Inner and Outer Approximations of Polytopes Using Boxes

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Joint Work with

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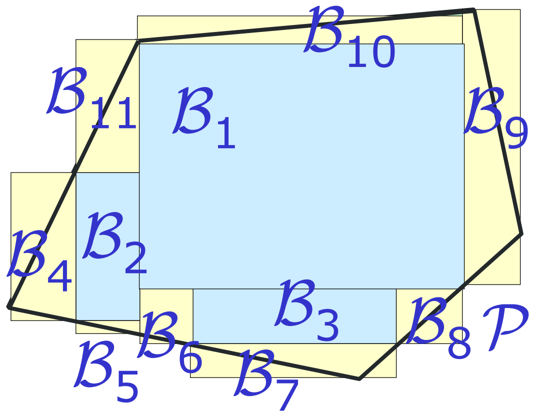
Approximation

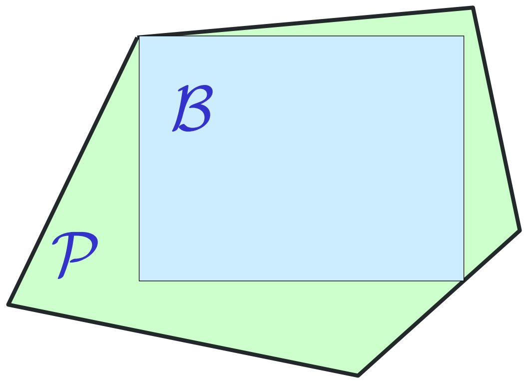


Given an H-polytope $\mathcal{P}$: $\{x\colon Ax \leq b\}$ look for two collections $\mathcal{I}$ and $\mathcal{E}$ of adjacent boxes s. t.:

1. the union of all boxes in $\mathcal{I}$ is contained in $\mathcal{P}$
2. the union of all boxes in $\mathcal{E}$ contains $\mathcal{P}$

minimize the volume error and minimize the total number of boxes

Single Inner Box



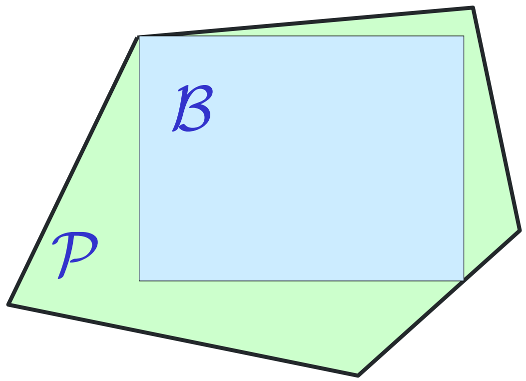
Main Idea: Maximize the volume of the box subj. to. all the vertices are in $\mathcal{P}$

$\mathcal{B}(x, x+y) = \{z \in \mathbb{R}^d : \; x \le z \le x+y\}$; $\mathcal{P} = \{x \in \mathbb{R}^d : \; Ax \le b\}$

$$\begin{array}{rl} \max_{x,y} & \prod_{j \in D} y_j \\ \text{subject to} & Ax + AV(S)y \le b \quad (\forall S \subseteq D) \\ & y > 0 \end{array}$$

where $D = \{1, \ldots, d\}$; $V(S) \in \{0,1\}^d$ is the incidence vector of $S \subseteq D$

Single Inner Box

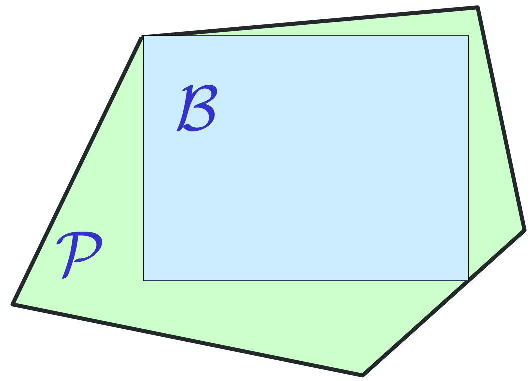


Lemma 1 *The constraints* $Ax + AV(S)y \leq b \ \forall S \subseteq D; y \geq 0$ *are equivalent to the set of constraints* $Ax + A^{+}y \leq b$*, where* A^{+} *is the positive part of* A*.*

Proof by lines, remembering that $y > 0$.

Lemma 2 $\max_{x,y} \prod_{j \in D} y_j$ *is equivalent to* $\max_{x,y} \sum_{j \in D} \ln y_j$*. Therefore the problem is convex and polynomially solvable.*

Single Inner Constrained Box



$$\mathcal{B}(x, x+\lambda r) = \{z : \; z \leq z \leq x + \lambda r\}; \; \mathcal{P} = \{x : \; Ax \leq b\}$$

Main Idea: If the edges of the box are constrained, maximizing the volume amount to maximizing one edge

$$\max_{x,\lambda} \quad \lambda$$
$$\text{subject to} \quad Ax + A^{+} r\lambda \leq b,$$

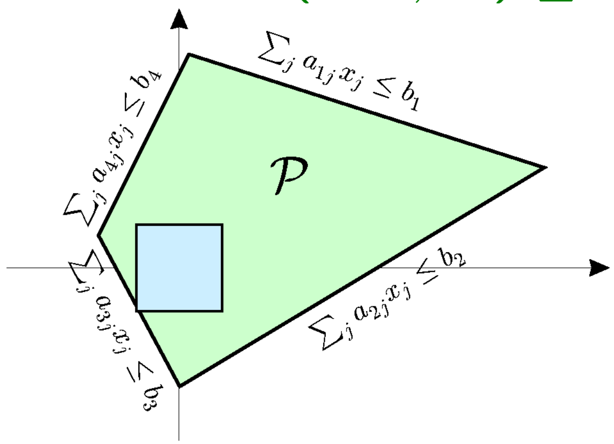
Complexity: $O(\mathbf{lp}(d+1, m))$

How to choose r? e (**1** vector), Inner Diameters, Outer Box

Single Greedy Inner Box

Assume that $0 \in \mathcal{P}$

How to find the max τ s.t. $\mathcal{B}(-e\tau, e\tau) \subseteq \mathcal{P}$?



$$\mathcal{B}(-e\tau, e\tau) = \{x \in \mathbb{R}^d : \ -e\tau \leq x \leq e\tau\},\ \tau \in \mathbb{R};$$

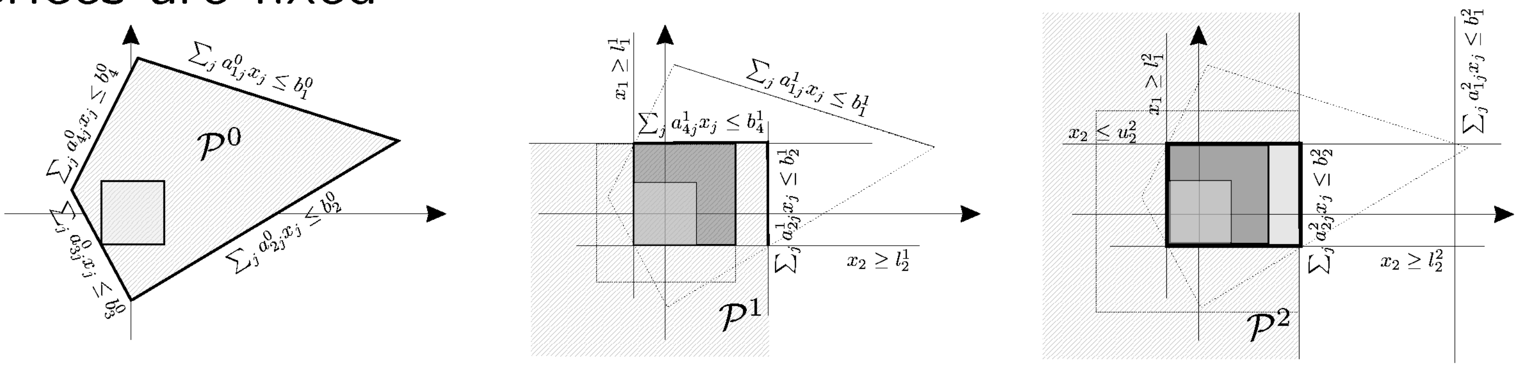
$\mathcal{P} = \{x \in \mathbb{R}^d : \ Ax \leq b\}$, a_{ij} is the j-th element in the i-th row of A

$\tau(\mathcal{P}) = \max\{\tau : \ \mathcal{B}(-\tau e, \tau e) \subseteq \mathcal{P}\} = \min\{\tau_i : \ i = 1, \ldots, m\}$ where

$$\tau_i = \begin{cases} \frac{b_i}{\sum_{j \in D} |a_{ij}|} & \text{if } \sum_{j \in D} |a_{ij}| > 0, \\ +\infty & \text{otherwise.} \end{cases}$$ because $z_i(\tau) = \max\left\{\sum_{j \in D} a_{ij}x_j : \ x \in \mathcal{B}\right\} \ldots$

Single Greedy Inner Box

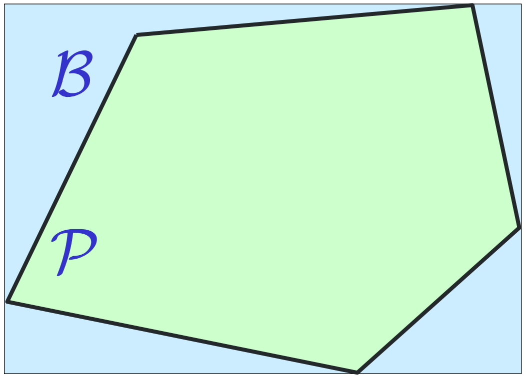
Main Idea: Starting from a point x_0 in $\mathcal{P}$, grow $\mathcal{B}$ until it bridges one of the constraints of $\mathcal{P}$. Then, fix a vertex, remove the active constraints and continue, until all the verices are fixed



(a) (b) (c)

Complexity $O(md^2)$, $m = \#$ rows of A

Single Outer Box



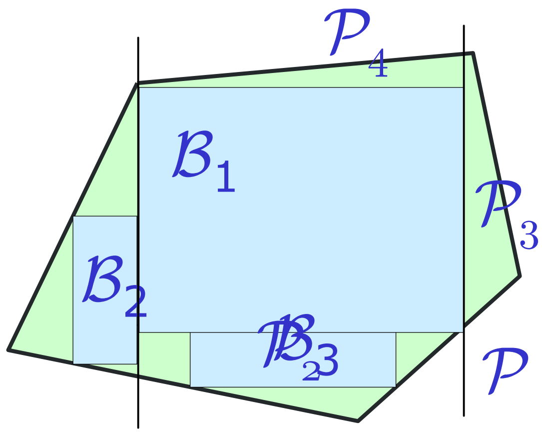
$$\mathcal{P} = \{x : \; Ax \leq b\}$$

Main Idea: Find the point u_j (l_j) in $\mathcal{P}$ with the biggest (smallest) j-th coordinate

$$\begin{aligned} l_j &= \min\{x_j : \; Ax \leq b\} \\ u_j &= \max\{x_j : \; Ax \leq b\} \end{aligned}$$

Complexity: $O(d \; \mathbf{lp}(m, d))$

Recursive Inner Approximation



Main Idea: Partition $\mathcal{P} \setminus \mathcal{B}$ in $2d$ polyhedra and compute the inner approximation of the rests

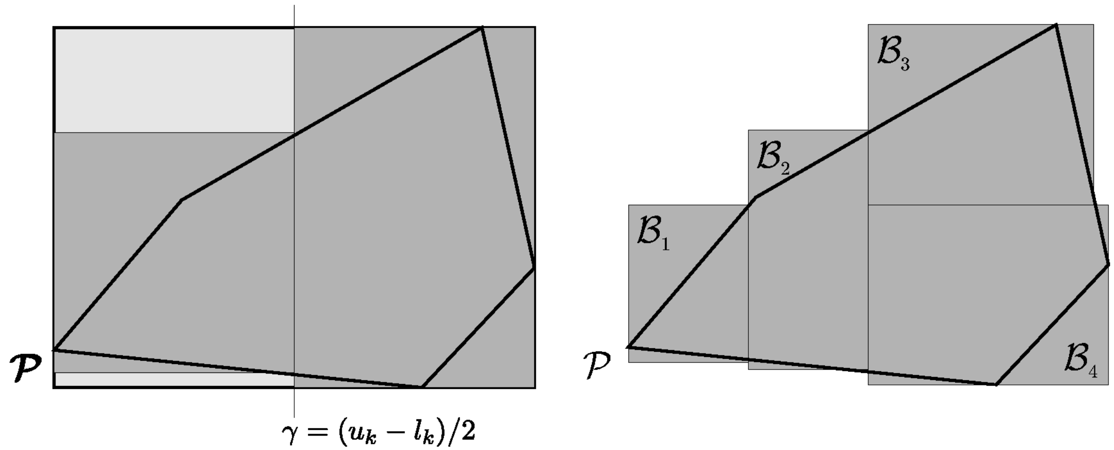
Stopping Condition: Prune a branch of the approximation if $\text{vol}(\mathcal{B}) < \epsilon$

Lemma 3 *The total number of recursive calls is bounded by* $2d \left\lfloor \frac{\text{vol}(\mathcal{P})}{\epsilon} \right\rfloor$.

Theorem 1 *Let* $\mathcal{I}_\epsilon = \{\mathcal{B}_t\}_{t=1}^{S(\epsilon)}$ *be the inner approximation of the polytope* $\mathcal{P}$ *for a given* $\epsilon > 0$. *Then,*

$$\lim_{\epsilon \to 0} \cup_{t=1}^{S(\epsilon)} \mathcal{B}_t \underset{a.e.}{=} \mathcal{P}.$$

Recursive Outer Approximation



Main Idea: Partition $\mathcal{P}$ in 2 polyhedra along the longest edge and compute the outer approximation of the two polyhedra

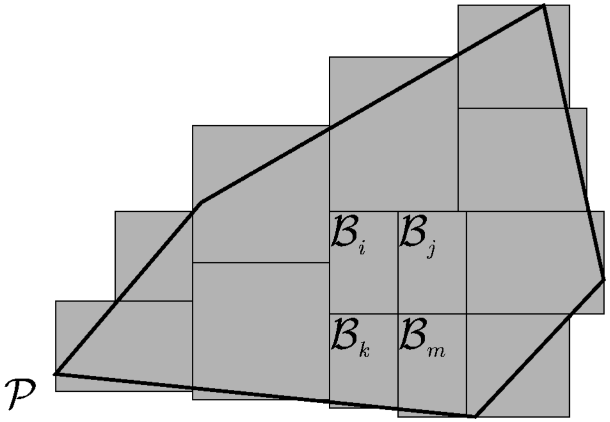
Stopping Condition: Prune a branch of the approximation if vol($\mathcal{B}$) $< \epsilon$

Lemma 4 *Let V denote the volume of the minimum volume outer box of $\mathcal{P}$. The total number of boxes is bounded by $\left\lceil \frac{4V}{\epsilon} \right\rceil$.*

Theorem 1 *Let $\mathcal{E}_\epsilon = \{\mathcal{B}_t\}_{t=1}^{T(\epsilon)}$ be the outer approximation of the polytope $\mathcal{P}$ for a given $\epsilon > 0$. Then*

$$\lim_{\epsilon \to 0} \cup_{t=1}^{T(\epsilon)} \mathcal{B}_t \underset{a.e.}{=} \mathcal{P}.$$

Fragmentation

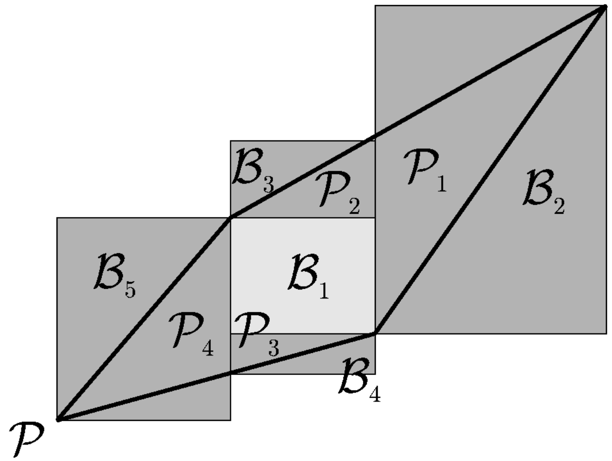


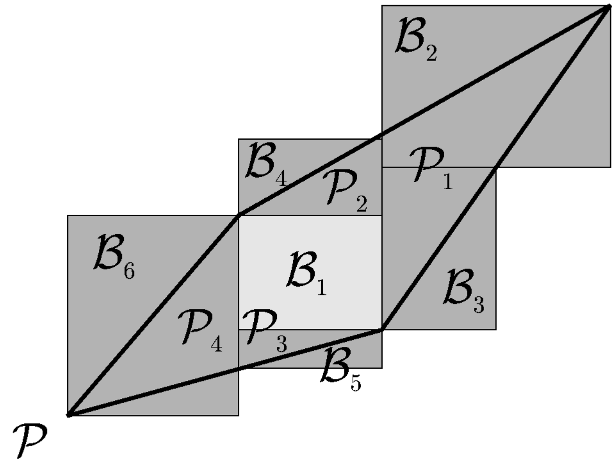
The multiple box outer approximation might generate too many polyhedra

Solution: Stop the approximation if $\mathcal{B}$ is in the interior of Pi.e. $\mathcal{B} \cap \delta\mathcal{P} = \emptyset$

Alternative: Combine inner and outer approximation

Recursive Inner-Outer Approximation

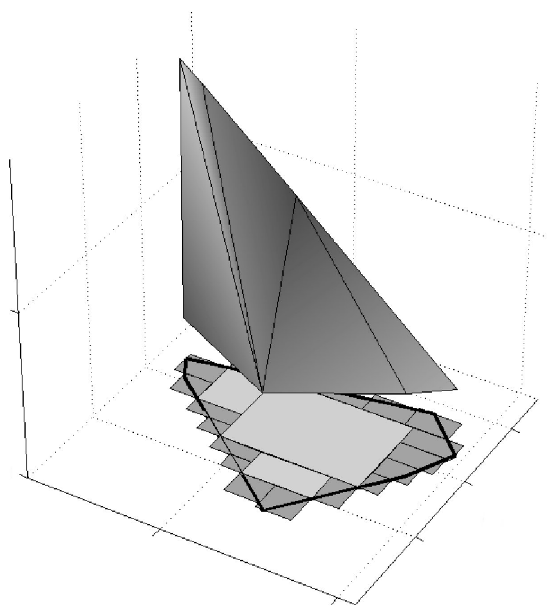


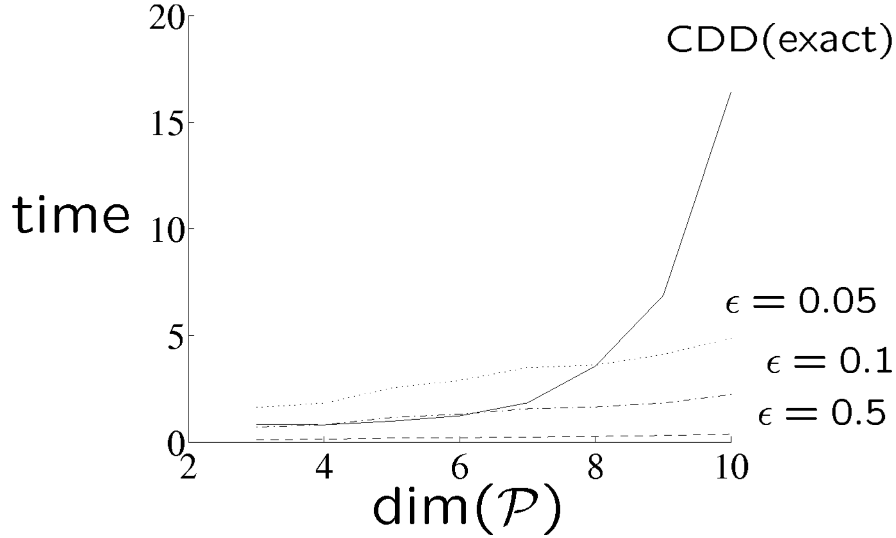


Main Idea: First perform an inner approximation, then compute the outer approximation of the rests

Computes in one shot both the inner and outer approximation

Extension: Approximate Projections





Main Idea: First perform an inner approximation, then compute the outer approximation of the rests

Computes in one shot both the inner and outer approximation

Conclusions

Algorithms to compute an inner and an outer approximation of a polytope

- Minimal volume error and number of boxes
- Alternative to the exact computation of the projection
- Good performance

Open Question: determine the projection of $\mathcal{P}$ (or a polyhedral approximation) using the approximation

Possible Extension: Use arbitrary polytopes (i.e. octagons) as approximant shapes