov space priors:**
  - Bayesian inversion [Siltanen et al. 2012]
- **X-let sparsity:**
  - Wavelet [Mirzargar et al. 2013]
  - Curvelet [Hyder & Sukanesh, 2011]
- **Adaptive sparsity via dictionary learning**
  - K-SVD [Aharon et al. 2006]

Related Work (Dictionary Learning)

- KSVD for limited-angle CT [Liao & Sapiro 2008]
  - Learns pixel values
  - Accounts for uniform noise

- Statistical iterative reconstruction [Xu et al. 2012]
  - Fixed and adaptive dictionaries
  - Updates pixel values using surrogate functionals
  - Handles Poisson noise

- Sinogram restoration [Shtok et al. 2011]
  - Weighted K-SVD
  - Handles Poisson noise

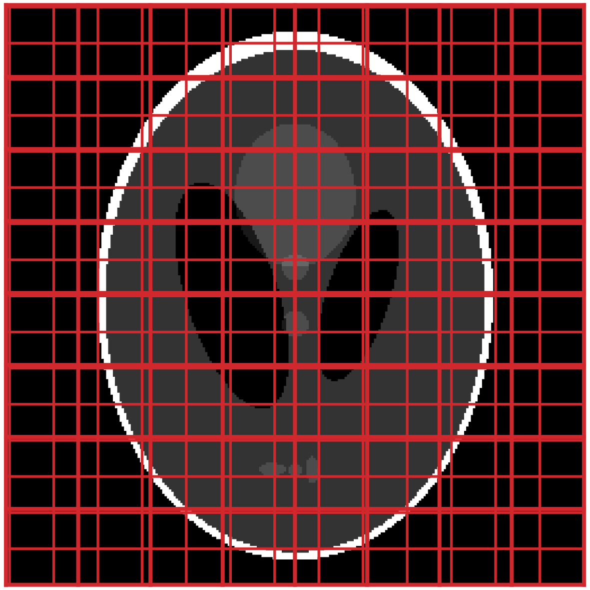
Common Pixel Representation

Continuous object

vs.

Finite grid reconstruction

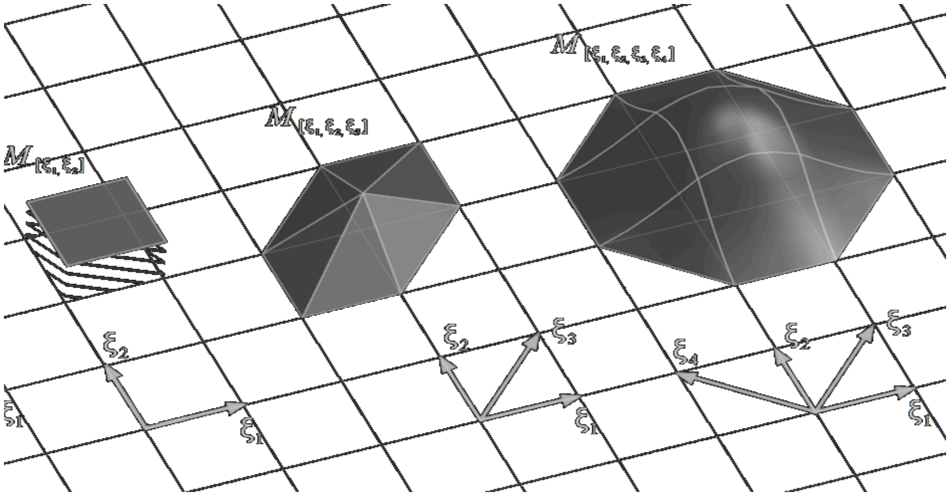
Image courtesy of C.G. Koay, https://science.nichd.nih.gov



Expansion Sets

- **Alternative for pixel-basis**
  - Blob functions [Lewitt 1990]
  - Kaiser-Bessel functions
  - Higher-order box-splines
    - Tensor-product linear B-spline
    - Tensor-product cubic B-spline
    - Zwart-Powell function

$$f(\mathbf{x}) = \sum_{n=1}^{N} \mathbf{c}_n \varphi(\mathbf{x} - \mathbf{x}_n)$$



Optimization Problem:

- **Integrate patch-based adaptive sparsity into spline framework:**

accounts for data-dependent noise

patch extractor

learned dictionary

$$\min_{\mathbf{c},\alpha} \; \boxed{||\mathbf{Hc} - \hat{\mathbf{p}}||^2_{\mathbf{W}}} + \lambda \boxed{\left(\sum_{k=1}^{K} ||E_k \mathbf{c} - \mathbf{D}\alpha_k||_2^2 + \mu_k ||\alpha_k||_0\right)}$$

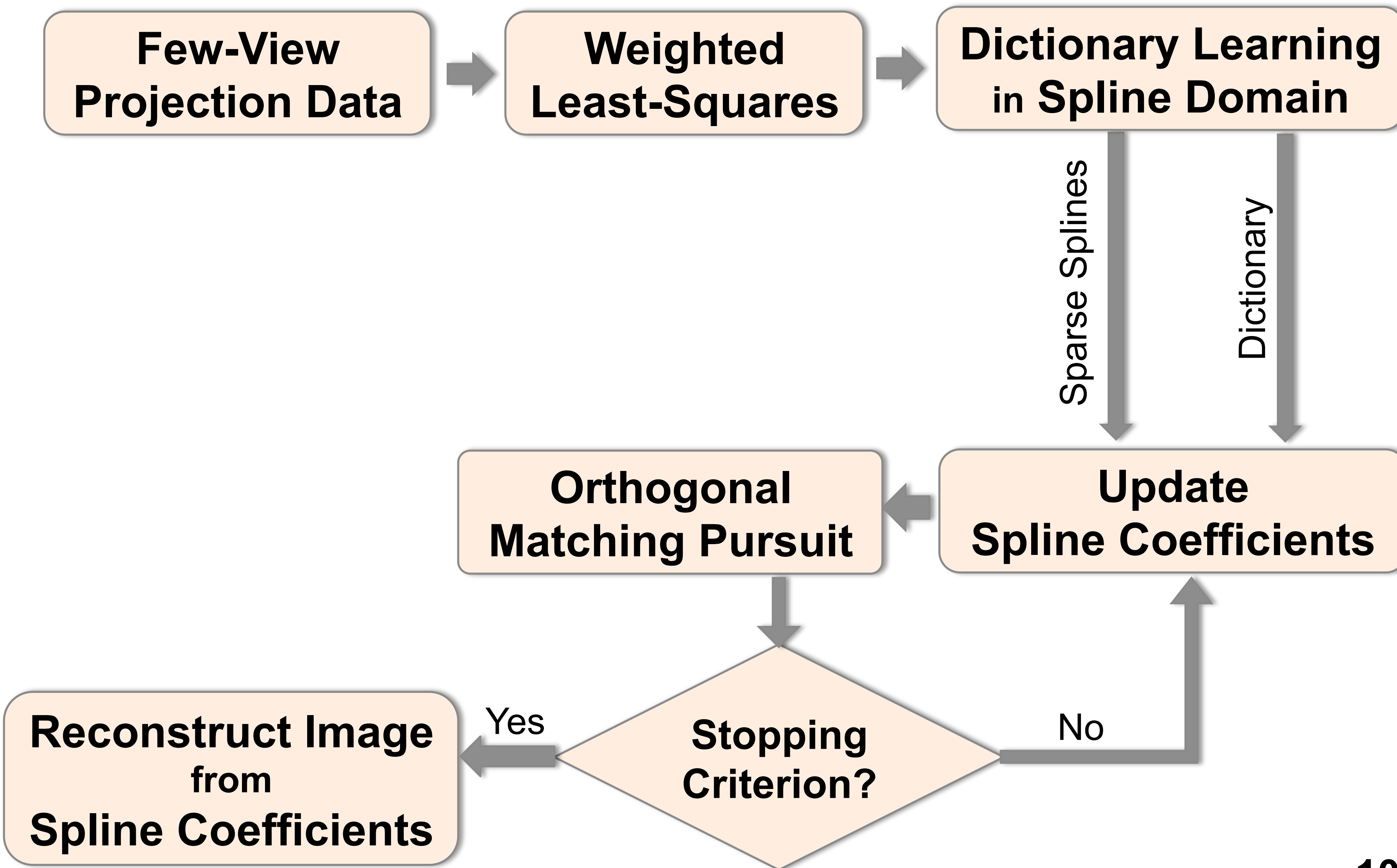
system matrix

spline coeff

Projection data

sparse representation of k^{th} patch

Proposed Approach



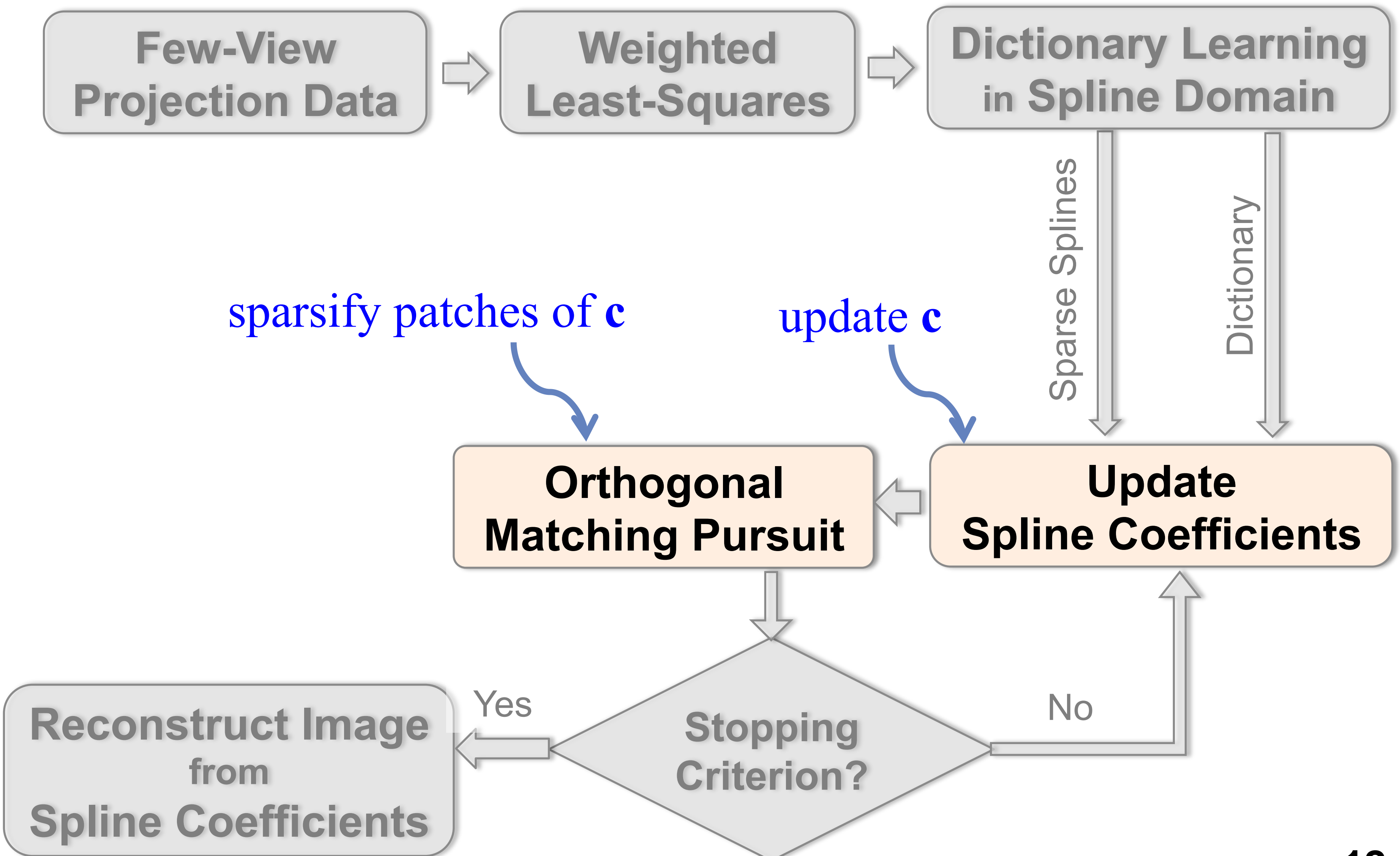
Update Splines

- **How to update the spline coefficients?**

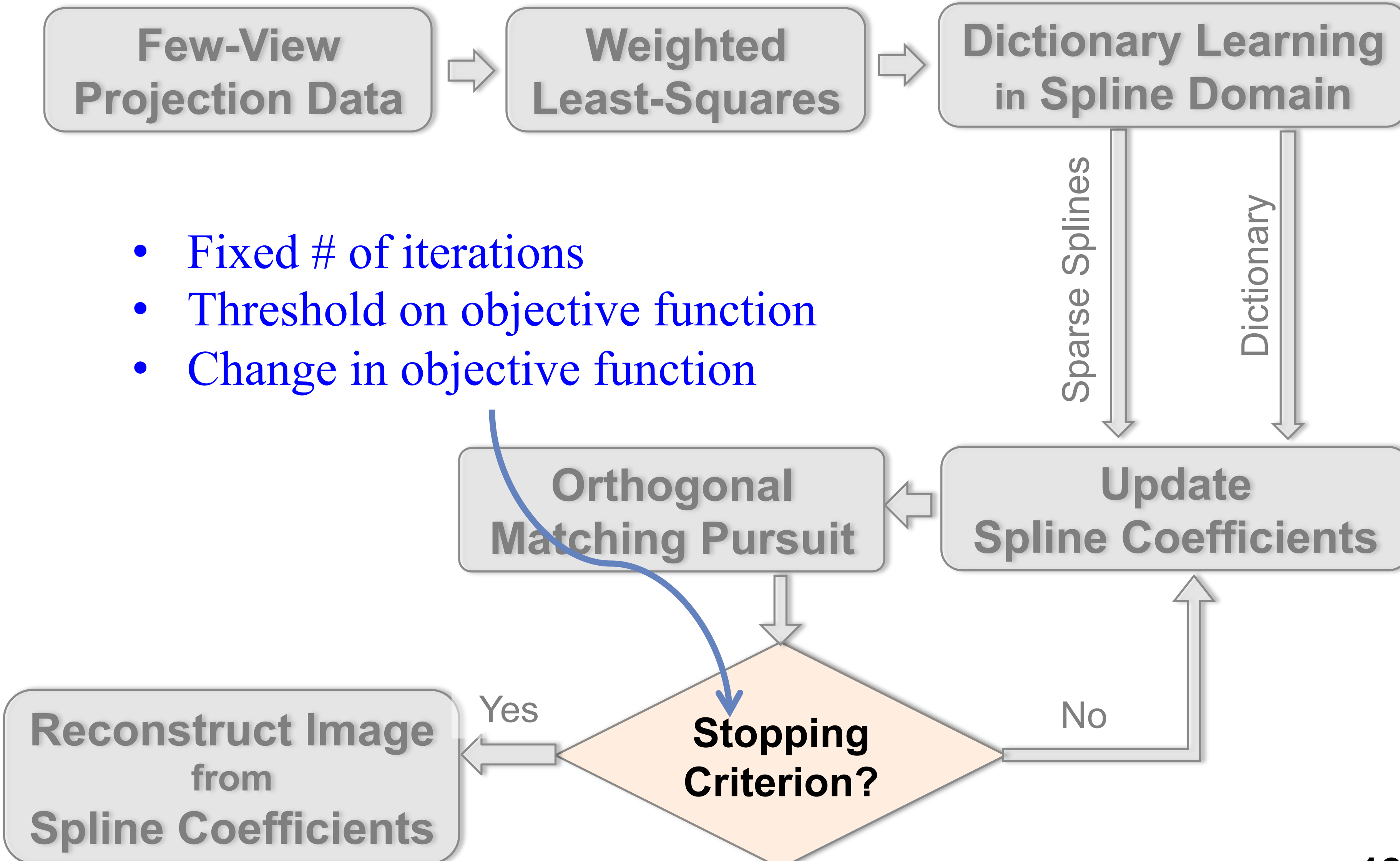
- **Differentiate the quadratic objective function:**

$$\left(\mathbf{H}^T\mathbf{W}\mathbf{H} + \lambda \sum_k E_k^T E_k\right)\mathbf{c} = \mathbf{H}^T\mathbf{W}\hat{\mathbf{p}} + \lambda \sum_k E_k^T \mathbf{D}\alpha_k$$

Proposed Approach

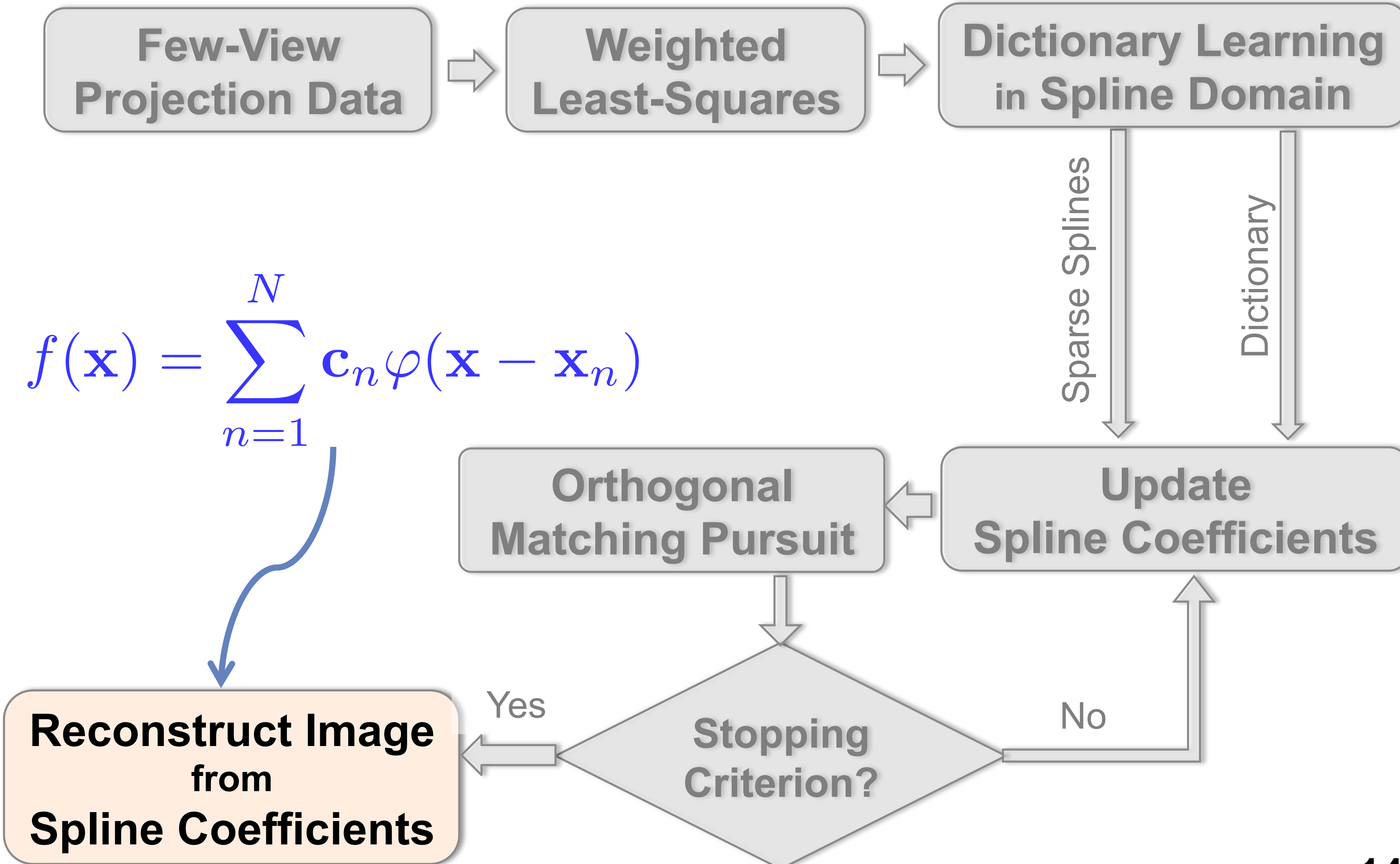


Proposed Approach



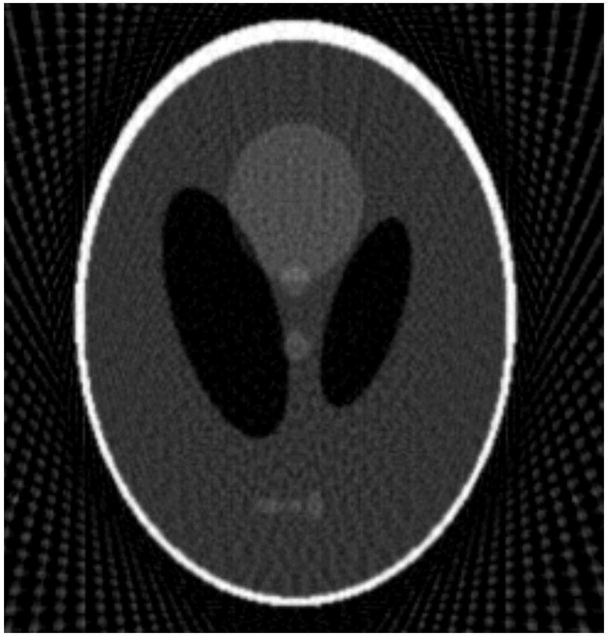
- Fixed # of iterations
- Threshold on objective function
- Change in objective function

Proposed Approach



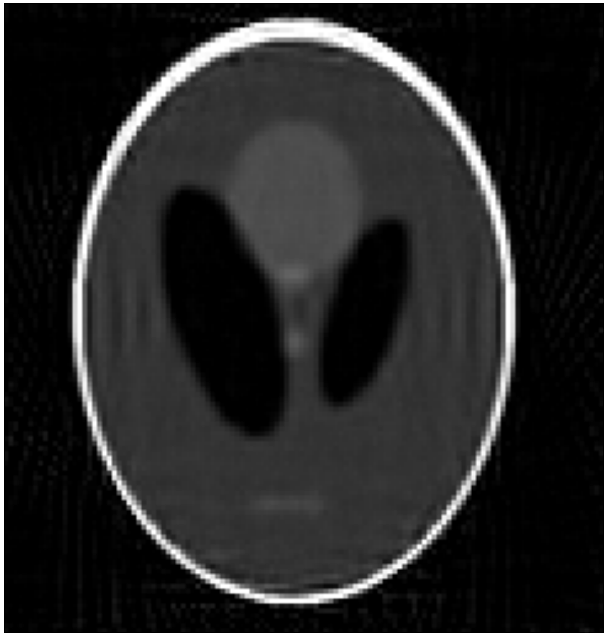
Results: pixel-basis vs. Linear

- 45 projection views:

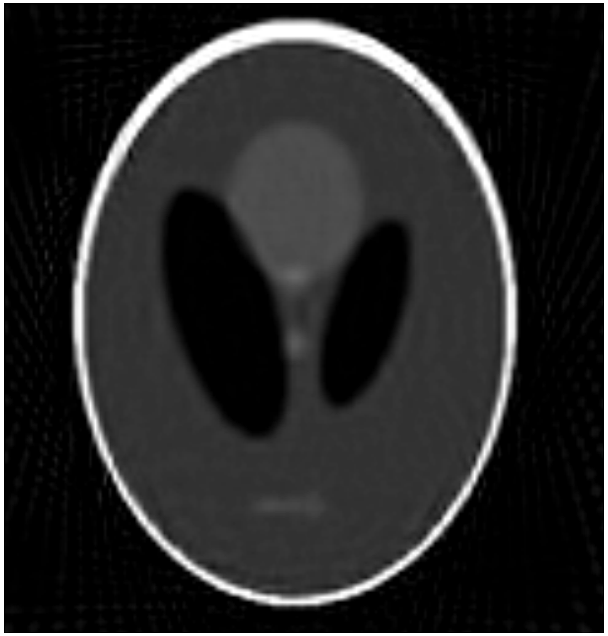


FBP

SNR: 10.49 dB



Pixel-basis
(first-order box-spline)
SNR: 10.52 dB



Linear
(second-order box-spline)
SNR: 14.46 dB

Results: LSQR vs. Spline Learning

- **60 projection views:**

Original

FBP (SNR: 15.51 dB)

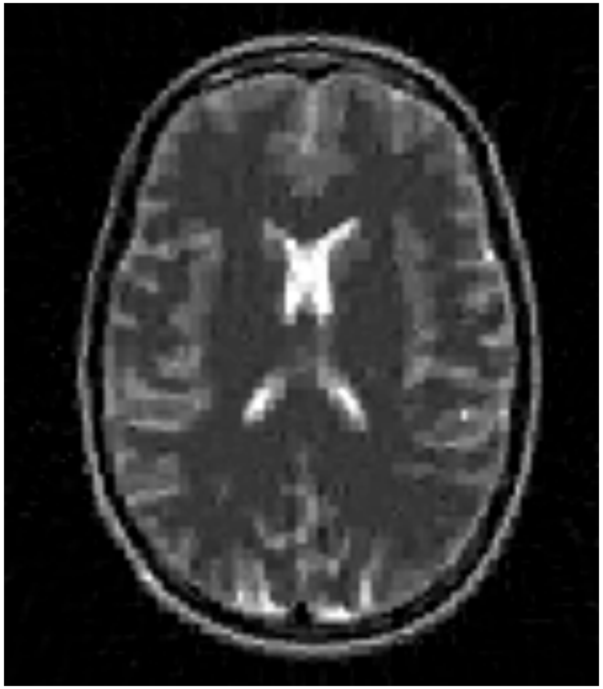
LSQR (SNR: 17.19 dB)

Spline Learning (SNR: 18.23 dB)

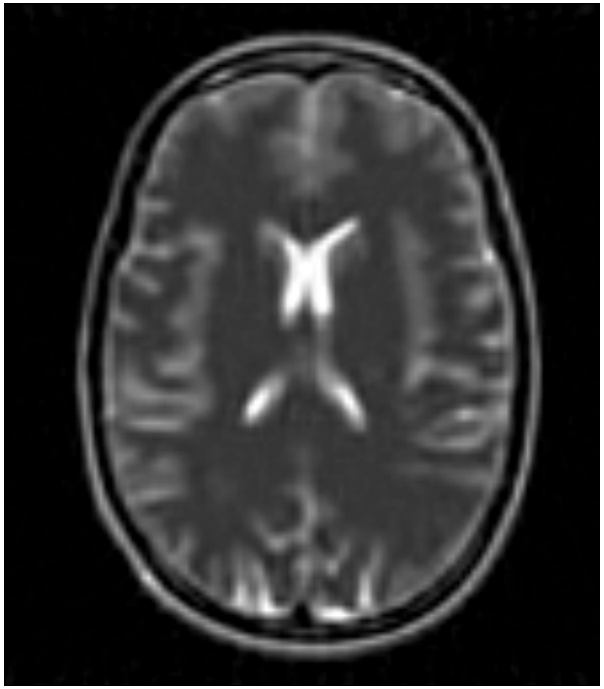
Results: Fixed vs. Learned Sparsity

- **60 projection views:**

Original



Wavelet
SNR: 15.72 dB



Spline Learning
SNR:17.58 dB

Results: Resilience to Reduction of Angles

90 views
SNR: 15.66 dB

60 views
SNR: 15.19 dB

45 views
SNR: 14.46 dB

Summary

- We proposed higher-order box-splines as

  alternatives for pixel-basis,

  integrated patch-based adaptive sparsity into this

  spline framework
- Superiority of higher-order splines
- Simply choice of tensor-product Linear B-spline

Future Work

- Mixed spline representations
- Analysis of approximation error as a function of grid resolution

References

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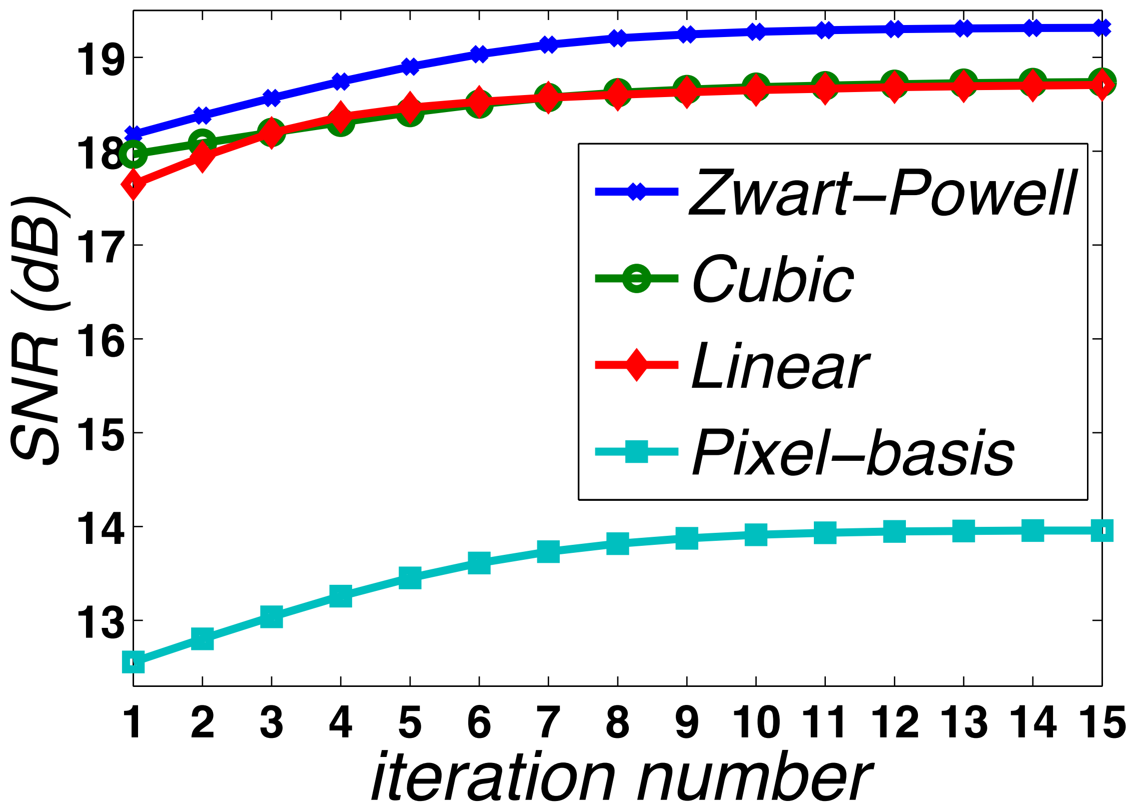
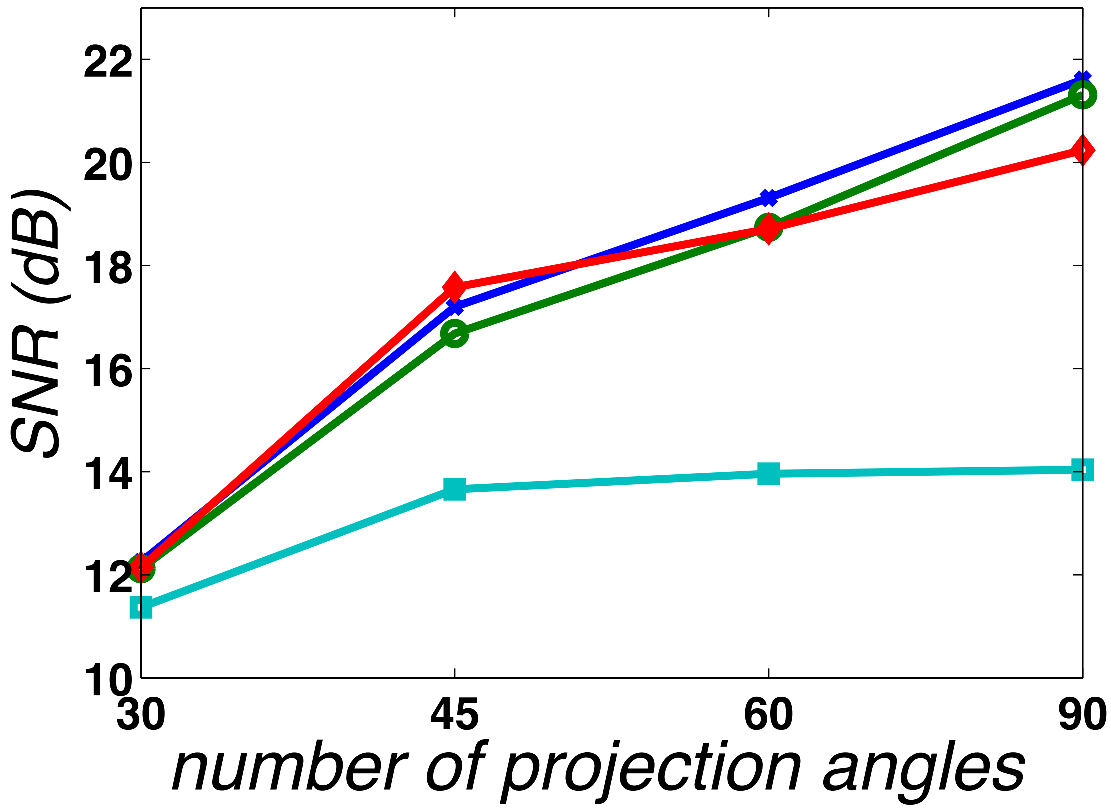
Thank you …

Questions?

Results: SNR vs. Iteration number



Results: Resilience



Results: Convergence

$$\min_{\mathbf{c},\alpha} \quad ||\mathbf{H}\mathbf{c} - \hat{\mathbf{p}}||^2_{\mathbf{W}} + \lambda \left(\sum_{k=1}^{K} ||E_k \mathbf{c} - \mathbf{D}\alpha_k||_2^2 + \mu_k ||\alpha_k||_0 \right)$$