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1. The Statement

Jensen's discovery of the covering lemma arose out of work on the singular cardinals problem. Cohen published his proof of the independence of the continuum hypothesis [4, 5] in 1963, and one year later Easton's thesis [13, 14] completely settled the question of the size of the continuum for regular cardinals. The continuum problem for singular cardinals remained open, and the *Singular Cardinal Hypothesis* (SCH), stating (in its simplest form) that $2^\lambda = \lambda^+$ for every singular strong limit cardinal, became one of the

The writing of this chapter was partially supported by grant number DMS-9970536 from the National Science Foundation.
The author would also like to thank Diego Rojas-Robello, Assaf Sharon and the referee for helpful suggestions.

most important problems in set theory. It was ten years before Silver made the first significant advance on the problem: In a sharp contrast to Easton's result, which stated that the only constraints on the size of the continuum for regular cardinals are the obvious ones, Silver [55] proved that the SCH cannot fail at a singular cardinal of uncountable cofinality unless it already fails at all but a nonstationary set of smaller cardinals. Silver's proof, which depends heavily on the use of the filter of closed unbounded subsets of λ, fails badly at cardinals of cofinality ω and attention turned immediately to understanding this case. A year later, in 1974, Jensen distributed a series of handwritten notes titled *Marginalia on a Theorem of Silver.*[1] These notes, later revised by Devlin and Jensen and published [7] under the same title, stated and proved the basic covering lemma for L:

1.1 Theorem. *If $0^\sharp$ does not exist then for any set x of ordinals there is a set $y \in L$ such that $x \subseteq y$ and $|y| = \max\{|x|, \omega_1\}$.*

It is an immediate corollary that $\neg 0^\sharp$ implies the SCH: theorem 1.1 implies that any function $f\colon \operatorname{cf}(\lambda) \to \lambda$ is determined by a covering set $y \supseteq \operatorname{ran}(f)$ in L of size at most $\max\{\omega_1, \operatorname{cf}(\lambda)\}$, together with a function from $\operatorname{cf}(\lambda)$ into y. Thus $\lambda^{\operatorname{cf}(\lambda)} \leq (\lambda^{\operatorname{cf}(\lambda)})^L \tau^{\operatorname{cf}(\lambda)} = \lambda^+ 2^{\operatorname{cf}(\lambda)} = \max\{\lambda^+, 2^{\operatorname{cf}(\lambda)}\}$, where $\tau = \max\{\omega_1, \operatorname{cf}(\lambda)\}$. This implies the more general form of the SCH, $\lambda^{\operatorname{cf}(\lambda)} = \lambda^+ 2^{\operatorname{cf}(\lambda)}$ for every singular cardinal λ, and this in turn implies $2^\lambda = \lambda^+$ if λ is a singular strong limit cardinal.

The most obvious direction in which to extend the covering lemma is by weakening the assumption $\neg 0^\sharp$ to allow larger cardinals in the universe. The first step in this direction was due to Dodd and Jensen, who constructed a core model K^{DJ} under the assumption that there is no inner model with a measurable cardinal [9, 10, 8]. The Dodd-Jensen core model is, in many ways, similar to L: it satisfies the GCH along with most of the combinatorial properties of L, and it satisfies the same covering lemma:

1.2 Theorem. *Assume that there is no inner model with a measurable cardinal, and let K^{DJ} be the Dodd-Jensen core model. Then for any set x of ordinals there is a set $y \in K^{\mathrm{DJ}}$ such that $y \supseteq x$ and $|y| = |x| + \omega_1$.*

The statement cannot be extended directly to larger cardinals, since Prikry forcing [45] gives a counterexample. However, Dodd and Jensen generalize theorem 1.2 to show that Prikry forcing is the only possible counterexample [11]:

1.3 Theorem. *Assume that $0^\dagger$ does not exist, but there is an inner model with a measurable cardinal, and that the model $L[U]$ is chosen so that $\kappa = \operatorname{crit}(U)$ is as small as possible. Then one of the following two statements holds:*

[1] It is worth pointing out that about 20 years later, during the 1990's, this same problem led to another of the major advances in set theory, Shelah's pcf theory ([52], see chapter [1]).

1. *For every set x of ordinals there is a set $y \in L[U]$ with $y \supseteq x$ and $|y| = |x| + \omega_1$.*

2. *There is a sequence $C \subseteq \kappa$, which is Prikry generic over $L[U]$, such that for all sets x of ordinals there is a set $y \in L[U, C]$ such that $y \supseteq x$ and $|y| = |x| + \omega_1$.*

Furthermore, the sequence C of clause (2) is unique up to finite initial segments.

Theorem 1.3 can easily be generalized to models with no inaccessible limit of measurable cardinals, but two problems have to be surmounted to extend it to larger cardinals: (i) it is necessary to construct a core model which can consistently contain larger cardinals and for which the basic argument of the proof of the covering lemma can be made to work, and (ii) it is necessary to find a useful statement of the covering lemma for this core model which can be proved from the basic argument. The construction and basic properties of the core model are given in chapters [46] and [56], or in [28]. In addition the current chapter includes, in section 4, an outline of the theory of the core model for sequences of measures and for non-overlapping extenders.

A statement of the full covering lemma for these models will be deferred to section 4 of this chapter, but Theorem 1.8 below states a simplified version which generalizes the result of Dodd and Jensen by showing that a singular cardinal which is regular in K is made singular by a set which approximates a Prikry-Magidor generic set (see [31, section 2.2]. This statement requires some preliminary definitions.

Say that a cardinal κ is *μ-measurable* if there is an embedding $i\colon V \to M$ such that the measure $\{ x \subseteq \kappa : \kappa \in i(x) \}$ associated with i is a member of M. This is the weakest large cardinal property which requires the existence of something more than normal ultrafilters.

For the rest of this subsection we assume that there is no inner model with a μ-measurable cardinal, and we assume that K is the core model. If κ is a cardinal of K and $\beta < o(\kappa)$ then use $\mathcal{U}(\kappa, \beta)$ to denote the measure of order β on κ in K. First we define what appears to be a rather weak notion of indiscerniblity:

1.4 Definition. Assume that κ is a singular cardinal which is regular in K. A closed unbounded subset C of κ is a *weak Prikry-Magidor set* for K if (i) $|C| < \kappa$, and (ii) if x is any closed unbounded subset of κ with $x \in K$ then $C - x$ is bounded in κ.

Any Prikry-Magidor generic subset of κ is a weak Prikry-Magidor set.

1.5 Theorem. *If there is no model with a μ-measurable cardinal then any weak Prikry-Magidor set $C \subseteq \kappa$ for K has the following two properties:*

1. *C is eventually contained in any set $a \in K$ such that $a \in \mathcal{U}(\kappa, \beta)$ for all $\beta < o(\kappa)$.*

2. *$C \cap \lambda$ is a weak Prikry-Magidor set over K for every sufficiently large limit point λ of C.*

1.6 Definition. A function σ is an *assignment function in K* for C if

1. There is $h \in K$ such that $\sigma(\nu) = h(\nu)$ for all sufficiently large $\nu \in C$.

2. C is a set of indiscernibles for $\mathcal{U}(\kappa, \sigma(\nu))$ in the sense that for any sequence $\langle a_\xi : \xi < \kappa \rangle \in K \cap {}^\kappa\mathcal{P}(\kappa)$ of subsets of κ, and for all sufficiently large $\nu \in C$, we have $\forall \xi < \nu \big(\nu \in a_\xi \iff a_\xi \in \mathcal{U}(\kappa, \sigma(\nu))\big)$.

If $o(\kappa) < \kappa$, as in Prikry-Magidor forcing, then we can always take σ to be the function $\sigma(\nu) = o(\nu)$. If $o(\kappa) \geq \kappa^+$ then Radin forcing (cf chapter[15]) can be used to add a set C which satisfies the definition of a weak Prikry-Magidor set except that κ remains regular, and hence $|C| = \kappa$. Clearly such a set does not have an assignment function, since any assignment function would be bounded in κ^+. Thus the following theorem would be false if the requirement that $|C| < \kappa$ were dropped from definition 1.4.

1.7 Theorem. *Any weak Prikry-Magidor set $C \subseteq \kappa$ for K has an assignment function in K. Furthermore*

1. *The assignment function σ is unique except for initial segments.*

2. *The assignment function is weakly increasing in the sense that $\sigma(\nu) \geq \limsup\{\, \sigma(\xi) + 1 : \xi \in C \cap \nu \,\}$ for every sufficiently large limit point ν of C.* $\dashv$

Any weak Prikry-Magidor set C which satisfies the stronger version of clause (2) obtained by changing the inequality to an equality, and in particular has $\sigma(\xi) = 0$ for each successor member ξ of C, is a Prikry-Magidor generic set.

We cannot hope to actually cover subsets of κ using indiscernibles for only a single cardinal κ, but the following theorem, which is our promised version of the covering lemma, generalizes Dodd and Jensen's theorem 1.3 to say that any small subset of κ can be approximated by a weak Prikry-Magidor set:

1.8 Theorem. *If the singular cardinal κ is regular in K then for any set $x \subseteq \kappa$ with $|x| < \kappa$ there is a weak Prikry-Magidor set $C \subseteq \kappa$ for K and a function $g\colon \kappa \to \kappa$ in K such that $x - \bigcup_{\nu \in C} (g(\nu) - \nu)$ is bounded in κ.*

The Weak Covering Lemma

No satisfactory statement of the full covering lemma is known for cardinals much larger than a single strong cardinal: the indiscernibles are too complicated to use to approximate arbitrary sets in the manner of theorem 1.3 or theorem 1.8. What remains is known as the weak covering lemma, which is proved by using the same basic proof as that used below a strong cardinal, but applying it only to subsets of the interval $(\lambda, (\lambda^+)^K)$, in which there cannot be any indiscernibles.

1.9 Definition. A class model M of set theory satisfies the *weak covering property* if ${\lambda^+}^M = \lambda^+$ for every singular cardinal λ of V.

The weak covering lemma, stating that K has the weak covering property, is among the most important consequences of the covering lemma. If K contains more than a few measurable cardinals then the weak covering property is needed to prove the basic properties of the core model, including the full covering lemma; indeed the weak covering property may be taken as part of the definition of what it means to be a "core model". The best results known to date are as follows:

1.10 Theorem. *1. If the sharp for a model with a class of strong cardinals does not exist, then there is a core model K of the form $L[\mathcal{E}]$ which satisfies the weak covering property (see [49]).*

2. If there are no models with a Woodin cardinal and there is a subtle cardinal θ, then the Steel core model $K_\theta = L_\theta[\mathcal{E}]$ below θ exists, and satisfies the weak covering property for $\lambda < \theta$ (see chapter [46]).

3. If $\mathrm{AD}^{L(\mathbb{R})}$ is false in every set generic extension, then either the Steel core model of clause (2) exists and satisfies the weak covering property, or else there is a ordinal α such that the Woodin core model K_{V_α} exists and satisfies the weak covering property above α.

The proof of clause (1) of theorem 1.10 will be sketched in section 4. The proof of clause (2) is given in [43] and [42], and part of the proof of clause (3) is given in [47].

It might be noted that the core model K_{V_α} of clause (3) contains all of V_α and thus only gives useful information about the universe above α. It is not clear what, if anything, can be done in the actual vicinity of a Woodin cardinal. Mitchell [41] reports some unsatisfactory results from applying the standard proof at a Woodin cardinal, but the following result of Woodin may suggest a more useful direction. The theorem only applies below 2^ω (assuming AC in V) but that is the region where the large cardinals implied by AD exist. This result also goes beyond the large cardinal limit of $\neg$AD imposed by theorem 1.10.

1.11 Theorem (Woodin [63]). *Suppose that the nonstationary ideal on ω_1 is ω_2-saturated, and suppose that M is a transitive inner model of* ZF + DC + AD *containing all reals and ordinals such that every set of reals in M is, in V, weakly homogeneously Souslin. Let X be a bounded subset of Θ^M such that $|X| = \omega_1$. Then there exists $Y \supseteq X$ in M such that $|Y|^M = \omega_1$.*

Here Θ^M is the supremum of the ordinals δ in M such that there is a map in M from the reals onto δ.

The Strong Covering Lemma

This concludes, until section 4, the discussion of cardinals larger than a measurable cardinal. We now return to the models L and $L[U]$ in order to look at another direction in which the original covering lemma has been extended. The *strong covering lemmas* use Jensen's proof but show that more can be extracted from it. Theorem 1.12, which is our version of the strong covering lemma for the Dodd-Jensen core model, is essentially taken from unpublished notes of Carlson, who proved it for L by using a variant, influenced by ideas of Silver, of Jensen's proof. The idea, as well as the name, comes from work of Shelah (see [52, theorem VII.0.1] and [53]) who obtains the strong covering property in a more general setting by assuming the ordinary covering property together with some extra combinatorial structure. We will describe his main application in the next section.

1.12 Theorem. *Assume that there is no inner model with a measurable cardinal. Then there is a class $\mathbf{C} \subseteq K^{\mathrm{DJ}}$, definable in K^{DJ}, such that the following statements hold:*

1. *If x is any uncountable set of ordinals then there is a set $X \in \mathbf{C}$ such that $x \subseteq X$ and $|x| = |X|$.*
2. *The class $\mathbf{C}$ is closed in V under increasing unions of uncountable cofinality; that is, if $\langle X_\nu : \nu < \eta \rangle$ is an increasing sequence of members of $\mathbf{C}$ and $\mathrm{cf}(\eta) > \omega$ then $\bigcup_{\nu<\eta} X_\nu \in \mathbf{C}$.*

Notice that clause 2 holds for all sequences $\langle X_\nu : \nu < \eta \rangle$, not only for those which are members of K.

The statement of theorem 1.12 remains valid if $L[U]$ exists but $0^\dagger$ does not, provided that K is replaced by the appropriate model $L[U]$ or $L[U, C]$ from theorem 1.3. The following statement, however, is stronger and is easier to generalize to larger core models (see section 4).

1.13 Theorem. *Assume that $0^\dagger$ does not exist, and that the measure U and Prikry sequence C are as in theorem 1.3(2). Then there is a class $\mathbf{C} \subseteq L[U, C]$ which satisfies clauses (1) and (2) of theorem 1.12, and in addition*

3. *For each set $X \in \mathbf{C}$ there is an ordinal $\rho < \max\{\omega_2, |X|^+\}$ and a function $h \in L[U]$ such that $X = \mathcal{H}^h(\rho \cup C)$, the smallest set containing $\rho \cup C$ and closed under h.*

4. *The class $\mathbf{C}$ is definable in $L[U]$ in the sense that there is a formula φ such that a set X is in $\mathbf{C}$ if and only if there is a set $A \in U$, a function $h \in L[U]$ and an ordinal ρ such that $L[U] \models \varphi(A, h, \rho)$ and $X = \mathcal{H}^h(\rho \cup (C \cap A))$.*

Clause (4) follows from the definability of forcing: the formula $\varphi(A, h, \rho)$ asserts that $(\varnothing, A) \Vdash \mathcal{H}^h(\check{\rho} \cup \dot{C}) \in \dot{\mathbf{C}}$, where the forcing is Prikry forcing for the measure U, $\dot{C}$ is a name for the resulting Prikry sequence, and $\dot{\mathbf{C}}$ is a name, derived from the proof of the covering lemma, for the class $\mathbf{C}$.

The following proposition gives a very useful property of the function h. It is also true for L, for the Dodd-Jensen core model, and for the core model for sequences of measures.

1.14 Proposition. *Let h be as in theorem 1.13 for $X \in \mathbf{C}$. Then h can be written as $h = \bigcup_{\nu<\alpha} h_\alpha$ for some functions $h_\nu \in X$ such that $h_\nu \subseteq h_{\nu'}$ whenever $\nu < \nu' < \alpha$.*

The Covering Lemma Without Second-Order Closure

The strong covering lemma can be viewed as asserting that if $0^\sharp$ does not exist then every sufficiently closed set is a member of L. The precise statement of the requirement that X be sufficiently closed has both first-order and second-order components. Magidor's covering lemma [30] for L weakens the conclusion of the covering lemma in order to eliminate the second-order components:

1.15 Theorem (Magidor [30]). *If $0^\sharp$ does not exist and x is a set of ordinals which is closed under the primitive recursive set functions, then there are sets $y_n \in L$ for $n < \omega$ such that $x = \bigcup_{n<\omega} y_n$.*

Magidor also extends theorem 1.15 to the Dodd-Jensen core model, by requiring closure under a larger set of functions in K^{DJ} and assuming that there is no inner model with an ω_1-Erdős cardinal. He points out that this assumption is necessary, since if there is an ω_1-Erdős cardinal in K then there is a generic extension M of K such that for any countable set $\mathcal{F}$ of functions in K there is a set $X \in M$ which is closed under the functions in $\mathcal{F}$, but is not a countable union of sets in K.

The following theorem was proved independently of theorem 1.15, but the same idea lies behind both theorems.

1.16 Theorem (Mitchell [32, 40], Jensen [12]). *If there is no model with a Woodin cardinal then every regular Jónsson cardinal is Ramsey in the core model K.*

Furthermore, if κ is δ-Jónsson for some uncountable ordinal $\delta < \kappa$ then κ is δ-Erdős in K.

A cardinal κ is said to be *δ-Jónsson* if any structure with universe κ and countably many predicates has an elementary substructure with order type δ; and κ is said to be *δ-Erdős* [3] if for any such structure and any closed unbounded subset C of κ there is a normal set of indiscernibles of order type δ contained in C.

A similar proof shows that Chang's conjecture implies that ω_2 is ω_1-Erdős in K, and together with a result of Silver (1967, unpublished) proves the equiconsistency of the two notions.

This concludes our discussion of the various statements of the covering lemma. In section 2 we will briefly describe some of the basic applications of the core model, and in section 3 we will outline the basic proof of the covering lemma and its variants under the hypothesis that $0^\dagger$ does not exist. The final section looks at larger cardinals, giving the statement and an outline of the proof of the covering lemma for sequences of ultrafilters or extenders. The basic proof is taken almost unchanged from section 3, but the analysis of the resulting system of indiscernibles is much more difficult.

2. Basic Applications

We pointed out earlier that the source of the covering lemma, as well as its first application, is the Singular Cardinal Hypothesis:

2.1 Theorem (Jensen [7]). *If $0^\sharp$ does not exist then $\lambda^{\mathrm{cf}(\lambda)} = \max\{\lambda^+, 2^{\mathrm{cf}(\lambda)}\}$ for every singular cardinal λ, and hence $2^\lambda = \lambda^+$ for every singular strong limit cardinal λ.*

Jensen's proof can be generalized to larger cardinals, but the full strength of the failure of the SCH was not discovered until Gitik combined the covering lemma with Shelah's pcf theory:

2.2 Theorem (Gitik [17, 20]). *The failure of the singular cardinal hypothesis is equiconsistent with $o(\kappa) = \kappa^{++}$.*

In section 4.3 we present Gitik's proof that the failure of the singular cardinal hypothesis implies that there is an inner model with $o(\kappa) = \kappa^{++}$. Gitik's proof that this is sufficient is given in [16]; in this Handbook [15] he describes a later method of forcing which is simpler and more general, but which gives slightly weaker results in this case.

The following theorem is Shelah's main application of the strong covering lemma 1.12. The sufficiency of slightly stronger large cardinal assumptions is proved in [54].

2.3 Theorem. *If M is a model containing K such that $M \models$ GCH, and r is a real such that $M[r] \models \neg$CH, then there is an inner model with an inaccessible cardinal. If, in addition, the cardinals of $M[r]$ are the same as those of M then there is a model with a measurable cardinal.*

The Weak Covering Lemma

By far the most important consequence of the covering lemma is the weak covering property, definition 1.9. Indeed it is arguably more accurate to turn the statement around: the covering lemma is an application, and not necessarily the most important application, of the weak covering lemma. Below a strong cardinal the proof of the weak covering lemma is a special case of the proof of the full covering lemma, so that the importance of the weak covering lemma is not immediately apparent. Beyond a strong cardinal, in Steel's core model, we do not know how to even begin the proof of the covering lemma without first proving, by an entirely different method using a weak large cardinal hypothesis, a slightly weaker version of the weak covering lemma.

Among the most important properties of the core model K (stated under the assumption that $0^\P$ does not exist) which follow from the weak covering lemma are the following:

- The construction of K from the countably complete core model K^{c}.
- If $i\colon K \to M$ is an elementary embedding, where M is well-founded, then i is an iterated ultrapower of K.
- If U is a normal K-ultrafilter on κ and $\mathrm{Ult}(K, U)$ is well-founded then $U \in K$. If $\mathrm{crit}(U) > \omega_2$ then the hypothesis that $\mathrm{Ult}(K, U)$ is well-founded can be omitted.

Many results which are usually regarded as consequences of the covering lemma in fact use only these basic properties of the core model. Among such results are the lower bounds in the following theorem:

2.4 Theorem. *1. The failure of the* GCH *at a measurable cardinal κ is equiconsistent with $o(\kappa) = \kappa^{++}$.*

2. If κ is weakly compact and $o^K(\kappa) < \kappa^{++}$ then $\kappa^{+^K} = \kappa^+$.

3. If κ is Jónsson, there is no model with a Woodin cardinal, and the Steel core model exists (in particular, if there is no model with a strong cardinal), then $(\kappa^+)^K = \kappa^+$; furthermore $(\lambda^+)^K = \lambda^+$ for stationarily many $\lambda < \kappa$ [61].

4. The consistency of a Woodin cardinal implies that of the existence of a saturated ideal on ω_1. If the Steel core model exists then the existence *of such an ideal implies in turn that there is a Woodin cardinal in an inner model [57].*

Sketch of Proof. We prove, as an example, the lower bound for clause 1. The upper bound is proved in Gitik [16]. Suppose that U is a measure on κ and $2^\kappa \geq \kappa^{++}$, but that $o(\kappa) < \kappa^{++}$ in K. Let $i^U : V \to M = \mathrm{Ult}(V, U)$, and consider $i = i^U \restriction K : K \to K^M$. Then i is an iterated ultrapower of K, so let $i = i_{0,\theta}$ where $i_{\nu,\nu'} : N_\nu \to N_{\nu'}$. If $\nu < \theta$ is a limit ordinal then there is $\xi_\nu < \nu$ and $U_\nu \in N_{\xi_\nu}$ such that $N_{\nu+1} = \mathrm{Ult}(N_\nu, i_{\xi_\nu,\nu}(U_\nu))$. Since $o(\kappa) < \kappa^{++} \leq \theta$, there is a stationary class $S \subseteq \kappa^{++}$ of ordinals of cofinality ω such that $\xi_\nu = \bar{\xi}$ and $U_\nu = \bar{U}$ are constant for $\nu \in S$. Now fix $\nu \in S \cap \lim(S)$. If $\vec{\kappa} = \langle \nu_n : n < \omega \rangle$ is a cofinal sequence in $S \cap \nu$ and $\kappa_n = \mathrm{crit}(i_{\nu_n,\nu})$, then the sequence $\vec{\kappa}$ generates the measure $i_{\bar{\xi},\nu}(\bar{U})$. Since ${}^\omega M \subseteq M$, the sequence $\vec{\kappa}$ and hence the measure $i_{\bar{\xi},\nu}(\bar{U})$ is a member of M. It follows that $i_{\bar{\xi},\nu}(\bar{U}) \in K^M$; but this is impossible since $i_{\bar{\xi},\nu}(\bar{U})$ is not in $N_{\nu+1}$ and hence not in $N_\theta = K^M$. This contradiction completes the proof that $o(\kappa) \not< \kappa^{++}$ in K. ⊣

The naïve proof of clause 2 uses the fact that κ is inaccessible; however Schindler [48] has used $\square_{\kappa,\kappa}$ in K to adapt it to successor cardinals, showing that if κ and κ^+ both have the tree property then κ is strong in K.

The main reason for the importance of the weak covering property is that it can be used to adapt to K techniques which Kunen (see [27, §21]) originally used in proving that $0^\sharp$ follows from the existence of a nontrivial elementary embedding from L into L. As applied to L these techniques make use of the fact that any proper class $\Gamma \prec L$ is isomorphic to L. The corresponding fact for K is that any class $\Gamma \prec K$ is isomorphic to K, provided that the class

$$\{\, \lambda : \mathrm{ot}(\Gamma \cap \lambda^{+K}) = \lambda^{+K} \,\} \tag{1.1}$$

is stationary. Cardinal calculations show that the classes Γ used in Kunen's arguments satisfy that

$$\{\, \lambda : 2^{<\lambda} = \lambda \wedge \mathrm{cf}(\lambda) < \lambda \wedge \left|\Gamma \cap \lambda^+\right| = \lambda^+ \,\} \tag{1.2}$$

is stationary. The weak covering lemma implies that the class (1.2) is contained in the class 1.1 and hence implies that $\Gamma \cong K$.

The Full Covering Lemma

The singular cardinal hypothesis has already been mentioned as a result which requires the full covering lemma. We now look at other such results.

2.5 Theorem (Dodd-Jensen [11], Mitchell [36]). *Let κ be a singular cardinal of cofinality λ which is regular in K. Then κ is measurable in K, and if $\lambda > \omega$ then $o(\kappa) \geq \lambda$ in K.*

The proof, using theorems 1.7 and 1.8, is easy, and a more careful analysis yields a classification of singular cardinals [37]:

2.6 Theorem (Mitchell [37]). *Assume that $\neg\exists\kappa\, o(\kappa) = \kappa^{++}$. Let κ be a singular cardinal which is regular in K. Then there is a weak Prikry-Magidor set C (with assignment function σ) witnessing the singularity of κ such that*

1. *If $\mathrm{cf}^K(\kappa) > \omega$ then C is a weak Prikry-Magidor set (definition 1.4).*
2. *If $\mathrm{cf}(\kappa) = \omega$ then let $\beta \leq o(\kappa)$ be the least ordinal such that $o(\nu) < \beta$ for all but boundedly many $\nu \in C$. Then*
   (a) *If β is a successor ordinal then C is Prikry generic over K.*
   (b) *If $\mathrm{cf}(\beta) < \kappa$ then $\mathrm{cf}(\beta) = \omega$, and C is a weak Prikry-Magidor sequence.*
   (c) *If $\mathrm{cf}^K(\beta) = \kappa$, witnessed by $\tau\colon \kappa \to \beta$, then there is a weak Prikry-Magidor sequence D with assignment function σ' such that the increasing enumeration $C = \langle\, c_n : n < \omega \,\rangle$ of C is definable recursively from D by letting c_{n+1} be the least member c of D such that $\sigma'(c) \geq \tau(c_n)$.*
   (d) *If $\mathrm{cf}^K(\beta) = \kappa^+$ then C is a sequence of accumulation points for κ (the definition of an accumulation point is given in definition 4.18).*

Further, the set C can be chosen to be maximal in a sense which makes it definable up to initial segment, except in case (2d) where any two such sequences eventually alternate.

A measure U on κ is a *weak repeat point* if for every set $A \in U$ there is $U' \lhd U$ with $A \in U'$. Results similar to the following theorem have been proved by Gitik [19, 22] for the nonstationary ideal.

2.7 Theorem (Mitchell [33]). *If the closed, unbounded filter on ω_1 is an ultrafilter, then there is a weak repeat point in K.*

2.8 Theorem (Sureson [59], Mitchell [35]). *The following four statements are equiconsistent, where $\delta < \kappa$ is a regular cardinal.*

1. *There is a κ-complete ultrafilter U on κ extending the closed, unbounded filter such that $\{\, \alpha : \mathrm{cf}(\alpha) = \delta \,\} \in U$.*
2. *There is a κ-complete ultrafilter U on κ with δ skies; that is, there is an increasing sequence $\langle\, \alpha_\nu : \nu < \delta \,\rangle$ of ordinals between κ and $i^U(\kappa)$ with the property that $i^U(f)(\alpha_\nu) < \alpha_{\nu'}$ for all $\nu < \nu' < \delta$ and all $f : \kappa \to \kappa$.*

3. *There is a κ^{+}-saturated normal filter F with $\{\,\alpha : \mathrm{cf}(\alpha) = \delta\,\} \in F$.*

4. *$o(\kappa) = \delta + 1$ if $\delta > \omega$, and $o(\kappa) = 2$ if $\delta = \omega$.*

The covering lemma is used to prove that each of clauses (1–3) imply that clause 4 holds in K. The forcing used in [35] to prove the other direction has been simplified and extensively generalized by Gitik; in particular it is used to give the upper bounds for the consistency strength of the failure of the SCH.

The $\mathbf{\Sigma}^1_3$ absoluteness theorem, theorem 2.9 below, was originally proved by Jensen assuming $\neg 0^{\dagger}$; Magidor (unpublished, see [58, §4]) has given a simpler proof but one which gives slightly less information. Clause (1) was proved under the assumption that $\neg\exists\kappa\, o(\kappa = \kappa^{++})$ by Mitchell [38] using Jensen's method. Steel and Welch [58] later proved clause (1) using Magidor's method, and Steel, using results of Hjorth, extended it [57, theorem 7.9] to prove clause (2).

We say that a model M is *$\mathbf{\Sigma}^1_3$-correct* if for any Σ^1_3 formula φ and any real $r \in M$ we have $M \models \varphi(r)$ if and only if $V \models \varphi(r)$.

2.9 Theorem ($\mathbf{\Sigma}^1_3$-absoluteness). *1. Suppose that there is no model of $o(\kappa) = \kappa^{++}$ and that $r^{\sharp}$ exists for every real r. Then any model M of* ZFC *such that $M \supseteq K$ is $\mathbf{\Sigma}^1_3$-correct.*

2. *Assume that there are two measurable cardinals and no inner model with a Woodin cardinal. Then any model $M \supseteq$ of* ZFC *is $\mathbf{\Sigma}^1_3$-correct.*

The conclusion can be equivalently stated as "$\mathbf{\Sigma}^1_3$ formulas are absolute for models containing K".

3. The Proof

This section outlines the proof of the Jensen and Dodd-Jensen covering lemmas up through a single measurable cardinal. Section 4 will continue, using the same basic ideas, to describe the covering lemmas for larger cardinals.

Subsection 3.1 briefly describes the basic tools, including fine structure, needed for the proof. Subsection 3.2 gives the proof of Jensen's covering lemma for L, theorem 1.1, (including the proof of the strong variant, theorem 1.12). Subsection 3.3 extends this proof to the Dodd-Jensen covering lemma, theorems 1.2 and 1.3. Finally section 3.4 looks at the two major variants on the covering lemma: Magidor's theorem 1.15, and the theorem 1.16 stating that Jónsson cardinals are Ramsey in K.

The proofs given in this section are not complete, but enough details are given that a reader with some understanding of fine structure should be able to fill in the rest. Complete proofs may be found in the original sources, [7, 8, 9, 10, 11, 26, 30], or in later references such as Devlin [6].

3.1. Fine Structure and Other Tools

This chapter has two incompatible aims: the first is to be accessible to a reader without a sophisticated knowledge of fine structure, and the second is to present a proof which is sufficiently complete that a reader with a understanding of fine structure can fill in the details.

One very interesting approach to this dilemma was invented by Silver (see [27, 30]), who gave a proof of the Jensen covering lemma which essentially eliminates any need for fine structure. He has extended this method to yield the Dodd-Jensen covering lemma, and it has been further extended and publicized by Magidor. In unpublished work, Magidor and Silver have used this approach at least up a model with a cardinal α such that $o(\alpha)$ is measurable. It is not known whether this approach works up to $o(\alpha) = \alpha^{++}$, and it seems unlikely that it will work for the newer models containing cardinals up to a Woodin cardinal. This rules out its use here, since this section is intended to serve as an introduction to covering lemmas for larger models.

The approach we have used is very close that that presented by Schindler and Zeman earlier in this Handbook [51]. We have attempted to make this chapter accessible without such an introduction: The hope is that this presentation will be sufficiently generic that a knowledgeable reader will be readily able to translate it to his preferred version, while at the same time it is sufficiently specific (without being too detailed) that it is understandable to a naïve reader. However any reader wanting a full understanding of the subject is encouraged to read [51] before or after this chapter.

Our presentation of fine structure, like Jensen's original papers, is based directly on master code structures. We follow current practice in using Jensen's J_α hierarchy, rather than the L_α-hierarchy. This newer hierarchy yields substantial advantages, some of which will be pointed out in the text, for a complete exposition of the fine structure; however the differences are not apparent at this level of detail and the naïve reader will lose little, if anything, by simply reading J_α as L_α.

One unfortunate exception to this equivalence comes from the fact that members $M = J_\alpha$ of the J-hierarchy are conventionally indexed by $\alpha = \mathrm{On}(M)$, which is always a limit ordinal. Thus the γth member of this hierarchy is $J_{\omega\cdot\gamma}$, which is nearly the same as L_γ. In particular $J_{\omega\cdot\gamma+n}$ does not exist for $0 < n < \omega$: the successor of a member $J_{\omega\cdot\gamma}$ of the hierarchy is $J_{\omega\cdot\gamma+\omega}$.

At some points in the arguments, primarily those involving the downward extension lemma, it did not seem possible to give the full proof without being more specific about the fine structure; In these cases we restrict ourselves to giving the proof in the simplest case, which is Σ_1 definability over J_α for a limit ordinal α. This case may seem very special, but in fact it essentially contains the general case. See [51] for a more complete discussion of fine structure.

As stated earlier, we take the basic models of our fine structure to be the sets J_α. We will call these models *mice* in anticipation of larger core models.

Two concepts are basic to the fine structure of a mouse $M = J_\alpha$: the Σ_n-projectum ρ_n^M and the Σ_n-Skolem function h_n^M. A third concept which is central to the proof of the covering lemma is the Σ_n-ultrafilter $\mathrm{Ult}_n(M, \pi, \kappa)$ of M, obtained by using the embedding π as an extender. A fourth concept, the use of substructures of mice, is used in the definition of fine structure and is central to the proof of $\square_\kappa$ and other combinatorial applications of fine structure; however it is more peripheral to the proof of the covering lemma, where it is only needed for the non-countably closed case.

We discuss these four concepts further before beginning the actual proof. We begin with the definition of the fine structure of J_α in the special case when $n = 1$ and α is a limit ordinal.

3.1 Definition. Assume that α is a limit ordinal, and that $M = (J_\alpha, A)$ is *amenable*, that is, $A \cap x \in J_\alpha$ for all $x \in J_\alpha$.

1. The Σ_1 *projectum* ρ_1^M of an amenable structure $M = (J_\alpha, A)$ is the least ordinal ρ such that there is Σ_1 subset x of ρ which is not a member of J_α, but is Σ_1 definable in M using a finite set $p \subseteq \alpha$ as a parameter.

2. The Σ_1 *standard parameter* p_1^M of M is the least finite sequence $p \in [\alpha]^{<\omega}$ of ordinals such that there is some set $x \subseteq \rho_1^M$ so that $x \notin J_\alpha$, but x is Σ_1 definable in M from parameters in $\rho_1^M \cup p$.

   The ordering of the parameters is lexicographical on descending sequences of ordinals; that is, $p < p'$ if $\max(p \triangle p') \in p'$.

3. The Σ_1 *standard master code* is the set A_1^M of pairs $(\ulcorner\varphi\urcorner, \xi)$ such that $\xi < \rho_1^M$ and $\ulcorner\varphi\urcorner$ is the Gödel number of a Σ_1 formula φ over M, with parameter p_1^M, such that $M \models \varphi(\xi)$.

4. The Σ_1 *Skolem function* h_1^M of M is defined as follows: fix an enumeration $\langle\, \exists z \varphi_n : n < \omega \,\rangle$ of the Σ_1 formulas of set theory. Then $h_1^M(\langle n, x\rangle)$ is defined if and only if there are z and y such that $M \models \varphi_n(x, y, z, p_1^M)$. In this case $h_1^M(\langle n, x\rangle) = y$ where (α', z, y) is the lexicographically least triple such that $(J_{\alpha'}, A \cap \alpha') \models \varphi_n(x, y, z, p_1^M)$.

5. The Σ_1-code $\mathfrak{C}_1(M)$ of M is the structure $(J_{\rho_1^M}, A_1^M)$.

It should be noticed that the Σ_1-Skolem function is itself Σ_1 definable over M. The Σ_1-Skolem function is a function of one variable; however we will frequently abuse the notation by writing it as a function with a variable number of arguments. Thus $h_1^M(x_1, x_2, x_3)$ should be understood to mean

$h_1^M(\ulcorner x_1, x_2, x_3 \urcorner)$ where $\ulcorner \dots \urcorner$ is an appropriate coding of finite sequences. In addition, we will abuse notation by writing $h_1``x$ to mean $h_1``^{<\omega}x$, the closure of x under the Skolem function function h_1.

We will rarely be using the function h_1^M as a Skolem function for any particular formula φ_n, and so will not normally mention the the parameter n explicitly, regarding it instead as being coded into the stated parameters.

If α is a successor ordinal, $\alpha = \gamma + 1$, then the definitions are the same, except that the hierarchy $\langle J_{\alpha'} : \alpha' < \alpha \rangle$ used for the definition of h_1^M is replaced by a hierarchy, with length ω, of the sets in $J_{\omega\cdot\gamma+\omega} - J_{\omega\cdot\gamma}$. The hierarchy depends on the specific fine structure being used. Jensen originally used the Levy hierarchy on L_γ, the k^{th} level of which contains the subsets of L_γ which are Σ_k definable in (L_γ, A). Later he invented the rudimentary functions and the hierarchy of sets $J_{\omega\cdot\alpha}$ in order to avoid technical complications caused by the use of the Levy hierarchy. See chapter [51] for a detailed presentation of the rudimentary functions and their use in setting up the fine structure.

One major advantage of the J_α-hierarchy over the L_α-hierarchy is that $[J_\alpha]^{<\omega} \subset J_\alpha$ even for successor ordinals α. Thus a finite set of ordinals $\alpha_0, \dots, \alpha_{k-1}$ can be freely treated as a single parameter, the finite sequence $\langle \alpha_0, \dots, \alpha_{k-1} \rangle$. In the case of the L_α-hierarchy some awkward and painful coding is necessary to achieve the same result.

We will not say more about the successor case, except to mention that arguments involving fine structure generally treat the case of successor α as simpler special cases of the arguments for limit ordinals α. This characterization of the case of successor α as "simpler" assumes, of course, an understanding of the detailed definition of the fine structure.

We now turn to consider the fine structure for $n > 1$. The central theme of fine structure is that it is never necessary to deal directly with Σ_{n+1} definability for any n greater than zero; instead a Σ_{n+1} formula is reduced to an equivalent Σ_1 formula over the Σ_n code of J_α. The definition of the Σ_n code $\mathfrak{C}_n(J_\alpha)$ is itself a good example of this theme.

3.2 Definition. We define the Σ_n-codes of J_α by recursion on $n < \omega$. We set $\mathfrak{C}_0(J_\alpha) = (J_\alpha, \varnothing)$, and for $n \geq 0$

$$\begin{array}{lll} \rho_{n+1}^{J_\alpha} = \rho_1^{\mathfrak{C}_n(J_\alpha)} & p_{n+1}^{J_\alpha} = p_1^{\mathfrak{C}_n(J_\alpha)} & h_{n+1}^{J_\alpha} = h_1^{\mathfrak{C}_n(J_\alpha)} \\ A_{n+1}^{J_\alpha} = A_1^{\mathfrak{C}_n(J_\alpha)} & \mathfrak{C}_{n+1}(J_\alpha) = \mathfrak{C}_1(\mathfrak{C}_n(J_\alpha)) & \end{array}$$

Finally, the projectum of J_α is defined to be $\operatorname{proj}(J_\alpha) = \rho^{J_\alpha} = \inf_n \rho_n^{J_\alpha}$. Since the sequence of projecti $\langle \rho_n^{J_\alpha} : n < \omega \rangle$ is nonincreasing, $\rho_n^{J_\alpha} = \operatorname{proj}(J_\alpha)$ for all sufficiently large $n < \omega$.

Note that if $n > 1$ then the Σ_n Skolem function h_n^M need not be Σ_n definable over M. Jensen's Σ_n-uniformization theorem states that this construction can be used to define a Skolem function for Σ_n formulas over J_α which is Σ_n-definable in J_α (though not uniformly so). This Σ_n-uniformization theorem was an important part of the motivation for Jensen's invention of fine structure, but it has turned out to have little direct importance because it is simpler and more useful to work directly with the fine structure.

We now consider the problem of recovering the original structure $M = (J_\alpha, A)$ from its code $M_1 = \mathfrak{C}_1(M) = (J_{\rho_1^M}, A_1^M)$. In order to do so, recall that the Skolem function h_1^M is Σ_1 definable in M from the parameter p_1^M, and that $A_1^M = \{\ulcorner\varphi_n(a, p_1^M)\urcorner : n < \omega \;\&\; a \in (\rho^{M_1})^{<\omega}\}$, the set of Gödel numbers of the Σ_1 theory of M with parameters from $\rho_1^M \cup p_1^M$. Now define X to be the set of equivalence classes $[\xi]_\sim$, where $\xi \in \rho_1^M \cap \operatorname{dom}(h_1^M)$ and $\xi \sim \xi'$ if and only if $\ulcorner h_1^M(\xi) = h_1^M(\xi')\urcorner \in A_1^M$. The membership relation E on X is defined by $[\xi] \mathrel{\mathrm{E}} [\xi']$ if and only if $\ulcorner h_1^M(\xi) \in h_\xi^M(\xi')\urcorner \in A_1^M$; and the subset $\bar{A} \subseteq X$ is defined by setting $[\xi] \in \bar{A}$ if and only if $\ulcorner h_1^M(\xi) \in A\urcorner \in A_1^M$.

It is straightforward to verify that we can define a Σ_1-elementary embedding $i\colon (X, \mathrm{E}, \bar{A}) \to (J_\alpha, \in, A)$ by setting $i([\xi]) = h_1^M(\xi)$. This embedding i is an isomorphism if and only if $M = h_1^M``\rho_1^M$, in which case we can say that this construction *recovers M from its Σ_1-code $\mathfrak{C}_1(M)$*.

3.3 Definition. The structure $M = (J_\alpha, A)$ is said to be *1-sound* if $J_\alpha = h_1^M``\rho_1^M$. Further, M is said to be *n-sound* if it is $n-1$-sound and $\mathfrak{C}_{n-1}(M)$ is 1-sound; and M is said to be *sound* if M is n-sound for all n.

We will say that the model J_α is *n-sound* or *sound*, respectively, if the structure $(J_\alpha, \varnothing)$ is n-sound or sound.

Notice that if M is sound then one can repeat the process described above n times in order to recover M from any of its codes $\mathfrak{C}_n(M)$. Thus the following lemma is the basic fact of fine structure:

3.4 Lemma. *If α is any ordinal then the structure J_α is sound.*

We will only consider the case when α is a limit ordinal, and begin with the proof that J_α is 1-sound.

Sketch of Proof. Let $Z = h_1^M``\rho_1^M \prec_1 J_\alpha$, and let i be the collapse map $i\colon \bar{M} \cong Z \prec_1 J_\alpha$. Since $\bar{M} \models$ "$V = L$", we must have $\bar{M} = J_{\bar{\alpha}}$ for some $\bar{\alpha} \leq \alpha$. Since i is Σ_1 elementary and $\rho_1^M \cup p_1^M \subseteq Z$, the set $A_1^{J_\alpha}$ is Σ_1 definable in $J_{\bar{\alpha}}$. Since $A_1^{J_\alpha} \notin J_\alpha$ it follows that $\bar{\alpha} = \alpha$.

Similarly, p_1^M is the least parameter which can be used to define A_1^M in J_α, and $i^{-1}(p_1^M) \leq p_1^M$, so $p_1^M = i(p_1^M)$. But every member of $\operatorname{dom}(i)$ is Σ_1 definable in $J_{\bar{\alpha}}$ from parameters in $\rho_1^M \cup p_1^M$, and it follows that i is the identity. Thus $h_1^M``\rho_1^M = Z = \operatorname{ran}(i) = J_\alpha$. This completes the proof that J_α is 1-sound. $\dashv$

It should be noted that this proof is closely related to the proof that the GCH holds in L. Both rely on the following condensation lemma:

3.5 Lemma. *If $Z \prec_0 J_\alpha$ then there is $\bar{\alpha} \leq \alpha$ such that $Z \cong J_{\bar{\alpha}}$.*

The proof here is somewhat more delicate than that of the GCH, as the sentence "$V = L$" needs to be carefully formulated so that it is satisfied by J_α even for successor α. The hypothesis of lemma 3.5, stating that Z is Σ_0-elementary, is meant be interpreted in the terms of the J_α-hierarchy as meaning that Z is closed under rudimentary functions. This is slightly stronger than the assertion that it is Σ_0-elementary in the sense of the Levy hierarchy. This observation is another example of the superiority of the J_α-hierarchy: when using the L_α-hierarchy, the hypothesis must be strengthened to require that Z be Σ_1-elementary.

In order to prove lemma 3.4 for arbitrary n we need an extension of lemma 3.5 in which the model J_α is replaced by the Σ_{n-1}-code $\mathfrak{C}_{n-1}(J_\alpha)$. This generalization is given by the *downward extension* or *condensation* property, stated in the following lemma, and is central to many applications of fine structure.

3.6 Lemma (Downward Extension). *Suppose that $i\colon (J_{\rho'}, A') \prec_0 \mathfrak{C}_n(J_\alpha)$. Then there is $\alpha' \leq \alpha$ such that $(J_{\rho'}, A') = \mathfrak{C}_n(J_{\alpha'})$, and i extends to a Σ_n-embedding $\tilde{\imath}\colon J_{\alpha'} \to J_\alpha$. Furthermore $\tilde{\imath}$ preserves the first n stages of the fine structure, so that $\tilde{\imath}\, h_k^{J_{\alpha'}} = h_k^{J_\alpha}\, \tilde{\imath}$ for all $k \leq n$.*

Sketch of Proof. Lemmas 3.4 and 3.6 are proved by a joint induction on n. First we assume that lemma 3.6 is true for $\mathfrak{C}_n(J_\alpha)$, and use this to prove that J_α is $n+1$-sound. The proof is essentially identical to the proof given above that J_α is 1-sound. The collapse map $i\colon J_{\bar{\alpha}} \cong Z \prec_0 J_\alpha$ becomes $i\colon (J_{\bar{\rho}}, \bar{A}) \cong Z \prec_0 (J_{\rho_n^M}, A_n^M) = \mathfrak{C}_n(J_\alpha)$. Since lemma 3.6 holds for $\mathfrak{C}_n(J_\alpha)$ this can be written as $i\colon \mathfrak{C}_n(J_{\alpha'}) \to \mathfrak{C}_n(J_\alpha)$ for some $\alpha' \leq \alpha$. Since $A_{n+1}^{J_\alpha}$ is Σ_1-definable in $\mathfrak{C}_n(J_{\alpha'})$ we must have $\alpha' = \alpha$, and since $i^{-1}(p_{n+1}^{J_\alpha}) \leq p_{n+1}^{J_\alpha}$, which is the least parameter which can be used to define $A_{n+1}^{J_\alpha}$, we must have $i(p_{n+1}) = p_{n+1}$. Hence i is the identity on $\mathfrak{C}_n(J_\alpha)$, so J_α is $n+1$-sound.

To complete the proof, we show that if J_α is $n+1$-sound, and lemma 3.6 holds for $\mathfrak{C}_n(J_\alpha)$, then lemma 3.6 also holds for $\mathfrak{C}_{n+1}(J_\alpha)$. Suppose that $i\colon (J_{\rho'}, A') \prec_0 \mathfrak{C}_{n+1}(J_\alpha)$.

Apply to the structure $(J_{\rho'}, A')$ the construction described before definition 3.3 to recover a structure (J_α, A) from its Σ_1-code $\mathfrak{C}_1(J_\alpha, A)$. The assumption that i is Σ_0-elementary implies that the construction succeeds at least to the extent of defining a model $(X, \mathrm{E}, \bar{A})$ and an embedding $i'\colon (X, E, \bar{A}) \to \mathfrak{C}_n(J_\alpha)$. The existence of the embedding i' ensures that (X, E) is well-founded, and therefore $X \cong J_{\rho''}$ for some ordinal ρ''. If A'' is the image of $\bar{A}$ under this isomorphism, then i' induces an embedding $\tilde{\imath}_n\colon (J_{\rho''}, A'') \to \mathfrak{C}_n(J_\alpha)$.

By the construction, the set A' encodes the Σ_1 theory of $(J_{\rho''}, A'')$, and since i is Σ_0 elementary and A encodes the Σ_1 theory of $\mathfrak{C}_n(J_\alpha)$ it follows that $\tilde{\imath}_n$ is a Σ_1-elementary embedding. By the induction hypothesis it follows that there is an ordinal α' such that $(J_{\rho''}, A'') = \mathfrak{C}_n(J_{\alpha'})$, and an embedding $\tilde{\imath}_0 \colon J_{\alpha'} \to J_\alpha$ which extends $\tilde{\imath}_n$ and which preserves the first n stages of the fine structure, as far as $\mathfrak{C}_n(J_{\alpha'})$.

Now it only remains to verify that $(J_{\rho'}, A') = \mathfrak{C}_1(J_{\rho''}, A'')$, which entails verifying that $\rho_1^{(J_{\rho''},A'')} = \rho'$ and $p_1^{(J_{\rho''},A'')} = \tilde{\imath}_n^{-1}(p_1^{\mathfrak{C}_n(J_\alpha)})$. The inequality $\rho_1^{(J_{\alpha''},A'')} \leq \rho'$ follows from the fact that $A' \notin J_{\rho''}$, which is proved by the argument of the Russell paradox: If $A' \in J_{\rho''}$ then so is $y = \{\, \nu : \ulcorner \nu \notin h(\nu) \urcorner \in A' \,\}$, where $h \colon \rho' \to J_{\rho''}$ is the Skolem function coded by A'. But this is impossible, as then $y = h(\nu)$ for some ν and then $\nu \in y \iff \nu \notin y$.

The inequality $\rho' \leq \rho_1^{(J_{\alpha''},A'')}$ follows from the fact that $A' \cap \xi \in J_{\rho'}$ for $\xi < \rho'$, which follows from the assumption that $(J_{\rho'}, A') \prec_0 \mathfrak{C}_{n+1}(J_\alpha)$. Thus $\rho' = \rho_1^{(j_{\alpha''},A'')}$, and this implies the inequality $p_1^{(J_{\rho''},A'')} \leq \tilde{\imath}_n^{-1}(p_1^{\mathfrak{C}_n(J_\alpha)})$ since $p_1^{(J''_\rho,A'')}$ is, by definition, the least parameter which can be used to define A''.

The final inequality $p_1^{(J_{\rho''},A'')} \geq \tilde{\imath}_n^{-1}(p_1^{\mathfrak{C}_n(J_\alpha)})$ is the point in the proof of lemma 3.6 which requires the joint induction with lemma 3.4: assume for the sake of contradiction that $p_{n+1}^{J_{\bar{\alpha}}} < \tilde{\imath}_n^{-1}(p_{n+1})^{J_\alpha}$ and apply lemma 3.4 to $J_{\bar{\alpha}}$. This implies that $\tilde{\imath}_n^{-1}(p_{n+1}^{J_\alpha})$ is Σ_1 definable in $(J_{\alpha'}, A')$ from $p_{n+1}^{J_{\bar{\alpha}}}$. It follows that $p_{n+1}^{J_\alpha}$ is Σ_1 definable in $\mathfrak{C}_n(J_\alpha)$ from $\tilde{\imath}_{n+1}(p_{n+1}^{J_{\bar{\alpha}}}) < p_{n+1}^{J_\alpha}$, but this contradicts the definition of $p_{n+1}^{J_\alpha}$.

This completes the proof of lemma 3.6, except for the claim that $\tilde{\imath}$ is Σ_{n+1}-elementary. To see this, notice that the embedding i'' constructed in the induction step is one quantifier stronger than i'. The map $\tilde{\imath}$ is obtained by repeating this process $n+1$ times, and hence the original Σ_0 embedding is strengthened to a Σ_{n+1}-elementary embedding $\tilde{\imath} \colon J_{\alpha'} \to J_\alpha$. ⊣

It should be noted that the statement that the embedding $\tilde{\imath}$ preserves the fine structure is stronger—and usually more useful—than the statement that $\tilde{\imath}$ is Σ_n-elementary.

If we define $\bar{h}_n^{J_\alpha} = h_1^{J_\alpha} \dots h_n^{J_\alpha}$, then $\bar{h}^{J_\alpha} \colon \rho_n^{J_\alpha} \to J_\alpha$ and an induction using lemma 3.4 shows that $J_\alpha = \bar{h}_n^{J_\alpha} “\rho_n^{J_\alpha}$. In order to avoid considering detailed fine structure as much as possible, we make the following convention:

Notation. Unless stated otherwise, we abuse notation by using $h_n^{J_\alpha}$ to denote the function $\bar{h}_n^{J_\alpha}$ described above, and we call it the Σ_n-Skolem function of J_α.

We end the discussion of lemma 3.6 with lemma 3.7, which is frequently useful in applications of the covering lemma and in particular proves, used along with the proof of the covering lemma itself, proposition 1.14 from the introduction.

3.7 Lemma. *The Σ_n-Skolem function $h_n^{J_\alpha}$ of J_α can be written as an increasing union $h_n^{J_\alpha} = \bigcup_{\nu<\eta} g_\nu$ of functions $g_\nu \in J_\alpha$, with $\eta \leq \rho_n^{J_\alpha}$.*

Sketch of Proof. First consider the case when $n = 1$ and α is a limit ordinal. Pick a sequence of ordinals α_ν cofinal in α, and define g_ν to be the function defined in J_{α_ν} by the same Σ_1 formula (with the same parameter $p_1^{J_\alpha}$) as was used to define $h_1^{J_\alpha}$ in J_α. Thus $g_\nu(x) = y$ if and only if $h_1^{J_\alpha}(x) = y$, both x and y are in J_{α_ν}, and in addition the witness to the Σ_1 fact "$h_1^{J_\alpha}(x) = y$" is a member of J_{α_ν}.

For $n > 1$, apply the construction above to $\mathfrak{C}_{n-1}(J_\alpha)$, noting that $\rho_n^{J_\alpha}$ is always a limit ordinal for $n > 0$. ⊣

The importance of lemma 3.6 to fine structure theory extends far beyond the arguments above; however its importance in the proof of the covering lemma is secondary to the that of the upward extension property, lemma 3.10, described below.

Embeddings of Mice

In this subsection we define a generalized ultrapower which is central to the proof of the covering lemma. This ultrapower, which is used to extend a given embedding $\pi\colon J_{\bar\kappa} \to J_\kappa$ to an embedding $\tilde\pi\colon J_{\bar\alpha} \to J_{\alpha'}$ with a larger domain, can be described in modern terms as the ultrapower by the extender $E_{\pi,\beta}$ of length β which is associated with the embedding π. It should be noted, however, that this construction of Jensen is older than, and in fact is ancestral to, the modern notion of an extender. Extenders are more completely described in chapter [31].

We first explain the extender construction by defining the Σ_0-ultrapower $\mathrm{Ult}(M, \pi, \beta)$ of a model M.

3.8 Definition. Assume that M and N are transitive models of a fragment of set theory, and that $\pi\colon N \to N'$ is a Σ_0-elementary embedding such that $\mathcal{P}(\nu) \cap M \subseteq N$ for all $\nu < \mathrm{On}(N)$ such that $\sup(\pi``\nu) < \beta$. Then

$$\mathrm{Ult}(M, \pi, \beta) = \big\{\, [a, f]_\pi : f \in M \text{ and } \mathrm{dom}(f) \in \mathrm{dom}(\pi) \\ \text{and } a \in [\beta]^{<\omega} \cap \pi(\mathrm{dom}(f)) \,\big\} \quad (1.3)$$

where $[a, f]_\pi$ is the equivalence class of the pair (a, f) under the relation

$$(a, f) \sim_\pi (a', f') \iff (a, a') \in \pi\big(\{\, (\vec\nu, \vec\nu') : f(\vec\nu) = f'(\vec\nu') \,\}\big). \quad (1.4)$$

The membership relation E_π and any other predicates of $\mathrm{Ult}(M, \pi, \beta)$ are defined similarly, and the embedding $i\colon M \to \mathrm{Ult}(M, \pi, \beta)$ is defined as usual by $i(x) = [a, C_x]_\pi$ where a is arbitrary and C_x is the constant function, $\forall z\, C_x(z) = x$.

The embedding $i\colon M \to \mathrm{Ult}(M, \pi, \beta)$ satisfies the Łos theorem for Σ_0 formulas:

3.9 Proposition. *If φ is a Σ_0 formula, then for any $f_0, \dots, f_n$ in M and $a_0, \dots, a_n$ in β we have $\mathrm{Ult}(M, \pi, \beta) \models \varphi([f_0, a_0], \dots, [f_n, a_n])$ if and only if $\langle a_0, \dots, a_n \rangle \in \pi\big(\{ \langle u_0, \dots, u_n \rangle : M \models \varphi(f_0(u_0), \dots, f_n(u_n)) \}\big)$.*

If $M = J_\alpha$ for some ordinal α then $\mathrm{ran}(i)$ is cofinal in $\mathrm{Ult}(M, \pi, \beta)$, and it follows that i is Σ_1-elementary. In particular, i preserves the Σ_1-Skolem function of J_α.

We will need to define Σ_n-ultrapowers, for arbitrary $n \in \omega$, so that they preserve Σ_{n+1}-Skolem functions. The obvious way to define such an ultrapower is to modify definition 3.8 by replacing the condition "$f \in M$" of (1.3) with "f is Σ_n-definable in J_α"; however doing so would require first proving Jensen's uniformization theorem, which states that there is a Σ_n-definable Skolem function for Σ_n formulas on J_α. A second possible approach is that of Silver, who showed that it is possible to define $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ by using compositions of the naïve Σ_n-Skolem function, and that the naïve Skolem function is preserved by the resulting embedding even though it is not defined by a Σ_n-formula. This is the simplest approach, as it avoids the use of fine structure, but it appears to have difficulties with models for larger cardinals.

Our approach will be closer to the first one, but will use the fine structure directly. The notion of Σ_n ultrapower which we use can be defined in two different, but equivalent, ways. One way is to define $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ directly, using definition 3.8, but allowing any function f of the form $f(x) = h_n(x, q)$ where h_n is the Σ_n-Skolem function mapping a subset of $J_{\rho_n^{J_\alpha}}$ onto J_α, and $q \in J_{\rho_n^{J_\alpha}}$ is an arbitrary parameter. The other way is indirect, by taking the ordinary Σ_0 ultrapower $i\colon \mathfrak{C}_n(J_\alpha) \to \mathrm{Ult}\big(\mathfrak{C}_n(J_\alpha), \pi, \beta\big)$ of the Σ_n code of J_α, and then extending this to a map $\tilde{i}\colon J_\alpha \to J_{\tilde{\alpha}}$. This approach has the advantage that most arguments can be carried out at the level of the Σ_n-code of J_α, which involves the easily understandable Σ_0-ultrapower and Σ_1-Skolem function.

The extension of π to an embedding $\tilde{\pi}$ with the larger domain J_α depends on lemma 3.10 below, which is the counterpart of the downward extension lemma 3.6 given earlier. One major difference between the upward and downward extension lemmas concerns the well-foundedness of the new structure. In the downward extension lemma, this structure is a substructure of a given well-founded structure and hence is automatically well-founded. In the upward extension lemma the well-foundedness of $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ must be explicitly assumed.

3.10 Lemma (Upward Extension). *Suppose that $\pi\colon J_{\bar\kappa} \to J_\kappa$, that $\beta \leq \kappa$, and either $\rho_n^{J_\alpha} > \min\{ \nu : \pi(\nu) \geq \beta \}$ or $\mathrm{ran}(\pi)$ is cofinal in β and $\pi(\rho_n^{J_\alpha}) \geq \beta$. Set $M_n = \mathfrak{C}_n(J_\alpha)$ and $\widetilde{M}_n = \mathrm{Ult}(M_n, \pi, \beta)$.*

1. *There is a structure* $\widetilde{M}_0$ *such that* $\widetilde{M}_n$ *is, formally, equal to* $\mathfrak{C}_n(\widetilde{M}_0)$. *If this structure* $\widetilde{M}_0$ *is well-founded then there is an ordinal* $\tilde{\alpha}$ *such that* $\widetilde{M}_0 = J_{\tilde{\alpha}}$ *and* $\widetilde{M}_n = \mathfrak{C}_n(J_{\tilde{\alpha}})$.
2. *There is an embedding* $\tilde{\pi}\colon J_\alpha \to \widetilde{M}_0$ *such that* $\pi \restriction J_{\bar{\beta}} = \tilde{\pi} \restriction J_{\bar{\beta}}$, *where* $\bar{\beta}$ *is the least ordinal such that* $\pi(\bar{\beta}) \geq \beta$ *if* $\beta < \kappa$, *or* $\bar{\beta} = \bar{\kappa}$ *if* $\beta = \kappa$.
3. *The embedding* $\tilde{\pi}$ *preserves the* Σ_k *codes for* $k \leq n$*: in particular,* $\tilde{\pi} \circ h_k^{J_\alpha}(x) = h_k^{\widetilde{M}_0} \circ \tilde{\pi}(x)$ *for all* x *for which either side is defined.*
4. *The embedding* $\tilde{\pi}$ *preserves the* Σ_1*-Skolem function of* M_n *in the sense that there is a function* $\tilde{h}$, *which is* Σ_1 *definable over* $\widetilde{M}_n$, *such that* $\tilde{\pi}\, h_{n+1}^M(x) = \tilde{h}\, \tilde{\pi}(x)$ *for all* $x \in M$ *such that either side is defined.*

The proof of the upward extension lemma is nearly the same as that of the downward extension lemma 3.6. The models $\widetilde{M}_{n-k}$, and embeddings $\tilde{\pi}_{n-k}\colon \mathfrak{C}_{n-k}(J_\alpha) \to \widetilde{M}_{n-k}$ are defined by recursion on $k \leq n$: The embedding $\tilde{\pi}_0\colon \mathfrak{C}_n(J_\alpha) \to \widetilde{M}_n$ is the Σ_0 ultrapower, and $\widetilde{M}_{n-(k+1)}$ is constructed from $\widetilde{M}_{n-k}$ by the same recovery process as was used for the downward extension lemma. This process uses the fact that $\widetilde{M}_{n-k}$ has the form $(\widetilde{M}_{n-k}, \mathrm{E}, \tilde{A}_{n-k})$ and satisfies the first-order sentences asserting that $(\widetilde{M}_{n-k}, \mathrm{E})$ is a model of $V = L$, and that $\tilde{A}_{n-k}$ is the Σ_n theory of a larger model $\widetilde{M}_{n-(k+1)}$ of which $\widetilde{M}_{n-k}$ is the Σ_1-code.

The embedding $\tilde{\pi}$ does not, in general, preserve fine structure below ρ_n^M; for example if β is a cardinal in L then $\rho^{\widetilde{M}} \geq \beta$, since every bounded subset of β in L is a member of J_β, but it may happen that $\rho_{n+1}^M < \bar{\beta}$. In this case $\tilde{\pi}(\rho_{n+1}^M) = \pi(\rho_{n+1}^M) < \beta$. It follows that the function $\tilde{h}$ will not, in general, be the Σ_1-Skolem function $h_1^{\widetilde{M}_n}$. It is defined by the same formula as $h_1^{\widetilde{M}_n}$, but using the image $\tilde{\pi}_0(p_1^{J_{M_n}})$ of the standard parameter of M_n instead of the standard parameter $p_1^{\widetilde{M}_n}$ of $\widetilde{M}_n$.

In order for the ultrapower $\mathrm{Ult}_n(M, \pi, \beta)$ described above to be defined, the set $\{\, (\vec{\nu}, \vec{\nu}') : f(\vec{\nu}) = f'(\vec{\nu}') \,\}$ must be in the domain of π for each pair (f, f') of functions in $\Sigma_n(M)$. This yields the following condition for the existence of $\mathrm{Ult}_n(M, \pi, \beta)$:

3.11 Proposition. *Let* M, π *and* β *be as above, and let* $\bar{\beta}$ *be the least ordinal such that* $\pi(\bar{\beta}) \geq \beta$ *(or* $\bar{\beta} = \mathrm{On}(N)$ *if* $\beta = \sup(\mathrm{ran}(\pi))$*). Then the ultrapower* $\mathrm{Ult}_n(M, \pi, \beta)$ *is defined if and only if either* $\rho_n^M > \bar{\beta}$, *or else* $\rho_n^M \geq \bar{\beta}$ *and* $\mathrm{ran}(\pi)$ *is cofinal in* β.

Equivalently, the ultrapower is defined if and only if either

1. *Every subset of* $\bar{\kappa}$ *which is* Σ_n *definable in* M *is a member of* $J_{\bar{\kappa}}$, *or*
2. $\mathrm{ran}(\pi)$ *is cofinal in* β *and every bounded subset of* $\bar{\kappa}$ *which is* Σ_n *definable in* M *is a member of* $J_{\bar{\kappa}}$.

3.2. Proof of the Covering Lemma for L

The main part of the proof of the covering lemma for L is a construction which shows that any set $X \prec_1 J_\kappa$, which is suitable in a sense to be made precise in definition 3.14, is a member of L. The concluding part of the proof is an analysis of the notion of suitability showing that every uncountable set of ordinals is contained in a suitable set of the same cardinality.

The construction, together with the definition 3.14 of suitability, will prove the following lemma:

3.12 Lemma. *1. If $X \prec_1 J_\kappa$ is suitable then there is a cardinal $\rho < \kappa$ of L and a function $h \in L$ such that $X = h``(\rho \cap X)$.*

2. If $X \prec_1 J_\kappa$ is suitable and $\rho < \kappa$ is a cardinal of L then $X \cap J_\rho$ is also suitable.

3.13 Corollary. *Any suitable set $X \prec_1 J_\kappa$ is a member of L.*

Proof. The proof is by induction on κ. Let $X \subseteq \kappa$ be suitable, and let h and ρ be as in clause (1). Then $X \cap J_\rho$ is suitable by clause (2) and hence is in L by the induction hypothesis, but then $X = h``(X \cap \rho) \in L$. $\dashv$

In order to describe the basic construction, we fix a cardinal κ of L and a set $X \prec_1 J_\kappa$ with $\sup(X) = \kappa \not\subseteq X$. Let $\pi\colon N \to X$ be the collapse map, so that $N = J_{\bar\kappa}$ for some ordinal $\bar\kappa$, and let (α, n) be the lexicographically largest pair such that $\mathrm{Ult}_n(J_\alpha, \pi, \kappa)$ is defined. There are two cases:

1. If $\mathcal{P}(\delta) \cap L \subseteq N$ for all $\delta < \bar\kappa$ then $J_\alpha = L$ and $n = 0$.
2. Otherwise α is the least ordinal such that there is a bounded subset of $\bar\kappa$ in $J_{\alpha+\omega} - J_{\bar\kappa}$, and n is the least integer such there is such a subset which is Σ_{n+1}-definable in J_α. That is, $\rho_{n+1}^{J_\alpha} < \bar\kappa \leq \rho_n^{J_\alpha}$, and $\rho_m^{J_{\alpha'}} \geq \bar\kappa$ whenever $\bar\kappa \leq \alpha' < \alpha$ and $m < \omega$.

The basic construction will succeed whenever $\widetilde{M} = \mathrm{Ult}_n(J_\alpha, \pi, \kappa)$ is well-founded; the definition 3.14 of suitability, given in the next subsection, is a generalization of this requirement. If case (1) occurs for some suitable set X then $\tilde\pi\colon L \to \mathrm{Ult}(L, \pi, \kappa) = L$ is a nontrivial embedding from L into L, which implies by Kunen's theorem (see chapter [31, Theorem 1.13]) that $0^\sharp$ exists. This contradicts our current assumption that the core model is equal to L, so we can assume that case (2) occurs for all suitable sets X. Then by lemma 3.10, $\mathrm{Ult}_n(J_\alpha, \pi, \kappa) = J_{\tilde\alpha}$ for some ordinal $\tilde\alpha$, and the following diagram commutes:

$$\begin{array}{ccc} J_\alpha & \xrightarrow{\tilde\pi} & \widetilde{\mathcal{M}} = \mathrm{Ult}_n(J_\alpha, \pi, \kappa) = J_{\tilde\alpha} \\ \uparrow & & \uparrow \\ J_{\bar\kappa} & \xrightarrow{\pi} & X \prec_1 J_\kappa \end{array} \tag{1.5}$$

Now let $\bar\rho = \rho^{J_\alpha}_{n+1}$. Then $\bar\rho < \bar\kappa$, and $J_\alpha = \bar h``\bar\rho$ where $\bar h = h^{J_\alpha}_{n+1}$, so

$$X = \pi``J_{\bar\kappa} = \pi``(\bar\kappa \cap \bar h``(\bar\rho)), \tag{1.6}$$

and furthermore

$$\tilde h \circ \tilde\pi = \tilde\pi \circ \bar h \tag{1.7}$$

where $\tilde h$ is the function given by lemma 3.10(4). Putting equations (1.6) and (1.7) together, we get

$$X = J_\kappa \cap (\widetilde\pi \circ \bar h``\bar\rho) = J_\kappa \cap (\tilde h \circ \widetilde\pi``\bar\rho) = J_\kappa \cap \tilde h``(X \cap \rho) \tag{1.8}$$

where $\rho = \sup(\pi``\bar\rho) < \kappa$. Since $\tilde h \in L$, this completes the basic construction.

Suitable Sets

Here is the formal definition of suitability:

3.14 Definition. Suppose $X \subseteq L$ and let $\pi\colon N \cong X$ be the inverse of the transitive collapse. Then X is *suitable* if $X \prec_1 J_\kappa$ for some ordinal κ and $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ is well-founded for all triples (α, n, β) such that the ultrapower is defined.

We write $\mathbf{C}$ for the class of suitable sets.

Proof of lemma 3.12. If $X \prec_1 J_\kappa$ is any set in $\mathbf{C}$ then the basic construction succeeds for X, and hence clause (1) of lemma 3.12 holds for X. Clause (2) of that lemma is clear. ⊣

It follows by corollary 3.13 that every suitable set is in L, so Jensen's covering lemma 1.1 for L will follow if we can show that every uncountable set is contained in a suitable set of the same cardinality. For the strong covering lemma 1.12 we additionally need to show that the class $\mathbf{C}$ is closed under increasing unions of uncountable cofinality. Notice that definition 3.14 is absolute, so that the class $\mathbf{C}$ is definable in L.

The countably closed sets give a easy, but useful, special case:

3.15 Definition. We will call a set $X \prec_1 J_\kappa$ *countably closed* if there is a set $Y \prec H(\lambda)$, for some $\lambda \geq \kappa$, such that ${}^\omega Y \subseteq Y$ and $X = Y \cap J_\kappa$.

If $|x|^\omega < \kappa$ then it is always possible to find a countably closed $X \supseteq x$ with $|X| = |x|^\omega$, so the following easily proved observation is often all that is needed.

3.16 Proposition. *Every countably closed set $X \prec_1 J_\kappa$ is suitable, and hence is a member of L.* ⊣

It follows that if $0^\sharp$ does not exist then every set x is contained in a set $y \in L$ such that $|y| \leq |x|^\omega$. This result gives much of the strength of the covering lemma, and moreover its proof highlights the most important ideas of the proof of the full covering lemma while omitting the most delicate part of the argument. This by itself would be sufficient reason to consider the countably closed case, but for core models involving measurable cardinals or non-overlapping extenders the countably closed case of the covering lemma is a necessary step in the proof of the full lemma: it is used to prove that the weak covering lemma, definition 1.9, holds in a variant K^{c} of the core model. The weak covering lemma for K^{c} is then used to prove the existence and essential properties of the true core model K, and only after this can the full covering lemma be proved for K.

The following lemma will conclude the proof of theorem 1.1 and of theorem 1.12 in the case $0^\sharp$ does not exist: the covering lemma and the strong covering lemma for L:

3.17 Lemma. *The class* $\mathbf{C}$ *is unbounded in* $[J_\kappa]^\delta$ *for every uncountable cardinal* δ*, and* $\bigcup_{\nu<\eta} X_\nu \in \mathbf{C}$ *whenever* $\langle X_\nu : \nu < \eta \rangle$ *is an increasing sequence of sets in* $\mathbf{C}$ *with* $\mathrm{cf}(\eta) > \omega$*.*

The proof of this lemma will take up the remainder of section 3.2.

Fix, for the moment, a set X which is not suitable, and let α, n and β be such that $\widetilde{M} = \mathrm{Ult}_n(J_\alpha, \pi, \beta)$ is defined but not well-founded. This ill-foundedness is witnessed by a descending E-chain, $\dots \mathrel{\mathrm{E}} z_2 \mathrel{\mathrm{E}} z_1 \mathrel{\mathrm{E}} z_0$, of members of $\widetilde{M}$, where E is the membership relation of $\widetilde{M}$. In order to prove lemma 3.17 we need to incorporate additional structure into such a witness:

3.18 Definition. A *witness* w *to the unsuitability* of $X \prec_1 J_\kappa$ is a ω-chain of Σ_0 elementary embeddings $i_k \colon m_k \to m_{k+1}$ such that

1. $i_k \in X$ and $m_k \in X$ for each $k < \omega$.
2. $\operatorname{dir\,lim}(\pi^{-1}[w]) = \mathfrak{C}_n(J_\alpha)$ for some ordinal α and some $n \in \omega$.
3. $\operatorname{dir\,lim}(w)$ is not the Σ_n-code of any well-founded model $J_{\tilde{\alpha}}$.
4. Write β_k for the critical point of i_k. Then the sequence $\langle \beta_k : k < \omega \rangle$ is nondecreasing.
5. For each k we have $m_k \in m_{k+1}$, and there is a function $f \in m_{k+1}$ such that $f``\beta_k = i_k``m_k$.

We will call $\beta = \sup_k(\beta_k)$ the *support* of the witness w, and we will call the pair (α, n) the *height* of w in X. We will say that a witness w is *minimal* in X if it has minimal height in X among all witnesses with the same support β.

There may be more than one minimal witness for X with the same support β. It is possible, with some care, to modify the definition so that this minimal witness is unique; however we do not need to do so.

3.19 Lemma. *A set $X \prec_1 J_\kappa$ is unsuitable if and only if it has a witness to its unsuitability. Furthermore, if w is a witness to the unsuitability of X then*

1. *If $w \subseteq X' \prec_1 X$ then w is also a witness to the unsuitability of X'.*
2. *If, in clause 1, w is a minimal witness for X then it is also a minimal witness for X', and furthermore any other minimal witness for X' with the same support is also a minimal witness for X.*
3. *If $X = Y \cap J_\kappa$, where $Y \prec_1 H(\tau)$ for some cardinal $\tau > \kappa$, then $w \notin Y$.*

We will give some immediate consequences of lemma 3.19 and then use it to finish the proof of the covering lemma. We will then give the proof of lemma 3.19.

3.20 Corollary. *If $\langle X_\nu : \nu < \eta \rangle$ is an increasing sequence of sets in* $\mathbf{C}$*, with* $\operatorname{cf}(\eta) > \omega$*, then* $X = \bigcup_{\nu<\eta} X_\nu \in \mathbf{C}$*.*

Proof. Otherwise there would be a witness w to the unsuitability of X; but since $\operatorname{cf}(\eta) > \omega$ this would imply that $w \subseteq X_\nu$ for some $\nu < \eta$ and hence w is a witness to the unsuitably of X_ν by clause 1. ⊣

We can use lemma 3.19 to give a proof of proposition 3.16, although a direct proof is somewhat simpler.

3.21 Corollary. *Every countably closed set $X \prec_1 J_\kappa$ is suitable, and hence is a member of L.*

Proof. By definition, X is countably closed if and only if $X = Y \cap J_\kappa$ for some $Y \prec_1 H(\tau)$ where ${}^\omega Y \subseteq Y$. Then any witness to the unsuitability of X would have to be a member of Y, contrary to clause 3 of lemma 3.19. ⊣

The following lemma will complete the proof of the covering lemma except for the proof of lemma 3.19.

3.22 Lemma. *The class* $\mathbf{C}$ *is unbounded in $[J_\kappa]^\delta$ for any cardinal δ with $\omega < \delta < \kappa$.*

Proof. Jensen's proof of this result begins by generically collapsing the cardinal κ onto δ^+. The proof given here is essentially the same, but the presentation is slightly different: instead of carrying out the generic collapse we work with the set $\operatorname{Col}(\delta^+, J_\kappa)$ of forcing conditions for the collapse. The members of $\mathrm{Col}(\delta^+, J_\kappa)$ are functions $\sigma\colon \xi \to J_\kappa$ with $\xi < \delta^+$. With the obvious notions of "closed" and "unbounded" this space satisfies Fodor's lemma: if $S \subseteq \mathrm{Col}(\kappa^+, J_\kappa)$ is a stationary set and F is a function with domain S such that $F(\sigma) \in \mathrm{ran}(\sigma)$ for all $\sigma \in S$, then F is constant on a stationary subset of S. The reason for using the space $\mathrm{Col}(\delta^+, J_\kappa)$ instead of $[J_\kappa]^\kappa$ is that $\mathrm{Col}(\delta^+, J_\kappa)$ also satisfies the following variant of Fodor's lemma:

3.23 Proposition. *Suppose that $S \subseteq \mathrm{Col}(\delta^+, J_\kappa)$ is a stationary set such that $\mathrm{cf}(\mathrm{dom}(\sigma)) > \omega$ for all $\sigma \in S$, and that F is a function defined on S such that $F(\sigma)$ is a countable subset of $\mathrm{ran}(\sigma)$ for all $\sigma \in S$. Then there is a stationary subset S' of S and a function $\sigma_0 \in S'$ such that*

$$\forall \sigma \in S' \big(\sigma_0 \subseteq \sigma \text{ and } F(\sigma) \subseteq \mathrm{ran}(\sigma_0)\big).$$

Proof. Let $f(\sigma) < \mathrm{dom}(\sigma)$ be the least ordinal η such that $F(\sigma) \subseteq \sigma``\eta$, and let $S_0 \subseteq S$ be a stationary set on which $f(\sigma)$ is constant. Pick any $\sigma_0 \in S_0$ and let $S' = \{\, \sigma \in S_0 : \sigma_0 \subseteq \sigma \,\}$. Then S' and σ_0 are as required. ⊣

Let S_0 be the set of functions $\sigma \in \mathrm{Col}(\delta^+, J_\kappa)$ such that $\mathrm{ran}(\sigma) \notin \mathbf{C}$, $\mathrm{cf}(\mathrm{dom}(\sigma)) > \omega$ and $\mathrm{ran}(\sigma) \prec_1 J_\kappa$. We will prove that S_0 is nonstationary, which implies that $\mathbf{C}$ is unbounded in $[J_\kappa]^\delta$

Suppose to the contrary that S_0 is stationary. It follows by lemma 3.19 that there is, for each $\sigma \in S_0$, a minimal witness w^σ to the unsuitability of $\mathrm{ran}(\sigma)$. Let β^s be the support of w^σ. By the ordinary Fodor's lemma there is a stationary set $S_1 \subseteq S_0$ such that $\beta = \beta^\sigma$ is constant for $\sigma \in S_1$, and by proposition 3.23 there is a stationary set $S_2 \subseteq S_1$ and $\sigma_0 \in S_2$ such that $\sigma_0 \subseteq \sigma$ and $w^\sigma \subseteq \mathrm{ran}(\sigma_0)$ for all $\sigma \in S_2$. It follows that w^{σ_0} is a minimal witness to the unsuitability of $\mathrm{ran}(\sigma)$ for each $\sigma \in S_2$. Now consider the class $\mathcal{Y}$ of sets $Y \prec_{\Sigma_1} H(\kappa^+)$ such that $w^{\sigma_0} \in Y$. Then

$$\mathcal{X} = \{\, \sigma \in \mathrm{Col}(\delta^+, J_\kappa) : \exists Y \in \mathcal{Y}\ \mathrm{ran}(\sigma) = Y \cap J_\kappa \,\}$$

contains a closed unbounded subset of $\mathrm{Col}(\delta^+, J_\kappa)$, and hence $S_2 \cap \mathcal{X} \neq \varnothing$. However, this contradicts lemma 3.19(3), and this contradiction completes the proof of lemma 3.22. ⊣

This completes the proof of the covering lemma, except for the proof of lemma 3.19:

Proof of lemma 3.19. First, notice that if w is a witness with support β to the unsuitability of X, then clauses 4 and 5 imply that $\mathrm{dir\,lim}(w) = \mathrm{Ult}\big(\mathrm{dir\,lim}(\pi^{-1}[w]), \pi, \beta\big)$, and hence clause 3.18(3) implies that X is in fact unsuitable.

Now suppose that X is unsuitable, so that there are α, n and β such that $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ is defined, but not well-founded. If we write $M_n = \mathfrak{C}_n(J_\alpha)$

then this means that $\mathrm{Ult}(M_n, \pi, \beta)$ is defined, but is not the Σ_n-code of any well-founded structure $J_{\tilde{\alpha}}$. We will find a witness w to the unsuitability of X, such that w has height and support less than or equal to (α, n) and β, respectively.

If $\mathrm{Ult}(M_n, \pi, \beta)$ is not well-founded then there are $f_k \in M_n$ and $a_k \in \beta$ so that $z_{k+1}\ \mathrm{E}\ z_k$, where $z_k = [a_k, f_k]_\pi = \tilde{\pi}(f_k)(a_k)$, and E is the membership relation of $\mathrm{Ult}(M_n, \pi, \beta)$. If on the other hand $\mathrm{Ult}(M_n, \pi, \beta)$ is well-founded, then, since $\tilde{\pi}\colon M_n \to \widetilde{M}_n = \mathrm{Ult}(M_n, \pi, \beta)$ is Σ_1-elementary, there is a (ill-founded) structure $\widetilde{M}$ such that $\widetilde{M}_n = \mathfrak{C}_n(\widetilde{M})$, along with a map $\tilde{h}_n$, the Σ_n-Skolem function of $\widetilde{M}$, mapping $\widetilde{M}_n$ onto $\widetilde{M}$. Then we can find $z_k = [a_k, f_k]_\pi$ so that $\tilde{h}_n(z_{k+1})\ \mathrm{E}\ \tilde{h}_n(z_k)$ for each $k < \omega$.

Write $M_n = (J_{\rho_n}, A_n)$ (if $n = 0$ then $\rho_n = \alpha$ and $A_n = \varnothing$, in which case we assume as usual that α is a limit ordinal). Let $\alpha_k < \rho_n$ be the least ordinal $\xi > \alpha_{k-1}$ such that $\{ f_1, \dots, f_k \} \subseteq J_\xi$, and let β_k be the least member of X such that $\{ a_0, \dots, a_k \} \subseteq \beta_k$. Finally let $\bar{\beta}_k = \pi^{-1}(\beta_k)$ and let $\bar{j}_k\colon \bar{m}_k \cong \mathcal{H}_{\Sigma_1}^{(J_{\alpha_k}, A_n \cap J_{\alpha_k})}(\bar{\beta}_k \cup \{ f_1, \dots, f_k \})$ be the transitive collapse of the Σ_1 hull of $\bar{\beta}_k \cap \{ f_1, \dots, f_k \}$ in $(J_{\alpha_k}, A_n \cap J_{\alpha_k})$, with $\bar{i}_k = \bar{j}_{k+1}^{-1}\bar{j}_k\colon \bar{m}_k \to \bar{m}_{k+1}$.

Then $\bar{m}_k, \bar{i}_k \in J_{\bar{\kappa}}$ for each $k < \omega$. Set $w = \langle \pi(\bar{m}_k), \pi(\bar{i}_k) : k < \omega \rangle$, with $m_k = \pi(\bar{m}_k)$ and $i_k = \pi(\bar{i}_k)$, and set $\beta' = \sup_k(\beta_k) \le \beta$.

If $\bar{w} = \langle \bar{m}_k, \bar{i}_k : k < \omega \rangle = \pi^{-1}[w]$ then $\operatorname{dir\,lim}(\bar{w}) \prec_0 M_n = \mathfrak{C}_n(J_\alpha)$ and hence, by lemma 3.6, $\operatorname{dir\,lim}(\bar{w})$ is the Σ_n-code of $J_{\alpha'}$ for some $\alpha' \le \alpha$, but w was constructed so that $\operatorname{dir\,lim}(w)$ is not the Σ_n code of any well-founded model. Finally, since $\alpha_{k+1} > \alpha_k$, the Skolem function mapping β_k onto $j_k``m_k \prec_{\Sigma_1} J_{\alpha_k}$, with parameters $\{ f_1, \dots, f_k \}$, is a member of $J_{\alpha_{\kappa+1}}$. This gives the functions f required by clause 3.18(5).

Thus w is the desired witness to the unsuitability of X.

To prove clause 3.19(3), note that by the absoluteness of well-foundedness we can find, working in Y, a sequence $a'_k < \beta'$ of ordinals and a sequence $f'_k \in m_{k+1}$ of functions such that if f''_k is the image $j_k(f'_k)$ of f'_k in $\operatorname{dir\,lim}(w)$ then the sets $z'_k = f''_k(a'_k)$ demonstrate, in the same way that $\langle z_k : k < \omega \rangle$ above did for $\widetilde{M}_n$, that $\operatorname{dir\,lim}(w)$ is not the Σ_n-code of a well-founded structure. Then the sets a'_k and f'_k are members of $Y \cap J_\kappa = X$, so the sets $\bar{z}'_k = \bar{i}_k\, \pi^{-1}(f'_k)(\pi^{-1}(\alpha'_k))$ demonstrate that $\operatorname{dir\,lim}(\pi^{-1}[w])$ is not the Σ_n-code of a well-founded structure, contradicting clause 3.18(2).

Clause 3.19(1), stating that any witness $w \subseteq X' \prec_1 X$ to the unsuitability of X is also a witness that X' is not suitable, is straightforward. Finally, to prove clause 3.19(2), suppose that w is minimal, and that w' is a minimal witness for X' having the same support β.

Let (α', n') and (α'', n) be the heights of w' and w, respectively, in X', and let $\bar\pi = (\pi^X)^{-1}\pi^{X'}$. Then $(\alpha', n') \leq (\alpha'', n)$ since w' is minimal, so

$$\begin{aligned}\operatorname{dir\,lim}((\pi^X)^{-1}\text{``}w') &= \mathrm{Ult}_{n'}(\operatorname{dir\,lim}((\pi^{X'})^{-1}\text{``}w'), \bar\pi, \bar\beta)\\ &= \mathrm{Ult}_{n'}(J_{\alpha'}, \bar\pi, \bar\beta)\\ &\subseteq \mathrm{Ult}_{n}(J_{\alpha''}, \bar\pi, \bar\beta) = \operatorname{dir\,lim}((\pi^X)^{-1}\text{``}w).\end{aligned}$$

Hence $\operatorname{dir\,lim}((\pi^X)^{-1}\text{``}w')$ is well-founded, and it follows that w' is a witness to the unsuitability of X with support β, and by the minimality of w we must have $\mathrm{Ult}_{n'}(J_{\alpha'}, \bar\pi, \bar\beta) = J_\alpha$ and $n' = n$. Hence the height of w' in X is (α, n), so that w' is also a minimal witness for X. ⊣

This completes the proof of the covering lemma for L. In the rest of this section we consider two variations on this proof. The first, and most important, extends the argument to models with a measurable cardinal in order to obtain the Dodd-Jensen covering lemma; the second variation applies the argument to unsuitable sets X, obtaining Magidor's covering lemma 1.15 and the absoluteness theorem for Jónsson cardinals, theorem 1.16.

3.3. Measurable Cardinals

The primary aim of this subsection is to prove the Dodd-Jensen covering lemma, and an important secondary aim is to prepare the way for section 4 which describes the covering lemma for larger core models. In accordance with this secondary aim we do not assume $\neg 0^\dagger$ except when it is explicitly specified. For simplicity we do assume that there is no model of $\exists\kappa\, o(\kappa) = \kappa^{++}$, but much of our discussion is true in general (though not always in detail) for the larger core models described in chapter [56].

In Dodd and Jensen's original papers [9, 10, 11], the minimal model $L[U]$ for a measurable cardinal is treated separately from the Dodd-Jensen core model K^{DJ}. The model $L[U]$ is the simplest natural analogue of L and was already well understood long before the core model was invented. While $L[U]$ has many of the properties of L, there is one vital difference: The existence of the model L is implied by the axioms of set theory, but the construction of the model $L[U]$ depends on being first given the filter U which will be the measure in the model $L[U]$.

If $L[U]$ does exist then K^{DJ} can easily be obtained by "iterating the measure U out of the universe":

$$K^{\mathrm{DJ}} = \textstyle\bigcap_{\nu\in\mathrm{On}} \mathrm{Ult}^\nu(L[U], U) = \bigcup_{\nu\in\mathrm{On}}(\mathrm{Ult}^\nu(L[U], U) \cap V_{i_\nu(\kappa)})$$

where $i_\nu\colon L[U] \to \mathrm{Ult}^\nu(L[U], U)$ is the ν-fold iteration of the ultrapower by U. In order to define an inner model which would exist even in the absence of a model $L[U]$, Dodd and Jensen defined the core model K^{DJ} to be $L[\mathcal{M}]$, where $\mathcal{M}$ is a class of approximations, called *mice*, to models of the form

$L[U]$. The mice are structures $M = J_\alpha[W]$ with the properties (i) $M \models$ "W is a measure", (ii) M is iterable (in the sense that every iterated ultrapower of M is well-founded) and (iii) M is sound and has projectum smaller than $\operatorname{crit}(W)$. Note that Condition (iii) implies that $J_{\alpha+\omega}[W] \models |\alpha| < \operatorname{crit}(W)$, so that W is not a measure in any model larger than M.

In [34], the Dodd-Jensen core model was extended to obtain a core model for sequences of measures. This extended core model had the form $K[\mathcal{U}] = L[\mathcal{U}, \mathcal{M}]$, where $\mathcal{U}$ was the sequence of measures in $K[\mathcal{U}]$ and $\mathcal{M}$ was a class of mice. The mice $M \in \mathcal{M}$ were models of the form $M = J_\alpha[\mathcal{U}']$ where the sequence $\mathcal{U}'$ was a concatenation $\mathcal{U}' = \mathcal{U}^\frown\mathcal{W}$ of the sequence of measures of $K[\mathcal{U}]$ with a sequence $\mathcal{W}$ of filters which are measures in M but not in $J_{\alpha+\omega}[\mathcal{U}']$. The sequence $\mathcal{W}$ corresponded to the measure W in a Dodd-Jensen mouse $J_\alpha[W]$.

The modern approach to mice, which we follow here, originated in attempts to extend the core model to cardinals approaching a supercompact cardinal. This program has many difficulties, some of which are still not solved, but a key to making a beginning was the observation that the original notion of a mouse was too simple. A key fact in the theory of the constructible sets is that all of the models J_α, as well as L itself, have the same structure, so that they differ only in length. It became clear in course of the investigation that the mice for an extended core model should similarly have the same structure as the full core model. That is, the mice are themselves constructed from smaller mice, like a well-founded version of Swift's well known flea, which

> hath smaller fleas that on him prey
> And these have smaller still to bit 'em;
> And so proceed *ad infinitum*.

This seemed prohibitively complicated, but a suggestion of S. Baldwin made it possible to realize the desired situation while simplifying, instead of complicating, the construction: The mice and the core model would be structures of the form $M = J_\alpha[\mathcal{E}]$ or $M = L[\mathcal{E}]$, respectively. The members of the sequence $\mathcal{E}$ would be extenders, but some would be only partial extenders, not measuring all of the sets in M. These partial extenders would be the full extenders of those mice which are members of M; thus the sequence $\mathcal{E}$ codes both the mice and the extenders in the structure M.

For the rest of this section we limit ourselves to sequences of measures, with no extenders, and we mark this restriction by using the letter $\mathcal{U}$ to denote the sequence instead of $\mathcal{E}$.

As hoped, this approach leads to a feasible fine structure for the extended core models, but surprisingly it also simplifies the fine structure for the previously existing core models. This is particularly surprising for the Dodd-Jensen core model, the mice of which have at most one measurable cardinal. It would seem at first glance that nothing could be simpler than a mouse of the form $J_\alpha[U]$, but the apparent simplicity of this model hides a complicated fine structure. For example, consider the key fact of the fine structure of L—and even of Gödel's proof of the continuum hypothesis—that every constructible subset of ω is a member of J_{ω_1}. This fact fails badly in the model $L[U]$: if $\kappa = \operatorname{crit}(U)$ then $J_\alpha[U] = J_\alpha$ for all $\alpha \leq \kappa + 1$. The first nonconstructible set to be constructed is $0^\sharp$, which is a subset of ω and is Δ_1 definable over $J_{\kappa+\omega}[U]$, so that $0^\sharp \in J_{\kappa+\omega\cdot 2}[U] - J_{\kappa+\omega}[U]$.

The newer fine structure avoids this problem because the subsets of ω in $L[\mathcal{U}]$ are all in $J_{\omega_1}[\mathcal{U}]$ and hence are constructed from the restriction $\mathcal{U}{\restriction}\omega_1$ of $\mathcal{U}$ to (partial) measures on countable ordinals. In fact, the first nontrivial member of $\mathcal{U}$ is the L-ultrafilter on the first Silver indiscernible c_0 which is induced by, and which constructs, the real $0^\sharp$. In the case $L[\mathcal{U}] = L[U]$, the sequence $\mathcal{U}$ has as its last nontrivial member the measure U, which is the only member of the sequence $\mathcal{U}$ which is a full measure on $L[\mathcal{U}]$.

The benefit of the new approach to the core model is suggested by the fact that the following two modifications are all that is necessary to adapt the definition 3.1 of fine structure for L to the core model.

1. An added predicate is needed to represent the sequence of measures, so that the Σ_1-code is a structure of the form $(J_\alpha[\mathcal{U}], \mathcal{U}{\restriction}\alpha, A)$ instead of (J_α, A).

2. For ordinals α such that $\mathcal{U}_\alpha \neq \varnothing$ it is necessary to begin the construction with a special *amenable code*, defined to be $\mathfrak{C}_0(J_\alpha[\mathcal{U}]) = (J_{\rho_0}, \mathcal{U}{\restriction}\rho_0, \mathcal{U}_\alpha)$ where, if $\kappa = \operatorname{crit}(\mathcal{U}_\alpha)$, then ρ_0 is defined to be κ^+ of $L[\mathcal{U}{\restriction}\alpha]$.

The "Skolem function" $h_0^{J_\alpha[\mathcal{U}]}$ mapping $\rho_0 = \rho_0^{J_\alpha[\mathcal{U}]} = \kappa^{+J_\alpha[\mathcal{U}]}$ onto $J_\alpha[\mathcal{U}]$ is derived from the function mapping functions $f\colon \kappa \to J_\kappa[\mathcal{U}]$ in $J_{\rho_0}[\mathcal{U}]$ to their equivalence classes $[f]_{\mathcal{U}_\alpha} \in J_\alpha[\mathcal{U}] = J_\alpha[\mathcal{U}{\restriction}\alpha] \subseteq \operatorname{Ult}(J_\kappa[\mathcal{U}], \mathcal{U}_\alpha)$.

The analogous structure in the fine structure of L was simply $\mathfrak{C}_0(J_\alpha) = (J_\alpha, \varnothing)$. The amenable code is needed here because the obvious structure $(J_\alpha[\mathcal{U}], \in, \mathcal{U}{\restriction}\alpha, \mathcal{U}_\alpha)$ is not amenable: $\mathcal{P}(\alpha)^{J_\alpha[\mathcal{U}]} \in J_\alpha[U]$, but $\mathcal{U}_\alpha \cap \mathcal{P}(\alpha)^{J_\alpha[\mathcal{U}]} = \mathcal{U}_\alpha \notin J_\alpha[\mathcal{U}]$. If $\mathcal{U}_\alpha = \varnothing$ then the amenable code is simply $(J_\alpha[\mathcal{U}], \mathcal{U}{\restriction}\alpha, \varnothing)$, as in L.

Some additional change is necessary for cardinals larger than measurable cardinals:

3. In models where iteration trees are needed instead of linear iterated ultrapowers, the standard parameter is augmented to included a *witness* to its minimality. This witness, which is discussed later, is used in the models of this section, but does not need to be explicitly included in the structure.

4. The amenable code is somewhat more complicated in the case of sequences $L[\mathcal{E}]$ involving extenders instead of only measures. See chapter [46] for details.

The proof that the fine structure given by this definition satisfies the necessary properties is, of course, more complicated than the proof in L. We begin with the definition of a mouse. We say that U is a *M-ultrafilter* on κ if it is a normal M-ultrafilter in the sense of Kunen, that is, U is a a normal ultrafilter on $\mathcal{P}^M(\kappa)$ and $U \cap X \in M$ whenever $X \in M$ and $M \models |X| = \kappa$. If M is a structure $J_\alpha[\mathcal{U}]$ and $\gamma < \alpha$ then we write $M|\gamma$ for the initial segment $J_\gamma[\mathcal{U}]$ of M.

3.24 Definition. A *mouse* is an premouse which is iterable and sound.

We define the terms *premouse* and *sound* by a simultaneous recursion on α:

1. A *premouse* is a model $J_\alpha[\mathcal{U}]$ (or $L[\mathcal{U}]$, allowing $\alpha = \mathrm{On}$) which satisfies the following three conditions:

   (a) For each γ such that $\mathcal{U}_\gamma \neq \varnothing$, there is a cardinal κ of $J_\alpha[\mathcal{U}{\restriction}\gamma]$ such that $(J_\gamma[\mathcal{U}{\restriction}\gamma], \mathcal{U}_\gamma) \models (\gamma = \kappa^{++}$ and $\mathcal{U}_\gamma$ is a normal $J_\gamma[\mathcal{U}{\restriction}\gamma]$-measure on $\kappa)$.

   (b) (Coherence) If $\mathcal{U}_\gamma \neq \varnothing$ then $\left(i^{\mathcal{U}_\gamma}(\mathcal{U}{\restriction}\gamma)\right){\restriction}\gamma + 1 = \mathcal{U}{\restriction}\gamma$.

   (c) (Soundness) The structure $(J_{\alpha'}[\mathcal{U}], \mathcal{U}{\restriction}\alpha', \mathcal{U}_{\alpha'})$ is sound for every ordinal $\alpha' < \alpha$.

   We say that a sequence $\mathcal{U}$ is *good* if $L[\mathcal{U}]$ is a premouse.

2. A premouse $\mathcal{M} = (J_\alpha[\mathcal{U}], \mathcal{U}{\restriction}\alpha, \mathcal{U}_\alpha)$ is said to be *n-sound* if $h^{\mathcal{M}}_m``\rho_m = J_\alpha[\mathcal{U}]$ for each $m \leq n$, where $h^{\mathcal{M}}_m$ and $\rho^{\mathcal{M}}_m$ are the Σ_m Skolem functions and Σ_m projectum of $\mathcal{M}$, respectively. The model $\mathcal{M}$ is *sound* if it is n-sound for all $n \in \omega$.

   We will say that $\mathcal{M}$ is *sound above* η if either $\mathcal{M}$ is sound or there is n such that $\rho^{\mathcal{M}}_{n+1} \leq \eta$, $\mathcal{M}$ is n-sound and $h^{\mathcal{M}}_{n+1}``\eta = J_\alpha[\mathcal{U}]$.

3. (Iterability) A premouse $J_\alpha[\mathcal{U}]$ is *iterable* if every iterated ultrapower of $J_\alpha[\mathcal{U}]$ is well-founded.

   Note that this definition of the term *iterable* needs to be supplemented by definition 3.30, given later, of an iterated ultrapower of a premouse.

Again, see chapter [46] for the somewhat more complicated conditions on the sequence $\mathcal{U}$ when it is allowed to contain extenders.

3.25 Remark. Notice that any premouse satisfies the GCH, since the soundness condition implies that whenever $x \subseteq \eta < \alpha$ and $x \in J_{\alpha+\omega}[\mathcal{U}] - J_\alpha[\mathcal{U}]$, then $J_{\alpha+\omega}[\mathcal{U}] \models |\alpha| \leq \eta$. This property is often called *acceptability* in the literature, where it is used as a placeholder for soundness in the definition of a premouse in order to avoid the use of simultaneous recursion as in definition 3.24.

If $J_\alpha[\mathcal{U}]$ is a n-sound premouse then the ultrapower $\mathrm{Ult}_n(J_\alpha[\mathcal{U}],\pi,\beta)$ of $M = J_\alpha[\mathcal{U}]$ by the extender derived from an embedding π is defined just like that for L, by taking the ultrapower $\mathrm{Ult}_0(\mathfrak{C}_n(M),\pi,\beta)$ of the Σ_n-code using functions in $\mathfrak{C}_n(M)$ and then using the upward extension property (lemma 3.10) to extend the embedding to all of M. In particular the upward extension property, Lemma 3.10 (which we will not restate here) is still valid for these models. Since not every premouse $J_\alpha[\mathcal{U}]$ is sound, the proposition 3.11 giving the conditions for the existence of $\mathrm{Ult}_n(J_\alpha[\mathcal{U}],\pi,\beta)$ needs to be supplemented with the requirement that $J_\alpha[\mathcal{U}]$ be n-sound.[2] In addition, we now have the possibility of taking an ultrapower by one of the ultrafilters $U = \mathcal{U}_\gamma$ in $M = J_\alpha[\mathcal{U}]$. The ultrapower $\mathrm{Ult}_n(M,U)$, like $\mathrm{Ult}_n(M,\pi,\beta)$, is obtained by taking the ordinary ultrapower $\mathrm{Ult}(\mathfrak{C}_n(M),U)$ of the nth code of M and using lemma 3.10.

3.26 Lemma. *Suppose that $M = J_\alpha[\mathcal{U}]$ is an n-sound iterable premouse, and U is a M-ultrafilter with $\mathrm{crit}(U) \geq \rho^M_{n+1}$. Then the embedding $i^U : M \to M' = \mathrm{Ult}_n(M,U)$ satisfies the following two properties:*

1. *$A^M_{n+1} \notin M'$, and hence $\rho^{M'}_{n+1} = \rho^M_{n+1}$.*
2. *$i^U(p^M_{n+1}) = p^{M'}_{n+1}$.*

Furthermore, any embedding $i\colon M \to M'$ which is given by the upward extension lemma from a Σ_0-elementary embedding from $\mathfrak{C}_n(M)$ and which satisfies clause 1 also satisfies clause 2.

The thing which makes the conclusion stronger than that of lemma 3.10 is the assertion that $\rho^{M'}_{n+1} = \rho^M_{n+1}$. This may be contrasted with diagram (1.5) in the proof of the covering lemma, in which $\tilde\pi\colon J_\alpha \to J_{\tilde\alpha} = \mathrm{Ult}(J_\alpha,\pi,\kappa)$ and $\rho^{J_\alpha} < \bar\kappa$, while $\rho^{J_{\tilde\alpha}}_{n+1} = \kappa = \tilde\pi(\bar\kappa)$.

A part of the proof of clause 2 will be deferred until after the discussion of iterated ultrapowers.

Sketch of Proof. To see that $\rho^{M'}_{n+1} = \rho^M_{n+1}$ we need to verify that the master code $A = A^M_{n+1} \subseteq \rho^M_{n+1}$ is not a member of M'. Suppose to the contrary that $A = [f]_U \in M' = \mathrm{Ult}_n(M,U)$. Then A can be written as $\{\,\beta < \rho^M_{n+1} : \{\,\xi : \beta \in f(\xi)\,\} \in U\,\}$, which is a member of M since the assumptions that U is a M-ultrafilter and $\rho^M_{n+1} \leq \kappa$ imply that

$$U \cap \{\,\{\,\xi < \kappa : \beta \in f(\xi)\,\} : \beta < \rho^M_{n+1}\,\} \in M.$$

[2]The ultrapower can also be defined on any iterated ultrapower of a mouse, and hence by lemma 3.26 the extra condition is not needed for an iterable potential premouse $\mathcal{M}$. However we will only need ultrapowers as described in the text.

This contradiction concludes the proof that $A \notin M'$.

One direction of clause 2 is straightforward: clearly $i^U(p^M_{n+1}) \geq p^{M'}_{n+1}$, since A^M_{n+1} can be defined using the parameter $i^U(p^M_{n+1})$. The hard part is to see that $i^U(p^M_{n+1}) \leq p^{M'}_{n+1}$. The proof proceeds by induction on n, and we will only present the case $n = 0$. If to the contrary $p^{M'}_1 < i^U(p^M_1)$, then there is an ordinal $i^U(\nu) \in i^U(p^M_1) - p^{M'}_1$ such that $p^{M'}_1 - i^U(\nu) = i^U(p^M_1) - (i^U(\nu)+1)$. Set $p = p^M - (\nu+1)$ and $p' = i^U(p) = p^{M'}_1 - i^U(\nu)$, and set $A' = \{\, \xi < i^U(\nu) : M' \models \Phi(\xi, p') \,\}$ where Φ is the universal Σ_1 formula. Any subset of $i(\nu)$ which is Σ_1-definable in M' from p' is rudimentary in A', so if we can show that $A' \in M'$, then it will follow that any set Σ_1 definable from parameters in $p' \cup \nu$ is also a member of M'. This will contradict the assumption that $p' = p^{M'}_1 - \nu$.

Set $A = \{\, \xi < \nu : \Phi(\xi, p) \,\}$. Then $A \in M$, so $i^U(A) \in M'$. Unfortunately it may not be the case that $A' = i^U(A)$, so we need to analyze this set further. Let $M = J^{\mathcal{U}}_\alpha$, and define a prewell-order R on A by $\xi' \mathrel{R} \xi$ if $\xi \in A$ and $\exists\gamma\, \big(J^{\mathcal{U}}_\gamma \models \Phi(\xi', p) \quad \& \quad \forall\gamma' < \gamma\ J^{\mathcal{U}}_\gamma \not\models \Phi(\xi, p)\big)$. Then R is also Σ_1 definable from p, and hence is a member of M. Define R' in M' similarly. Then the prewell-ordering R' on A' is an initial segment of the preordering $i(R)$ on $i(A)$. Now $i(R) \in M'$, and if $i(R)$ is a prewell-ordering then all initial segments of $i(R)$ are also in M'. The following definition will be used to show that $i(R)$ is a prewell-ordering:

3.27 Definition. A *solidity witness that* $\nu \in p^M_1$ is a function $\tau \in M$ which maps A into the ordinals of M so that $\forall \xi', \xi \in A\ \big(\xi' \mathrel{R} \xi \iff \tau(\xi') \leq \tau(\xi)\big)$.

If τ is a solidity witness that ν is in p^M_{n+1} then $i(\tau)$ is an order preserving embedding from $i(R)$ into the ordinals of M'. Since M' is well-founded it follows that $i(R)$ is a prewell-order.

The general proof of the existence of a solidity witness will be deferred until after the introduction of iterated ultrapowers; however we note here that the construction of a solidity witness τ can be carried out in any admissible set containing R; in fact this is the central element of the standard proof that well-foundedness is absolute. This leads to two easy cases, in which the solidity witness for a mouse $M = J_\alpha(\mathcal{U}^M)$ can be found in an admissible initial segment $M|\gamma = J_\gamma(\mathcal{U}^M \restriction \gamma)$ of M. If M has a measurable cardinal $\mu \geq \nu$ then $(\nu^+)^M$ exists, so there is a solidity witness in the admissible set $M|(\nu^+)^M$; and if M has a full measure $\mathcal{U}_\gamma$ with $\operatorname{crit}(\mathcal{U}_\gamma) < \nu < \gamma$ and $R \in \operatorname{Ult}(M, \mathcal{U}_\gamma)$ then M has a solidity witness in the admissible set $M|\gamma$. ⊣

Note that the hypothesis that U is a normal M-ultrafilter holds whenever $U = \mathcal{U}^M_\gamma$ for some $\gamma \leq \alpha$. If $\gamma < \alpha$ then a slightly stronger result holds, since the hypothesis that $\kappa \geq \rho^M_{n+1}$ can be eliminated (with some adjustment to the conclusion). Even then, however, not all of the fine structure of M is preserved by the ultrapower i^U. First, and most important, the ultrapower $M' = \mathrm{Ult}_n(M, U)$ is never sound above $\kappa = \mathrm{crit}(U)$, even if M is, since $\kappa \notin (i^U h^M_{n+1})``\mathrm{crit}(U) = h^{M'}_{n+1}``\mathrm{crit}(U) = h^{M'}_{n+1}``\rho_{n+1}$. The model $M'$ *is* sound above $\kappa + 1$. Second, the two projecti $\rho^M_n$ and $\rho^M_{n+1}$ need not be preserved by the embedding $i$: if $\mathrm{crit}(U) = \rho^M_{n+1}$ then $\rho^{M'}_{n+1} = \rho^M_{n+1} < i^U(\rho^M_{n+1})$; and in any case $\rho^{M'}_n = \sup(i^U ``\rho^M_n)$, which may be smaller then $i^U(\rho^M_n)$.

The existence of unsound premice is an important difference between the fine structure of L and that of larger core models. The counterpart of lemma 3.4, which states that J_α is sound, is given by the following lemma:

3.28 Lemma. *Any iterable premouse $M = J_\alpha[\mathcal{U}]$ is an iterated ultrapower of a mouse.*

The mouse is given by the following definition:

3.29 Definition. The nth *core* of a premouse M, written $\mathrm{core}_n(M)$, is the model obtained by decoding the nth code $M_n = \mathfrak{C}_n(M)$ of M. The *core* of M, written $\mathrm{core}(M)$, is $\mathrm{core}_n(M)$ where n is least such that $\rho^M_n = \rho^M$.

Note that the definition of $\mathfrak{C}_n(M)$ is not hindered by the possibility that M is not sound. The structure $\mathrm{core}(M)$ will be equal to the transitive collapse of the substructure of M containing those elements which are, in an appropriate sense, definable in M. In particular, if we define $\mathrm{core}_1(M)$ to be the model obtained by decoding the Σ_1-code $M_1 = \mathfrak{C}_1(M)$, and then decoding M_1, then $\mathrm{core}_1(M)$ is the transitive collapse of $h^M_1 ``\rho^M_1$, the set of $x \in M$ which are Σ_1 definable using parameters from $\rho^M_1 \cup p^M_1$.

Any further sketch of the proof will clearly depend on the definition and properties of iterated ultrapowers. These were described in chapter [31], but are complicated here by the fact that they may involve ultrapowers of differing degrees and since they may involve filters $\mathcal{U}^{M_\nu}_\gamma$ which are not full ultrafilters on M_ν. Both of these situations result in the *drops* mentioned in the following definition.

The situation is slightly simpler in the case when $0^\dagger$ does not exist, so that the premice $J_\alpha[\mathcal{U}]$ have at most one full ultrafilter, than it is in the more general case needed in section 4. At stage ν of the iterated ultrapowers being considered here there are only two possible choices. One is to use the single full ultrafilter in the model M_ν, which will be the last member of the sequence $\mathcal{U}^{\mathcal{M}_\nu}$; this is case 3a of the definition. The other is to use one of the earlier filters in the sequence $\mathcal{U}^M$. This earlier filter is not a full ultrafilter in M_ν and hence must be applied to a smaller mouse in M_ν on which it is an ultrafilter; this is case 3c. Case 3b does not arise in the absence of $0^\dagger$.

3.30 Definition. An *iterated ultrapower of a premouse* $M = J_\alpha[\mathcal{U}]$ is a sequence of models M_ν for $\nu \leq \theta$, together with a finite set $D \subseteq \theta + 1$, called the set of *drops*, and embeddings $i_{\nu,\nu'} \colon M_\nu \to M_{\nu'}$ defined for all pairs $\nu < \nu' \leq \theta$ such that $D \cap (\nu, \nu'] = \varnothing$.

All of these are determined by a sequence of filters $U_\nu = \mathcal{U}_{\gamma_\nu}^{M_\nu} \in M_\nu$, with a strictly increasing sequence of critical points $\mathrm{crit}(U_\nu)$, as follows:

1. $M_0 = J_\alpha[\mathcal{U}]$.
2. If ν is a limit ordinal then $M_\nu = \operatorname{dir\,lim}_{\nu_0 \le \nu' < \nu} M_{\nu'}$ where $\nu_0 = \sup(D \cap \nu)$. Note that this direct limit exists since the finiteness of D implies that $D \cap \nu$ is bounded in ν, so that the embeddings $i_{\nu'',\nu'} \colon M_{\nu''} \to M_{\nu'}$ exist for all sufficiently large $\nu'' < \nu' < \nu$.
3. If $\nu + 1 \le \theta$ then $M_{\nu+1}$ is determined by the choice of the ultrafilter $U_\nu = \mathcal{U}_{\gamma_\nu}^{M_\nu}$, where $\gamma_\nu > \gamma_{\nu'}$ for all $\nu' < \nu$. There are three cases:
   (a) If U_ν is a full ultrafilter on M_ν then set $n_\nu = n$ where n is the largest number such that $\mathrm{Ult}_n(M_\nu, U_\nu)$ is defined. If $n_\nu = n_{\nu'}$ for all sufficiently large $\nu' < \nu$, then $M_{\nu+1} = \mathrm{Ult}_{n_\nu}(M_\nu, U_\nu)$. In this case $i_{\nu,\nu+1}$ is the canonical embedding.
   Note that since the critical points of the ultrafilters U_ν are increasing, M_ν is sound above $\mathrm{crit}(U_\nu)$ and hence the soundness hypothesis of lemma 3.26 is satisfied.
   (b) If U_ν is a full ultrafilter on M_ν, but $n_\nu < n_{\nu'}$ for all $\nu' \in \nu - \max(D \cap \nu)$, then $M_{\nu+1} = \mathrm{Ult}_{n_\nu}(M_\nu, U_\nu)$. In this case we add $\nu + 1$ to D, so that $i_{\nu,\nu+1}$ is not defined.
   Note that this happens when $\mathrm{crit}(U_\nu) \ge \rho_{n_{\nu'}}^{M_\nu}$, but $\mathrm{crit}(U_{\nu'}) < \rho_{n_{\nu'}}^{M_{\nu'}}$, for $\sup(D \cap \nu) < \nu' < \nu$. This case is known as a *drop in degree*.
   (c) If U_ν is not a full ultrafilter on M_ν, then let $M^*_{\nu+1}$ be the largest initial segment of M_ν on which U_ν is an ultrafilter. Thus $M^*_{\nu+1} = J_{\alpha^*_\nu}[\mathcal{U}_\nu]$ where α^*_ν is the least ordinal $\beta < \alpha$ such that there is a subset x of $\mathrm{crit}(U_\nu)$ in $J_{\beta+\omega}[\mathcal{U}_\nu] - J_\beta[\mathcal{U}_\nu]$ which is not measured by U_ν.
   In this case, which is known as a *normal drop*, we set $M_{\nu+1} = \mathrm{Ult}(M^*_{\nu+1}, U_\nu)$, and we add $\nu+1$ to D so that $i_{\nu,\nu+1}$ is not defined.

We say that M is *iterable* if every model in any iterated ultrapower of M is well-founded and no attempt to create an iterated ultrapower leads to infinitely many drops.

Here again the situation becomes more complicated in the case of extenders, where *iteration trees* are needed instead of the linear iterated ultrapowers described above. See chapter [46] or [56].

3.31 Remark. In practice we will frequently make the trivial modification that $U_\nu = \mathcal{U}_{\gamma_\nu}^{M_\nu} = \varnothing$ is also allowed in an iterated ultrapower, and set $M_{\nu+1} = M_\nu$ in this case. This gives what is known as *padded* iterated ultrapowers.

3.32 Lemma. *The formula asserting that a set M is a mouse is absolute for models N containing ω_1.*

Proof. The statement that M is a premouse is first-order over M, as is the assertion that M is sound, so we only need verify that the iterability of M is absolute. If M is countable in N then this can be proved using the Shoenfield absoluteness theorem, as the statement that there is an ill-founded iterated ultrapower of M, with the iterated ultrapower indexed by a countable well-order, is a Σ^1_2 statement. The proof for general M is similar to the proof of Shoenfield's theorem: for each countable ordinal α one builds a "tree of attempts to find a ill-founded iterated ultrapower of length at most α", that is to say, a tree T_α such that the infinite branches of T_α correspond exactly to the ill-founded iterated ultrapowers of M of length at most α. If there is an ill-founded iteration of length $\alpha < \omega_1$ in V, then T_α has an infinite branch and hence is ill-founded. Assuming that the tree T_α can be constructed in N just as it was constructed in V (this relies on the fact that $\omega_1 \subseteq N$, so that α is countable in N) the tree T_α is in N. By the absoluteness of well-foundedness it is ill-founded there, so there is, in N, an infinite branch of T_α which specifies an ill-founded iterated ultrapower of M.

Since this is a very important technique, we suggest here one method of constructing such a tree. In order to simplify the construction we first ignore the possibility of drops. A node at the nth level of the tree T_α will be a 4-tuple $p = \langle x, \vec{U}, \vec{M}, \vec{\xi}\rangle$ such that

1. $\{0, \alpha\} \subseteq x \in [\alpha + 1]^{<\omega}$,
2. $\vec{M}$ is a finite iterated ultrapower of $M_0 = M$ indexed by the ordinals in x and using the ultrafilters $\vec{U}$. That is, if $\nu \in x$ and $\nu' = \min(x - (\nu + 1))$ then $U_\nu \in M_\nu$ and $M_{\nu'} = \mathrm{Ult}(M_\nu, U_\nu)$.
3. $\vec{\xi}$ is a descending sequence of ordinals, $\vec{\xi}$ has length n, and $\vec{\xi} \in M_\alpha$.

We will say that a node $p' = \langle x', \vec{U}', \vec{M}', \vec{\xi}\rangle$ at the $n + 1$st level of T_α is *below p in T_α* if $x' \supseteq x$ and for each $\nu \in x$ there is $\sigma_\nu \colon M_\nu \to M'_\nu$ such that

1. σ_0 is the identity.
2. $\sigma_\nu(U_\nu) = U'_\nu$.
3. If $\nu \in x$, $\nu' = \min(x - (\nu + 1))$ and $\nu'' = \min(x' - (\nu + 1))$ then $\sigma_{\nu'}([f]_{U_\nu}) = i'_{\nu',\nu''}([\sigma_\nu(f)]_{U'_\nu})$, where $i'_{\nu',\nu''} \colon M_{\nu'} \to M_{\nu''}$ is the embedding associated with the iteration $\vec{M}'$.
4. $\xi'_k = i'_{\nu,\nu'}(\sigma_\nu(\xi_k))$ for each $k < n$, where $\nu = \max(x)$ and $\nu' = \max(x')$.

In order to see how an ill-founded iteration $\langle M_\nu : \nu < \alpha \rangle$ yields an infinite branch in T_α we need the concept of a *support*:

3.33 Definition. If $\langle M_\nu : \nu \leq \eta \rangle$ is an iterated ultrapower, then the notion of a *support* is defined by recursion on η: a finite set $y \subseteq \eta + 1$ is a support for $z \in M_\eta$ if

1. $\{0, \eta\} \subseteq y$.
2. If η is a limit ordinal then there is $\nu < \eta$ in y and $z' \in M_\nu$ such that $z = i_{\nu,\eta}(z')$ and $y \cap (\nu + 1)$ is a support for z' in $\vec{M} \restriction (\nu + 1)$.
3. If $\eta = \nu + 1$ then $\nu \in y$ and $y \cap \eta$ is a support for $\{U_\nu, f\}$, where $M_\eta = \mathrm{Ult}(M_\nu, U_\nu)$ and $z = [f]_{U_\nu}$.

Suppose that $y \subseteq \eta + 1$ is a support in $\langle M_\nu : \nu \leq \eta \rangle$. We will construct a finite iterated ultrapower $\langle M'_\nu : \nu \in x \rangle$, together with embeddings $\sigma_\nu \colon M'_\nu \to M_\nu$ for $\nu \in x$, with the key property that the range of each embedding σ_ν is exactly the set of $z \in M_\nu$ such that $y \cap (\nu + 1)$ is a support for z.

The index set x for the iteration is the set of $\nu \in y$ such that $\nu = 0$, $\nu = \eta$, or $\nu + 1 \in y$. The models M'_ν and embeddings $\sigma_\nu \colon M'_\nu \to M_\nu$ for $\nu \in x$, and the ultrafilters $U'_\nu \in M'_\nu$ for $\nu \in x \cap \eta$, are defined by recursion on ν. To start out, $M'_0 = M_0$ and σ_0 is the identity. If $\nu \in x \cap \eta$ then $U'_\nu = \sigma_\nu^{-1}(U_\nu)$, which exists because $y \cap (\nu + 1)$ is a support for U_ν in M_ν, and $M'_{\nu'} = \mathrm{Ult}(M_\nu, U_\nu)$ where $\nu' = \min(x - (\nu + 1))$. If $\nu' = \nu + 1 \in x$ then $\sigma_{\nu+1}$ is defined by setting $\sigma_{\nu+1}([f]_{U'_\nu}) = [\sigma_\nu(f)]_{U_\nu}$. Otherwise ν' is a limit ordinal and $\sigma_{\nu'}([f]_{U'_\nu}) = i_{\nu+1,\nu'}([\sigma_\nu(f)]_{U_\nu})$.

Now if $\langle M_\nu : \nu \leq \alpha \rangle$ is an ill-founded iterated ultrapower, then let $\vec{\xi} = \langle \xi_n : n < \omega \rangle$ be an infinite descending sequence of ordinals in the final model M_α, and pick an increasing sequence $\{ y_n : n < \omega \}$ such that y_n is a support for $\vec{\xi} \restriction n$. Then the construction in the last paragraph gives a sequence $\langle (x_n, \vec{U}_n, \vec{M}_n, \vec{\xi}_n) : n \in \omega \rangle$ which is an infinite branch in the tree T_α.

To go the other direction, if $\langle \langle x_n, \vec{U}_n, \vec{M}_n, \vec{\xi}_n \rangle : n \in \omega \rangle$ is an infinite branch of T_α then we can obtain, by a direct limit construction, an iterated ultrapower which is indexed by $\bigcup_n x_n$ and hence has length at most α. Clause 4 of the definition implies that the direct limit maps $\sigma_{n,\alpha} \colon M_{n,\alpha} \to M_\alpha$ satisfy $\sigma_{n,\alpha}(\vec{\xi}_n) = \sigma_{n',\alpha}(\vec{\xi}_{n'}) \restriction n$ for all $n < n' < \omega$. Thus $\bigcup_n \sigma_{n,\alpha}(\vec{\xi}_n)$ is an infinite descending sequence of ordinals which witnesses that the final model M_α is ill-founded.

In order to allow for iterated ultrapowers including drops, the definition of T_α must be modified: first, the definition of a node allows iterations with drops, and in the definition of the tree ordering the maps σ_ν is required to preserve the drops; furthermore any $\nu \in x' - x$ at which $\vec{M}'_\nu$ drops are required to be larger than $\max(x \cap \alpha)$. Finally, clause 4 of that definition is modified to state that either clause 4 holds as stated previously, or else the iteration $\vec{M}'$ has a drop in $x' - x$.

An infinite branch of T_α describes an presumptive iteration which is indexed by a subset of α, and hence has length at most α. If a new drop is added at infinitely many levels in the branch then the presumptive iteration has infinitely many drops; otherwise the presumptive iteration is a real iteration and hence has a last model, but the levels of the branch beyond the last drop provide a witness $\vec{\xi}$ that the last model of the iteration is ill-founded. In either case this presumptive iteration demonstrates that the model M_0 is not iterable. $\dashv$

Comparisons of mice In the case of L, the only mice are the structures J_α, and hence it is trivial that, given two mice M and N, one is an initial segment of the other. Under appropriate conditions the same crucial fact is true of the mice for higher core models, but the proof requires the use of iterated ultrapowers to compare the two mice. We describe this process below, using the notation of definition 3.30 for the iterated ultrapowers. Superscripts M and N are used to distinguish the iterated ultrapower on M from that on N.

3.34 Definition (Comparison for premice). We will say that two premice M and N *strongly agree up to* τ if $\operatorname{crit}(\mathcal{U}^M_{\gamma'}) \geq \tau$ and $\operatorname{crit}(\mathcal{U}^N_{\gamma'}) \geq \tau$ for all $\gamma' \geq \gamma$, where γ is the least ordinal such that $\mathcal{U}^M_\gamma \neq \mathcal{U}^N_\gamma$. Assume that the iterable premice M and N are sound above τ and strongly agree up to τ. Then the comparison of M and N is defined by the use of iterated ultrapowers on M and N, which we distinguish by means of superscripts M and N.

Start the comparison by setting $M_0 = M$ and $N_0 = N$. Now suppose that M_ν and N_ν have been defined. If either of the models N_ν or M_ν is an initial segment of the other then the comparison is complete, and the iterated ultrapower is terminated with $\theta = \nu$. Otherwise let γ_ν be the least ordinal γ such that $\mathcal{U}^{N_\nu}_\gamma \neq \mathcal{U}^{M_\nu}_\gamma$, and set $U^M_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu}$ and $U^N_\nu = \mathcal{U}^{N_\nu}_{\gamma_\nu}$. Now use these ultrafilters to define $M_{\nu+1}$ and $N_{\nu+1}$ as in the definition 3.30 of an iterated ultrapower.

Note that definition 3.34 uses padded iterated ultrapowers, since it may be that $\mathcal{U}^{N_\nu}_\gamma \neq \mathcal{U}^{M_\nu}_\gamma$ because one of the two is equal to $\varnothing$.

The coherence property of premice ensures that both the indices γ_ν and the critical points of the ultrafilters U^M_ν and U^N_ν are strictly increasing.

3.35 Lemma. *This comparison process always stops after fewer than τ^+ steps, with one of M_θ and N_θ an initial segment of the other.*

Proof. Suppose to the contrary that the comparison continues for τ^+ steps, and let $i_{\nu,\nu'}\colon N_\nu \to N_{\nu'}$ and $j_{\nu,\nu'}\colon M_\nu \to M_{\nu'}$ be the iteration embeddings.

Set $\nu_0 = \max(D^N \cap D^M)$, the last place at which either iteration drops. Then for each $\nu < \tau^+ - \nu_0$ at least one of $i_{\nu,\nu+1}$ and $j_{\nu,\nu+1}$ is nontrivial; let κ_ν be the critical point of this embedding. If one of these embeddings is trivial then set $x_\nu = \varnothing$; otherwise pick $x_\nu \subseteq \kappa_\nu$ so that $x_\nu \in M_\nu \cap N_\nu$ and $x_\nu \in U^M_\nu \iff\!\!\!\!\!\!/\;\; x_\nu \in U^N_\nu$.

Now for each limit ordinal $\nu \in \tau^+ - \nu_0$ there is some $\eta_\nu < \nu$ such that κ_ν and x_ν are in the range of $i_{\eta_\nu,\nu}$, say $\kappa_\nu = i_{\eta_\nu,\nu}(\kappa'_\nu)$ and $x_\nu = i_{\eta_\nu,\nu}(x'_\nu)$. By Fodor's lemma there is a stationary set $S_0 \subseteq \tau^+$ on which η_ν is constant, say $\eta_\nu = \eta$, and since $|N_\eta| < \tau^+$ there is a stationary $S_1 \subseteq S_0$ on which κ'_ν and x'_ν are also constant, say $\kappa'_\nu = \kappa'$ and $x'_\nu = x'$. Then for any $\nu' < \nu$ in S_1 we have $i_{\nu,\nu'}(\kappa_\nu) = i_{\nu,\nu'} i_{\eta,\nu}(\kappa') = i_{\eta,\nu'}(\kappa') = \kappa_{\nu'}$, and similarly $i_{\nu,\nu'}(x_\nu) = x_{\nu'}$. In particular $i_{\nu,\nu+1}$ is not the identity for $\nu \in S_1$, since otherwise we would have $\kappa_{\nu'} = i_{\nu,\nu'}(\kappa_\nu) = i_{\nu+1,\nu'} i_{\nu,\nu+1}(\kappa_\nu) = i_{\nu+1,\nu'}(\kappa_\nu) = \kappa_\nu$. Similarly there is a stationary set $S_2 \subseteq S_1$ such that if $\nu < \nu'$ are in S_2 then $j_{\nu,\nu'}(\kappa_\nu) = \kappa_{\nu'}$ and $j_{\nu,\nu'}(x_\nu) = x_{\nu'}$. But this is impossible, for if $\nu < \nu'$ are in S_2 then $x_\nu \in U^N_\nu \iff \nu \in i_{\nu,\nu'}(x_\nu) = x_{\nu'} = j_{\nu,\nu'}(x_\nu) \iff \kappa_\nu \in U^M_\nu$, contrary to the choice of x_ν. ⊣

The next few results analyze some of the possible outcomes of this comparison. All results assume that M and N satisfy the requirements for definition 3.34: that is, they strongly agree up to τ and are sound above τ.

3.36 Lemma. *Suppose that M_θ is a proper initial segment of N_θ. Then M is sound, and the only ultrafilters U^M_ν used in the iteration of M are full ultrafilters with* $\operatorname{crit}(U^M_\nu) < \rho^{M_\nu}$. *Thus $D^M = \varnothing$.*

Sketch of Proof. Since M_θ is an initial segment of the premouse N_θ, the definition of a premouse implies that it is sound. Any model $\operatorname{Ult}(M_\nu, U)$ obtained by taking the ultrapower by an ultrafilter U with $\operatorname{crit}(U) \geq \rho^{M_\nu}$ is unsound, and this unsoundness is preserved by any further iterated ultrapowers.

If $D^M \neq \varnothing$ then let $\nu + 1 = \max(D^M)$. Then $M_{\nu+1} = \operatorname{Ult}(M^*_{\nu+1}, U^M_\nu)$ where $\operatorname{crit}(U^M_\nu) \geq \rho^{M^*_{\nu+1}}$, and hence $M_{\nu'}$ is unsound for all $\nu' \geq \nu + 1$. Similarly, if any of the models $M_{\nu+1}$ arise as ultrapowers by an ultrafilter $U^M_\nu \in M_\nu$ with $\operatorname{crit}(U^M_\nu) \geq \rho^{M_\nu}$, or if $M = M_0$ is unsound, then all succeeding models $M_{\nu'}$ are unsound. In either case this contradicts the fact that M_θ is sound. ⊣

3.37 Lemma. *Suppose that $M_\theta = N_\theta$, that $D^M = D^N = \varnothing$, and that $\tau \geq \max\{\rho^M, \rho^N\}$. Then $M = N$.*

Sketch of Proof. Since $D^M = D^N = \varnothing$, both $i^M_{0,\theta}\colon M \to M_\theta$ and $i^N_{0,\theta}\colon N \to N_\theta$ are defined. Since $\tau \geq \rho^M$ we have $M = h^M``\tau$. By lemma 3.26, we have $i^M_{0,\theta} h^M = h^{M_\theta}$ so it follows that $M \cong h^{M_\theta}``\tau$. Similarly $N \cong h^{N_\theta}``\tau$, and since $N_\theta = M_\theta$ it follows that $M = N$. ⊣

3.38 Corollary. *At least one of D^M and D^N are empty.*

Sketch of Proof. Suppose to the contrary that both iterated ultrapowers drop. Then lemma 3.36 implies that neither of M_θ and N_θ is a proper initial segment of the other, so $M_\theta = N_\theta$. Now let $\nu+1$ be the largest member of $D^M \cup D^N$, and suppose for example that $\nu+1 \in D^M$. Then the remainder of the iterated ultrapower can be regarded as a comparison of $M^*_{\nu+1}$ with either N_ν or $N^*_{\nu+1}$, depending on whether $\nu+1 \in D^N$. Furthermore, since D^N contains some ordinal $\nu' \leq \nu+1$, this comparison satisfies the hypothesis of lemma 3.37, with $\tau = \text{crit}(U^M_\nu)$, so $M^*_{\nu+1} = N_\nu$ or $M^*_{\nu+1} = N^*_{\nu+1}$. This is impossible since $U^M_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu} \neq \mathcal{U}^{N_\nu}_{\gamma_\nu}$. ⊣

3.39 Lemma. *Suppose that M and N are mice with* $\max\{\rho^N, \rho^M\} \leq \tau$, *and that M and N strongly agree up to τ. Then one of M and N is an initial segment of the other.*

Sketch of Proof. We prove the lemma by induction on the lengths of the mice M and N. First suppose that $M_\theta = N_\theta$. If $D^M = D^N = \varnothing$ then $M = N$ by lemma 3.37. Otherwise, suppose that $D^N \neq \varnothing$ and let $\nu+1$ be the largest member of D^N. If this is a drop in degree, then the remainders $\langle M_\xi : \nu \leq \xi < \theta \rangle$ and $\langle N_\xi : \nu \leq \xi < \theta \rangle$ of the two iterated ultrapowers form the comparison of M_ν with N_ν. This comparison has no drops, so we can apply lemma 3.37 to conclude that $M_\nu = N_\nu$, contradicting the fact that $\nu < \theta$. Similarly, if this is a normal drop, then the remainders of the two iterated ultrapowers form the comparison of M_ν with the mouse $N^*_{\nu+1}$ to which the iteration on N drops at that point. Again, lemma 3.37 shows that $M_\nu = N^*_{\nu+1}$, which is an initial segment of N_ν, so that the comparison would have terminated at $\nu < \theta$.

Thus we can assume without loss of generality that M_θ is a proper initial segment of N_θ. It follows by lemma 3.36 that the iteration of M uses only ultrafilters with critical point smaller than the projectum; however since the M and N strongly agree to τ the comparison uses only ultrafilters with critical point larger than τ, which in turn is larger then the projectum ρ^M. Thus M is never moved in the comparison, that is, $M_\theta = M$. ⊣

We are now ready to sketch a proof of lemma 3.28 together with the existence of solidity witnesses:

3.40 Lemma. *If M is an iterable premouse then M is an iterated ultrapower of the mouse* core(M)*, and M has a solidity witness for each $\nu \in p^M_k$ and $k < \omega$.*

Sketch of proof of lemmas 3.28 and 3.40. The proof is an induction over n, showing for each n that $\text{core}_n(M)$ is an iterated ultrapower of $\text{core}_{n+1}(M)$ and that p^M_{n+1} has solidity witnesses. We will give the proof for the case $n = 0$, beginning by showing that $M = \text{core}_0(M)$ is an iterated ultrapower of $N = \operatorname{core}_1(M)$ (with critical point at least ρ_1^M) and will assume for the moment that N has solidity witnesses for all $\nu \in p_1^N$. Let $\pi\colon N \to M$ be the collapse map, and let $i\colon N \to N_\theta$ and $j\colon M \to M_\theta$ be the iterated ultrapowers comparing N and M. We begin by assuming that both maps have critical point at least ρ_1^M. Then neither side of the comparison can drop: if, say, the iterated ultrapower on M dropped then M_θ would be a proper initial segment of N_θ since A_1^M is definable in N_θ, but not in M_θ; however this contradicts lemma 3.36. Furthermore $N_\theta = M_\theta$ since A_1^M is definable in, but not a member of, each. Now the existence of solidity witnesses for p_1^N implies that $i(p_1^N) = p_1^{N_\theta} = j\pi(p_1^N)$, but this implies that $ih_1^N(\xi) = j\pi h_1^N(\xi)$ for all $\xi < \rho_1^M$. Since N is 1-sound it follows that $i = j\pi$.

If M is not an iterated ultrapower of N, then $M \neq M_\theta$ and j is not the identity. Let ν be the least stage in the ultrapower that $j_{\nu,\nu+1}$ is nontrivial, let U'_ν be the ultrafilter used at this point, and let $\eta = \operatorname{crit}(j)$ be its critical point. Then $\eta \notin j\pi h_1^N``\eta = ih_1^N``\eta$, and it follows that η is also the critical point of an ultrapower in i: that is, $i_{\nu,\nu+1} = i^{U_\nu}$ for an ultrafilter U_ν in N_ν, as in diagram (1.9):

$$\begin{tikzcd}
N \arrow[rr, "i_{0,\nu}"] \arrow[dr, "\pi"'] & & N_\nu \arrow[r, "i^{U_\nu}"] & N_{\nu+1} \arrow[r, "i_{\nu+1,\theta}"] & N_\theta \arrow[d, equal] \\
& M \arrow[r, "j_{0,\nu}"] & M_\nu \arrow[r, "j^{U'_\nu}"] & M_{\nu+1} \arrow[r, "j_{\nu+1,\theta}"] & M_\theta
\end{tikzcd} \tag{1.9}$$

Let $x \subseteq \eta$ be a set in $N_\nu \cap M_\nu$ such that $x \in U_\nu \not\Longleftrightarrow x \in U'_\nu$. Then $x = h_1^{N_\nu}(\xi)$ for some $\xi < \eta$. We claim that $x = h_1^{M_\nu}(\xi)$ as well: to see this, note that $j_{\nu,\theta}(h_1^{M_\nu}(\xi)) = h_1^{M_\theta}(\xi) = h_1^{N_\theta}(\xi) = j_{\nu,\theta}(x)$, and each of $i_{\nu,\theta}$ and $j_{\nu,\theta}$ are the identity on η. Thus $i_{\nu,\theta}(x) = j_{\nu,\theta}(x)$, but this is impossible since then $x \in U_\nu \iff \nu \in i_{\nu,\theta}(x) \iff \nu \in j_{\nu,\theta}(x) \iff x \in U'_\nu$, contradicting the choice of x.

This completes the proof that M is an iterated ultrapower of $\mathfrak{C}_1(M)$, except for verifying that neither i nor j have critical point smaller than ρ_1^M. If this is false, then it must be that $\rho_1^M = \mu^{+M}$, where $\mu = \operatorname{crit}(\mathcal{U}_\gamma^M)$ for some $\gamma > \rho_1^M$, and $(\mu^{++})^N = (\mu^{++})^M \cap h_1^M``\rho_1^M < (\mu^{++})^M$. If $\gamma \geq (\mu^{++})^N$ is least such that $\mathcal{U}_\gamma^M$ is a measure on μ, then the measures in N with critical point μ are exactly the measures in $\mathcal{U}^M{\restriction}\gamma$. It follows that none of these measures will be applied in the comparison, and hence $\operatorname{crit}(i) \geq \rho_1^M$.

To see that $\operatorname{crit}(j) \geq \rho_1^M$ we need to use another basic result, the proof of which will be delayed until after the current proof is completed.

3.41 Lemma (Dodd-Jensen Lemma). *Suppose that N is an iterable premouse, $i\colon N \to P$ is an iterated ultrapower, and $k\colon N \to P$ is any Σ_0-elementary embedding. Then the range of k is cofinal in M, i does not drop, and $i(\alpha) \geq k(\alpha)$ for all $\alpha \in N$.*

Now if $\mathrm{crit}(j) = \mu < \mathrm{crit}(i)$, then $i(\mu) = \mu < j(\mu) = j\pi(\mu)$. This contradicts lemma 3.41 with $P = N_\theta$, $k = i\pi$, and $\alpha = \mu$, and this contradiction completes the proof that M is an iterated ultrapower of $\mathrm{core}_1(M)$.

The proof that M has solidity witnesses is similar. Fix $\nu \in p_1^M$, set $p = p_1^M - (\nu + 1)$, and let $\pi\colon N \to M$ be the transitive collapse of the Σ_1 hull h_1^M "ν of $\nu \cup p$ in M. Recall that $A = \{\, \xi < \nu : M \models \Phi(\xi, p)\,\}$, the Σ_1 theory of $\nu \cup p$ in M, and R is the prewell-ordering of A defined by letting $\xi\, R\, \xi'$ if there is γ such that $J_\gamma^{\mathcal{U}} \models \Phi(\xi, p)$ and $\forall \gamma' < \gamma\ J_{\gamma'}^{\mathcal{U}} \models \neg\Phi(\xi', p)$ where Φ is the universal Σ_1 formula. Clearly A and R are Σ_1 definable in N. Let $i\colon N \to N_\theta$ and $j\colon M \to M_\theta$ be the iterated ultrapowers comparing N and M. By the same argument as before, $\mathrm{crit}(i) \geq \nu$. We claim that N_θ is a proper initial segment of M_θ, for if not, then lemma 3.41 implies $\mathrm{crit}(j) \geq \nu$ by the same argument as before, and hence A is a member of M_θ; however $A \notin N_\theta$ and hence $M_\theta \not\subseteq N_\theta$.

Now there is a $\tau \in M_\theta$ mapping A into the ordinals of N_θ such that $\forall \xi, \xi' \in A\ (\tau(\xi) \leq \tau(\xi') \iff \xi\, R\, \xi')$, namely the map defined by letting $\tau(\xi)$ be the least ordinal γ such that $J_\gamma^{\mathcal{U}^{M_\theta}} \models \Phi(\xi, i\pi^{-1}(p))$. This map is Σ_1 definable in N_θ and hence is a member of M_θ.

If j is the identity, then $M_\theta = M$ and hence $\tau \in M$ is the desired solidity witness. If j is not the identity, then it does not drop by lemma 3.36 so either M has a full measure U with $\mathrm{crit}(U) \geq \nu$ or else there is a full measure $U \in M$ with $\mathrm{crit}(U) < \nu$ but $R \in \mathrm{Ult}(M, U)$. We observed following definition 3.27 that either of these implies that M has a solidity witness that ν is in p_1^M, and this completes the proof of lemma 3.40. ⊣

We outline the proof of lemma 3.41. A full proof is given in section 4 of chapter [56].

Sketch of the proof of lemma 3.41. Suppose that i and k are as in the hypothesis. We will define iterations $i_n^*\colon N_n \to N_{n+1}$ with $i_0^* = i\colon N \to M$, so that $i^* = \dots i_3^* i_2^* i_1^* i_0^*\colon N_0 \to N_\omega$ is an iterated ultrapower on N. We will then see that if the conclusion fails then i^* contradicts the assumption that N is iterable.

The maps i_n^* and k_n are defined by recursion on n, by setting $i_0^* = i$ and $k_0 = k$, and defining $i_{n+1}^* = k_n(i_n^*)$ and $k_{n+1} = i_n^* * k_n$ using the copy construction given in the next paragraph:

Suppose that $i\colon N \to M$ is any iterated ultrapower and $k\colon N \to M$ is a Σ_0 elementary embedding. Then we define a copy map $k(i)$ and a map $i * k$ so that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "k(i)"] & N' \\ N \arrow[u, "k"] \arrow[r, "i"] & M \arrow[u, "i*k"'] \end{tikzcd} \tag{1.10}$$

The definition is by recursion on length of the iterated ultrapower i, with the induction step using the basic box

$$\begin{tikzcd} M_\nu \arrow[r, "i^{k_\nu(U_\nu)}"] & M_{\nu+1} \\ N_\nu \arrow[u, "k_\nu"] \arrow[r, "i^{U_\nu}"] & N_{\nu+1} \arrow[u, "k_{\nu+1}"'] \end{tikzcd} \tag{1.11}$$

where $i^{U_\nu}\colon N_\nu \to N_{\nu+1}$ is the νth stage of the iteration on N, $k_0 = k$, and $k_{\nu+1}$ is defined by $k_{\nu+1}([f]_{U_\nu}) = [k_\nu(f)]_{i^{k_\nu(U_\nu)}}$. The maps of diagram (1.10) are defined by letting $k(i)$ be the iteration using the maps $i^{k_\nu(U_\nu)}$ on the top row of diagram (1.11), and setting $i * k = k_\theta$ where θ is the length of the iteration.

Now suppose that the map i does contain a drop. Then each of the maps i^*_n contains a drop, and hence $i^* = \dots i^*_3 i^*_2 i^*_1 i^*_0\colon N_0 \to N_\omega$ is a presumptive iteration on N containing infinitely many drops, contrary to the assumption that N is iterable. Hence i does not drop.

Now we will show that if either of the other clauses of the conclusion is false, then there are ordinals $\alpha_n \in N_n$ such that $i_{n,n+1}(\alpha_n) > \alpha_{n+1}$, so the ordinals $\alpha'_n = \dots i^*_{n+1} i^*_n(\alpha_n)$ form an infinitely descending sequence of ordinals in N_ω, again contradicting the assumption that N is iterable. In the case that the range of k is bounded in $M = N_1$ a simple induction shows that the ordinals $\alpha_{n+1} = \sup(\operatorname{ran}(k_n)) \in N_{n_1}$ are as required. If $i(\alpha) > k(\alpha)$ for some $\alpha \in N$, then the ordinals $\alpha_n \in N_n$ are defined by recursion on n, setting $\alpha_0 = \alpha$ and $\alpha_{n+1} = k_n(\alpha_n)$. Then by induction on n we have $\alpha_{n+1} = k_n(\alpha_n) < i_n(\alpha_n)$ and $k_{n+1}(\alpha_{n+1}) = k_{n+1}k_n(\alpha_n) < k_{n+1}i_n(\alpha_n) = i_{n+1}k_n(\alpha_n) = i_{n+1}(\alpha_{n+1})$. ⊣

The Dodd-Jensen Core Model. The discussion above is valid for the core model up through $o(\kappa) = \kappa^{++}$. We now turn our attention to the Dodd-Jensen core model, and for this purpose we assume that $0^\dagger$ does not exist, and hence no mouse has any full measures except possibly for the final nontrivial measure on its sequence.

3.42 Definition. We will define the Dodd-Jensen core $K^{\text{DJ}} = L[\mathcal{U}]$ by recursion as follows: Suppose $K^{\text{DJ}}_\kappa = J_\kappa[\mathcal{U}\restriction\kappa]$ has been defined. Let $\mathcal{M}$ be the set of mice $M = J_{\alpha^M}[\mathcal{U}^M]$ such that M has no full measures, $\rho^M = \kappa$ and $\mathcal{U}^M\restriction\kappa = \mathcal{U}\restriction\kappa$. Then $K^{\text{DJ}}_{\bar\kappa} = \bigcup\mathcal{M}$, where $\bar\kappa = \sup\{\, \alpha^M : M \in \mathcal{M}\,\}$.

To justify this definition of K^{DJ}, notice that lemma 3.39 implies that of any two members M_0 and M_1 of $\mathcal{M}$, one is an initial segment of the other. Thus $K^{\text{DJ}}_{\bar\kappa} = J_{\bar\kappa}[\mathcal{U}\restriction\bar\kappa]$ where $\mathcal{U}\restriction\bar\kappa = \bigcup\{\mathcal{U}^M : M \in \mathcal{M}\,\}$. The ordinal $\bar\kappa$ will be equal to κ^+ in K^{DJ}.

If there is no inner model with a measurable cardinal, then the core model K is equal to K^{DJ}. If there is an inner model with a measurable cardinal, but $0^\dagger$ does not exist, then $K = L[U]$, where the model $L[U]$ is chosen so that the critical point of the measure U of $L[U]$ is as small as possible.

Note that $K^{\text{DJ}} = L$ if $0^\sharp$ does not exist. If $K = L[U]$, with U a measure on κ, then we can write $L[U] = L[\mathcal{U}]$ where, setting $\gamma = \kappa^{++K^{\text{DJ}}}$, the sequence $\mathcal{U}$ has a largest member $\mathcal{U}_\gamma = U$ and $\mathcal{U}\restriction\gamma = \mathcal{U}^{\text{DJ}}\restriction\gamma$.

The fact that a mouse is required to be iterable, while a level J_α of the L hierarchy is only required to be well-founded, makes the definition of a mouse logically more complicated than that of the sets J_α. For example, the statement $\exists\alpha\,(\omega, \mathrm{E}) \cong (J_\alpha, \in)$ is a Π^1_1 statement about the set $\mathrm{E} \subseteq \omega^2$, while the statement that (ω, E) is isomorphic to a mouse $J_\alpha[\mathcal{U}]$ is Π^1_2.

However $(K^{\text{DJ}})^M = K^{\text{DJ}} \cap M$ whenever M is a transitive model of ZF containing ω_1, since the definition of a mouse is absolute for models containing ω_1 by lemma 3.32.

It is not the case, as it is for L, that there is a sentence $V = K$ such that every class model of $V = K$ is equal to K. This can be seen by considering the class L, which may or may not be equal to K^{DJ}, depending on whether $0^\sharp$ exists. A more general example can be obtained by taking any mouse M in K^{DJ} with a measure U on κ, and let $\widetilde{M} = \mathrm{Ult}_{\mathrm{On}}(M, U)$. Then $i^U_{\mathrm{On}}(\kappa) = \mathrm{On}$, and the initial segment $\widetilde{M}|\,\mathrm{On} = V^{\widetilde{M}}_{\mathrm{On}}$ of $\widetilde{M}$ is a class model of ZFC + $V = K$ which is not equal to K^{DJ}. Notice, for example, that the critical points of the iteration form a closed and unbounded class of indiscernibles for $\widetilde{M}|\,\mathrm{On}$ which is definable in $L[M]$, and hence in K^{DJ}.

Vickers and Welch have observed [60] that the existence of a Ramsey cardinal implies that it is consistent that there is a proper class $X \prec K^{\text{DJ}}$ such that $X \ncong K^{\text{DJ}}$.

The proof of Jensen's covering lemma for L relied on Kunen's result that the existence of a nontrivial embedding $i\colon L \to L$ implies that $0^\sharp$ exists. The proof of the covering lemma for K^{DJ} will rely on an analogous result, lemma 3.47, stating that if $0^\dagger$ does not exist and there is a nontrivial elementary embedding $i\colon K^{\text{DJ}} \to M$, then $K = L[U]$ for some measure U in K. In particular, if there is a nontrivial embedding $i\colon K^{\text{DJ}} \to K^{\text{DJ}}$, then $K \neq K^{\text{DJ}}$, so that (assuming $0^\dagger$ does not exist) $K = L[U]$ for some measure U in K. However Jensen has shown that it is not necessarily true, as one might expect, that U is the ultrafilter associated with i, or even that $\mathrm{crit}(U) = \mathrm{crit}(i)$. Notice that the model N of this proof gives an example of a mouse which is not a member of K, and of a mouse which is added by a set forcing.

3.43 Theorem. *Suppose that $L[U]$ satisfies that U is a measure on κ, and let G be $L[U]$-generic for the Levy collapse of $\lambda = \kappa^{+(L[U])}$ onto ω. Then in $L[U, G]$ there is a fully iterable premouse N with measurable cardinal less than κ such that $J_\lambda(U)$ is an ultrapower of N. Hence there is an elementary embedding $i\colon K^{\text{DJ}} \to K^{\text{DJ}}$ with* $\mathrm{crit}(i) < \kappa$.

Proof. We will prove that the theorem is true in $\mathrm{Ult}(L[U], U)$, so that by elementarity it is true in $L[U]$ as well. Let $j : L[U] \to M = \mathrm{Ult}(L[U], U)$, let

$$\kappa' = j(\kappa), \quad \lambda' = j(\lambda), \quad \text{and} \quad U' = j(U),$$

and let G be M-generic for the Levy collapse of λ' onto ω. Let $\mathcal{L}$ be the infinitary language with a constant $\mathbf{x}$ for each member x of $J_{\lambda'}[U', G]$ and one additional constant $\mathbf{W}$ which will denote a measure on κ. Let T be the theory with sentences

$$\forall z\Big(z \in \mathbf{x} \iff \bigvee_{y \in x} (z = \mathbf{y})\Big)$$

for each $x \in J_{\lambda'}[U', G]$, together with a sentence asserting that $\mathbf{W}$ is an amenable measure in $J_\lambda[\mathbf{W}]$ such that $\mathrm{Ult}(J_\lambda[\mathbf{W}], \mathbf{W}) \cong J_{\lambda'}[U']$. This theory is consistent because it is true in $L[U]$, so by the Barwise compactness theorem it has a model $\mathcal{A}$. Let $W = \mathbf{W}^{\mathcal{A}}$ and let N be the premouse $J_\lambda[W]$. Then $\mathrm{Ult}(N, W) = J_{\lambda'}[U']$ by the construction of W, and it follows that N is fully iterable since $J_{\lambda'}[U']$ is fully iterable.

Now K^{DJ} is equal to the initial segment of $\mathrm{Ult}_{\mathrm{On}}(N, W)$ below On, its measurable cardinal, and hence it follows easily that there is an embedding from K^{DJ} into K^{DJ} with critical point κ. ⊣

A major difference between the proof of the covering lemma for L and the proof of the covering lemma for $L[U]$ is that the proof of the analog 3.47 of Kunen's result itself uses the covering lemma. In order to avoid circularity we prove the covering lemma in two steps: the first step proves enough of the strength of the covering lemma to prove lemma 3.47, using the following observation:

3.44 Lemma. *If U is a countably complete normal K^{DJ}-ultrafilter then U is a measure in $L[U]$, and $K^{\mathrm{DJ}} \subseteq L[U]$.*

Proof. Let $\kappa = \mathrm{crit}(U)$ and set $\gamma = (\kappa^{++})^{K^{\mathrm{DJ}}}$. Take $\mathcal{U}$ to be the sequence such that $K^{\mathrm{DJ}} = L[\mathcal{U}]$ and let $\mathcal{U}'$ be the good sequence defined by $\mathcal{U}' \restriction \gamma = \mathcal{U} \restriction \gamma$ and $\mathcal{U}'_\gamma = U$. We will show that $\mathcal{P}(\kappa)^{L[\mathcal{U}']} = \mathcal{P}(\kappa)^{K^{\mathrm{DJ}}}$, which implies that U is a measure in $L[\mathcal{U}']$, and hence in $L[U]$.

Suppose the contrary, and let α be the least ordinal such that there is a set $x \in \mathcal{P}(\kappa) \cap J_{\alpha+\omega}[\mathcal{U}'] - K^{\mathrm{DJ}}$. Now the model $J_\alpha[\mathcal{U}']$ need not be sound, but the iterability of $L[\mathcal{U}]$ and the countable completeness of U ensure that $J_{\alpha'}[\mathcal{U}]$ is iterable. Hence lemma 3.28 implies that $J_\alpha[\mathcal{U}']$ is an iterated ultrapower of a mouse N, but this is impossible since any such mouse would be a member of K^{DJ}.

To see that $K^{\mathrm{DJ}} \subseteq L[U]$, suppose the contrary and use iterated ultrapowers to compare K^{DJ} with $K^{L[U]}$. Note that the latter is equal to $L[U]$, and can be written $L[\mathcal{U}'']$, where $\mathcal{U}''$ is constructed like the sequence $\mathcal{U}'$ above, but using $(K^{\mathrm{DJ}})^{L[U]}$ instead of the true K^{DJ}. If the iterated ultrapower beginning with K^{DJ} does not drop, then it must be trivial since there are no full measures in K^{DJ}, and this would imply $K^{\mathrm{DJ}} \subseteq L[U]$. If it does drop, on the other hand, then that on $L[U]$ does not drop, and the final model $L[U_\theta]$ of that iteration is an initial segment of the final model of the iteration on K^{DJ}. Then that iteration would construct a class of indiscernibles for $L[U_\theta]$, which implies that $0^\dagger$ exists and in fact is a member of K^{DJ}, which is absurd. ⊣

The second part of the proof will use the following analog of the condensation lemma for L:

3.45 Lemma. *Assume that there is no inner model with a measurable cardinal, and that $X \prec K^{\mathrm{DJ}}$ is a class such that* $\mathrm{ot}(X \cap \lambda^{+K^{\mathrm{DJ}}}) = \lambda^{+K^{\mathrm{DJ}}} = \lambda^+$ *for a proper class of cardinals λ of K^{DJ}. Then $X \cong K^{\mathrm{DJ}}$.*

Sketch of Proof. Let N be the transitive collapse of X, and suppose to the contrary that $N \neq K^{\mathrm{DJ}}$. Then there is a mouse $M \in K^{\mathrm{DJ}} - N$. Now use iterated ultrapowers to compare the models M and N, and let M_θ and N_θ be the final models of the iterated ultrapower of M and N, respectively. Then N_θ must be a proper initial segment of M_θ, since there is a set $x \subseteq \rho^M$ which is definable in M but not in N. Then x is definable in M_θ, but not in N_θ so N_θ is a proper initial segment of M_θ and it follows by corollary 3.38 that the iterated ultrapower on N does not drop. Since N does not contain any full ultrafilters it follows that the iterated ultrapower on N must be trivial, that is, $N = N_\theta$, and this implies that $\mathrm{ot}(X \cap \lambda^{+K^{\mathrm{DJ}}}) = (\lambda^+)^N = (\lambda^+)^{M_\theta} < \lambda^+$ for every cardinal $\lambda > |M_\theta|$, contradicting the hypothesis. This contradiction completes the proof of proposition 3.45. ⊣

The assumption that $(\lambda^+)^{K^{\mathrm{DJ}}} = \lambda^+$ is actually unnecessary here: any iterated ultrapower of K^{DJ} is a member of K^{DJ}, since it is a finite sequence of drops, separated by an iterated ultrapower by a single ultrafilter. In particular $M_\theta \in K^{\mathrm{DJ}}$.

The set Γ from Kunen's proof satisfies $|\Gamma \cap \lambda^+| = \lambda^+$ on a stationary set. Hence the importance of the weak covering property:

3.46 Definition. A cardinal λ is *countably closed* if $\eta^\omega < \lambda$ for all $\eta < \lambda$, and a model M has the *countably closed weak covering property* if for all sufficiently large countably closed singular cardinals λ we have $(\lambda^+)^M = \lambda^+$.

3.47 Lemma. *Suppose that $0^\dagger$ does not exist, and that K satisfies the countably closed weak covering property. If there is a nontrivial elementary embedding $i\colon K \to M$, then $K = L[U]$ where U is a measure in $L[U]$ with* $\mathrm{crit}(U) = \mathrm{crit}(i)$.

Proof. Set $\kappa = \mathrm{crit}(i)$. We can assume that $M = \{\, i(f)(\kappa) : f \in K \,\}$, for otherwise we could factor i as

$$i : K \xrightarrow{i'} M' \cong \{\, i(f)(\kappa) : f \in K \,\} \prec M,$$

and work with $i' \colon K \to M'$ instead of $i \colon K \to M$.

First we show that $K \neq K^{\mathrm{DJ}}$, so that $K = L[U]$ for some measure U. Suppose to the contrary that $K = K^{\mathrm{DJ}}$. We claim that in this case M also equals K^{DJ}: to see this, assume $M \neq K^{\mathrm{DJ}}$ and use iterated ultrapowers to compare M with K^{DJ}. The iteration on M is trivial, and the iteration on K^{DJ} drops and generates a closed and unbounded class I of indiscernibles for M. Then our assumption on i implies that $i^{-1}[I]$ is a class of indiscernibles for K^{DJ}, which would generate a countably complete K^{DJ}-measure U_{ω_1} on the limit of the first ω_1 members of I. By lemma 3.47, it follows that U_{ω_1} is a measure in $L[U_{\omega_1}]$, contrary to our assumption that $K = K^{\mathrm{DJ}}$.

Thus $i \colon K^{\mathrm{DJ}} \to K^{\mathrm{DJ}}$, and we can apply the proof, sketched in chapter [31, Theorem 1.13], of Kunen's corresponding result for L. This involves defining a continuously descending sequence of classes Γ_α, beginning with $\Gamma_0 = \mathrm{ran}(i)$ and setting $\Gamma_{\alpha+1} = \{\, x \in \Gamma_\alpha : i_\alpha(x) = x \,\}$ where i_α is the transitive collapse of $\Gamma_\alpha \prec K$. The classes Γ_α contain all of their limit points of cofinality greater than κ, and if $\kappa < \nu \in \Gamma_\alpha$ then $|\Gamma_\alpha \cap \nu^+| = \nu^+$. Since $\nu^+ = \nu^{+K}$ by the weak covering property, it follows by lemma 3.45 that $\Gamma_\alpha \cong K^{\mathrm{DJ}}$ for each ordinal α. Now the same argument as for L shows that if we set $\kappa = \mathrm{crit}(i)$ and $\kappa_\alpha = \min(\Gamma_\alpha - \kappa)$ then the class of ordinals $(\kappa_\alpha : \alpha \in \mathrm{On})$ is a closed and unbounded class of indiscernibles for K^{DJ}. It follows that $\{\, \kappa_\alpha : \alpha < \omega_1 \,\}$ generates a normal K^{DJ}-measure U_{ω_1} on κ_{ω_1}, and since U_{ω_1} is countably complete it follows by lemma 3.44 that U_{ω_1} is a measure in $L[U_{\omega_1}]$.

This completes the proof that $K = L[U]$ for some measure U. We must have $\mathrm{crit}(U) \geq \mathrm{crit}(i)$, for otherwise Kunen's argument for L implies directly that $0^\dagger$ exists. To see that $\mathrm{crit}(U) \leq \mathrm{crit}(i)$, assume to the contrary that $\lambda = \mathrm{crit}(U) > \kappa = \mathrm{crit}(i)$ and observe that it is true for $L[U]$, as it is for L, that $\mathcal{H}^{L[U]}(\Gamma) \cong L[U]$ for any proper class Γ of ordinals. Now $M = L[U']$, with $\lambda' = \mathrm{crit}(U') \geq \lambda = \mathrm{crit}(U)$, so there is an iterated ultrapower $j \colon K \to M$. Let $\Gamma = \{\, \nu : i(\nu) = j(\nu) \,\}$. Then Γ is a proper class, it contains its limit points of cofinality greater than λ, and $\kappa \notin \mathcal{H}^K(\Gamma) = \mathrm{On} \cap \Gamma$.

We will complete the proof by showing that this is impossible. First, note that the family of proper classes $\Gamma \prec L[U]$ which contain all of their limit points of cofinality greater than λ is closed under intersections of size at most λ. Hence there is such a class Γ' such that $\Gamma' \cap \lambda$ is as small as possible. Now if $k \colon L[U''] \cong \mathcal{H}^K(\Gamma')$ is the transitive collapse then, since $\mathrm{crit}(U)$ is as small as possible, $\mathrm{crit}(U'') = \mathrm{crit}(U) = \lambda$. However there is some $\eta < \lambda$ so that $k(\eta) > \eta$. Then $k(\eta) \in \kappa \cap \mathcal{H}^K(\Gamma') - \mathcal{H}^K(\Gamma'')$, where $\Gamma'' = \{\, \nu : k(\nu) = \nu \,\}$, contrary to the choice of Γ'. $\dashv$

Part 1 of the Proof

The proof of the Dodd-Jensen covering lemma can be divided into two parts. The first part is a direct generalization of the proof of the covering lemma for L: it involves defining the basic construction for suitable sets $X \prec K_\kappa$, and showing that the class of suitable sets is unbounded. One of the two major novelties in this stage of the proof is the possible use of an iterated ultrapower in the construction. The second part of the proof, which has no analog for L, is used to analyze the indiscernibles generated by this iterated ultrapower. In the case when this iterated ultrapower is infinite, these indiscernibles will yield a sequence C which is Prikry generic over $K = L[U]$.

The other major novelty arises from the fact that lemma 3.47, which is needed in the proof, has the hypothesis that K satisfies the countably closed weak covering property. Thus we will, during part one of the proof, simultaneously prove two results, the first of which is the hypothesis to the second.

3.48 Lemma. *Assume that* $0^\dagger$ *does not exist.*

1. *The core model* K *satisfies the countably closed weak covering property.*
2. *If* K *satisfies the countably closed weak covering property, then it also satisfies the full Dodd-Jensen covering lemma, theorems 1.12 and 1.13.*

Most of this proof will be reused in proving the covering lemma for sequences of measures; however certain segments of the proof are substantially simplified by our assumption that $0^\dagger$ does not exist. This extra assumption is equivalent to the assumption that no premouse has more than one full ultrafilter.

The following definition will be valid up to a strong cardinal. We write K_κ for $J_\kappa[\mathcal{U}] = K \cap V_\kappa$.

3.49 Definition. Let $X \prec_1 K_\kappa$, with transitive collapse $\pi\colon \overline{K} \to X$, where $\overline{K} = L_{\bar\kappa}[\overline{\mathcal{U}}]$. We say that X is *suitable* if $\mathrm{Ult}_n(M, \pi, \beta)$ is iterable whenever $n \in \omega$, $\beta \leq \kappa$, and $M = J_\alpha[\mathcal{U}']$ is an iterable premouse (possibly with $\alpha = \mathrm{On}$) such that $\mathrm{Ult}_n(M, \pi, \beta)$ is defined and $\mathcal{U}' \restriction \bar\beta = \overline{\mathcal{U}} \restriction \bar\beta$ where $\bar\beta$ is the least ordinal such that $\pi(\bar\beta) \geq \beta$.

As in the proof of the covering lemma for L, we say that $X \prec_1 K_\kappa$ is *countably closed* if $X = Y \cap K_\kappa$, where ${}^\omega Y \subseteq Y$ and $Y \prec H(\tau)$ for some $\tau > \kappa$.

3.50 Lemma. *(i) Every countably closed set* $X \prec_1 K_\kappa$ *is suitable, and (ii) the class of suitable sets* X *is closed under increasing unions of uncountable cofinality, and is unbounded in* $H(\delta)^{(K_\kappa)}$ *for any uncountable cardinal* δ.

Sketch of proof. The only difference between the proof of this lemma and the corresponding lemma for L is that we need to check that the model $\widetilde{M} = \mathrm{Ult}_n(M, \pi, \beta)$ is iterable rather than merely well-founded. This is straightforward for clause (i). For clause (ii) this involves changing definition 3.18 of a witness to the unsuitability of X: Clause 3.18(2), stating that $\operatorname{dir}\lim(\pi^{-1}(w)) = \mathfrak{C}_n(J_\alpha)$ for some ordinal α, is modified to require that $\operatorname{dir}\lim(\pi^{-1}(w))$ be the Σ_n code for some mouse. Clause 3.18(3), is modified to state that the the witness w either the structure $\widetilde{M}_0$ of which $\operatorname{dir}\lim(w)$ is the Σ_n-code is ill-founded, or else there is an ill-founded iteration of this structure.

The proof for both clauses relies on the fact that if there is an ill-founded iteration then there is one of countable length. ⊣

We are now ready to describe the basic construction. As in the proof of the covering lemma for L, we are given a suitable set $X \prec_1 K_\kappa$, and we let $\pi\colon \overline{K} = J_{\bar\kappa}[\overline{\mathcal{U}}] \cong X \prec_1 K_\kappa$ be the transitive collapse. We assume that X is not transitive, so that π is not the identity, and furthermore we assume that either X is countably closed or else K satisfies the countably closed weak covering property.

In order to postpone some complications which arise in the proof of the covering lemma for sequences of measures, we make the following additional assumption:

$$\begin{gathered}\text{If } K = L[U], \text{ where } U \text{ is a measure on a cardinal } \mu \text{ of } K,\\ \text{then either } \mu^{+K} \subseteq X \text{ or else } \kappa \leq \mu^{+K}.\end{gathered} \tag{1.12}$$

This assumption does not involve any loss of generality: the case $\mu^{+K} \subseteq X$ shows that any set x of size at most μ^{+K} is contained in a set $y' \in K$ of size μ^{+K}, and then the case $\kappa \leq \mu^{+K}$ shows that x can be covered by a set $y \subseteq y'$ which satisfies theorem 1.3. The case $\mu^{+K} \subseteq X$ is a relativization of the proof for L and requires no new ideas.

In the proof for L, the next step was to set $M = J_\alpha$, where $\alpha \geq \bar\kappa$ was the least ordinal such that there is a bounded subset of $\bar\kappa$ in $J_\alpha - J_{\bar\kappa}$. If it happens that $\overline{\mathcal{U}} = \mathcal{U}{\restriction}\bar\kappa$ then we can similarly take $J_\alpha[\mathcal{U}]$, but in general we need to modify the construction by using iterated ultrapowers to compare the models $\overline{K} = J_{\bar\kappa}[\overline{\mathcal{U}}]$ and $K = L[\mathcal{U}]$. A key step of the proof is showing (see lemma 3.51(2)) that $\overline{K}$ is never moved in this comparison, so that the final model of the iterated ultrapower on K is a model $M_\theta = J_{\alpha_\theta}[\mathcal{U}_\theta]$ such that $\overline{\mathcal{U}} = \mathcal{U}_\theta{\restriction}\bar\kappa$.

Thus we obtain the following variant of diagram 1.5, where each of the subset symbols indicate containment as an initial segment:

$$\begin{tikzcd} K \arrow[r, rightsquigarrow, "i"] & M_\theta \supseteq M \arrow[r, "\tilde\pi"] & \mathrm{Ult}_n(M,\pi,\kappa) = \widetilde{M} \\ & \overline{K} = J_{\bar\kappa}[\overline{\mathcal{U}}] \arrow[u, hook] \arrow[r, "\pi"] & X \prec K_\kappa \arrow[u, hook] \end{tikzcd} \tag{1.13}$$

We will write $M = J_\alpha[\mathcal{U}^M]$ and $\widetilde{M} = J_{\tilde\alpha}[\widetilde{\mathcal{U}}]$. As in diagram (1.5), $\overline{K}$ is the transitive collapse of the set X and π is the inverse of the collapse map. The iterated ultrapower is indicated by the wavy line from K to M_θ. Since this iteration drops whenever it is nontrivial (see lemma 3.51(3)), the wavy line does not represent an embedding.

Once the model $M_\theta = J_{\alpha_\theta}[\mathcal{U}^{M_\theta}]$ has been constructed, diagram (1.13) is completed like diagram 1.5: Let $M = J_\alpha[\mathcal{U}^{M_\theta}]$, where (α, n) is the largest pair $(\alpha, n) \leq (\alpha_\theta, n_\theta)$ such that $\mathrm{Ult}_n(J_\alpha[\mathcal{U}^{M_\theta}], \pi, \kappa)$ is defined. Thus α is the least ordinal such that there is a bounded subset x of κ which is definable in $J_\alpha[\mathcal{U}_\theta]$ but is not a member of $\overline{K}$. Finally, set $\widetilde{M} = \mathrm{Ult}_n(M, \pi, \kappa)$.

3.51 Lemma. *Assume $\neg 0^\dagger$. Let $X \prec_1 K_\kappa$ be a suitable set which is not transitive, so that the collapse $\pi\colon \overline{K} \cong X \prec_1 K_\kappa$ is not the identity. Finally, assume that either X is countably complete or else K has the countably closed weak covering property.*

1. *$\mathcal{P}^K(\eta) \not\subseteq \overline{K}$, where $\eta = \mathrm{crit}(\pi)$.*
2. *In the comparison of K and $\overline{K}$, the iterated ultrapower on the model $\overline{K}$ is trivial.*
3. *Either $\overline{K}$ is an initial segment of K, or else $1 \in D$, so that the iterated ultrapower on K drops immediately.*
4. *$\overline{K}$ is an initial segment of the final model M_θ of the iteration of K.*
5. *$\widetilde{M} \in K$.*

Sketch of Proof. If clause (1) fails then $U' = \{\, x \in \mathcal{P}^{\overline{K}}(\eta) : \eta \in \pi(x) \,\}$ is a K-ultrafilter. If X is countably closed then U' is countably complete, so lemma 3.44 implies that U' is a measure in $L[U']$. If K has the countably closed weak covering property then $\mathrm{Ult}(K, U')$ is well-founded since it can be embedded into $\mathrm{Ult}(K, \pi, \kappa)$, which is well-founded by the definition of suitability, so lemma 3.47 implies that $K = L[U]$ where $\mathrm{crit}(U) \leq \mathrm{crit}(U')$. Thus, under the hypothesis of either clause of lemma 3.48, $K = L[U]$ for some measure U with $\mathrm{crit}(U) \leq \mathrm{crit}(i)$; but this is impossible: if $\mathrm{crit}(U) < \mathrm{crit}(i)$ then it would follow that $0^\dagger$ exists, contrary to the the hypothesis, while if $\mathrm{crit}(U) = \mathrm{crit}(i)$ then $\mathrm{crit}(U)$ would be definable in K_κ as the only measurable cardinal, and hence would be in X.

To see that the ultrapower on $\overline{K}$ is trivial, first note that the extra assumption (1.12) on X implies that any full measure in $\overline{K}$ is contained in K_η. Thus the iterated ultrapower on $\overline{K}$ must be trivial unless it drops. By lemma 3.38 this would imply that the iterated ultrapower on K does not drop, and its final model M_θ is an initial segment of the final model above $\overline{K}$; but this is absurd, since $\overline{K}$ is a set and M_θ is a proper class.

To verify clause (3), note that if $\overline{K}$ is not an initial segment of K then the iterated ultrapower on K is nontrivial; however, again using (1.12), any full ultrafilter in K with critical point smaller than η would also be in $\overline{K}$, and hence would not be used in the iteration. Thus the iterated ultrapower on K must drop immediately.

Clause (4) follows from clause (2), so it only remains to check clause (5), stating that $\widetilde{M} \in K$. If $\widetilde{M}$ has no full measure $U = \mathcal{U}_\gamma^{\widetilde{M}}$ with $\text{crit}(U) < \kappa$, then $\widetilde{M}$ is iterable because of the suitability of X. Now $\widetilde{M}$ is sound above κ, because of its construction as an ultrapower $\text{Ult}(M, \pi, \kappa)$. On the other hand, the projectum ρ of $\widetilde{M}$ cannot be smaller that κ, as otherwise $\widetilde{M}$ would be an iterated ultrapower of a mouse M' of size at most ρ, but this is impossible since then $M' \in K_\kappa \subseteq M$. It follows that $\widetilde{M}$ is sound, and hence is a member of K.

Now it will be sufficient to show that there is no full measure U in $\widetilde{M}$ with $\mu = \text{crit}(U) < \kappa$. First, we observe that any such measure U would have to satisfy $\kappa = \mu^{+K}$: otherwise $\kappa > \mu^{+K}$, so $U = \mathcal{U}_\gamma^{\widetilde{M}}$ for some $\gamma < \kappa$. Then $\tilde{\pi}^{-1}(U) = \mathcal{U}^M_{\pi^{-1}(\gamma)}$, with $\pi^{-1}(\gamma) < \bar{\kappa}$. By the construction it follows that $\tilde{\pi}^{-1}(U) \in \overline{K}$, so $U \in K$, contradicting the special assumption (1.12).

Hence the following lemma will prove that there is no full measure U in $\widetilde{M}$ with $\mu = \text{crit}(U) < \kappa$, and hence will complete the proof of lemma 3.51:

3.52 Lemma. *κ is not a successor cardinal in K.*

Proof. We will first assume that there is no measure in $\widetilde{M}$ with critical point below κ. We will show that if κ is a successor in K then there is $\eta < \kappa$ such that $X = \tilde{h}``(X \cap \eta)$, which implies that κ is singular in K and hence is not a successor. To do so we will need to consider the indiscernibles generated by the iteration i.

If $M \neq M_\theta$ then M is a proper initial segment of the potential premouse M_θ. It follows that M is sound and is hence a mouse. In this case the proof proceeds exactly as in that of the covering lemma for L, and leads to the conclusion that $X = \tilde{h}``(X \cap \rho)$, where $\tilde{h}$ comes from lemma 3.10 and $\rho = \pi(\rho^M_{m+1})$ where m is least such that $\rho^M_{m+1} < \bar{\kappa}$.

Thus we can assume that $M = M_\theta$. Then lemma 3.51(3) states that $1 \in D$, so $D \neq \varnothing$. Let $\nu_0 + 1 < \theta$ be the largest member of D. Then M_θ is an iterated ultrapower (without drops) of the potential premouse $M^*_{\nu_0+1}$, which is an initial segment of M_{ν_0}. All of the remaining ultrapowers have the same degree n, and $M^*_{\nu_0+1}$ is n-sound. Let $\overline{C} = \{\, i_{\nu_0,\nu}(\kappa_{\nu_0}) : \nu_0 < \nu < \theta \,\}$, where $i_{\nu_0,\nu} \colon M^*_{\nu_0+1} \to M_\nu$. Then $\overline{C}$ is a sequence of indiscernibles for M_θ.

Let $\bar\rho$ be the Σ_n-projectum of M, which is equal to the Σ_n-projectum of $M^*_{\nu_0+1}$, and let $\bar h$ be the Σ_n-Skolem function of M_θ. Then

$$M_\theta = \bar h``(\bar\rho \cup \overline{C}) \tag{1.14}$$

by the soundness of $M^*_{\nu_0+1}$ and lemma 3.26. Now let $\rho = \sup(\pi``\bar\rho)$ and $C = \pi``\overline{C}$. If $\tilde h = \tilde h^X$ is the function given by lemma 3.10, then it follows that $X = K_\kappa \cap \tilde\pi``M_\theta = K_\kappa \cap \tilde\pi h``(\bar\rho \cup \overline{C}) = \tilde h\pi``(\bar\rho \cup \overline{C}) \subseteq \tilde h``(\rho \cup C)$.

Now $\overline{C}$ cannot be unbounded in $\bar\kappa$: $\bar\kappa$ is not a limit cardinal in M since κ is not a limit cardinal in K, but each each member of $\overline{C}$ is a cardinal in M. Thus $X \subseteq \tilde h``\eta$ where $\eta = \sup(\rho \cup C) < \kappa$, as claimed.

This completes the proof in the case that there is no measure $U \in \widetilde{M}$ with $\operatorname{crit}(U) < \kappa$. If there is such a measure then, as was pointed out at the end of the proof of lemma 3.51, $\kappa = \mu^{+K}$ where $\mu = \operatorname{crit}(U)$. In this case set $M' = \operatorname{Ult}_n(M, \overline{U})$ where $\overline{U} = \tilde\pi^{-1}(U)$. The same argument as above shows that $\widetilde{M}' = \operatorname{Ult}(M', \pi, \kappa) \in K$. In this case M' and $\widetilde{M}'$ should be used in place of M and $\widetilde{M}$ in proof above. Note that M' is the result of carrying out one more step in the iteration i of which M_θ is the last model. $\dashv$

This completes the proof of lemma 3.52, and hence of lemma 3.51, and we can now finish the proof of clause 1 of lemma 3.48:

3.53 Corollary. *If $0^\dagger$ does not exist then K satisfies the countably closed weak covering property.*

Sketch of Proof. Suppose to the contrary that λ is a countably closed singular cardinal, and that $\kappa = \lambda^{+K} < \lambda^+$. Then $\operatorname{cf}(\kappa) \leq \lambda$, and since λ is singular it follows that $\operatorname{cf}(\kappa) < \lambda$. Since λ is countably closed it follows that $\operatorname{cf}(\kappa)^\omega < \lambda$, so there is a set $Y \prec H(\kappa^+)$ with $Y^\omega \subseteq Y$ and $|Y| = \operatorname{cf}(\lambda)^\omega < \kappa$ such that $Y \cap \kappa$ is cofinal in κ. Thus $X = Y \cap K_\kappa$ is countably closed, and hence is suitable, contradicting lemma 3.52. $\dashv$

Part 2 of the Proof: Analyzing the Indiscernibles

We have now constructed all of the elements of diagram 1.13 and we have proved the countably closed weak covering lemma. In order to complete the proof of lemma 3.48(2), and hence of theorems 1.12 and 1.13, the strong covering lemma below $0^\dagger$, we need to study in more detail the indiscernibles C introduced in the proof of lemma 3.52. The use of indiscernibles from an iterated ultrapower as a Prikry sequence is discussed in section 2.2 of chapter [31].

Fix, for the moment, an arbitrary suitable set X. We need to find $f \in K$ and $\eta < \kappa$ such that either $X = f``(\eta \cap X)$ or else $C$ is a Prikry sequence and $X = f``(C \cup (\eta \cap X))$. Furthermore, we want to show that the Prikry sequence C, if it is exists, is unique modulo finite differences.

Equation (1.14) states that $M = h``(\rho \cup \overline{C})$. This statement can be strengthened:

$$\forall \xi \in \bar{\kappa} - \overline{C} \;\; \xi \in \bar{h}``(\bar{\rho} \cup (\overline{C} \cap \xi)). \tag{1.15}$$

Now let $\rho = \sup(\pi``\bar{\rho})$ and $C = \pi``\overline{C}$. If $\tilde{h} = \tilde{h}^X$ is the function given by lemma 3.10, then it follows that $X = \tilde{h}``((X \cap \rho) \cup C)$, and if $\xi \in X \cap \kappa$ then $\xi \in \tilde{h}``((X \cap \rho) \cup (C \cap \xi + 1))$.

If $\overline{C}$ is finite then we can define $f(x) = \tilde{h}^X(x, C)$, so that $f \in K$ and $X = f``(X \cap \rho^X)$. Thus the first of the desired alternatives hold.

For the remainder of the proof, we will assume that C is infinite. We use a superscript X to designate the results of applying this construction to the arbitrary suitable set X.

3.54 Definition. Let $\mathbf{C}$ be the class of suitable sets X such that either C^X is finite or else $K = L[U]$, the set C^X is a Prikry sequence for U, and C^X is maximal in the sense that $C - C^X$ is finite whenever C is any other Prikry sequence for $L[U]$.

Notice that C^X and $C^{X'}$ differ only finitely for any two sets $X, X' \in \mathbf{C}$ such that C^X and $C^{X'}$ are both infinite.

The following lemma will complete the proof of the Dodd-Jensen covering lemma:

3.55 Lemma. *If* $0^\dagger$ *does not exist then the class* $\mathbf{C}$ *is closed under increasing unions of uncountable cofinality, and is unbounded in* $[K_\kappa]^\delta$ *whenever* κ *is a cardinal of* K *and* δ *is an uncountable regular cardinal.*

The proof of this lemma will take up the rest of subsection 3.3. We already know that the class of unsuitable sets is nonstationary, and by the comments above, we can assume that C^X is infinite for all but a nonstationary set of sets X.

First we show, assuming lemma 3.55 is true for all cardinals $\mu < \kappa$, that $\mathrm{ot}(C^X) = \omega$ on all but a nonstationary set. Assume the contrary; then there is $\mu < \kappa$ such that the ω^{th} member C^X of C^X is equal to μ for stationarily many sets X. Since the induction hypothesis states that the covering lemma holds for $C \cap K_\mu$, there is X with $\mu^X = \mu$ such that $X \cap K_\mu \in \mathbf{C}$, but this implies that $K = L[U]$ where U is a measure on μ. Now this measure U must be in X, and is generated by $C^X \cap \mu$. Thus the iterated ultrapower from diagram (1.13) which was used to generate $\overline{C}$ would not continue past $(\pi^X)^{-1}(\mu)$, and hence $\overline{C}^X \subseteq (\pi^X)^{-1}(\mu)$. This contradicts the assumption that $C^X \not\subseteq \mu$, and completes the proof that C^X has order type ω except on a nonstationary set.

The proof of lemma 3.55 is based on the following observation:

3.56 Proposition. *Suppose that X is suitable and $X = Y \cap K_\kappa$ where $Y \prec H(\kappa^+)$ and $C^X \in Y$. Then $X \in \mathbf{C}$.*

Proof. First we show that U is a measure in $L[U]$. Since the members of $\overline{C} = \pi^{-1}(C^X)$ come from the iteration of the unique full measure of M_θ, they generate the final measure $\overline{U}$ of the measure sequence of that model. Thus the filter U is the final measure in the measure sequence $\widetilde{\mathcal{U}}$ of $\widetilde{M} = J_{\tilde{\alpha}}[\widetilde{\mathcal{U}}]$. It follows that $\widetilde{\mathcal{U}} \in Y$ since $U \in Y$. If γ is the least ordinal such that U is not a measure in $J_\gamma[\widetilde{\mathcal{U}}]$, then $\gamma \in Y$ and hence $(\pi^N)^{-1}(J_\gamma[\widetilde{\mathcal{U}}]) = J_{\bar{\gamma}}[\mathcal{U}^M]$ is in the transitive collapse N of Y. Evidently $\bar{\gamma} \geq \mathrm{On}(M)$ since $\overline{U}$ is a measure in M, but this is impossible since there is a bounded subset of $\bar{\kappa}$ which is definable in M but is not in $\overline{K} = H(\bar{\kappa})^N$.

Thus C^X is a Prikry sequence for the measure U, and if C^X fails to satisfy definition 3.54 then it is because there is another Prikry sequence C' such that $C' - C$ is infinite. Then by elementarity there is such a sequence C' which is a member of Y. Then $C' \subseteq X$, so any member α of $C' - C^X$ is in $\tilde{h}^X``(\tilde{\rho}^X \cup (C \cap \alpha)) \subseteq h^X``\alpha$, and since $\tilde{h}^X \in K = L[U]$ it follows that $C' - C^X$ is finite since C' is a Prikry sequence. $\dashv$

3.57 Corollary. *If X is countably closed then $X \in \mathbf{C}$.* $\dashv$

Now we will deal with the proof of lemma 3.55 for non-countably closed sets X. This proof uses the ideas of the proof of lemma 3.50, stating that the class of suitable sets is unbounded, but is significantly more difficult. We use the notation $a \subseteq^* b$ to mean that $a - b$ is finite, and $a =^* b$ to mean that $a \subseteq^* b$ and $b \subseteq^* a$.

3.58 Lemma. *If X_0, X_1 are suitable and $X_0 \subseteq X_1$, then $C^{X_1} \cap X_0 \subseteq^* C^{X_0}$.*

Proof. We will use subscripts 0 and 1 to distinguish objects defined from X_0 or X_1; for example we write $\pi_0 = \pi^{X_0}$ and $\pi_1 = \pi^{X_1}$. We will find a function h^*, definable in $M_1 = M^{X_1}$, such that $\xi \in h^*``\xi$ for all but boundedly many $\xi \in \pi_1^{-1}((C_1 \cap X_0) - C_0)$. Since this can only hold for finitely many $\xi \in \pi_1^{-1}(C_1)$, this will imply that $C_1 \cap X_0 \subseteq^* C_0$.

To this end, let ν be any member of $X_0 \cap (C_1 - C_0)$ and set $\nu_0 = \pi_0^{-1}(\nu)$. Then $\nu_0 \notin \overline{C}_0$, so $\nu_0 \in h_0``\nu_0$ where h_0 is the Skolem function of $M_0 = M^{X_0}$. Now let $\tau = \pi_1^{-1} \circ \pi_0 \colon \overline{K}_0 \to \overline{K}_1$, and let

$$\tilde{\tau}\colon M_0 \to M^* = \mathrm{Ult}(M_0, \tau, \bar{\kappa}_1).$$

Then $\nu_1 = \tau(\nu_0) \in h^*``\nu_1$ where h^* is given by lemma 3.10. But M^* is sound above $\bar{\kappa}_1$, and agrees with $\bar{K}_1$ up to $\bar{\kappa}_1$, so by lemma 3.39 one of M^* and M_1 is an initial segment of the other. Since every bounded subset of $\bar{\kappa}$ in M^* is a member of $\bar{K}_1$, it must be that M^* is an initial segment of

M_1 and it follows that h^* is definable in M_1 from some parameter q. Since $\nu \in X_0 \cap (C_1 - C_0)$ was arbitrary, it follows that $X_0 \cap (C_1 - C_0)$ is finite, that is, $C_1 \cap X_0 \subseteq^* C_0$. ⊣

3.59 Corollary. *The class* $\mathbf{C}$ *is uncountably upward closed.*

Proof. Suppose that $X = \bigcup_{\xi<\eta} X_\xi$ is an increasing union of sets $X_\xi \in \mathbf{C}$ such that $\mathrm{cf}(\eta) > \omega$. Then X is suitable since the class of suitable sets is closed under uncountable increasing unions, and $C^X \subseteq X_\xi$ for some $\xi < \eta$ so $C^X \subseteq^* C^{X_\xi}$ by lemma 3.58. In particular, the fact that C^{X_ξ} is a Prikry sequence for the measure U implies that C^X is a Prikry sequence for the same measure.

To complete the proof that $X \in \mathbf{C}$ we need to show that C^X is maximal, and since C^{X_ξ} is maximal it is sufficient to show that $C^{X_\xi} \subseteq^* C^X$. Now if ν is any member of $C^{X_\xi} - C^X$, then $\nu \in \tilde{h}``\nu$ where $\tilde{h}$ is the function given by lemma 3.10. But since $C^{X_\xi}$ is a Prikry sequence over $K$ and $\tilde{h} \in K$, it follows that $C^{X_\xi} - C^X \subseteq \{\, \nu \in C^{X_\xi} : \nu \in \tilde{h}``\nu \,\}$ is finite, so $C^{X_\xi} \subseteq^* C^X$. ⊣

The following lemma completes the proof of the Dodd-Jensen covering lemma. We give a proof which is somewhat different from that given by Dodd and Jensen in [11], as that proof does not easily adapt to larger core models.

3.60 Lemma. *If* δ *is a regular cardinal and* κ *is a cardinal of* K *then* $\mathbf{C}$ *is unbounded in* $[K_\kappa]^\delta$.

Proof. As in the proof of the covering lemma for L, we work in the space $\mathrm{Col}(\delta, K_\kappa)$. If $\sigma \in \mathrm{Col}(\delta, K_\kappa)$ then we will sometimes identify σ with $\mathrm{ran}(\sigma)$, especially when it appears as a superscript. Let S be the set of functions $\sigma \in \mathrm{Col}(\delta, K_\kappa)$ such that $\mathrm{cf}(\mathrm{dom}(\sigma)) > \omega$ and $\mathrm{ran}(\sigma)$ is suitable, but $\mathrm{ran}(\sigma) \notin \mathbf{C}$, and suppose for the sake of contradiction that S is stationary in $\mathrm{Col}(\delta, K_\kappa)$. By lemma 3.23 there is $\sigma_0 \in S$ and a stationary set $S_0 \subseteq S$ such that $\sigma \supseteq \sigma_0$ and $C^\sigma \subseteq \mathrm{ran}(\sigma_0)$ for all $\sigma \in S_0$. Thus $C^\sigma \subseteq^* C^{\sigma_0}$ for all $\sigma \in S_0$ by lemma 3.58. Set $C_0 = C^{\sigma_0}$.

As in the proof of the covering lemma for L, we define, for each member of S_0, a witness $w(\sigma)$ to the fact that $\mathrm{ran}(\sigma) \notin \mathbf{C}$:

3.61 Claim. *There is a function* w *mapping each member* σ *of* S_0 *to a countable subset of* $\mathrm{ran}(\sigma)$ *such that for any* $\sigma_1, \sigma_2 \in S_0$ *such that* $\sigma_1 \subseteq \sigma_2$ *and* $w(\sigma_2) \subseteq \mathrm{ran}(\sigma_1)$ *we have* $C^{\sigma_1} \subseteq^* C^{\sigma_2}$.

First we show that the lemma follows from this claim. By applying lemma 3.23 a second time, we can find $\sigma_1 \in S_0$ and a stationary set $S_1 \subseteq S_0$ so that $\sigma_1 \subseteq \sigma$ and $w(\sigma) \subseteq \mathrm{ran}(\sigma_1)$ for all $\sigma \in S_1$. If σ is any member of S_1 then $C^{\sigma_1} \subseteq^* C^\sigma$ by claim 3.61 and $C^\sigma \subseteq^* C^{\sigma_1}$ by lemma 3.58. Thus $C^\sigma =^* C^{\sigma_1}$ for all $\sigma \in S_1$.

Since S_1 is stationary, there is $\sigma \in S_1$ such that $\operatorname{ran}(\sigma) = Y \cap K_\kappa$ for some $Y \prec H(\kappa^+)$ with $C^{\sigma_1} \in Y$. Since $C^\sigma =^* C^{\sigma_1}$ it follows that $C^\sigma \in Y$. This implies $\operatorname{ran}(\sigma) \in \mathbf{C}$ by lemma 3.56, contradicting the fact that $\sigma \in S_1 \subseteq S_0$. This contradiction shows that S_0 is not stationary, and hence $\mathbf{C}$ is unbounded.

Proof of claim 3.61. We will fix $\sigma \in S_1$ for the moment in order to define $w(\sigma)$. The critical point is that $C^\sigma \subseteq^* C^{\sigma_0}$, so that C^σ is determined, up to a finite set, by $D = C^{\sigma_0} - C^\sigma$. If D is finite then we can set $w(\sigma) = \varnothing$, so we will assume that D is infinite. Let $\langle\, d_k : k < \omega \,\rangle$ enumerate D in increasing order, and set $\bar{d}_k = \pi^{-1}(d_k)$. Then $\bar{d}_k \in h^M``d_k$, where h^M is the Skolem function for the premouse M of diagram (1.13).

To define the function w we modify the definition 3.18 of a witness to the unsuitability of X by replacing clause 3 with the statement that there is a function h which is Σ_n definable over $\operatorname{dir\,lim}(w)$ such that $d \in h``d$ for all $d \in D$.

To see that this witness function $w(\sigma)$ satisfies claim 3.61, let $\sigma_1 \subseteq \sigma$ be a member of S_0 with $w(\sigma) \subseteq \operatorname{ran}(\sigma_1)$. Write π_1 for π^{σ_1}, and set $\tau = \pi_1^{-1}\pi \colon \overline{K}^{\sigma_1} \to \overline{K}^{\sigma}$. If $\bar{\mathfrak{m}} = \operatorname{dir\,lim}\big(\pi_1^{-1}(w(\sigma))\big)$ then the map τ extends to an elementary embedding $\tilde{\tau} \colon \bar{\mathfrak{m}} \to \mathfrak{m}$, so $\bar{\mathfrak{m}}$ is also a premouse. The measure on $\bar{\kappa}_1$ in $\bar{\mathfrak{m}}$ is generated by the indiscernibles $\tau^{-1}(\overline{C}^{\sigma}) = \pi_1^{-1}(C^\sigma)$, and since $C^\sigma \subseteq^* C^{\sigma_1}$ it follows that this measure is equal to the measure in M^{σ_1}. Thus $\bar{\mathfrak{m}}$ strongly agrees with M^{σ_1} up to $\bar{\kappa}_1$. Since both premice are sound above $\bar{\kappa}_1$ it follows that one is an initial segment of the other. Now if M^{σ_1} were a proper initial segment of $\bar{\mathfrak{m}}$, then there would be a bounded subset x of $\bar{\kappa}_1$ in $\bar{\mathfrak{m}} - \overline{K}^{\sigma_1}$. This is not the case, since every bounded subset of $\bar{\kappa}_1$ in $\bar{\mathfrak{m}}$ is in some $\bar{\mathfrak{m}}_k \in \overline{K}^{\sigma_1}$, so $\bar{\mathfrak{m}}$ must be an initial segment of M^{σ_1}. Hence, the Skolem function of $\bar{\mathfrak{m}}$ is definable in M^{σ_1}, and thus every sufficiently large member d of $\pi_1^{-1}(D)$ is in $h^{M^{\sigma_1}}``d$. It follows that $D \cap C^{\sigma_1}$ is finite, which is to say that $C^{\sigma_1} \subseteq^* C^\sigma$, as was to be proved. $\dashv$

This completes the proof of lemma 3.60, and hence of the Dodd-Jensen covering lemma, theorems 1.2 and 1.3. $\dashv$

3.4. Unsuitable Covering Sets

As we have seen, the proof of the covering lemma for L shows, assuming $\neg 0^\sharp$, that every suitable set is in L. This striking fact suggests that the proof may also have something to say about sets X which are not suitable. Some restrictions on X are certainly needed: for example, if X is a Cohen generic subset of some uncountable regular cardinal τ then any unbounded $y \subseteq \tau$ in L intersects both X and its complement $\tau - X$. Thus we will retain the first order part of the definition of suitability: we assume that

$X \prec_1 J_\kappa$ (or $X \prec_1 K_\kappa$ in the case of larger core models) but omit the second-order condition that $\widetilde{M}^X$ is well-founded and (in the case of K) that C^X is a maximal Prikry sequence. This one idea leads to two separate results: For L, or more generally if there is no ω_1-Erdős cardinal in K, it gives Magidor's covering lemma, 1.15, while in the presence of larger cardinals it gives theorem 1.16 stating that Jónsson and Ramsey cardinals relativize to K.

We recall the statement of Magidor's covering lemma 1.15 for L. This statement follows [30] in using the hypothesis that X is primitive recursively closed instead of $X \prec_1 J_\kappa$, but we do not verify that this weaker condition is sufficient.

3.62 Theorem. *Suppose that $0^\sharp$ does not exist and that X is a primitive recursively closed subset of J_κ, and let $\delta = \inf(\kappa - X)$. Then there are functions $h_i \in L$ for $i < \omega$ such that $X \cap \kappa = \bigcup_{i<\omega} h_i$ "δ.*

Sketch of Proof. Like the covering lemma, this theorem is proved by induction on κ. Suppose that $X \prec_1 J_\kappa$. If $\mathrm{cf}(\kappa) = \omega$, say $\kappa = \bigcup_{n<\omega} \kappa_n$, then $X = \bigcup_{n<\omega}(J_{\kappa_n} \cap X)$, so the truth of the theorem for X follows from its truth for each of the sets $X \cap J_{\kappa_n}$ for $n < \omega$. Thus we can assume that $\mathrm{cf}(\kappa) > \omega$. In addition we can assume that κ is a cardinal in L, that X is cofinal in κ, and that $\kappa \not\subseteq X$. Note that we do not assume that $|X| < \kappa$.

The proof begins exactly like that of the covering lemma, with the transitive collapse $\pi \colon N = J_{\bar\kappa} \cong X \subseteq J_\kappa$. Thus $\delta = \mathrm{crit}(\pi) < \kappa$.

If X is suitable then $X \in L$ by the proof of the covering lemma, so we can assume that X is not suitable. We recall the construction, given in lemma 3.19, of a witness to the unsuitability of X. There is a triple (α, n, β), with $\beta < \kappa$, such that $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ is defined but not well-founded. Let (α, n, β) be the least such triple, in the lexicographic ordering, and pick $f_i \in J_\alpha$ and $a_i \in [\beta]^{<\omega}$ for $i < \omega$ so that $[a_{i+1}, f_{i+1}]_\pi \mathrel{\mathrm{E}} [a_i, f_i]_\pi$, where E is the membership relation in the ultrapower. Then $\beta = \sup(\bigcup_i a_i)$, and since $\mathrm{cf}(\kappa) > \omega$ it follows that $\beta < \kappa$. We will show that there are functions $h_i \in L$ such that $X = \bigcup_{i<\omega} h_i$"$(X \cap \beta)$. The truth of the theorem for X then follows by applying the induction hypothesis to the set $X \cap \beta$.

In order to simplify notation we will assume that $n = 0$ and that α is a limit ordinal. We make two observations:

1. We can choose $\langle f_i : i < \omega \rangle$ so that $J_\alpha = \mathcal{H}_{\Sigma_1}^{J_\alpha}(\beta \cup \{ f_i : i < \omega \})$. If this is not true for the original choice of functions f_i, then let $\mathcal{M}' = J_{\alpha'}$ be the transitive collapse of $\mathcal{H}_{\Sigma_1}^{J_{\alpha_i}}(\beta \cup \{ f_i : i < \omega \})$. Then $\alpha' \leq \alpha$ and $\mathrm{Ult}(J_{\alpha'}, \pi, \beta)$ is ill-founded, so $\alpha' = \alpha$ by the minimality of α. The original functions f_i may be moved in the collapse, but we can replace them by their images under the collapse.

2. If $\alpha_i < \alpha$ is the least ordinal such that $f_i \in J_{\alpha_i}$ and $\mathcal{M}_i$ is the transitive collapse of $\mathcal{H}_{\Sigma_1}^{J_{\alpha_i}}(\bar\kappa \cup \{ f_0, \ldots, f_i \})$, then $\widetilde{\mathcal{M}}_i = \mathrm{Ult}(\mathcal{M}_i, \pi, \kappa)$

is well-founded. To see this, note that $\mathrm{Ult}(J_{\alpha_i}, \pi, \kappa)$ is well-founded by the minimality of the triple (α, n, β), and $\widetilde{\mathcal{M}}_i$ is a substructure of $\mathrm{Ult}(J_{\alpha_i}, \pi, \kappa)$.

It follows that $\widetilde{\mathcal{M}}_i = \mathrm{Ult}(\mathcal{M}_i, \pi, \kappa) \in L$ for each $i < \omega$, and $X = \pi``\bar{\kappa} = J_\kappa \cap \bigcup_{i<\omega} h_i``(X \cap \beta)$ where $h_i \in L$ is the function given by lemma 3.10. This completes the proof of theorem 3.62. ⊣

This argument, applied to K^{DJ} in the absence of a model with a ω_1-Erdős cardinal, yields Magidor's generalization of theorem 1.15 to K^{DJ}, while applied to larger core models K it yields the absoluteness to K of Jónsson and Ramsey cardinals, theorem 1.16. This extension of the argument to K requires that the iterated ultrapower constructed in definition 3.30 be modified by adding a second type of drop: Suppose that $M_\nu = J_{\alpha_\nu}[\mathcal{U}_\nu]$ has been defined, and let $\bar{\beta}_\nu$ be the largest ordinal such that $\mathcal{U}_\nu \restriction \bar{\beta}_\nu = \overline{\mathcal{U}} \restriction \bar{\beta}_\nu$. The next model, $M_{\nu+1}$, is defined normally, following definition 3.30, except in the special case when $D \cap \nu = \varnothing$ and there is a triple (α, n, β) with $\beta \le \pi(\beta_\nu)$ such that $\mathrm{Ult}_n(J_\alpha[\mathcal{U}_\nu], \pi, \beta)$ is defined but not iterable. In this case put ν into D and set $M_{\nu+1} = J_\alpha[\mathcal{U}_\nu]$, where (α, n, β) is the least such triple. It is still true that if $X = Y \cap K_\kappa$ and N is the transitive collapse of Y then K^N is an initial segment of the final model M_θ of this iteration: either as in the original proof because M_θ defines a bounded subset of $\bar{\kappa}$ which is not in $H(\kappa)^{(N)} = \overline{K}$, or else because $\mathrm{Ult}_n(M_\theta, \pi, \kappa)$ is not iterable, while $\mathrm{Ult}(K^N, \pi, \kappa)$ can be embedded into $K_{\sup(Y)}$ which is iterable.

Let C^X be the set of putative indiscernibles generated by this proof, that is, the image under π of the critical points (after the last drop) of the iterated ultrapower. Then we get, as in the proof of theorem 3.62, a set of functions $h_k \in K$ for $k < \kappa$ so that

$$\begin{aligned}\widetilde{M} &= \bigcup\{\, \mathcal{H}^{h_k}(\rho \cup C^X) : k \in \omega \,\}\\ &= \bigcup\{\, h_k``(\rho \cup \vec{c}) : k < \omega \wedge \vec{c} \in [C^X]^{<\omega} \,\}\end{aligned}$$

where $\rho = \inf(\kappa - X)$.

If C^X is finite or countable then this gives X as a countable union of sets in K, so we can assume that C^X is uncountable. There is no reason to expect C^X to be a set of indiscernibles for K, but it *is* a set of indiscernibles for any structure in the range of $\tilde{\pi}$. This observation explains the importance of the following proposition:

3.63 Proposition. *Suppose that $X = Y \cap K_\kappa$ where $\kappa \in Y$, Y is cofinal in κ, and $Y \cap K_\lambda \prec_1 K_\lambda$ for some cardinal $\lambda > \kappa$. Then $\mathcal{P}(\kappa) \cap Y \subseteq \mathrm{ran}(\tilde{\pi}^X)$.*

Proof. Let $\pi^Y : N^Y \cong Y$ be the transitive collapse of Y, so that $N^Y \cap K_{\bar{\kappa}} = \overline{K}$ and $\pi^Y \restriction \overline{K} = \pi^X$. Fix any member z of $\mathcal{P}^K(\kappa) \cap Y$, let $\mathfrak{m} \in Y$ be the least mouse such that $z \in \mathfrak{m}$, and set $\bar{\mathfrak{m}} = (\pi^X)^{-1}(\mathfrak{m}) \in N^Y$. By lemma 3.39 one of $\bar{\mathfrak{m}}$ and M^X is an initial segment of the other. Every bounded subset of $\bar{\kappa}$ in $\bar{\mathfrak{m}}$ is a member of N^X since $X = Y \cap K_\kappa$, but there is a bounded subset of $\bar{\kappa}$ which is definable in M^X and not a member of N^X. It follows that M^X is not a proper initial segment of $\bar{\mathfrak{m}}$, so $\bar{\mathfrak{m}}$ must be an initial segment of M^X. It follows that $\bar{z} = (\pi^Y)^{-1}(z) \in M^X$, and hence $z = \tilde{\pi}^Y(\bar{z}) = \tilde{\pi}^X(\bar{z}) \in \operatorname{ran} \tilde{\pi}^X$. $\dashv$

3.64 Corollary. *Suppose that $X = Y \cap K_\kappa$ where $Y \prec H(\lambda)$ for some $\lambda > \kappa$, and that $\mathcal{A} \in Y \cap K$ is a structure with universe κ. Then C^X is a set of indiscernibles for $\mathcal{A}$.*

Furthermore, if $D \in K \cap Y$ is a closed and unbounded subset of κ then $C^X - D$ is bounded in $\sup(C^X)$. $\dashv$

We will use this proposition to show that any Jónsson cardinal is Ramsey in K. The argument that every δ-Jónsson cardinal is δ-Erdős in K is similar, as is Magidor's argument that theorem 1.15 holds for K^{DJ} unless there is a ω_1-Erdős cardinal in K^{DJ}.

Let $\mathcal{A}$ be any structure in K with universe κ. Since X is Jónsson there are sets Y and X as in the hypothesis of proposition 3.64 such that $|X| = \kappa$ but $\kappa \not\subseteq X$.

It follows from the construction of the set C^X that $\left|C^X\right| = \kappa$. To show that κ is Ramsey in K we will show that there is a $\rho < \kappa$ and a set $C \in K$ of indiscernibles for $\mathcal{A}$ such that $(C^X - \rho) \subseteq C$. To this end let U be the filter on κ generated by C^X, that is, $z \in U$ if and only if $C^X - z$ is bounded in κ. Let $\mathfrak{m}$ be the least mouse with projectum κ such that $\mathcal{A} \in \mathfrak{m}$. Then $\mathfrak{m} \in Y$, so U is a normal ultrafilter on $\mathfrak{m}$. Furthermore $\mathrm{Ult}(\mathfrak{m}, U)$ is iterable since U is countably complete, so $\mathrm{Ult}(\mathfrak{m}, U) \in K$ and hence $U \cap \mathfrak{m} \in K$. Let h be the Skolem function of $\mathfrak{m}$ and define C to be the set of $\nu < \kappa$ such that, for each $k < \omega$ and each set $z \in \kappa^{1+k} \cap h``\nu$,

$$
\begin{aligned}
z \in U &\iff \nu \in z && \text{if } k = 0 \\
z \in U^{1+k} &\iff \{\, \vec{\gamma} \in (\kappa - \nu)^k : \nu^\frown\vec{\gamma} \in z \,\} \in U^k && \text{if } k > 0.
\end{aligned}
$$

Then C is a set of indiscernibles for $\widetilde{M}^X$, and hence for $\mathcal{A}$. Furthermore $C \in K$, and $C^X - C$ is bounded in κ since $h \in Y$. Then C is the required set of indiscernibles in K for $\mathcal{A}$, and since $\mathcal{A}$ was arbitrary this completes the proof that every Jónsson cardinal is Ramsey in K.

4. Sequences of Measures

This section concerns the covering lemma in the presence of models containing large cardinals. Most of the section will concentrate on the core model for sequences of measures; the remainder will describe, with less detail, what is known about the covering lemma up to a strong cardinal and then for overlapping extenders in the Steel core model up to and beyond a Woodin cardinal. We begin with a general survey, which is followed by a precise statement of the covering lemma for sequences of measures and some indications as to its proof.

The two large cardinal properties which critically affect the statement of the covering lemma are measurable cardinals and Woodin cardinals. Measurable cardinals are critical because they provide, via Prikry forcing, the first counterexample to the full covering property. Woodin cardinals are critical because they provide, via stationary tower forcing, a counterexample to the weak covering property as described in section 4 of chapter [31].

The Covering Lemma and Sequences of Measures

The Dodd-Jensen covering lemma elegantly accommodates the covering lemma to models $L[U]$ with a single measure; indeed the hypotheses $\neg\exists 0^\dagger$ and $K = L[U]$ are as well understood as are the hypothesis $\neg\exists 0^\sharp$ of the Jensen covering lemma. The situation for larger core models is both more complicated and less elegant. We begin this section by describing some of these complications, in rough order of the size of the core model at which they first appear.

The first three observations are relevant even in models in which $o(\kappa) \leq 1$ for all κ, that is, when no cardinal has more than one measure. To simplify the notation for this case we use an increasing enumeration $\vec{\kappa} = \langle \kappa_\nu : \nu < \theta \rangle$ of the measurable cardinals in $K = L[\mathcal{U}]$, and write U_ν for the full measure on κ_ν. A system of indiscernibles for this model K is a sequence $\mathcal{C} = \langle C_\nu : \nu < \theta \rangle$, with $C_\nu \subseteq \kappa_\nu$. Each set C_ν is either finite or a Prikry sequence, but in addition the sequence $\mathcal{C}$ as a whole is uniformly a system of indiscernibles:

$$\forall \vec{x} \in K \Big((\forall \nu < \theta\, x_\nu \in U_\nu) \implies |\bigcup \{ C_\nu - x_\nu : \nu < \theta \}| < \omega \Big). \tag{1.16}$$

This leads to our first observation:

1. The sets C_ν need not be infinite: formula (1.16) is meaningful even if some or all of the sets C_ν are finite.

The only constraint on the function $f(\nu) = |C_\nu|$ when $o(\kappa_\nu) = 1$ for all ν is that $\operatorname{ot} C_\nu \leq \omega$. For any predetermined function f, there is a straightforward modification of Prikry forcing which can be used to obtain a sequence such that $|C_\nu| = f(\nu)$ for all $\nu \in \operatorname{dom}(\mathcal{U})$: the conditions are pairs $(a, \vec{A})$ such that $A_\nu \in U_\nu$, $a_\nu \subseteq \kappa_\nu$ and $|a_\nu| \leq f(\nu)$ for each $\nu < \theta$, and $\bigcup_\nu a_\nu$ is finite. The order is defined by $(a', \vec{A}') \leq (a, \vec{A})$ if $a'_\nu \supseteq a_\nu$, $A'_\nu \subseteq A_\nu$, and $a'_\nu - a_\nu \subseteq A_\nu$ for each $\nu < \theta$.

As a consequence the relation between $L[\mathcal{U}]$ and $L[\mathcal{U}, \mathcal{C}]$ is more complicated than that between $L[U]$ and $L[U, C]$:

2. The function $f(\nu) = |C_\nu|$ need not be a member of K.

As an example, suppose that $\theta \geq \omega$ and let $a \subseteq \omega$ be a real which is Cohen generic over K. Then each of the measures U_n can be extended to a measure in $K[a]$, so we can modify Prikry forcing as described above to obtain a system $\mathcal{C}$ of indiscernibles for $K[a]$ such that $|C_n| = 1$ (or, alternatively, $|C_n| = \omega$) for each $n \in a$ and $C_n = \varnothing$ for each $n \in \omega - a$. Thus $a \in K[\mathcal{C}]$. If $|C_n| = \omega$ for each $n \in \omega$ then a is definable in $K[\mathcal{C}]$ as $\{ n \in \omega : \mathrm{cf}(\kappa_n) = \omega \}$. If $|C_n| = 1$ for $n \in a$ then the covering lemma can be used to show that the system $\mathcal{C}$, and hence the set a, is definable in $K[\mathcal{C}]$ up to a finite set.

Note that a can be any set so long as the measures U_n can be extended to measures in $K[a]$.

In the Dodd-Jensen covering lemma for $L[U]$, the Prikry sequence C, if it exists, does not depend on the set x to be covered. This is not true for longer sequences:

3. If there is an inaccessible limit of measurable cardinals in K, then there is a cardinal preserving generic extension $K[G]$ of K in which each measure in K has a Prikry sequence, but there is no sequence $\mathcal{C} = \langle C_\nu : \nu < \kappa \rangle$ of Prikry sequences which satisfies (1.16) [35, theorem 1.3].

An inaccessible limit of measurable cardinals is needed to obtain such a sequence: it is shown in [39, thm 4.1] that if there is no model with an inaccessible limit of measurable cardinals then, as in the Dodd-Jensen covering lemma, there is a single sequence $\mathcal{C}$ which can be used to cover any set x.

Since the remaining observations only apply in the presence of cardinals κ with $o(\kappa) > 1$, we now revert to the notation for sequences of measures described in chapter [31] and in the last section: the core model K is a structure of the form $L[\mathcal{U}]$, where $\mathcal{U}$ is a sequence of filters such that each member $\mathcal{U}_\gamma$ of the sequence is a normal measure on $L[\mathcal{U}{\restriction}\gamma] \cap \mathcal{P}(\kappa)$, where $\gamma = \kappa^{++}$ in $L[\mathcal{U}{\restriction}\gamma]$. Not all of the filters $\mathcal{U}_\gamma$ are full measures in K, but we only need to consider those measures which are are full.

We frequently write $\mathcal{U}(\alpha, \beta)$ for the βth full measure on α in $L[\mathcal{U}]$; that is, $\mathcal{U}(\alpha, \beta) = \mathcal{U}_{\gamma_\beta}$ where $\langle \gamma_\nu : \nu < o(\alpha) \rangle$ is the increasing enumeration of the ordinals γ such that $\mathcal{U}_\gamma$ is a full measure on α in $L[\mathcal{U}]$. We write $o(\alpha)$, as above, for least ordinal β such that $\mathcal{U}(\alpha, \beta)$ is undefined. The sequence $\mathcal{U}$ has the following *coherence* property: if $i\colon K \to \mathrm{Ult}(K, \mathcal{U}(\alpha, \beta))$ then $o^{i(\mathcal{U})}(\alpha) = \beta$ and $i(\mathcal{U})(\alpha, \beta') = \mathcal{U}(\alpha, \beta')$ for all $\beta' < \beta$.

For the next three observations we assume that the core model does not contain any extenders, so that K always satisfies $o(\alpha) \leq \alpha^{++}$.

Corresponding to a sequence $\mathcal{U}$ of measures we will use $\mathcal{C}$ to denote a system of indiscernibles: if $\gamma \in \mathrm{dom}(\mathcal{C})$ then $\mathcal{C}_\gamma \subseteq \mathrm{crit}(\mathcal{U}_\gamma)$ is a set of indiscernibles for the measure $\mathcal{U}_\gamma$ (or, in the other notation, $\mathcal{C}(\alpha,\beta) \subseteq \alpha$ is a set of indiscernibles for $\mathcal{U}(\alpha,\beta)$). The precise definition of a system of indiscernibles will be given later, in definitions 4.15 through 4.18 and in the covering lemma, theorem 4.19.

4. The sets $\mathcal{C}(\kappa,\beta)$ may have order type greater than ω. In general, the set $\bigcup_{\beta<o(\kappa)}\mathcal{C}(\kappa,\beta)$ of indiscernibles for measures on a cardinal κ is a closed subset of κ which may have any order type up to $\min\{\kappa,\omega^{o(\kappa)}\}$.

In [29] Magidor generalizes Prikry forcing in order to add such a sequence of indiscernibles and hence change the cofinality of a cardinal κ to any smaller regular cardinal λ, provided that $o(\kappa) \geq \lambda$ in the ground model. This forcing is discussed briefly in chapter [31] and extensively in chapter [15].

For longer sequences of measures, and in particular when $o(\kappa) > \kappa$, it is important in applications that the domain of the sequence $\mathcal{C}^X$ is contained in the covering set X. For this reason, we assume a slightly different context for the covering lemma for sequences than was used for the Dodd-Jensen covering lemma. Let $\kappa = \sup(x)$ be a cardinal of K, where x is the set which we are trying to cover. We will look for a covering set $X \supseteq x$ such that $X \prec_1 K_{\breve{\kappa}}$, where $\breve{\kappa}$ is the least cardinal of K such that $\breve{\kappa} \geq \max\{\kappa, o(\kappa)\}$.

This requirement that $\operatorname{dom}(\mathcal{C}^X) \subseteq X$ leads to two somewhat technical problems in the study of longer sequences of measures:

5. It need not be that every suitable set X can be written as $X = h``(\rho;\mathcal{C}^X)$ where $\rho = \min(\kappa - X)$ and $h \in K$.

The notation $h``(\rho;\mathcal{C}^X)$ (which is defined in definition 4.17) corresponds to the notation $h``(\rho \cup C)$ used when $K = L[U]$, but takes account of the fact that $\mathcal{C}^X$ is a function rather than a set. Recall that the strong version of the Jensen covering lemma states that if $0^\sharp$ does not exist then every suitable set X is in L, and the Dodd-Jensen covering lemma for $L[U]$ states (assuming there is a Prikry sequence C) that any such set can be written as $h``(\rho \cup C)$ where $\rho = \min(\kappa - X)$.

The covering lemma for longer sequences states that $X = h``(\rho \cap X;\mathcal{C}^X)$ for some $\rho < \kappa$, however the induction used to show that ρ can be taken to be $\inf(\kappa - X)$ breaks down for sequences of measures: it depends on the fact that $X \cap K_\rho$ is suitable as a subset of K_ρ, but in the case of sequences of measures it may be that ρ is measurable, but is not a member of X. In this case $X' = X \cap K_{\breve{\rho}}$ is not suitable since $\operatorname{dom}(\mathcal{C}^{X'})$ is not contained in X'.

So far this limitation has not caused problems in applications, nor has the next difficulty:

6. If $o(\alpha) \geq \alpha^+$ then it is not known whether countable completeness of a set X is enough to ensure that X is a suitable covering set. What

is known is stated in theorem 4.19, which requires that the set X be $\operatorname{cf}(\kappa)$-closed. In particular it is not known whether there always exist suitable covering sets of size less than $\operatorname{cf}(\kappa)^+$.

The problem, again, comes from the requirement that $\operatorname{dom}(\mathcal{C}) \subseteq X$, but in this case it is the measures on κ which are in question. These measures are generated by cofinal subsets of $\bigcup_\beta \mathcal{C}(\kappa, \beta)$, so the assumption that X contains its subsets of size at most $\operatorname{cf}(\kappa)$ implies that these subsets, and hence the corresponding measures, are in X.

Extenders

If $o(\kappa) > \kappa^{++}$ then the core model is built using extenders, and we will write $K = L[\mathcal{E}]$ to denote the core model. Below $0^{\P}$, the sharp for a strong cardinal, the extenders do not overlap and the covering lemma as stated for sequences of measures remains true with two modifications. One of these is primarily notational, but the following situation is unexpected:

7. If $\operatorname{cf}(\kappa) = \omega$ and $\{\, \alpha < \kappa : o(\alpha) \geq \alpha^{+n} \,\}$ is unbounded in κ for all $n < \omega$, then the fact that a set X is countably closed does not ensure that X contains all of the extenders on κ which are generated by the system $\mathcal{C}^X$ of indiscernibles for X.

   If $\operatorname{cf}(\kappa) \geq \omega_1$ and ${}^{\operatorname{cf}(\kappa)}X \subseteq X$, however, in this case X does contain all such extenders. As a result the covering lemma up to $0^{\P}$ is similar to the result of substituting "ω_1" and "countable" for "ω" and "finite" in the covering lemma for sequences of measures.

Both parts of this observation are due to Gitik. In [18] he defines a game which can be used to reconstruct a extender E on a cardinal of uncountable cofinality from the sequences of ordinals which generate the constituent ultrafilters, and in [21] he constructs a model in which this is not possible for extenders on a cardinal of cofinality ω.

The set $0^{\P}$ marks the introduction of overlapping extenders, and thus of a dramatic shift in our understanding of the covering lemma:

8. If $0^{\P}$ exists then we cannot prove much more than the weak covering lemma and the absoluteness theorem for Jónsson and Ramsey cardinals (theorem 1.16).

The basic construction of the proof of the covering lemma does still go through for overlapping extenders, with considerably increased technical difficulties [43, 42], but it uses iteration trees rather than the linear iteration of definition 3.30. The indiscernibles generated by such iterations are very poorly understood, and the proofs for the known results above $0^{\P}$ rely on avoiding indiscernibles rather than on analyzing them.

9. No core model with cardinals very much larger than $0^{\P}$ is known to exist and satisfy the weak covering lemma without an additional assumption that there is an subtle cardinal in the universe. It is known that there is no model for a Woodin cardinal which satisfies the weak covering lemma in set generic extensions.

Even the weak covering lemma is false for any model with a Woodin cardinal δ, since nonstationary tower forcing can be used to collapse successors of many singular cardinals below δ. The situation between $0^{\P}$ and a Woodin cardinal is still under investigation.

Of course any statement such as these must rely on implicit assumptions about what it means to be a "core model". Section 4 of chapter [31] explores the assumptions lying behind the statement here.

The use of $0^{\P}$ as the dividing line is an oversimplification: it is possible to use tricks to push some of the results somewhat further. In fact Schindler [49] has constructed a core model under the assumption that there is no sharp for a model with a class of strong cardinals. More importantly, there are some suggestions that it is the presence of actual overlapping extenders in K which cause the difficulty, not partial extenders such as those which appear in the countable mouse $0^{\P}$. Schimmerling and Woodin have shown that in certain special cases the core model can be proved to have the full covering property, even though it contains inner models with several Woodin cardinals. See [47], where Schimmerling and Woodin show that this result is not limited to the Steel core models, but has consequences for the existence of core-like models.

This concludes our summary. The next subsection contains a more detailed discussion of the covering lemma for sequences of measures.

4.1. The Core Model for Sequences of Measures

See chapter [31] for a discussion of the inner models for sequences of measures, and section 3.3 of this chapter on the Dodd-Jensen core model for its discussion of the core model K in particular. Recall that $K = L[\mathcal{U}]$, where $\mathcal{U}$ is a coherent sequence with members $\mathcal{U}_\gamma$ which are $J_\gamma[\mathcal{U}\restriction\gamma]$-measures. We will define the sequence $\mathcal{U}$, and hence the core model, by recursion on γ. The main problem in designing this recursion is to ensure that the final model $L[\mathcal{U}]$ is iterable: when $\mathcal{U}\restriction\gamma$ has been defined, then the decision whether to set $\mathcal{U}_\gamma = U$ for some measure U must take into account the requirement that any iterated ultrapower of the as yet undefined model $L[\mathcal{U}]$ must be well-founded. This is accomplished by defining two core models: the first, the countably complete core model K^{c}, has the weak covering property and is iterable because its full measures are countably complete; the second, the true core model K, has the full covering property and is iterable because it is an elementary substructure of K^{c}.

4.1 Definition. Either the core model K, or the countably complete core model K^{c}, are defined as $L[\mathcal{U}]$ where the sequence $\mathcal{U}$ is defined by recursion on γ as follows. Assume that $\mathcal{U}{\restriction}\gamma$ has already been defined:

1. If there is a mouse $M = J_{\gamma'}[\mathcal{U}']$ such that $\mathcal{U}'{\restriction}\gamma = \mathcal{U}{\restriction}\gamma$, the projectum of M is smaller than γ, and no measure in $\mathcal{U}' - \mathcal{U}$ is full in M, then set $\mathcal{U}{\restriction}\gamma' = \mathcal{U}'$.

2. If there is no mouse as in clause 1, and if $J_\gamma[\mathcal{U}] \models \gamma = \kappa^{++}$ for some $\kappa < \gamma$ such that there is a $J_\gamma[\mathcal{U}]$-ultrafilter U on κ with $i^U(\mathcal{U}){\restriction}\gamma+1 = \mathcal{U}{\restriction}\gamma$, then set $\mathcal{U}_\gamma = U$, provided it satisfies an iterability condition depending on which model is being constructed:

   (a) For the model K^{c}, the ultrafilter U is added to the sequence only if U is countably complete and $\mathrm{cf}(\mathrm{crit}(U)) = \omega_1$.

   (b) For the true core model K, the ultrafilter U is added to the sequence only if $\mathrm{Ult}(L[\mathcal{W}], U)$ is well-founded for every iterable inner model $L[\mathcal{W}]$ such that $\mathcal{W}{\restriction}\gamma = \mathcal{U}$.

3. If neither of the previous clauses apply then $\mathcal{U}_\gamma = \varnothing$.

The construction in clause 1 apparently depends on the choice of the mouse M to be added; however if two mice $J_{\alpha_0}[\mathcal{W}_0]$ and $J_{\alpha_1}[\mathcal{W}_1]$ satisfy clause 1, then one of them is an initial segment of the other. Thus clause 1 could be equivalently restated by specifying that $\mathcal{U}{\restriction}\gamma$ is to be extended to the longest good sequence $\mathcal{U}' \supseteq \mathcal{U}{\restriction}\gamma$ such that $L_{\gamma'}[\mathcal{U}']$ is iterable, the largest cardinal in $L[\mathcal{U}']$ below $\sup(\mathrm{dom}(\mathcal{U}'))$ is smaller than γ, and no measures in $\mathcal{U}' - \mathcal{U}{\restriction}\gamma$ are full in $L[\mathcal{U}']$.

It can be shown that there is never more than one choice of the measure $\mathcal{U}_\gamma$ satisfying clause 2. One way of doing so is to pick a mouse M with projectum κ containing a set x on which two candidate measures U and U' differ, and compare $\mathrm{Ult}(M, U)$ and $\mathrm{Ult}(M, U')$. Another is by using a *bicephelus*, which is a structure $\mathcal{B} = (J_\gamma[\mathcal{U}], \mathcal{U}, U, U')$ which is like a mouse except that both of U and U' are used as the top measure $\mathcal{U}_\gamma$ of $\mathcal{U}$. As in the proof of [31, Theorem 3.22], an iterated ultrapower is used to compare $\mathcal{B}$ with itself and conclude that in fact U is equal to U'. The construction is simpler than that of [31, Theorem 3.22] since $\mathcal{B}$ is a perfectly normal mouse except for the doubled top measures U and U', which are used only as predicates, not in the construction of $J_\gamma[\mathcal{U}]$.

4.2 Lemma. *The model K^c is iterable.*

Proof. We present a proof which seems slightly oblique compared to the original proof, but which extends naturally to models with sequences of extenders. First we show that if $\sigma\colon M \to K^{\mathrm{c}}_\theta$ is an elementary embedding, where M is a countable transitive set and θ is a sufficiently large cardinal, and if U is a full measure in M, then σ can be extended to obtain an elementary embedding $\tilde{\sigma}\colon \operatorname{Ult}(M,U) \to K^{\mathrm{c}}_{\theta}$ such that $\sigma = \tilde{\sigma} i^U$. To define $\tilde{\sigma}$ let $A = \bigcap \sigma``U$. Then $A \neq \varnothing$ since $\sigma``U$ is a countable subset of the countably complete ultrafilter $\sigma(U)$, so choose $\lambda \in A$ and define $\tilde{\sigma}([f]_U) = \sigma(f)(\lambda)$. Then $\tilde{\sigma}$ is elementary because $\operatorname{Ult}(M,U) \models \varphi([f]_U)$ if and only if $M \models \{\,\nu : \varphi(f(\nu))\,\} \in U$, and by the choice of λ this holds if and only if $K^{\mathrm{c}}_{\theta} \models \varphi(f(\lambda))$.

Now suppose that K^{c} is not iterable. Then for sufficiently large θ there is a countable elementary substructure $X \prec V_\theta$ containing an iteration witnessing this failure. If $\sigma\colon M \cong X \cap K^{\mathrm{c}}_{\theta}$ is the transitive collapse then M is not iterable, and there is a countable iteration of M witnessing this failure. If this iteration does not contain any drops then the construction of the last paragraph can be repeated countably many times to obtain embeddings $\tilde{\sigma}_\nu\colon M_\nu \to K^{\mathrm{c}}_{\theta}$ of the models M_ν of this iteration into K^{c}_{θ}, but this is impossible because the final ill-founded model M_δ of the iteration is embedded by $\tilde{\sigma}_\delta$ into the well-founded set K^{c}_{θ}. If the iteration does contain a drop, with the first drop occurring at ν_0, then $\tilde{\sigma}_{\nu_0} \restriction M^*_{\nu_0+1}$ embeds $M^*_{\nu_0+1}$ into an iterable mouse $\widetilde{M} = \tilde{\sigma}_{\nu_0}(M^*_{\nu_0+1})$ of K^{c}_{θ}. The remainder of the iteration on $M = M_0$ can then be copied to obtain an ill-behaved iteration on $\widetilde{M}$, which contradicts the fact that $\widetilde{M}$ is iterable. ⊣

We will show that K is iterable by giving, in lemma 4.11, a characterization of K as the transitive collapse of an elementary substructure of K^{c}. This characterization, which depends on the weak covering lemma for K^{c}, begins with the following preliminary definitions generalizing the fact that the weak covering lemma for K^{DJ} implies that any elementary substructure $X \prec K^{\mathrm{DJ}}$ with $|X \cap \lambda^+| = \lambda^+$ is isomorphic to K^{DJ}.

4.3 Definition. An iterable premouse $M = L[\mathcal{U}]$ is said to be *universal* if, whenever M is compared with any other iterable premouse M', the iterated ultrapower on M' does not drop and the final model in that iteration is a (possibly proper) initial segment of the final model of the iteration on M.

Note that a universal premouse M must be a proper class, since if $M = J_\alpha[\mathcal{U}]$ is an iterable premouse which is a set then lemma 3.28 implies that M is the iterated ultrapower of a mouse $M' = J_{\alpha'}[\mathcal{U}']$. Thus $L[\mathcal{U}']$ is an iterable premouse, and M comes out shorter than $L[\mathcal{U}']$ when they are compared because the iteration on $L[\mathcal{U}']$ consists of an initial drop to the mouse M', followed by the iterated ultrapower of M' to M.

4.4 Proposition. *If M is a iterable class premouse and $\lambda^{+M} = \lambda^+$ for a stationary class of cardinals, then M is universal.*

Proof. If M is not universal then there is an iterable premouse M' and iterated ultrapowers $i\colon M \to P$ and $i'\colon M' \to P'$ such that P is a proper initial segment of P'. Thus the class of ordinals of P is the class Ω of actual ordinals, and since P' is longer, we have $\Omega \in P'$. The iteration i does not drop by lemma 3.36. We can assume that i' also does not drop, since if it did drop we could consider only the tail of the iteration starting after the last drop. If ν_0 and κ are chosen so that $\Omega = i'_{\nu_0,\Omega}(\kappa)$ then the class Γ of ordinals λ such that $i_{0,\lambda}``\lambda \subseteq \lambda$ and $\lambda = i'_{\nu_0,\lambda}(\kappa)$ is closed and unbounded. Fix $\lambda \in \Gamma$ such that $\lambda^{+^M} = \lambda^+$. Then $\lambda^{+^{P'}} = \lambda^{+^{M'_\lambda}} = i'_{\nu_0,\lambda}(\kappa^{+^{M'_{\nu_0}}}) < \lambda^+$. This implies that $\lambda^{+^P} = \lambda^{+^{M_\lambda}} < \lambda^+$. We will obtain a contradiction by showing that $\kappa^{+^{M_\lambda}} = \lambda^+$. If $i_{0,\lambda}(\lambda) = \lambda$ then then this follows immediately since $\lambda^{+^{M_\lambda}} = i_{0,\lambda}(\lambda^+) \geq \lambda^+$. Otherwise there is a unique $\nu < \lambda$ such that $i_{0,\nu}(\lambda) = \lambda$ and $i_{\nu+1,\lambda}(\lambda) = \lambda$, but $\operatorname{crit}(i_{\nu,\nu+1}) = \operatorname{cf}^{M_\nu}(\lambda)$. Then $\lambda^{+^{M_\lambda}} = \lambda^{+^{M_{\nu+1}}} = \lambda^{+^{M_\nu}} = \lambda^{+^M}$, with the first and last equalities being proved as in the case when λ is never moved. ⊣

We will prove the following lemma later, simultaneously with the full covering lemma:

4.5 Lemma (Weak Covering Lemma for K^c). *Suppose that there is no inner model of $\exists\kappa\, o(\kappa) = \kappa^{++}$ and that λ is a singular strong limit cardinal. Then $(\lambda^+)^{K^c} = \lambda^+$.*

4.6 Definition. A class Γ is *thick* if Γ is a proper class, there is some τ such that Γ contains its limit points of cofinality greater than τ, and $|\Gamma \cap \lambda^+| = \lambda^+$ for all sufficiently large singular strong limit cardinals $\lambda \in \Gamma$.

4.7 Proposition. *Any set sized intersection of thick classes is thick.* ⊣

The following observation explains the definition of a thick class, and also the decision to require that every measurable cardinal in K^c have cofinality ω_1.

4.8 Proposition. *Suppose that W is an iterable, class length premouse, τ is an ordinal, and Γ is a thick class such that for any singular cardinal $\lambda \in \Gamma$ with $\operatorname{cf}(\lambda) > \tau$ we have that $\operatorname{cf}^W(\lambda)$ is not measurable, and $\lambda^{+^W} = \lambda^+$.*

Then any (nondropping) iterated ultrapower $i\colon W \to N$ is continuous at points $\xi \in \Gamma$ of cofinality greater than τ. Hence, the class Γ' of ordinals $\xi \in \Gamma$ such that $i(\xi) = \xi$ is thick.

Proof. If $i(\xi) > \sup(i``\xi)$, then it must be that at some stage ν in the iteration, $W_{\nu+1} = \operatorname{Ult}(W_\nu, U_\nu)$ where U_ν is a measure in W_ν on a cardinal κ_ν with $\operatorname{cf}^{W_\nu}(i_\nu(\xi)) = \kappa_\nu$. Since $\kappa_\nu = i_\nu(\operatorname{cf}^W(\xi))$ it follows that $\operatorname{cf}^W(\xi)$ is measurable in W. By the assumption on W, it follows that ξ is not a member of Γ of cofinality greater than τ.

For the final sentence, consider the class X of cardinals ξ such that ξ is a limit point of Γ of cofinality at least τ, and $i``\xi \subseteq \xi$. Then X is a proper class, and Γ' contains all of the limit points of X of cofinality at least τ.

Finally, if $\lambda \in \Gamma'$ and $\lambda^{+W} = \lambda^+$ then $i(\lambda^+) = \lambda^+$, and an argument like that in the last paragraph shows that $|\{\, \xi \in \lambda^+ : i(\xi) = \xi \,\}| = \lambda^+$. Thus $|\Gamma' \cap \lambda^+| = \lambda^+$, so Γ' is thick. $\dashv$

4.9 Lemma. *Assume that K^c satisfies the weak covering lemma, that λ is a limit cardinal and Γ is thick. Then for each $x \in \mathcal{P}^{K^c}(\lambda)$ there is $y \in \mathcal{H}^{K^c}(\Gamma \cup \lambda)$, the Skolem hull of $\Gamma \cup \lambda$, such that $x = y \cap \lambda$.*

Proof. Let $\pi\colon N \cong \mathcal{H}^{K^c}(\Gamma \cup \lambda)$ be the transitive collapse. Then $\xi^{+N} \geq |\Gamma \cap \xi^+| = \xi^+$ for all singular limit cardinals ξ of sufficiently large cofinality, so N is universal by lemma 4.4. It follows that the comparison of N with K^c will result in iterated ultrapowers with no drops and a common final model P. Furthermore, the critical point of each of the associated embeddings $i\colon N \to P$ and $j\colon K^c \to P$ is at least λ, since N and K^c agree on all measures with critical point below λ. Thus $x \in \mathcal{P}^P(\lambda) = \mathcal{P}^N(\lambda) \subseteq \operatorname{dom}(\pi)$, and the set $y = \pi(x)$ satisfies the requirements. $\dashv$

4.10 Lemma. *If K^c satisfies lemma 4.5, Γ is thick and $\lambda > \omega_1$ is a strong limit cardinal of cofinality ω_1, then $\lambda \in \mathcal{H}^{K^c}(\lambda \cup \Gamma)$.*

Proof. Let $X = \mathcal{H}^{K^c}(\lambda \cup \Gamma)$, and suppose to the contrary that $\lambda \notin X$. Let $\pi\colon N \cong X$ be the transitive collapse, and let $U = \{\, x \subseteq \lambda : \lambda \in i(x) \,\}$. Then U is a K^c-ultrafilter, since $\mathcal{P}^{K^c}(\lambda) \subseteq N$. Furthermore U is countably complete. To see this, let A be any countable subset of U. Since $\operatorname{cf}(\lambda^{+K^c}) = \operatorname{cf}(\lambda^+) = \lambda^+ > \omega$ there is $B \in K^c$ of size κ such that $A \subseteq B$, and since $\operatorname{cf}(\lambda) > \omega$ it follows that there is $B' \in K^c$ of size less than κ such that $A \subseteq B' \subseteq B$. Then $\bigcap A \supseteq \bigcap (B' \cap U)$, but the latter is nonempty since $B' \cap U \in K^c$ and U is a K^c-ultrafilter.

Thus $U \in K^c$, which is impossible since $o(U) = o^{K^c}(\lambda)$. The contradiction shows that $\lambda \in X$. $\dashv$

4.11 Lemma. *If K^c satisfies lemma 4.5 then K is isomorphic to the class $X = \bigcap \{\, \mathcal{H}^{K^c}(\Gamma) : \Gamma$ is thick$\,\}$.*

Sketch of Proof. Let $\pi\colon \widetilde{K} \cong X$ be the transitive collapse of X. First we show that X is a proper class. Suppose to the contrary that X is a set and let $\kappa = \sup(X \cap \mathrm{On})$. Now use proposition 4.7 to define a descending sequence $\langle\, \Gamma_\nu : \nu < \omega_1 \,\rangle$ of thick classes such that $X = \mathcal{H}^{K^c}(\Gamma_0) \cap V_{\kappa+1}$, the sequence of ordinals $\kappa_\nu = \inf(\mathcal{H}^{K^c}(\Gamma_\nu) - \kappa)$ is strictly increasing, and $\lambda = \sup\{\, \kappa_\nu : \nu < \omega_1 \,\}$ is a strong limit cardinal. We will show that $\lambda \in X$, contradicting the choice of κ.

Suppose the contrary, and pick $\Gamma_{\omega_1} \subseteq \bigcap_{\nu<\omega_1} \Gamma_\nu$ so that $\lambda \notin \mathcal{H}^{K^c}(\Gamma_{\omega_1})$. By lemma 4.10 there is a parameter $a \in \Gamma_{\omega_1}$, an ordinal $\xi < \lambda$, and a formula φ such that λ is the unique η such that $K^c \models \varphi(a, \xi, \eta)$. Let $\tau \in K$ be the Skolem function for φ (with parameter a) so that $K^c \models \forall \iota\, (\exists \eta\, \varphi(a, \iota, \eta) \implies \varphi(a, \iota, \tau(\iota)))$.

Now, notice that if $\nu < \omega_1$ then $\tau``\kappa_\nu \cap \lambda \subseteq \kappa_\nu$: otherwise there is some $\nu' > \nu$ and $\xi' < \kappa_\nu$ so that $\kappa_\nu \leq \tau(\xi') < \kappa_{\nu'}$. Then the least such ξ' is definable from κ_ν, $\kappa_{\nu'}$ and a, so $\xi' \in \mathcal{H}^{K^c}(\Gamma_\nu) \cap \kappa_\nu = X \cap \kappa$, but this is impossible since in that case $\tau(\xi') \in \mathcal{H}^{K^c}(\Gamma_{\nu'})$.

Now let ξ_0 be the least ordinal ξ such that $\lambda = \tau(\xi)$, and fix $\nu_0 < \omega_1$ so $\xi_0 < \kappa_{\nu_0}$. Then $\lambda = \min(\tau``\kappa_{\nu_0} - \kappa_{\nu_0}) \in \mathcal{H}^{K^c}(\Gamma_{\nu_0})$. However, this implies that $\xi_0 \in \mathcal{H}^{K^c}(\Gamma_{\nu_0}) \cap \kappa_{\nu_0} = X \cap \kappa \subseteq \mathcal{H}^{K^c}(\Gamma_{\omega_1})$. Hence $\lambda \in \mathcal{H}^{K^c}(\Gamma_{\omega_1})$, contrary to the choice of Γ_{ω_1}.

Now we show that $\widetilde{K} = K$. Else, fix λ such that $\widetilde{K}_\lambda \neq K_\lambda$, and fix a thick class Γ small enough that if $\pi\colon W \cong \mathcal{H}^{K^c}(\Gamma)$ is the collapse map then $\pi``\widetilde{K}_\lambda = X \cap K^c_{\pi(\lambda)}$. Note that W, with the class $\pi^{-1}[\Gamma]$, satisfies the conditions of proposition 4.8. If we consider the least place at which K differs from $\widetilde{K}$, and hence from W, then there are two possibilities: $\widetilde{K}$ is missing a mouse which is in K, or $\widetilde{K}$ is missing a full measure which is in K. The first is impossible, since it would contradict the universality of W. Thus there must be a measure $U = \mathcal{U}_\gamma$ on some cardinal $\kappa < \lambda$ in K such that $U \notin W$, but K and W agree up to $\gamma = \kappa^{++W}$. Now consider the following diagram:

$$\begin{tikzcd} W \arrow[r, "i^U"] \arrow[dr, "j"'] & \mathrm{Ult}(W,U) \arrow[d, "k"] \\ & \overline{N} \end{tikzcd}$$

where $\overline{N}$ is the common final model of the iterated ultrapowers coming from the comparison of the universal models W and $\mathrm{Ult}(W,U)$, and j and k are the embeddings from these iterated ultrapowers. Let Γ' be the set of $\xi \in \pi^{-1}[\Gamma]$ such that $ki^U\pi^{-1}(x) = j\pi^{-1}(x)$. Then Γ' is thick, but $\pi(\kappa) \notin \mathcal{H}^W(\Gamma')$, contradicting the assumption that $\pi(\kappa) \in X$. $\dashv$

The trick used at the end of the last proof, using an approximation W which agrees with the relevant initial segment of K but which satisfies the hypothesis of proposition 4.8, is often necessary. The following theorem gives another example:

4.12 Theorem. *If K^c satisfies the weak covering lemma then any universal iterable premouse M is an iterated ultrapower of K.*

Proof. Suppose that M is a counterexample, and let ν be the first stage in the comparison with K at which the iterated ultrapower on M becomes nontrivial. Thus $M_\nu = M$ and $M_{\nu+1} = \mathrm{Ult}(M, U)$, and the ultrafilter $U = \mathcal{U}^{M_\nu}_\gamma$ is not in the νth model N_ν in the iterated ultrapower on K.

Fix η large enough that $i_\nu(\eta) > \gamma$, where $i_\nu\colon K = N_0 \to N_\nu$ is the embedding coming from the iterated ultrapower on K, and as in the last proof choose W satisfying the hypothesis of proposition 4.8 which agrees with K up to η, so that $W_\eta = K_\eta$. Thus the description in the last paragraph of the comparison between K and M applies equally to the comparison between W and M, and for the rest of the proof we use the latter.

Since both W and M are universal, the comparison of these two models gives iterated ultrapowers of length $\theta \leq \mathrm{On}$ with no drops and with a common final model P, as in the left half of the following diagram:

$$\begin{tikzcd}
W = N_0 \arrow[rr, "i_\nu"] & & N_\nu \arrow[r, "i^U"] \arrow[d, "i_{\nu,\theta}"'] \arrow[dr, "s"] & \mathrm{Ult}(N_\nu, U) \arrow[d, "t"] \\
M \arrow[r, equal, "j_\nu"] & M_\nu \arrow[r, "j_{\nu,\theta}"'] & P & Q
\end{tikzcd} \tag{1.17}$$

Since there are no drops, the models N_ν, P and M_ν all have the same subsets of κ, so U is an N_ν-ultrafilter. Furthermore, $\mathrm{Ult}(N_\nu, U)$ is well-founded: otherwise $\mathrm{Ult}(P, U)$ would be ill-founded, but the iterated ultrapower $j_{\nu,\theta}$ can be copied to an iterated ultrapower on $\mathrm{Ult}(M_\nu, U)$ with last model $\mathrm{Ult}(P, U)$, so the iterability of M_ν implies that $\mathrm{Ult}(P, U)$ is well-founded. Now compare N_ν and $\mathrm{Ult}(N_\nu, U)$. Again, both models are universal so this comparison gives embeddings s and t as in diagram (1.17) with the same final model Q.

Let $\Gamma = \{\, \xi > \eta : ti^U i_\nu(\xi) = si_\nu(\xi) \,\}$. Then Γ is thick, so theorem 4.11 implies that $K_\eta = W_\eta \subseteq \mathcal{H}^W(\Gamma)$. It follows that $i_\nu(K_\eta) \subseteq \mathcal{H}^{N_\nu}(\Gamma \cup \kappa)$, so $ti^U \restriction W_{i_\nu(\eta)} = s \restriction W_{i_\nu(\eta)}$. In particular, $s(\kappa) > \kappa$ since $ti^U(\kappa) > \kappa$, so the iteration s begins with an ultrapower by some measure $U' = \mathcal{U}_\gamma^{N_\nu} \in N_\nu$ with critical point κ. But then $U = U'$, since if x is any subset of κ in N_ν then $x \in U \iff \kappa \in ti^U(x) = s(x) \iff x \in U'$. This contradicts the assumption that $U \notin N_\nu$. ⊣

4.13 Corollary. *If K^c satisfies the weak covering lemma and U is a normal ultrafilter on K such that $\mathrm{Ult}(K, U)$ is well-founded, then there is some γ such that $U = \mathcal{U}_\gamma$, where $K = L[\mathcal{U}]$.*

Proof. Apply theorem 4.12 to $\mathrm{Ult}(K, U)$. ⊣

Note that the hypothesis that M is universal cannot be eliminated from theorem 4.12: The model N constructed in the proof of Jensen's theorem 3.43 provides a counterexample in which N is a set, and a similar argument, starting with an assumption somewhat weaker than two measurable cardinals, gives a counterexample in which N is a proper class. However, such situations can only occur below ω_2: If $\lambda \geq \omega_2$ then the covering lemma implies that $\mathrm{cf}(\lambda^{+K}) > \omega$, and it follows that any K-ultrafilter on λ is countably complete.

4.2. The Covering Lemma up to $o(\kappa) = \kappa^{++}$

We use the following setting for the covering lemma for sequences of measures: We take κ to be a cardinal of K which is singular in V, and we consider covering sets $X \prec_1 K_{\breve{\kappa}}$ where $\breve{\kappa}$ is a cardinal of K with $\breve{\kappa} \geq \max\{\kappa, o(\kappa)\}$. The covering lemma asserts that for suitable sets X there is a system $\mathcal{C}^X$ of indiscernibles, a function h in K, and a $\rho < \kappa$ such that $X = h``(X \cap \rho; \mathcal{C})$, the smallest set containing $X \cap \rho$ and closed under h and $\mathcal{C}$. We will call such a set X a *covering set*.

Some definitions are required before we can give a precise statement of the covering lemma. Here is a general definition for a system of indiscernibles:

4.14 Definition. 1. If U is a measure, then $\mathrm{crit}(U)$ is the cardinal κ such that U is a measure on κ.

2. If $\gamma < \gamma'$, with $\gamma' \in \mathrm{dom}(\mathcal{U})$, then $\mathrm{Coh}_{\gamma,\gamma'}$ is the least function f in the ordering of $L[\mathcal{U}]$ such that $\gamma = [f]_{\mathcal{U}_{\gamma'}}$ in $\mathrm{Ult}(L[\mathcal{U}], \mathcal{U}_{\gamma'})$.

4.15 Definition. If $\mathcal{U}$ is a sequence of measures then a *system of indiscernibles* for $M = L[\mathcal{U}]$ is a function $\mathcal{C}$ such that

1. $\mathrm{dom}(\mathcal{C}) \subseteq \mathrm{dom}(\mathcal{U})$, and $\mathcal{C}_\gamma \subset \mathrm{crit}(\mathcal{U}_\gamma)$ for all $\gamma \in \mathrm{dom}(\mathcal{C})$.
2. For any function $f \in M$ there is a finite set a of ordinals such that if $\gamma \in \mathrm{dom}(\mathcal{U})$ and $\lambda = \mathrm{crit}(\mathcal{U}_\gamma)$ then

$$\forall \nu \in (\mathcal{C}_\gamma - \sup(a \cap \lambda)) \forall x \in f``(\nu \times \{\lambda\}) \, (\nu \in x \iff x \cap \lambda \in \mathcal{U}_\gamma).$$

The indiscernible sequences rising from the covering lemma have some additional structure:

4.16 Definition. If $\mathcal{C}$ is a system of indiscernibles for M, then $\mathcal{C}$ is said to be *h-coherent* if $h \in M$ is a function and the following conditions hold:

1. For all $\nu \in \bigcup_\gamma \mathcal{C}_\gamma$ there is unique $\gamma \in h``\nu$ such that $\nu \in \mathcal{C}_\gamma$.
2. Suppose that $\nu \in \mathcal{C}_\gamma \cap \mathcal{C}_{\gamma'}$ and $\gamma \in h``\nu$. If $\gamma' \neq \gamma$ then $\mathrm{crit}(\mathcal{U}_{\gamma'}) < \mathrm{crit}(\mathcal{U}_\gamma)$, and $\mathrm{crit}(\mathcal{U}_{\gamma'}) \in \mathcal{C}_{\gamma''}$ for some $\gamma'' < \gamma$ with $\mathrm{crit}(\mathcal{U}_{\gamma''}) = \mathrm{crit}(\mathcal{U}_\gamma)$.
3. Suppose $\gamma_\nu = \mathrm{Coh}_{\gamma',\gamma}(\nu)$, where $\gamma' < \gamma$ with $\mathrm{crit}(\mathcal{U}_{\gamma'}) = \mathrm{crit}(\mathcal{U}_\gamma)$; and suppose that $\nu \in \mathcal{C}_\gamma$ with $\gamma' \in h``\nu$. Then $\mathcal{C}_{\gamma_\nu} = \mathcal{C}_{\gamma'} \cap (\nu - \nu')$, where $\nu'$ is least such that $\gamma \in h``\nu'$.

For a simple example, consider a set $C \subseteq \kappa$ which is Magidor generic over M, making $\mathrm{cf}(\kappa) = o^M(\kappa) = \lambda$ for some cardinal $\lambda < \kappa$. In this case we can take h to be the function such that $h(\beta)$ is the index of the βth full measure on κ, that is, such that $\mathcal{U}(\kappa, \beta) = \mathcal{U}_{h(\beta)}$ for all $\beta < o(\kappa)$. Then

$\mathcal{C}_{h(\beta)} = \{\,\nu \in C : o(\nu) = \beta\,\}$. If we take C to be Radin generic, with $o(\kappa) < \kappa^+$, then we could define h so that $\mathcal{U}(\kappa, \sigma(\xi)) = \mathcal{U}_{h(\xi)}$, where σ is the canonical function taking κ onto $o(\kappa) < \kappa^+$. If C is Radin generic with $o(\kappa) = \kappa^+$, on the other hand, then there is no $h \in M$ and h-coherent system $\mathcal{C}$ of indiscernibles such that $C = \bigcup_\gamma \mathcal{C}_\gamma$, for having such a system $\mathcal{C}$ would require that h maps κ onto κ^+.

The function $h``(x;\mathcal{C})$ provides a weak sense in which a covering set X is generated by a function $h \in K$ and a sequence $\mathcal{C}$ of indiscernibles:

4.17 Definition. Suppose that $\mathcal{C}$ is a system of indiscernibles and x is a set. Then $h``(x;\mathcal{C})$ is the smallest set $X$ such that $x \subseteq X$ and $X = h``(X \cup \bigcup_{\gamma \in X} \mathcal{C}_\gamma)$.

Definition 4.17 is too weak, since it does not provide any bounds on the size of the sets $\mathcal{C}_\gamma$. The functions defined below are used in clause (4) of theorem 4.19 to describe a stronger sense in which X is generated by $\mathcal{C}$:

4.18 Definition. If $\mathcal{C}$ is a g-coherent system of indiscernibles, and X is a set then we define

1. $s^{\mathcal{C}}(\gamma, \xi)$ is the least member of $\mathcal{C}_\gamma - (\xi + 1)$.
2. $s^{\mathcal{C}}_*(\gamma, \xi)$ is the least member of $\bigcup_{\gamma' \geq \gamma} \mathcal{C}_{\gamma'} - (\xi + 1)$.
3. If λ is measurable in K and $\gamma \leq \lambda^{++K}$ then an ordinal ξ is an *accumulation point* of $\mathcal{C}$ in X for γ if the ordinals γ, ξ are in X, and the set $\bigcup\{\, \mathcal{C}_{\gamma''} : \mathrm{crit}(\mathcal{U}_{\gamma''}) = \lambda \text{ and } \gamma'' \geq \gamma' \,\}$ is unbounded in $X \cap \xi$ for all $\gamma' < \gamma$ in $X \cap g``\nu$.
4. $a^{\mathcal{C},X}(\gamma, \xi)$ is the least accumulation point of $\mathcal{C}$ in X for γ above ξ.

This definition of an accumulation point does not seem to be entirely satisfactory, since it depends on the set X and the function g as well as on the system $\mathcal{C}$; however Clause 5 of theorem 4.19 gives a sense in which the functions $s^{\mathcal{C}}$ and $a^{\mathcal{C},X}$ are, up to finite differences, independent of g, X and $\mathcal{C}$.

4.19 Theorem (Covering for sequences of measures). *Assume there is no model of $\exists\kappa\, o(\kappa) = \kappa^{++}$. Let κ be a cardinal of the core model K, and let $\breve\kappa$ be a cardinal of K such that $\breve\kappa \geq \max\{\kappa, o(\kappa)\}$. Finally, let X be a set such that $\kappa \not\subseteq X = Y \cap K_{\breve\kappa}$ for some set Y such that $Y \prec_1 H(\breve\kappa^+)$ and ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$.*

Then there is an ordinal $\rho < \kappa$, a function $h \in K$, and a function $\mathcal{C}$ such that

1. $\mathcal{C}$ is an h-coherent system of indiscernibles for K.

2. $\mathrm{dom}(\mathcal{C}) \subseteq X$ and $\bigcup_\gamma \mathcal{C}_\gamma \subseteq X$.

3. *$X = h``(X \cap \rho; \mathcal{C})$, and hence $X \subseteq h``(\rho; \mathcal{C})$.*

4. *For all $\nu \in X \cap \kappa$, either $\nu \in h``(X \cap \nu)$ or else there is γ such that $\nu \in \mathcal{C}_\gamma$. In the latter case there is $\xi \in X \cap \nu$ such that either*

   (a) *$\nu = s^{\mathcal{C}}(\gamma, \xi) = s_*^{\mathcal{C}}(\gamma, \xi)$, or else*

   (b) *$\nu = a^{\mathcal{C},X}(\gamma', \xi)$ for some $\gamma' > \gamma$ in $h``(X \cap \nu)$.*

   Furthermore, clause (a) holds if ν is a limit point of X.

5. *If X' is another set satisfying the hypothesis of the theorem, then there is a finite set a of ordinals such that for any $\xi, \gamma \in X \cap X'$ with $a \cap \mathrm{crit}(\mathcal{U}_\gamma) \subseteq \xi$ and $\xi > \max\{\rho^X, \rho^{X'}\}$ we have*

$$s^{\mathcal{C}}(\gamma, \xi) = s^{\mathcal{C}'}(\gamma, \xi)$$
$$s_*^{\mathcal{C}}(\gamma, \xi) = s_*^{\mathcal{C}'}(\gamma, \xi)$$
$$a^{\mathcal{C},X}(\gamma, \xi) = a^{\mathcal{C}',X'}(\gamma, \xi)$$

   whenever either is defined.

To see that theorem 4.19 implies the Dodd-Jensen covering lemma as a special case, notice that if $K = L[U]$ then $\mathcal{C}$ contains only a single set C of indiscernibles for the unique measure U. Then clause 4 asserts that $\mathrm{ot}(C) \leq \omega$, and clause 5 asserts that C is maximal.

4.20 Remark. As with the Dodd-Jensen covering lemma, the hypothesis ${}^{\mathrm{cf}(\kappa)}X \subseteq X$ can be weakened: If $\mathrm{cf}(\kappa) < \delta < \kappa$ and δ is the successor of a regular cardinal, then there is an unbounded class $\mathbf{C} \subseteq \mathcal{P}_\delta(K_{\breve{\kappa}})$ of sets X satisfying the conclusion of theorem 4.19 such that if $\vec{X}$ is an increasing chain of members of $\mathbf{C}$ such that $\mathrm{cf}(\kappa) < \mathrm{cf}(\mathrm{len}(\vec{X})) < \kappa$ then $\bigcup \vec{X} \in \mathbf{C}$.

4.21 Remark. The assumption that ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$ is used to ensure that the measures on κ generated by $\mathcal{C}$ are members of X. As was pointed out in observation 6, this assumption can be weakened to ${}^{\omega}Y \subseteq Y$ if $o(\kappa) < \kappa^+$.

Similarly, if $o(\kappa) < \kappa^+$ then remark 4.20 can be improved to state that $\mathbf{C}$ is closed under increasing unions of uncountable cofinality.

4.22 Remark. If every measurable limit point of X is a member of X then the condition $\rho < \kappa$ can be strengthened to $\rho = \inf(\kappa - X)$, so that $X = h``(\rho; \mathcal{C})$. In particular, $\rho = \inf(\kappa - X)$ whenever $o(\alpha) < \inf(\kappa - X)$ for all $\alpha < \kappa$.

Introduction to the Proof

Before beginning to sketch the proof of the covering lemma we pause to look at three complications and digressions:

1. It was pointed out earlier that in order to ensure that $\operatorname{dom}(\mathcal{C}) \subseteq X$ we are assuming that $X \prec_1 K_{\check{\kappa}}$, rather than $X \prec_1 K_\kappa$ as in the last section. This change, however, appears only in the very last step of the proof: until then we work only with $X \cap K_\kappa$ and use arguments which closely parallel those of the Dodd-Jensen covering lemma.

Similarly, this final step is the only place where the closure condition ${}^{\operatorname{cf}(\kappa)}Y \subseteq Y$ is used: up until then countable closure, ${}^{\omega}Y \subseteq Y$, is all that is needed.

2. The proof we give is for the $\operatorname{cf}(\kappa)$-closed case, with ${}^{\operatorname{cf}(\kappa)}Y \subseteq Y$, as in the statement of lemma 4.19. With one exception, the extension of the proof to the stronger result of remark 4.20 is relatively straightforward, using the ideas outlined in the proof of the Dodd-Jensen covering lemma. The exception is Lemma 4.28, and we will digress from the main line of the proof to state lemma 4.29, the analogue of lemma 4.28 for the unclosed case, and to sketch its proof. The reader may, if desired, skip this digression.

3. As was explained in section 4.1 an essential complication arises from the special role which the weak covering lemma plays in the definition of the core model. Beyond $0^\dagger$, the core model K is constructed in two stages: The first stage constructs the *countably complete* core model K^{c}, for which iterability is guaranteed (below the sharp for a class of strong cardinals) by the fact that every full measure of K^{c} is countably complete in V. After lemma 4.5 is proved for K^{c}, the true model K is shown to be an elementary substructure of K^{c}, so that the iterability of K^{c} implies the iterability of K.

Part 1 of the Proof

Here we give part one of the proof of the covering lemma for the true core model K. At the end of this subsection we will show how to adapt this proof to prove the weak covering lemma 4.5 for K^{c}. As in the proof of the Dodd-Jensen covering lemma we begin with the following diagram:

$$\begin{array}{ccccc} M_0 \overset{i}{\rightsquigarrow} M_\theta \supseteq & M & \xrightarrow{\tilde{\pi}} & \widetilde{M} = \operatorname{Ult}_n(M,\pi,\kappa) & \\ & \uparrow\!\!\downarrow & & \uparrow\!\!\downarrow & \\ & \overline{K} & \xrightarrow{\pi} & X \cap K_\kappa \prec_1 K_\kappa & \end{array} \tag{1.18}$$

The construction of this diagram is identical to the construction for the Dodd-Jensen covering lemma: M_θ is obtained as the last model of the iterated ultrapower of $M_0 = K$ arising from the comparison of K with the transitive collapse $\overline{K}$ of $X \cap K_\kappa$; and $\widetilde{M} = \operatorname{Ult}_n(M,\pi,\kappa)$ where M is the largest initial segment of M_θ, and n is the largest integer, such that the ultrapower is defined.

The proof of the analogue of lemma 3.51, which states that the construction of diagram (1.18) succeeds, is the same as for the Dodd-Jensen covering lemma except for two items. The first is clause 3.51(3):

4.23 Claim. *Either $\theta = 0$ and M is a proper initial segment of $M_0 = K$, or else 1 is in the set D of drops in the iteration on M_0. That is, either the iteration is trivial or it drops immediately.*

Proof. Set $\eta = \operatorname{crit}(\pi)$ and let ρ be least such that $\mathcal{P}^K(\rho) \not\subseteq \overline{K}$. As in the proof of lemma 3.17 it will be sufficient to show that $\rho \leq \eta$ and that any ultrafilter U in $K - \overline{K}$ has critical point $\operatorname{crit}(U) \geq \rho$.

We will show the second half first: suppose to the contrary that $U \in K - \overline{K}$ and $\tau = \operatorname{crit}(U) < \min\{\eta, \rho\}$. Evidently $\eta > \tau^{+\overline{K}}$, since otherwise $\tau^{+K} > \tau^{+\overline{K}}$, which contradicts the assumption that $\mathcal{P}^K(\tau) \subseteq \overline{K}$. Then $\eta \geq \tau^{++\overline{K}}$, and since $K \models o(\tau) < \tau^{++}$ it follows that $o^{\overline{K}}(\tau) < \eta$. Thus $o(\tau) = \pi(o^{\overline{K}}(\tau)) = o^{\overline{K}}(\tau)$ and $o(\tau) \subseteq \operatorname{ran}(\pi)$, which implies that every measure on τ in K is in $\overline{K}$, contradicting the choice of U.

Now suppose that $\rho > \eta$, that is, that $\mathcal{P}^K(\eta) \subseteq \overline{K}$. Then the filter $U = \{\, x \subseteq \tau : \eta \in \pi(x) \,\}$ is a normal ultrafilter on $\mathcal{P}^K(\tau)$, and hence is a member of K. Now factor π into $\pi\colon \overline{K} \xrightarrow{i^U} \operatorname{Ult}(\overline{K}, U) \xrightarrow{k} K$ and apply the argument from the last paragraph to the map k to conclude that every ultrafilter on η in K is in $\operatorname{Ult}(\overline{K}, U)$. In particular $U \lhd U$, which is impossible since $\lhd$ is well-founded. ⊣

The second item to consider is clause 3.51(2):

4.24 Claim. *The model $\overline{K}$ is not moved in the comparison of $\overline{K}$ with K.*

Proof. Since the iterated ultrapower on K drops, that on $\overline{K}$ does not. Suppose for the sake of contradiction that the claim is false, and let ν be the least stage at which the iterated ultrapower on $\overline{K}$ is nontrivial. Thus $N_\nu = N_0 = \overline{K}$, and $N_{\nu+1} = \operatorname{Ult}(\overline{K}, \overline{U})$ for some measure $\overline{U} = \mathcal{U}_\gamma^{\overline{K}}$ which is not in the νth model M_ν of the iteration on K. If $\overline{U}$ is an ultrafilter on M_ν then set $\overline{M}_\nu = M_\nu$; otherwise let $\overline{M}_\nu$ be the largest initial segment of M_ν such that every set in $\overline{M}_\nu$ is measured by $\overline{U}$. In either case $\overline{U}$ is a $\overline{M}_\nu$-ultrafilter, and $\overline{M}_\nu$ is a mouse with projectum at most $\operatorname{crit}(\overline{U})$.

First we show that $\operatorname{Ult}_n(\overline{M}_\nu, \overline{U})$ is iterable, where n is largest for which the ultrapower is defined. To see this, let $\mu = \operatorname{crit}(\overline{U})$ and note that $\pi(\overline{U}) = \mathcal{U}_{\pi(\gamma)}$ is a measure on $\pi(\mu)$ in K, while $\widetilde{M}_\nu = \operatorname{Ult}_n\big(\overline{M}_\nu, \pi, \pi(\mu) + 1\big)$ is an initial segment of K by the same argument as for $\widetilde{M} = \widetilde{M}_\theta$. Since $\mathcal{U}_{\pi(\gamma)}$ is a full measure in K it follows that $\operatorname{Ult}_n(\widetilde{M}_n, \mathcal{U}_{\pi(\gamma)})$ is iterable. Then $\operatorname{Ult}_n(M_\nu, \overline{U})$ must also be iterable, since it can be embedded into $\operatorname{Ult}_n(\widetilde{M}_n, \mathcal{U}_{\pi(\gamma)})$ and hence any witness to the contrary could be copied to a witness that $\operatorname{Ult}_n(\widetilde{M}_\nu, \mathcal{U}_{\pi(\gamma)})$ is not iterable.

Thus we can use iterated ultrapowers to compare the models M_ν and $\operatorname{Ult}_n(M_\nu, \overline{U})$. An argument like that for lemma 3.39 shows that neither of the two iterated ultrapowers drops and that they have the same last model N, giving rise to the following diagram, where s and t are the embeddings of the two iterated ultrapowers.

$$\begin{tikzcd} M_\nu \arrow[rr, "s"] \arrow[dr, "i^U"'] & & N \\ & \mathrm{Ult}_n(M_\nu, \overline{U}) \arrow[ur, "t"'] & \end{tikzcd} \tag{1.19}$$

Furthermore diagram (1.19) commutes, since every member of M_ν can be written as $h^{M_\nu}_{n+1}(\xi)$ for some $\xi < \mu$, and both of the embeddings s and $ti^{\overline{U}}$ are the identity on μ and both embeddings map $h^{M_\nu}_{n+1}$ to h^{N}_{n+1} by lemma 3.26. It follows that $\mathrm{crit}(s) = \mathrm{crit}(ti^{\overline{U}}) = \mu$, so that the ultrapower s on M_ν starts with an ultrapower using a measure $\mathcal{U}^{M_\nu}_\gamma$. Furthermore, for every set $x \subseteq \kappa$ in M_ν we have $x \in \overline{U} \iff \mu \in ti^{\overline{U}}(x) \iff \mu \in i(x) \iff x \in \mathcal{U}^{M_\nu}_\gamma$. Thus $U = \mathcal{U}^{M_\nu}_\gamma \in M_\nu$. $\dashv$

This completes part one of the proof of the covering lemma for K, and we are now ready to prove the weak covering lemma 4.5 for K^{c}:

Proof of lemma 4.5. The proof is similar to the proof of the weak covering lemma for K^{DJ}. Suppose to the contrary that λ is a singular cardinal with $\mu^\omega < \lambda$ for all $\mu < \lambda$, and that $\kappa = \lambda^{+K^{\mathrm{c}}} < \lambda^+$. Then $\mathrm{cf}(\kappa) < \lambda$, and hence there is a set $X \prec_1 K_\kappa$, cofinal in κ, such that $\lambda \not\subseteq X$, ${}^\omega X \subseteq X$ and if $\eta = \min(\lambda - X)$ then $\mathrm{cf}(\eta) = \omega_1$. The final condition is obtained by constructing X as the union of an increasing chain of sets of length ω_1.

Now apply the construction above of part one of the proof to the set X, using K^{c} for K. The constraint $\mathrm{cf}(\eta) = \omega_1$ is needed to ensure that the measure U of the first paragraph of the proof of claim 4.23 would be in K^{c} if it existed.

Now, as in the proof of the weak covering lemma for K^{DJ}, the fact that $\kappa = (\lambda^+)^{K^{\mathrm{c}}}$ implies that the set of indiscernibles generated by the construction is bounded by $\lambda + 1$. It follows that $X = h^X``(X \cap \lambda)$, which is impossible since it would imply that $\mathrm{cf}^{K^{\mathrm{c}}}(\lambda^{+K^{\mathrm{c}}}) \leq \lambda$. This contradiction completes the proof of lemma 4.5. $\dashv$

We now turn to the main subject of this section, the analysis of indiscernibles which will complete the proof of the full covering lemma.

Part 2 of the Proof: Analyzing the Indiscernibles

As in the proof of the Dodd-Jensen covering lemma, the model $\widetilde{M} = \mathrm{Ult}_n(M, \pi, \kappa)$ of diagram (1.18) is a mouse in K. It follows that $\widetilde{M}$ is an initial segment of K; that is, $\widetilde{M} = J_{\tilde\alpha}[\mathcal{U}{\restriction}\tilde\alpha]$ for some ordinal $\tilde\alpha < \kappa^+$.

Still following the proof of the Dodd-Jensen covering lemma, let ν_0 be the largest member of the set D of drops, and let $\bar\rho < \bar\kappa$ be the Σ_n-projectum of $M^*_{\nu_0+1}$ and hence of M_ν for every ordinal ν in the interval $\nu_0 < \nu \le \theta$. The ordinal ρ required by lemma 4.19 must satisfy

$$\rho \ge \sup(\pi``\bar\rho). \tag{A}$$

In the proof of the Dodd-Jensen covering lemma we could set $\rho = \sup(\pi``\bar\rho)$, but in the present proof there are several other things which can go wrong, and each of these will determine a separate lower bound for ρ. Rather than specifying ρ at this point we will, at various points during the course of the proof, specify a series (A)–(E) of lower bounds on ρ. At any point in the proof we will assume that ρ is an ordinal less than κ which satisfies all the lower bounds specified up to that point.

Let $\overline{\mathcal{C}}$ be the system of indiscernibles on $\overline{K}$ given by the iteration of K, and define $\widetilde{\mathcal{C}}$ with $\operatorname{dom}(\widetilde{\mathcal{C}}) = \tilde\pi``\operatorname{dom}(\overline{\mathcal{C}})$ by setting $\widetilde{\mathcal{C}}_{\tilde\pi(\gamma)} = \pi``\overline{\mathcal{C}}_\gamma - \rho$ for each $\gamma \in \operatorname{dom}(\overline{\mathcal{C}})$. This is nearly the desired set of indiscernibles: it is an $\tilde h$-coherent system of indiscernibles at least for $\operatorname{ran}(\tilde\pi)$, and $X \cap K_\kappa = K_\kappa \cap \pi``h(\bar\rho \cap X; \overline{\mathcal{C}}) = K_\kappa \cap \tilde h(X \cap \rho; \widetilde{\mathcal{C}})$ where $\tilde h$ is the Skolem function of $\widetilde{M}$.

In order to convert $\widetilde{\mathcal{C}}$ into a system of indiscernibles for K we will show that $\widetilde{\mathcal{C}}$ generates a sequence of normal ultrafilters $\mathcal{U}^*_\gamma$ on K such that $\operatorname{Ult}(K, U)$ is well-founded. It will follow that $\mathcal{U}^*_\gamma$ is equal to some full measure $\mathcal{U}_{\tau(\gamma)}$ in K, and will define a sequence $\mathcal{C}$ of indiscernibles for K by setting $\mathcal{C}_{\tau(\gamma)} = \mathcal{C}^*_\gamma$. Finally, in order to show that $\mathcal{C}$ is a sequence of indiscernibles, we will use the assumption ${}^{\operatorname{cf}(\kappa)}Y \subseteq Y$ to show that the range of τ is contained in Y, and obtain the required function h by combining $\tilde h$ with a function g obtained by applying the covering lemma to $X \cap (\breve\kappa - \kappa)$.

The coherence function $\operatorname{Coh}_{\gamma',\gamma}$ was defined in definition 4.14. Note that this definition does make sense even though the measures $\mathcal{U}_{\gamma'}$ and $\mathcal{U}_\gamma$ are partial in K, and are full measures only in $\widetilde{M}$.

4.25 Definition. Define the relation $\nu \in_\gamma x$, for $x \in K$ and γ an ordinal, as follows:

$$\nu \in_\gamma x \iff \begin{cases} \nu \in x & \text{if } \nu \in \widetilde{\mathcal{C}}_\gamma \\ x \cap \nu \in \mathcal{U}_{\gamma''} & \text{if } \nu \in \mathcal{C}_{\gamma'} \text{ where } \gamma < \gamma' \text{ and } \gamma'' = \operatorname{Coh}_{\gamma,\gamma'}(\nu) \\ \text{undefined} & \text{otherwise} \end{cases}$$

4.26 Definition. If $\gamma \in \operatorname{dom}(\widetilde{\mathcal{C}})$ then define

$$\mathcal{C}^+_\gamma = \bigcup\bigl\{\widetilde{\mathcal{C}}_{\gamma'} : \gamma' \ge \gamma \;\;\&\;\; \operatorname{crit}(\widetilde{\mathcal{U}}_{\gamma'}) = \operatorname{crit}(\mathcal{U}_\gamma)\bigr\}.$$

If $\mathcal{C}^+_\gamma$ is cofinal in $\operatorname{crit}(\mathcal{U}_\gamma)$, then we write $\mathcal{U}^*_\gamma$ for the set of $x \in \mathcal{P}^K(\operatorname{crit}(\mathcal{U}_\gamma))$ such that $\nu \in_\gamma x$ for all sufficiently large $\nu \in \mathcal{C}^+_\gamma$.

In order to show that the filters $\mathcal{U}^*_\gamma$ are K-ultrafilters we use the idea of an *indiscernible sequence*:

4.27 Definition. A sequence $\vec{\alpha} = \langle \alpha_n : n < \omega \rangle$ is a $\widetilde{\mathcal{C}}$*-indiscernible sequence* for $\vec{\gamma} = \langle \gamma_n : n < \omega \rangle$ if $\vec{\alpha}$ is strictly increasing, $\alpha_n \in \widetilde{\mathcal{C}}_{\gamma_n}$ for all but finitely many $n < \omega$, and either (i) $\sup_n(\gamma_n) = \sup_n(\alpha_n)$, or (ii) $\operatorname{crit}(\mathcal{U}_{\gamma_n}) = \sup_{n\in\omega}(\alpha_n)$ for all $n < \omega$.

The following lemma corresponds to the argument that C^X is a Prikry sequence in the proof of the Dodd-Jensen covering lemma.

4.28 Lemma. *If $\vec{\alpha}$ is a $\widetilde{\mathcal{C}}$-indiscernible sequence for $\vec{\gamma}$ in $\widetilde{\mathcal{C}}$ then for any function $f \in K$ there is $n_0 < \omega$ such that:*

1. *If $n_0 \le n < n' < \omega$ and $\operatorname{crit}(\mathcal{U}_{\gamma_{n'}}) < \gamma_m \le \min\{\gamma_n, \gamma_{n'}\}$, then for all $\xi < \alpha_n$ we have $\alpha_n \in_{\gamma_m} f(\xi) \iff \alpha_{n'} \in_{\gamma_m} f(\xi)$.*
2. *If $n_0 < n$ and $\gamma_n < \kappa$ then for all $\xi < \alpha_n$ we have $\alpha_n \in f(\xi)$ if and only if $f(\xi) \cap \operatorname{crit}(\mathcal{U}_{\gamma_n}) \in \mathcal{U}_{\gamma_n}$.*

Proof. Suppose that the lemma fails for some $\widetilde{\mathcal{C}}$-indiscernible sequence $\vec{\alpha}$ for $\vec{\gamma}$. The assertion that clause 2 fails uses parameters $\vec{\alpha}$ and $\{\,\gamma_n : \gamma_n < \kappa\,\}$, both of which are contained in X, and since ${}^\omega Y \subseteq Y$ it follows that both parameters are members of Y. By elementarity it follows that there is such a function f which is a member of Y. Then $f \in \operatorname{ran}(\tilde{\pi})$ by proposition 3.63, so $\tilde{\pi}^{-1}(f)$ is in M and contradicts the fact that $\bar{\mathcal{C}}$ is a sequence of indiscernibles for M.

For clause 1, define $\gamma_{n',n} = \operatorname{Coh}_{\gamma_{n'},\gamma_n}(\alpha_n)$ whenever this is defined. Then the statement "$\alpha_n \in_{\gamma_m} f(\xi)$" is equivalent to the statement "Either $\gamma_n = \gamma_m$ and $\alpha_n \in f(\xi)$ or else $\gamma_{m,n}$ is defined and $f(\xi) \cap \alpha_n \in \mathcal{U}_{\gamma_{m,n}}$", so the statement that the lemma does not hold for $\vec{\alpha}$, $\vec{\gamma}$ and f can be stated using as parameters $\vec{\alpha}$, the ordinals $\gamma_{n',n}$, and $\{\,(n,n') \in \omega^2 : \gamma_n = \gamma_{n'}\,\}$. All of these are contained in X, so the same argument as in the last paragraph yields a contradiction. ⊣

Before using lemma 4.28 to show that the sets $\mathcal{U}^*$ are K-ultrafilters, we digress to look at the analog of lemma 4.28 for the case when X is not countably closed.

Digression for non-countably closed sets X. It was pointed out in the introduction to the proof of theorem 4.19 that lemma 4.28 is the one point in the proof where a new idea, beyond those presented in the proof of the Dodd-Jensen covering lemma, is needed in order to strengthen theorem 4.19 as in remark 4.20 by removing the assumption that $X = Y \cap K_{\check{\kappa}}$ for some countably closed set Y. Lemma 4.29 below substitutes for lemma 4.28 in this case. Lemma 4.29 and its proof may be skipped without affecting the proof of the covering lemma as stated in theorem 4.19.

4.29 Lemma. *Suppose that $\delta = \tau^+$ where τ is a uncountable regular cardinal, and let* $\mathbf{C}$ *be the set of $X \in \mathcal{P}_\delta(K_{\breve\kappa})$ such that $\mathcal{C}^X$ satisfies lemma 4.28. Then* $\mathbf{C}$ *is unbounded in $X \in \mathcal{P}_\delta(K_{\breve\kappa})$ and is closed under unions of increasing sequences of uncountable cofinality.*

Note that the requirement on δ is stronger than is needed for the corresponding results in the proof of the Dodd-Jensen covering lemma, for which δ could be any uncountable cardinal.

Proof sketch. The proof of the following analogue of lemma 3.58 is straightforward:

4.30 Lemma. *If $X_0 \subseteq X_1$, and $\vec\alpha$ and $\vec\gamma$ are sequences with range contained in X_0 such that $\vec\alpha$ is a $\mathcal{C}^{X_1}$-indiscernible sequence for $\vec\gamma$, then $\vec\alpha$ is also a $\mathcal{C}^{X_0}$-indiscernible sequence for $\vec\gamma$.* $\dashv$

As in the Dodd-Jensen covering lemma, it easily follows that $\mathbf{C}$ is closed under increasing unions of uncountable cofinality. Thus we only need to prove that $\mathbf{C}$ is unbounded.

Let S be the set of $\sigma \in \mathrm{Col}(\delta, K_\lambda)$ such that $\mathrm{cf}(\mathrm{dom}(\sigma)) = \tau$ and $\mathrm{ran}(\sigma)$ fails to satisfy lemma 4.28. As in the Dodd-Jensen covering lemma, we will finish the proof of lemma 4.29 by showing that S is nonstationary. Suppose toward a contradiction that S is stationary, and for each function $\sigma \in S$ let $\vec\alpha^\sigma$ and $\vec\gamma^\sigma$ be sequences which witness that lemma 4.29 fails for $X^\sigma = \mathrm{ran}(\sigma)$; that is, $\vec\alpha^\sigma$ is a $\mathcal{C}^\sigma$-indiscernible sequence for $\vec\gamma^\sigma$, but $\vec\alpha$ and $\vec\gamma$ fail to satisfy one of clauses 1 or 2 of lemma 4.28. Now continue following the proof of lemma 3.60, which was the analog in the proof of the Dodd-Jensen covering lemma of lemma 4.29: Let A^σ be the set

$$\left\{ \vec\alpha^\sigma, \{ \gamma^\sigma_n : \gamma^\sigma_n < \kappa \}, \{ (n', n) : \gamma^\sigma_{n',n} \text{ is defined} \}, \{ (n', n) : \gamma^\sigma_{n'} = \gamma^\sigma_n \} \right\}$$

of parameters used in the proof of lemma 4.28, and find $\sigma_0 \in S$ and a stationary set $S_0 \subseteq S$ so that $\sigma \supseteq \sigma_0$ and $A^\sigma \subseteq \mathrm{ran}(\sigma_0)$ for all $\sigma \in S_0$.

Recall that the key point in the proof of lemma 3.60 was that, because $C^\sigma \subseteq^* C^{\sigma_0}$ for every $\sigma \in S_0$, each of the sets C^σ were determined (up to a finite set) by the subset $D^\sigma = C^{\sigma_0} - C^\sigma$ of C^{σ_0}. The key step in the current proof is to use Fodor's lemma and the hypothesis that $\delta = \tau^+$ to find a set Z which fills the role of C^{σ_0}. Toward this end, choose an function $k\colon \tau \cong \mathrm{dom}(\sigma_0) < \tau^+$. Since $\mathrm{cf}(\tau) > \omega$ there is, for each $\sigma \in S$, an ordinal $\xi^\sigma < \tau$ such that $\bigcup A^\sigma \subseteq \sigma_0 k``\xi^\sigma$. By Fodor's lemma there is a stationary subset $S'_0 \subseteq S_0$ such that $\xi^\sigma$ is constant, say $\xi^\sigma = \xi$ for each $\sigma \in S'_0$. Set $Z = (\sigma_0 \circ k)``\xi$, so that $|Z| < \tau$ and $\bigcup A^\sigma \subseteq Z$ for every $\sigma \in S'_0$.

The rest of the proof parallels the proof of lemma 3.60 for the Dodd-Jensen covering lemma. First define, for each $\sigma \in S'_0$, a set $w(\sigma)$ which witnesses that the restriction of $\widetilde{\mathcal{C}}$ to Z is as large as possible. This set is obtained by modifying definition 3.18 as follows: Set the support β^σ of $w(\sigma)$ to be $\beta^\sigma = \max\{\sup(Z), \rho^\sigma + 1\} < \kappa$, and replace the requirement that $w(\sigma)$ be countable with the condition $|w(\sigma)| = |Z|$. Finally, modify Clause (3) of definition 3.18 to state that $\mathfrak{m}^\sigma = \operatorname{dir\,lim}(w(\sigma))$ is the Σ_n-code of a mouse of K, and there is a function $f = f^\sigma$ which is Σ_1-definable in $\mathfrak{m}^\sigma$ such that (i) for any $\alpha, \gamma \in Z$ such that $\gamma < \kappa$ and $\alpha \notin \widetilde{\mathcal{C}}^\sigma(\gamma)$, there is a set $x \in f``\alpha$ such that $x \in \mathcal{U}(\gamma)$ but $\alpha \notin x$, and (ii) for any $\alpha < \alpha'$ in $Z$ which are not members of the same set $\mathcal{C}^\sigma(\gamma)$, there is $x \in f``(\alpha \cap w(\sigma))$ such that $\alpha \in x$ and $\alpha' \notin x$.

Thus $w(\sigma)$ gives a complete description of the restriction of $\mathcal{C}^\sigma$ to Z.

Now, since $|w(\sigma)| = |Z| < \tau$ and $\operatorname{cf}(\operatorname{dom}(\sigma)) = \tau$ for every member σ of S_0', lemma 3.23 implies that there is $\sigma_1 \in S_0'$ and a stationary set $S_1 \subseteq S_0'$ such that if $\sigma \in S_1$ then $\sigma_1 \subseteq \sigma$ and $w(\sigma) \subseteq \operatorname{ran}(\sigma_1)$. By shrinking S_1 further, if necessary, we can ensure that $\beta^\sigma = \beta$ is constant for $\sigma \in S_1$.

Now since S_1 is unbounded and the sequences $\vec{\alpha}^{\sigma_1}$ and $\vec{\gamma}^{\sigma_1}$ do not satisfy the conclusion of lemma 4.28, there is some $\sigma \in S_1$ with a function $f \in \operatorname{ran}(\sigma)$ which witnesses this failure. It follows that $\vec{\alpha}^{\sigma_1}$ is not a $\widetilde{\mathcal{C}}^\sigma$-indiscernible sequence, and by the definition of $w(\sigma)$ it follows that there is a function f' which is Σ_1-definable in $\operatorname{dir\,lim}(w(\sigma))$ which witnesses this failure. Now $w(\sigma) \subseteq \operatorname{ran}(\sigma_1)$, and it follows that $\operatorname{dir\,lim}(w(\sigma)) \subseteq \widetilde{M}^{\sigma_1}$. To see this, notice that $\bar{\mathfrak{m}} = \operatorname{dir\,lim}((\pi^{\sigma_1})^{-1}(w(\sigma)) \subseteq M$, since every subset of $\bar{\rho} = (\pi^{\sigma_1})^{-1}(\rho^{\sigma_1})$ in $\mathfrak{m}$ is a member of M and there is a subset of $\bar{\rho}$ definable in M which is not a member of M.

Thus $\vec{\alpha}^{\sigma_1}$ is not a $\widetilde{\mathcal{C}}^{\sigma_1}$-indiscernible sequence for $\vec{\gamma}^{\sigma_1}$. This contradicts the choice of $\vec{\alpha}^{\sigma_1}$ and $\vec{\gamma}^{\sigma_1}$, and hence completes the proof of lemma 4.29. ⊣

Continuation of the main proof. This completes the digression for non-countably closed covering sets X, and we now return to the basic proof of the covering lemma.

4.31 Lemma. *Suppose that $\mathcal{C}^+(\gamma)$ is cofinal in $\alpha = \operatorname{crit}(\mathcal{U}_\gamma)$. Then $\mathcal{U}^*_\gamma$ is a normal ultrafilter on K, and $\operatorname{Ult}(K, \mathcal{U}^*_\gamma)$ is well-founded. Hence $\mathcal{U}^*_\gamma = \mathcal{U}_{\tau(\gamma)}$ in K for some ordinal $\tau(\gamma)$.*

Furthermore, for any function $f \in K$ there is $\eta < \alpha$ such that

$$\forall \nu, \gamma \Big(\eta < \nu < \alpha < \gamma \;\;\&\;\; \nu \in \mathcal{C}^+_\gamma \implies \forall \xi < \nu \, (\nu \in_\gamma f(\xi) \iff f(\xi) \in \mathcal{U}^*_\gamma) \Big). \tag{1.20}$$

We break up the proof of lemma 4.31 into two parts, depending on the cofinality of α.

4.32 Lemma. *The conclusion of lemma 4.31 holds whenever* $\operatorname{cf}(\alpha) = \omega$. *Furthermore if $\alpha < \kappa$ then $\tau(\gamma) = \gamma$, and if $\alpha = \kappa$ then $\tau(\gamma) \in Y$.*

Proof. In this case everything except the existence of $\tau(\gamma)$ follows immediately from lemma 4.28, as any counterexample could be witnessed by a $\widetilde{\mathcal{C}}$-indiscernible sequence. The assertion that $\tau(\gamma) = \gamma$ if $\alpha < \kappa$ follows from clause 2 of that lemma, and the assertion that $\tau(\gamma) \in Y$ if $\gamma > \kappa$ follows from its proof.

The existence of $\tau(\gamma)$ follows from corollary 4.13, which states that $\mathcal{U}^*_\gamma$ is equal to some full measure $\mathcal{U}_{\gamma'}$ on the K-sequence provided that $\mathrm{Ult}(K, \mathcal{U}^*_\gamma)$ is well-founded. If it is not well-founded then there are functions $f_n \in K \cap H(\alpha^+)$ such that $[f_{n+1}]_{\mathcal{U}^*_\gamma} < [f_n]_{\mathcal{U}^*_\gamma}$ for each $n < \omega$. As in the proof of lemma 4.28 we can assert this condition on the functions f_n by a statement in Y, and by elementarity there must be such a sequence in Y. This is impossible, as it would imply that $\mathrm{Ult}(M, \tilde{\pi}^{-1}(\mathcal{U}_\gamma))$ is ill-founded, but $\tilde{\pi}^{-1}(\mathcal{U}_\gamma) = \mathcal{U}^M_{\pi^{-1}(\gamma)} \in M$, and hence $\mathrm{Ult}(M, \tilde{\pi}^{-1}(\mathcal{U}_\gamma))$ must be well-founded since M is an iterable model obtained by an iteration on K. $\dashv$

Before proving lemma 4.31 when $\mathrm{cf}(\alpha) > \omega$ we need to make the following important observation:

4.33 Lemma. *Suppose $\vec{\alpha}$ is an increasing sequence with $\alpha_n \in \widetilde{\mathcal{C}}_{\gamma_n}$ for each $n < \omega$, and that $\mathrm{crit}(\mathcal{U}_{\gamma_n}) = \alpha$ for all $n < \omega$. If $\alpha' = \lim_n \alpha_n < \alpha$ then $\alpha' \in \widetilde{\mathcal{C}}_{\gamma'}$ for some $\gamma' \geq \limsup\{\, \gamma_n + 1 : n < \omega \,\}$.*

Proof. We can assume that $\gamma_n < \limsup_{m<\omega}(\gamma_m + 1)$ for all $n < \omega$. We want to use lemma 4.32, using α' for α. This can be done by using $X \cap K_{\check{\alpha}'}$, which is a suitable set for the covering lemma at α'. The iteration used in the construction of diagram (1.18) for $X \cap K_{\check{\alpha}'}$ is an initial segment of that for X: let θ' be the least ordinal such that $\mathrm{crit}(i_{\theta',\theta}) > \pi^{-1}(\alpha')$ where the embeddings $i_{\xi',\xi}$ come from the iteration of K with last model M_θ. The first θ' stages of this iteration are exactly those which are used in the proof of the covering lemma for α', using the suitable set $X \cap K_{\check{\alpha}'}$. By lemma 4.32 it follows that this sequence generates measures $\mathcal{U}_{\gamma'}$, with critical point α', on the K-sequence; and furthermore $\mathcal{U}_{\gamma'} \in X$ since it is generated by countable sequences contained in X. Now the embedding $i_{\theta',\theta'+1}$ comes from an ultrapower of $M_{\theta'}$ using a measure $\mathcal{U}^{M'_\theta}_{\bar{\gamma}'}$ larger than all of those in N. Thus $\bar{\gamma}' > \pi^{-1}(\gamma'_n)$ for each $n < \omega$. But $\gamma_n = \tilde{\pi} i_{\theta',\theta}(\pi^{-1}(\gamma'_n))$ and $\alpha' \in \widetilde{\mathcal{C}}_{\gamma'}$ where $\gamma' = \tilde{\pi} i_{\theta',\theta}(\bar{\gamma}')$. Thus $\gamma' > \gamma_n$ for each $n < \omega$. $\dashv$

Proof of 4.31 for $\mathrm{cf}(\alpha) > \omega$. Suppose that $\mathrm{cf}(\alpha) > \omega$. We will first prove that for any function $f \in K$ there is η satisfying (1.20).

Suppose to the contrary that f is a function for which no η exists as required. Define sequences ξ_n, ν_n and γ_n so that $\vec{\gamma}$ is nondecreasing, $\xi_n < \nu_n \in \mathcal{C}^+_{\gamma_n}$, and $\nu_n \in_{\gamma_n} f(\xi_n) \;\not\Longleftrightarrow\; f(\xi_n) \in \mathcal{U}^*_{\gamma_n}$ but for all $\nu \in \mathcal{C}^+_{\gamma_n} - \nu_{n+1}$ we have $\nu \in_{\gamma_n} f(\xi_n) \iff f(\xi_n) \in \mathcal{U}^*_{\gamma_n}$.

Now set $\alpha' = \sup_n(\nu_n)$. By lemma 4.33 $\alpha' \in \mathcal{C}_{\gamma'}$ for some $\gamma' \geq \sup_n(\gamma_n + 1)$, and if we set $\gamma'_n = \mathrm{Coh}_{\gamma_n,\gamma'}(\nu_n)$ then $\vec{\nu}$ is a $\widetilde{\mathcal{C}}$-indiscernible sequence for $\vec{\gamma}'$. Hence the lemma fails at α', contradicting lemma 4.32. A similar argument shows that $\mathcal{U}^*_\gamma$ is normal.

Finally $\mathcal{U}^*_\gamma$ is countably complete when $\mathrm{cf}(\alpha) > \omega$, so $\mathrm{Ult}(K, \mathcal{U}^*_\gamma)$ is well-founded. Hence corollary 4.13 implies that $\mathcal{U}^*_\gamma = \mathcal{U}_{\tau(\gamma)}$ for some ordinal $\tau(\gamma)$. ⊣

We are now ready to specify the second and third of the lower bounds on ρ:

$$\rho > \sup\{\, \gamma \in \mathrm{dom}(\mathcal{U}^*) \cap \kappa : \mathcal{U}_\gamma \neq \mathcal{U}^*_\gamma \,\} \tag{B}$$

Condition (B) holds for all sufficiently large $\rho < \kappa$ since Lemma 4.28 implies that $\{\, \mathrm{crit}(\mathcal{U}^*_\gamma) : \mathcal{U}_\gamma \neq \mathcal{U}^*_\gamma \,\}$ is finite.

$$\rho > \sup \bigcup \{ \widetilde{\mathcal{C}}_\gamma : \kappa < \gamma \ \ \& \ \ \mathcal{U}^*_\gamma \text{ is not defined} \} \tag{C}$$

To see that the right-hand side of condition (C) is smaller than κ, suppose to the contrary that there is a cofinal set $C \subseteq \kappa$ such that for each $\nu \in C$ there is $\gamma_\nu > \kappa$ such that $\nu \in \widetilde{\mathcal{C}}_{\gamma_\nu}$ and $\mathcal{U}^*_{\gamma_\nu}$ is not defined. By taking a subsequence if necessary, we can assume that $\langle\, \gamma_\nu : \nu \in C \,\rangle$ is nondecreasing, but this implies that $\mathcal{U}^*_{\gamma_\nu}$ is defined for each $\nu \in C$.

Conditions (B) and (C) enable us to complete the definition of $\mathcal{C}$:

4.34 Definition. If $\gamma \in \mathrm{dom}\, \widetilde{\mathcal{C}}$ then let $\tau(\gamma)$ be the ordinal such that $\mathcal{U}^*_\gamma = \mathcal{U}_{\tau(\gamma)}$.

Define $\mathcal{C}$ by setting $\mathcal{C}_{\tau(\gamma)} = \widetilde{\mathcal{C}}_\gamma$ for all γ such that $\tau(\sigma)$ is defined.

Condition (A) ensures that $\tau(\gamma) = \gamma$ for all $\gamma < \kappa$. We now make our single use of the assumption that X is $\mathrm{cf}(\kappa)$-closed, that is, that $X = Y \cap K_{\breve{\kappa}}$ for some $Y \prec H(\lambda)$ with ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$

4.35 Claim. *If $\mathcal{U}^*_\gamma$ is defined then $\mathcal{U}^*_\gamma \in X$, and hence $\tau(\beta) \in X$.*

Proof. As in the proof of lemma 4.28, the filter $\mathcal{U}^*_\gamma$ is generated in X by any cofinal subsequence of $\mathcal{C}^+_\gamma$. Since ${}^{\mathrm{cf}(\kappa)}\kappa \subseteq Y$, there is such a subsequence in Y. ⊣

4.36 Remark. If $o(\kappa) < \kappa^{+K}$ then the assumption that X is $\mathrm{cf}(\kappa)$-closed is unnecessary, for in that case there is a partition of κ into disjoint sets $\langle\, A_\beta : \beta < o(\kappa) \,\rangle$ such that $A_\beta \in \mathcal{U}(\kappa, \beta)$ for each $\beta < o(\kappa)$. If $\vec{A} \in Y$ then there is $\eta < \kappa$ so that $\mathcal{C}(\kappa, \beta) - A_\beta \subseteq \eta$ for all $\beta < o(\kappa)$. If $\nu \in \mathcal{C}(\kappa, \beta) - \eta$ then β is definable from $\vec{A}$ as the unique ordinal β such that $\nu \in A_\beta$, so claim 4.35 holds for all $X = Y \cap K_{\breve{\kappa}}$ with $\vec{A} \in Y$.

It is easy to see from the construction that $\mathcal{C}$ is a sequence of indiscernibles for K. Thus $\mathcal{C}$ satisfies clauses 1 and 2 of theorem 4.19.

4.37 Claim. *There is a function $g \in K$ such that $X \subseteq g``(X \cap \kappa)$.*

Proof. Apply the proof of the covering lemma to the full set $X \prec_1 K_{\breve{\kappa}}$ (rather than to $X \cap K_\kappa$). Notice that, as in the proof of lemma 4.5, there are no measurable cardinals in the interval $(\kappa, \breve{\kappa}]$ and hence any indiscernibles which come up in the construction must be smaller than κ. It follows, just as in the proof of the covering lemma for L, that there is a function $g \in K$ such that $X = g``(X \cap \kappa)$. $\dashv$

We now put the fourth lower bound on ρ:

$$\rho > \sup\{\, \nu : \exists\beta\,(\nu \in \mathcal{C}(\kappa, \beta) \wedge \beta \notin g``(X \cap \nu))\,\}. \tag{D}$$

The following claim justifies this bound:

4.38 Claim. *There is an ordinal $\eta < \kappa$ such that $\gamma \in g``(X \cap \nu)$ whenever $\gamma > \kappa$ and $\eta < \nu \in \mathcal{C}_\gamma$*

Proof. Define, in K, a disjoint sequence of sets $\langle\, A_\gamma : \gamma \in \operatorname{ran}(g)\,\rangle$ such that $A_\gamma \in \mathcal{U}_\gamma$ whenever $\gamma \in \operatorname{ran}(g)$ and $\mathcal{U}_\gamma$ is a full ultrafilter on κ in K. By lemma 4.31 there is $\eta < \kappa$ so that for all $\gamma \in \operatorname{dom}(\mathcal{C}) - \kappa$ and all $\nu \in \mathcal{C}_\gamma - \eta$ and $\xi < \nu$ we have $\nu \in A_{g(\xi)} \iff A_{g(\xi)} \in \mathcal{U}_\gamma$. Since the diagonal union $B = \{\, \nu < \kappa : \exists\xi < \nu \;\; \nu \in A_{g(\xi)}\,\}$ is a member of each measure $\mathcal{U}_{g(\xi)}$, we can also assume $\bigcup\{\, \mathcal{C}_\gamma - \eta : \gamma \in \operatorname{ran}(g)\,\} \subseteq B$. It follows that this choice of η will satisfy the statement of the lemma. $\dashv$

This completes the proof of the first three clauses of theorem 4.19. For the rest of the proof of the theorem it will be convenient to use the notation $\mathcal{U}(\alpha, \beta)$, which explicitly names the critical point of the measure, rather than the notation $\mathcal{U}_\gamma$. In doing so we will consistently adjust the notation described earlier by replacing γ with the pair (α, β): for example, we will write $s^{\mathcal{C}}(\alpha, \beta, \xi)$ instead of $s^{\mathcal{C}}(\gamma, \xi)$, we will say that $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ instead of $\vec{\gamma}$, and we will write $\mathrm{Coh}_{\alpha,\beta,\beta'}$ for the coherence function relating $\mathcal{U}(\alpha, \beta)$ and $\mathcal{U}(\alpha, \beta')$.

It will also be useful to have a notion of an indiscernible sequence which, like that of a Prikry sequence, depends directly on the sequence $\mathcal{U}$ of measures rather than on a system of indiscernibles.

4.39 Definition. We say that $\vec{\nu}$ is an *indiscernible sequence* for $(\vec{\alpha}, \vec{\beta})$ if (i) $\vec{\nu}$ is a strictly increasing sequence of ordinals of length ω, (ii) either $\sup_n(\nu_n) = \sup_n(\alpha_n)$ or else $\alpha_n = \sup_n(\nu_n)$ for all n, and (iii) for any function $f \in K$ there is $n_0 < \omega$ such that $\forall n > n_0 \forall \xi < \alpha_n \left(\nu_n \in f(\xi) \iff f(\xi) \cap \alpha_n \in \mathcal{U}_{\alpha_n, \beta_n}\right)$.

Notice that lemma 4.28 implies that any $\widetilde{\mathcal{C}}$-indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta}')$ where $\beta'_n = \tau(\beta_n)$.

In the rest of this proof we will say $\vec{\nu} <^* \vec{\nu}'$ to mean that $\nu_n < \nu'_n$ for all but finitely many $n < \omega$; and we will use $>^*$, $\leq^*$ and $\geq^*$ similarly.

We will first prove clause 4 of theorem 4.19 in the case when $\mathrm{cf}(\nu) = \omega$. Suppose that $\nu \in \mathcal{C}_{\alpha,\beta}$; we want to show that there is $\xi < \nu$ so that $\nu = s(\alpha, \beta, \xi) = s_*(\alpha, \beta, \xi)$. If this is not so then there is a cofinal sequence of ordinals $\nu_n \in \mathcal{C}_{\alpha,\beta_n}$ with $\beta_n \geq \beta$, and this contradicts lemma 4.33 which implies that $\beta \geq \limsup_n(\beta_n + 1)$.

Now let $\nu \in \mathcal{C}_{\alpha,\beta_0}$ be arbitrary and let β_1 be the largest ordinal such that ν is an accumulation point in X for (α, β_1). Then $\bigcup_{\beta_1 \leq \beta < o(\alpha)} \mathcal{C}(\alpha, \beta)$ is bounded in $\nu \cap X$, say by $\xi \in X \cap \nu$. If $\beta_0 \geq \beta_1$ then $\nu = s(\alpha, \beta_0, \xi) = s_*(\alpha, \beta_0, \xi)$, so we can suppose that $\beta_0 < \beta_1$. We claim that there are only finitely many accumulation points in X for (α, β_1) in the interval (ξ, ν), so that $\nu = a(\alpha, \beta_1, \xi')$ for some ξ' in $[\xi, \nu) \cap X$. If, to the contrary, there are infinitely many such accumulation points, then let ν' be the least member of the interval $(\xi, \nu]$ which is a limit of accumulation points for (α, β_1). Then $\mathrm{cf}(\nu') = \omega$ and it follows from the last paragraph that $\nu' \in \mathcal{C}(\alpha, \beta')$ for some $\beta' \geq \beta_1$, contradicting the choice of ξ. This contradiction completes the proof of clause 4, except for the last sentence which states that $\nu = s(\alpha, \beta_0, \xi_0) = s_*(\alpha, \beta_0, \xi_0)$ whenever ν is a limit point of X. We will defer the proof of this for the case $\mathrm{cf}(\nu) > \omega$ until after the proof of clause 5, on which its proof depends.

Notice that any increasing ω-sequence $\vec{\nu}$ of indiscernibles from $\mathcal{C}$ is an indiscernible sequence for some $\vec{\alpha}, \vec{\beta}$. To see this, suppose that $\nu_n \in \mathcal{C}(\alpha_n, \beta_n)$, with $\alpha_n, \beta_n \in g``\nu_n$. If $\alpha_n \leq \sup_n(\nu_n)$ for each n, then $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$. Otherwise $\alpha_n = \alpha$ is constant for sufficiently large n with $\alpha_n > \alpha' = \sup_n(\nu_n)$, and lemma 4.33 implies that $\alpha' \in \mathcal{C}(\alpha, \beta)$ for some $\beta < o(\alpha)$ such that $\beta_n > \beta$ for all sufficiently large $n < \omega$. Then $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha}', \vec{\beta}')$ where $\alpha'_n = \alpha'$ and $\beta'_n = \mathrm{Coh}_{\alpha,\beta_n,\beta}(\alpha')$ for all n such that $\alpha_n = \alpha$.

The proof of clause 5 relies on lemma 4.28(2), which implies for any sequences $\vec{\nu}, (\vec{\alpha}, \vec{\beta}) \in X$ that $\vec{\nu}$ is a $\mathcal{C}^X$-indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ if and only if it is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ in the sense of definition 4.39; and similarly for X'. Suppose that clause 5 is false for the function $s^{\mathcal{C}}$. Then we can assume, without loss of generality, that there are infinite sequences $\vec{\alpha}, \vec{\beta}$ and $\vec{\xi}$ in $X \cap X'$ such that for each $n < \omega$ we have (i) $\nu'_n = s^{\mathcal{C}'}(\alpha_n, \beta_n, \xi_n)$ exists, (ii) $\xi_{n+1} \geq \nu'_n$, and (iii) $s^{\mathcal{C}}(\alpha_n, \beta_n, \xi_n)$ either does not exist or is strictly larger than $s^{\mathcal{C}'}(\alpha_n, \beta_n, \xi_n)$. Then $\vec{\nu}'$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$, and since $Y \prec H(\lambda)$ it follows that Y satisfies that there is an indiscernible sequence $\vec{\nu}$ for $(\vec{\alpha}, \vec{\beta})$ such that $\nu_n > \xi_n$ for all n. Thus, by lemma 4.28, $s^{\mathcal{C}}(\alpha_n, \beta_n, \xi_n)$ exists for all but finitely many $n < \omega$, so we can set $\nu_n = s^{\mathcal{C}}(\alpha_n, \beta_n, \xi_n)$. By the choice of $\vec{\alpha}, \vec{\beta}$ and $\vec{\xi}$, we must have $\vec{\nu} >^* \vec{\nu}'$, but then again Y satisfies that there is an indiscernible sequence ν'' for $(\vec{\alpha}, \vec{\gamma})$ such that $\vec{\xi} <^* \vec{\nu}'' <^* \vec{\nu}$, so that $\vec{\nu}''$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ in $\mathcal{C}$, which contradicts the choice of $\vec{\nu}$.

The proof of clause 5 for the function $s^{\mathcal{C}}_*$ is similar, except that $\vec{\nu}'$ is an indiscernible sequence for some $(\vec{\alpha}, \vec{\beta}')$ with $\vec{\beta}' \geq^* \vec{\beta}$, instead of for $(\vec{\alpha}, \vec{\beta})$ itself.

The proof that clause 5 holds for the function $a^{\mathcal{C}}$ is similar but slightly more complicated. We say that $\vec{\nu}$ is an *accumulation point sequence* for $(\vec{\alpha}, \vec{\gamma})$ if for all sequences $\vec{\gamma}' <^* \vec{\gamma}$ and $\vec{\nu}' <^* \vec{\nu}$ there are sequences $\vec{\nu}''$ and $\vec{\beta}''$ with $\vec{\nu}' <^* \vec{\nu}'' <^* \vec{\nu}$ and $\vec{\beta}' \leq \vec{\beta}''$ such that $\vec{\nu}''$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta}'')$. By using the elementarity of Y and the fact that being an indiscernible sequence is absolute between Y and V, it follows that being an accumulation point sequence is also absolute between Y and V. The rest of clause 5 follows as for the functions $s^{\mathcal{C}}$ and $s^{\mathcal{C}}_*$.

This completes the proof of clause 5, and we now return to the proof of the final sentence of clause 4, which states that if ν is any limit point of X such that $\nu \in \mathcal{C}(\alpha, \beta)$ for some $\beta < o(\alpha)$ then $\nu = s(\alpha, \beta, \xi)$ for some $\xi \in X \cap \nu$. Let Z be the set of ordinals $\nu \in X \cap \lim(X)$ such that $\nu \notin h``\nu$ and there is no α, β and ξ in X such that $\nu = s(\alpha, \beta, \xi) = s_*(\alpha, \beta, \xi)$.

We specify the last lower bound on ρ:

$$\rho \geq \sup(Z). \tag{E}$$

This is justified by the following claim:

4.40 Claim. *The set Z is finite.*

Sketch of Proof. Suppose to the contrary that $\vec{\nu}$ is an increasing ω-sequence of members of Z. Then $\alpha = \sup_n(\nu_n) \notin Z$ since clause 4 holds for ordinals of cofinality ω, so $\vec{\nu}$ is an indiscernible sequence in $\mathcal{C}$ for some pair $(\vec{\alpha}, \vec{\beta})$ with $\alpha_n \leq \alpha$. Since $\nu_n \in Z$, $\nu_n = a^{\mathcal{C}}(\alpha_n, \beta'_n, \xi_n)$ for some β'_n with $\beta_n < \beta'_n \leq o(\alpha_n)$ and $\xi_n \in X \cap \nu_n$.

Now proceed as in the proof of lemma 4.33 for each of the ordinals ν_n. Let $C_n = \bigcup\{\, \mathcal{C}(\alpha_n, \beta) : \beta_n \leq \beta < \beta'_n \,\}$ and for each $\nu \in C_n$ let $\beta_{n,\nu}$ be the ordinal β such that $\nu \in \mathcal{C}(\alpha_n, \beta)$. Set $\vec{\beta}_n = \langle\, \beta_{n,\nu} : \nu \in C_n \,\rangle$. If the sequences $\vec{C}_n$ and $\vec{\beta}_n$ are in X then we can use the argument of lemma 4.33 to conclude that $\beta_n \geq \sup_{\nu \in C_n} \beta_{n,\nu}$, contrary to assumption.

To deal with the general case, pick a set $X' = Y' \cap K_{\check{\kappa}}$ as in the hypothesis of the covering lemma 4.19 so that $C_n \in Y'$ and $\vec{\beta}_n \in Y'$ for each $n < \omega$. Then the argument in the last paragraph shows that it cannot be true that $\nu_n \in \mathcal{C}^{X'}(\alpha_n, \beta_n)$ and at the same time $\nu \in \mathcal{C}^{X'}(\alpha_n, \beta_{n,\nu})$ for unboundedly many $\nu \in C_n$. But by clause 5, for sufficiently large $n < \omega$ we have $\nu_n \in \mathcal{C}^{X}(\alpha_n, \beta_n) \implies \nu_n \in \mathcal{C}^{X'}(\alpha_n, \beta_n)$ and $\nu \in \mathcal{C}^{X}(\alpha_n, \beta_{n,\nu}) \implies \nu \in \mathcal{C}^{X'}(\alpha_n, \beta_{n,\nu})$ for all $\nu \in C_n$. This contradiction completes the proof of claim 4.40. $\dashv$

This completes the proof of the last sentence of clause (4), which is the end of the proof of theorem 4.19, the covering lemma for sequences of measures.

4.3. The Singular Cardinal Hypothesis

We will now use theorem 4.19 to establish the lower bound for the strength of a failure of the singular cardinal hypothesis:

4.41 Theorem (Gitik [17]). *If there is a singular cardinal κ with $2^\kappa > \max\{\kappa^+, 2^{\mathrm{cf}(\kappa)}\}$ then there is a cardinal κ with $o(\kappa) \geq \kappa^{++}$ in K.*

The proof combines the use of the covering lemma with two theorems from Shelah's pcf theory. The first can be found as Conclusion 5.10(2) on page 410 of [52].

4.42 Theorem. *If κ is the least cardinal satisfying $\kappa^{\mathrm{cf}(\kappa)} > \kappa^+ + 2^{\mathrm{cf}(\kappa)}$ then $\mathrm{pp}(\kappa) \geq \kappa^{++}$, $\mathrm{cf}(\kappa) = \omega$, and $\forall \mu < \kappa\, \mu^\omega \leq \max\{\mu^+, 2^\omega\}$.*

We will assume that $o^K(\kappa) < \kappa^{++}$, where κ is given by the conclusion of theorem 4.42, and derive a contradiction. Note that the conclusion implies that $\kappa > 2^\omega$ and $\mu^\omega = \mu$ for each cardinal μ of uncountable cofinality in the interval $2^\omega \leq \mu < \kappa$.

The statement that $\mathrm{pp}(\kappa) \geq \kappa^{++}$ implies that there is a sequence $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$ of regular cardinals smaller than κ, together with a sequence $\vec{f} = \langle f_\alpha : \alpha < \kappa^{++} \rangle$ of functions in $\prod \vec{\kappa}$ which is $<^*$-increasing and $<^*$-cofinal in $\prod \vec{\kappa}$. We will call such a sequence a *scale* and will use it to derive the contradiction. The first part of the proof will use the covering lemma to obtain from the given scale a scale in which each of the functions f_α is what Gitik calls a *diagonal sequence*. The exact meaning of this term will be given in lemma 4.44 after some notation has been established, but a typical example, requiring $o(\kappa_{n+1}) \geq \kappa_n$ for each n, would be a sequence $f \in \prod \vec{\kappa}$ such that $f(n+1) = s_*(\kappa_{n+1}, f(n), \kappa_n)$ for each $n < \omega$. This construction requires separate covering sets for each sequence f_α, and relies heavily on the fact that any two such covering sets agree (on their common domain) for all but finitely many κ_n. The final contradiction, however, requires finding an appropriate collection of covering sets which agree for some particular fixed κ_n, and for this a second result of Shelah will be needed. A proof is in Jech [25, lemma 24.10].

4.43 Lemma. *If $\langle f_\alpha : \alpha < \kappa^{++} \rangle$ is a scale, then for each $\alpha < \kappa^{++}$ with $\mathrm{cf}(\alpha) = \kappa^+$ there is an* exact upper bound *(eub) of $\langle f_{\alpha'} : \alpha' < \alpha \rangle$; that is, a function $g \in \prod \vec{\kappa}$ such that $f_{\alpha'} <^* g$ for all $\alpha' < \alpha$, and for any function $g' <^* g$ there is $\alpha' < \alpha$ such that $g' <^* f_{\alpha'}$.*

In what follows we say that a set X is a *covering set* if it satisfies the hypothesis of the covering lemma, theorem 4.19. All covering sets have cardinality 2^ω unless stated otherwise. We will be using a number of different covering sets, and will heavily use the next indiscernible function $s^X_*(\kappa, \beta, \xi)$ and next accumulation point function $a^X(\gamma, \beta, \xi)$ from that lemma. These functions depend on the choice of covering set X, but by clause 5 of theorem 4.19 there is, for any two covering sets X and X', an $n_0 < \omega$ such that the functions defined using the two sets agree (whenever the arguments are in both sets) above κ_{n_0}. Keeping this in mind, we will normally simplify the notation by omitting the superscripts X. In addition we will use a standard, fixed covering set X for many of our calculations, but we will want this set to include a number of objects which are not defined until later in the course of the proof. To see that we can do so without loss of generality, note that if some desired object is not a member of X then we can choose a new, larger covering set X' which does include it. If we were to redo the proof up to this point using X' instead of X then there is some $n < \omega$ such that the X agrees with X' about indiscernibles above κ_n, and hence about everything defined in the proof so far which lies above κ_n. In this case we can throw out a finite initial segment $\vec{\kappa}\restriction n+1$ of the sequence $\vec{\kappa}$. By restricting the functions f_α in the original scale to this reduced sequence we obtain a scale for which X and X' agree. This will cause no problems so long as it occurs only finitely often.

We begin by assuming that $\{\vec{\kappa}, \vec{f}\} \subset X$. Set $\kappa'_n = \min(h^X``\kappa_n)$, so $\kappa_n \leq \kappa'_n \leq \kappa$. If $\kappa'_n > \kappa_n$ then κ_n will be an indiscernible for κ'_n. Let $\beta_n \leq o(\kappa_n)$ be the largest ordinal such that κ_n is an accumulation point for (κ'_n, β_n) in $\mathcal{C}^X$, noting that the definition 4.18 of an accumulation point makes perfectly good sense even if $\kappa'_n = \kappa_n$. Pick $g^*(n) < \kappa_n$ in X large enough that $\kappa'_n \in h^X``g^*(n)$, $\bigcup_{\beta \geq \beta_n} \mathcal{C}_{\kappa'_n,\beta} \subset g^*(n)$, and $\mathcal{C}^X$ has no accumulation points for (κ'_n, β_n) in $\kappa_n - g^*(n)$. The latter is possible because it follows from $\operatorname{cf}(\kappa_n) > \omega$ that there are only boundedly many accumulation points for (κ'_n, β_n) below κ_n

Now choose, for each $\alpha < \kappa^{++}$, a covering set X_α with $f_\alpha \in X_\alpha$. Since $o(\kappa) < \kappa^{++}$ implies that there are only κ^+ many possible Skolem functions h^{X_α}, there is a function h such that $\{\, \alpha < \kappa^+ : h^{X_\alpha} = h \,\}$ is cofinal in κ^{++}. By throwing away the rest of the sequence $\vec{f}$ we can assume without loss of generality that $h^{X_\alpha} = h$ for all $\alpha < \kappa^{++}$. Similarly we can assume that there is an $n_0 < \omega$ such that $\rho^{X_\alpha} = \rho$ is constant and that the ordinals κ'_n, β_n and $g^*(n)$ computed using any X_α are the same as those computed using X for all $n > n_0$. By cutting off the start of the sequence $\vec{\kappa}$ we can assume that $n_0 = 0$. We will also assume that $h \in X$.

Define, for each $\alpha < \kappa^{++}$, a function f'_α by taking $f'_\alpha(n)$ to be the least ordinal $\xi \leq f_\alpha(n)$ such that $\kappa'_n \cap h^X``(\{\kappa'_n\} \cup (\xi+1)) \not\subseteq f_\alpha(n)$. The functions $f'_\alpha$ are unbounded in $\prod_n \kappa_n$: to see this, let $g$ be any member of $\prod \vec{\kappa}$ and pick $\alpha$ so that $f_\alpha(n) > \sup(\kappa_n \cap h``g(n))$ for almost all n. Then $f'_\alpha >^* g$.

Thus we can assume that $f'_\alpha = f_\alpha$ for all α, which implies that $f_\alpha(n)$ is an indiscernible in $\mathcal{C}_{\kappa'_n,\beta}$ for some $\beta < o(\kappa'_n)$. We now show that we can assume that the functions f_α are what Gitik calls *diagonal sequences*:

4.44 Lemma. *Under the assumptions of theorem 4.42 there is a sequence*

$\vec{\kappa}$ of regular cardinals and a scale $\langle f_\alpha : \alpha < \kappa^{++} \rangle$ in $\prod \vec{\kappa}$ such that, using the notation introduced above, $\operatorname{cf}(\beta_n) = \kappa_{n-1}$. Furthermore, if we fix continuous, cofinal functions $t_n \colon \kappa_{n-1} \to \beta_n$, then each of the functions f_α satisfies $f_\alpha(n) = s_*\big(\kappa'_n, t_n(f_\alpha(n-1)), g^*(n)\big)$ for almost all n.

Note that the cofinalities are computed in V, and the maps t_n need not be in K.

Proof. Each of the ordinals β_n is a limit ordinal, for if $\beta_n = \beta+1$ then $\mathcal{C}_{\kappa'_n,\beta}$ is cofinal in κ_n and $\mathcal{C}_{\kappa'_n,\beta+1} \cap \kappa_n \subseteq g^*(n)$, but this implies that $\operatorname{cf}(\kappa_n) = \omega$, contrary to assumption.

For any $\alpha < \kappa^{++}$ we know that each of the ordinals $f_\alpha(n)$ is equal to either $a(\kappa'_n, \beta, \gamma)$ or $s(\kappa'_n, \beta, \gamma)$ for some $\gamma < f(n)$ and $\beta \in h``f_\alpha(n)$. Since there is always some $\beta' \in \beta_n \cap h``f_\alpha(n)$ such that $s_*(\kappa'_n, \beta, g^*(n))$ is larger than either of these, we can assume that $f_\alpha(n) = s_*(\kappa'_n, \beta_{\alpha,n}, g^*(n))$ for some $\beta_{\alpha,n} \in \beta_n \cap h``f_\alpha(n)$.

4.45 Claim. *For any $\delta < \kappa$ there are at most finitely many $n < \omega$ such that $\operatorname{cf}(\beta_n) < \delta$; and there are at most finitely many n such that $\operatorname{cf}(\beta_n) = \kappa_n$.*

Proof. First suppose that $\operatorname{cf}(\beta_n) < \delta < \kappa$ for all n in an infinite set A, and let $\sigma_n \colon \operatorname{cf}(\beta_n) \to \beta_n$ be cofinal maps. For each $s \in \prod_{n\in A} \delta_n$ define $g_s \in \prod_{n \in A} \kappa_n$ (up to a finite set) by $g_s(n) = s_*(\kappa'_n, \sigma_n(s(n)), g^*(n))$. Then the maps g_s are cofinal in $\prod_{n\in A} \kappa_n$, but this is impossible since $\prod_{n\in A} \kappa_n$ has cofinality κ^{++} and there are at most $\delta^\omega < \kappa$ many functions g_s.

Now suppose that $\operatorname{cf}(\beta_n) = \kappa_n$ for all n in an infinite set A. We will use the assumption that $h \in X$ to show that for all $\alpha < \kappa^{++}$ we have $f_\alpha(n) < \sup(X \cap \kappa_n)$ for all but finitely many $n \in A$. This is impossible since $|X| = 2^\omega < \kappa_n = \operatorname{cf}(\kappa_n)$, and hence $X \cap \kappa_n$ is bounded in κ_n.

Recall that each f_α is covered by the covering set X_α in the sense that $f_\alpha(n) = \gamma_k$ for some sequence $\langle \gamma_0, \dots, \gamma_k \rangle$ of indiscernibles in $\mathcal{C}^{X_\alpha}$ such that for each $i \le k$ either $\gamma_i = s(\alpha_i, \eta_i, \xi_i)$ or $\gamma_i = a(\alpha_i, \eta_i, \xi_i)$ for some α_i, η_i and ξ_i in $h``\big(\rho \cup \vec{\gamma}{\restriction}i\big)$. Let $i \le \kappa$ be least such that $\gamma_i \ge \sup(X \cap \kappa_n)$. If $\alpha_i < \kappa_n$ then $\gamma_i < \alpha_i < \sup(X \cap \kappa_n)$, contrary to the choice of i, so it must be that $\alpha_i \ge \kappa_n$ which implies $\alpha_i = \kappa'_n$. In that case we have $\eta_i < \sup(\beta_n \cap h``(X \cap \kappa_n))$ and $\xi_i < \sup(X \cap \kappa_n)$. Thus we can find $\beta > \eta_i$ in $X \cap \beta_n$ and $\xi > \xi_i$ in $X \cap \kappa_n$, and then $\gamma_i \le s_*(\kappa'_n, \beta, \xi) < \sup(X \cap \kappa_n)$ for all n sufficiently large that X agrees with X_α at κ_n. This again contradicts the choice of i. $\dashv$

Now define D to be the smallest set such that each of the ordinals κ_n is in D and D is closed under the function σ defined as follows: Suppose that $\gamma \in D$, let γ' be largest such that either $\gamma' = \gamma$ or $\gamma \in \mathcal{C}_{\gamma',\beta}$ for some β, and let $\beta = \beta(\gamma) > 0$ be the the largest ordinal such that γ is an accumulation point for (γ', β). If $\operatorname{cf}(\beta) > \omega$ then define $\sigma(\gamma) = \operatorname{cf}(\beta)$; otherwise leave $\sigma(\gamma)$ undefined.

4.46 Claim. $\mathrm{ot}(D) = \omega$.

Proof. Otherwise let δ be the least limit point of D. Then the set $D' = \{\, \gamma \in D - \delta : \sigma(\gamma) < \delta \,\}$ is infinite. Now consider a covering set $X' \supset X$ of size δ^ω with $\delta \subseteq X$. Then $\sigma^{X'}(\gamma) = \sigma^X(\gamma)$ for infinitely many of the $\gamma \in D'$, and X' is cofinal in any γ with $\sigma^{X'}(\gamma) < \delta$. This contradicts the fact that every $\gamma \in D$ is regular. $\dashv$

If $\gamma \in D$ and $\sigma(\gamma)$ is defined then let $t_\gamma \colon \sigma(\gamma) \to \beta(\gamma)$ be continuous, increasing and cofinal. Note that we do not assume that $t_\gamma \in K$. Also let $g^{**}(\gamma) < \gamma$ be large enough that $\mathcal{C}_{\gamma',\beta} \cap \gamma \subseteq g^{**}(\gamma)$ for all $\beta \geq \beta(\gamma)$. Define $\mathcal{F}$ to be the set of functions $f \in \prod D$ such that for all but finitely many $\gamma \in D$ such that $\sigma(\gamma)$ is defined we have $f(\gamma) = s(\gamma', t_\gamma(f(\sigma(\gamma)), g^{**}(\gamma)))$.

4.47 Claim. *For each $\alpha < \kappa^{++}$ there is $f \in \mathcal{F}$ such that $f_\alpha <^* f \restriction \vec{\kappa}$.*

Proof. Fix $\alpha < \kappa^{++}$, and work in the covering set X_α for f_α. Define a sequence of functions $g_k \in \prod D$ by recursion on $k \in \omega$ as follows: set $g_0(\kappa_n) = f_\alpha(\kappa_n)$ for $n < \omega$, and $g_0(\gamma) = 0$ for $\gamma \in D - \vec{\kappa}$. Now define g_{k+1} so that the inequalities

$$\begin{aligned} g_{k+1}(\gamma) &\geq s_*(\gamma', t_\gamma(g_k(\sigma(\gamma))), g^{**}(\gamma)) \\ s_*(\gamma', t_\gamma(g_{k+1}(\sigma(\gamma))), g^{**}(\gamma)) &\geq g_k(\gamma) \end{aligned}$$

hold for all $\gamma \in D$ such that $\sigma(\gamma)$ is defined. Then the function $f \in \prod D$ defined by $f(\gamma) = \sup_k(g_k(\gamma))$ is as required. $\dashv$

Define a tree order on D by letting the immediate successors of a ordinal $\gamma' \in D$ be the ordinals $\gamma \in D$ such that $\gamma' = \sigma(\gamma)$. This tree is infinite and finitely branching, so it has an infinite branch. This branch is an infinite subset $D' \subseteq D$ such that $\sigma(\gamma) = \max(D' \cap \gamma)$ for each $\gamma \in D' - \{\min(D')\}$. This set D' satisfies all of the original assumptions on $\vec{\kappa}$, and $\prod D'$ has a scale $\vec{f'}$ of functions satisfying the equation

$$f'_\alpha(n) = s_*(\kappa'_n, t_n(f_\alpha(n-1)), g^*(n)) \tag{1.21}$$

for all $\alpha < \kappa^{++}$ and all sufficiently large $n < \omega$, so the scale $\vec{f'}$ consists of diagonal sequences. This completes the proof of lemma 4.44. $\dashv$

Now modify the sequence $\vec{f}$ by replacing the functions f_α such that $\mathrm{cf}(\alpha) = \kappa^+$ with an exact upper bound (given by lemma 4.43) of the sequence $\langle\, f_{\alpha'} : \alpha' < \alpha \,\rangle$. These functions need not satisfy (1.21), but they do satisfy the following:

4.48 Claim. *If $\mathrm{cf}(\alpha) = \kappa^+$ then $f_\alpha(n) = a(\kappa'_n, t_n(f_\alpha(n-1)), g^*(n))$ for all but finitely many $n < \omega$. Furthermore $\mathrm{cf}(f_\alpha(n)) > \mathrm{cf}(f_\alpha(n-1))$ for infinitely many $n < \omega$.*

Note that we do not exclude the possibility that $f_\alpha(n)$ is also equal to $s_*(\kappa'_n, \beta_{\alpha,n}, g^*(n))$.

Proof. Let $\vec{\xi}$ be any sequence such that $\xi_{n-1} < f_\alpha(n-1)$ for almost all $n < \omega$. Since f_α is an exact upper bound of $\vec{f}\restriction\alpha$, there is $\alpha' < \alpha$ so that $f_\alpha(n-1) > f_{\alpha'}(n-1) > \xi_n$ for almost all n. Then $f_\alpha(n) > f_{\alpha'}(n) \geq s_*(\kappa'_n, t_n(\xi_n), g^*(n))$ for almost all n.

This shows that there is an accumulation point sequence $\vec{\eta}$ for the sequence $\langle\, \big(\kappa'_n, t_n(f_\alpha(n-1))\big) : n < \omega \,\rangle$ such that $\vec{\eta} \leq^* f_\alpha$. We now show that no accumulation point sequence $\vec{\eta}$ can satisfy $g^*(n) <^* \eta(n) <^* f_\alpha(n)$ for infinitely many n. To this end let $\vec{\eta}$ be any sequence such that $f_\alpha(n) > \eta_n$ for almost all n, and pick $\alpha' < \alpha$ so large that $f_{\alpha'}(n) > s_*(\kappa_n, t_n(\eta_{n-1}), g^*(n))$ for almost all n. Then $s_*(\kappa'_n, t_n(f_{\alpha'}(n-1)), g^*(n)) = f_{\alpha'}(n) > \eta_n$ for almost all n.

Finally, if $\mathrm{cf}(f_\alpha(n)) \leq \mathrm{cf}(f_\alpha(n-1))$ for all but finitely many $n < \omega$ then the set $\{\, \mathrm{cf}(f_\alpha(n)) : n < \omega \,\}$ is bounded by some $\delta < \kappa$, but in this case $\prod_n f_\alpha(n)$ would have cofinality at most $\delta^\omega < \kappa$, when in fact it has true cofinality $\mathrm{cf}(\alpha) = \kappa^+$. ⊣

There is a certain tension implicit in the statement of claim 4.48: the first sentence appears to say that $\{\, s_*(\kappa'_n, t_n(\xi), g^*(n)) : \xi < f_\alpha(n-1) \,\}$ is cofinal in $f_\alpha(n)$, but that would contradict the second sentence. This is not yet an actual contradiction because the covering set in which $s_*(\kappa'_n, t_n(\xi), g^*(n))$ is evaluated varies with ξ. In the remainder of the proof we will realize this contradiction. Towards this end, pick a set $A \subseteq \omega$ such that

$$\{\, \alpha < \kappa^{++} : \mathrm{cf}(\alpha) = \kappa^+ \ \&\ A = \{\, n : \mathrm{cf}(f_\alpha(n)) > \mathrm{cf}(f_\alpha(n-1)) \,\} \,\} \tag{1.22}$$

has cardinality κ^{++}, and then use the case $(2^\omega)^+ \to (\omega_1)^2_\omega$ of the Erdős-Rado theorem to find an uncountable subset S of the set (1.22) and an $n_0 \in \omega$ such that $f_{\alpha'}(n) < f_\alpha(n)$ for all $\alpha' < \alpha$ in S and all $n > n_0$. Let $\langle\, \delta_\iota : \iota < \omega_1 \,\rangle$ enumerate S, set $g_\iota = f_{\delta_\iota}$ and write τ_n for $\sup_{\iota<\omega_1}(g_\iota(n))$.

Enlarge the covering set X, if necessary, so that $S \cup \{S\} \subseteq X$. Note that $\{\, g_\iota(n) : \iota < \omega_1 \,\} \subseteq X$ for each n, so X is cofinal in τ_n. It follows by the final sentence of theorem 4.19.4 that $\tau_n = s(\kappa'_n, \beta'_n, g^*(n))$ for some β'_n, for all but finitely many $n < \omega$, and that $\beta'_n \geq \sup_{\iota<\omega_1}(t_n(g_\iota(n-1)))$. As a consequence we can work with indiscernibles for τ_n rather than for κ'_n, as follows: Let $t^*_n \colon \tau_n \to \beta'_n$ be defined by $t^*_n(\xi) = \mathrm{Coh}_{\kappa'_n, t_n(\xi), \beta'_n}(\tau_n)$. Then $s_*(\kappa'_n, t_n(\gamma), g^*_n) = s_*(\tau_n, t^*_n(\gamma), g^*_n)$.

For each $n < \omega$ let Y_n be a covering set containing all of the data so far which has $\tau_{n-1} \subseteq Y_n$ and $|Y_n| = |\tau_{n-1}|^\omega \leq \tau^+_{n-1} \leq \kappa_{n-1} < \tau_n$. Define maps $d_{\iota,n}(\gamma)$ by setting $d_{\iota,n}(\gamma) = s^{Y_n}_*(\tau_n, t^*_n(\gamma), g^*(n))$ for each ordinal $\gamma < g_\iota(n-1)$, and set $d^*_{\iota,n} = \sup\{\, d_{\iota,n}(\gamma) : \gamma < g_\iota(n-1) \,\}$. Notice that $d^*_{\iota,n} < g_\iota(n)$ for all $n \in A$. This is clear if the nondecreasing sequence $\langle\, d_{\iota,n}(\gamma) : \gamma < g_\iota(n) \,\rangle$ is eventually constant; and if it is not constant then $\mathrm{cf}(d^*_{\iota,n}) = \mathrm{cf}(g_\iota(n-1)) < \mathrm{cf}(g_\iota(n))$, and since $d^*_{\iota,n} \leq g_\iota(n)$ it follows that $d^*_{\iota,n} < g_\iota(n)$.

Fix, for each $\iota < \omega_1$, some $\alpha_\iota < \delta_\iota$ and $n_\iota < \omega$ so that $d^*_{\iota,n} < f_{\alpha_\iota}(n) < g_\iota(n)$ for all $n \geq n_\iota$ in A. Then for each $\iota < \omega_1$ we have, for sufficiently large $n < \omega$,

$$\begin{aligned} s^{Y_n}_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)) = d_{\iota,n}(f_{\alpha_\iota}(n-1) &\leq d^*_{n,\gamma} \\ &< f_{\alpha_\iota} = s^{X_{\alpha_\iota}}_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)). \end{aligned} \tag{1.23}$$

By enlarging X if necessary, we can assume that all ordinals mentioned in the inequality (1.23) are in X. Then for all $n < \omega$ there is ι_n such that for all $\iota > \iota_n$

$$s^{Y_n}_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)) = s^X_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)) \tag{1.24}$$

and for every $\iota < \omega_1$ there is $n_\iota < \omega$ such that for all $n > n_\iota$

$$s^{X_{\alpha_\iota}}_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)) = s^X_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)). \tag{1.25}$$

Now fix $\iota > \sup_{n<\omega}(\iota_n)$, and then pick $n > n_\iota$ large enough that inequality (1.23) holds for this n and ι. Then all three of (1.23), (1.24) and (1.25) hold, and this contradiction completes the proof of theorem 4.41.

4.4. The Covering Lemma for Extenders

This subsection is unevenly divided into three parts. The largest part concerns the covering lemma up to $0^{\P}$, which is understood nearly as well as that for sequences of measures. A smaller part covers the covering lemma for the Steel core model, for which little is known beyond the weak covering lemma, and the final part describes what is known beyond this. No proofs are given. See [27] or chapter [56] for definitions and basic properties of extenders.

Up to a Strong Cardinal

This subsection covers the covering lemma when $o(\kappa) > \kappa^{++}$ but $0^{\P}$ does not exist; that is, when the core model contains extenders, but not overlapping extenders. More information may be found in [23].

It was remarked in the introduction to section 4 that the extension of the covering lemma to this region involves two significant changes: one which is easy and largely notational, and another which is rather surprising. We will begin with the notational considerations, which come into play whenever extenders are present. These considerations are all that is needed for theorem 4.50, which deals with extenders of length less than $\kappa^{+\omega}$ where

κ is the critical point of the extender. We will consider the more surprising change following this theorem.

The first observation is that since there are no overlapping extenders, the notations $\mathcal{E}(\alpha,\beta)$ and $o(\alpha)$ are still meaningful: $\mathcal{E}(\alpha,\beta)$ is the βth full extender on α, and $o(\alpha)$ is the order type of the set of full extenders on α. Some care is required in the use of this notation: it is not true, as it is for sequences of measures, that $E = \mathcal{E}(\alpha,\beta)$ implies $o^{i^E(\mathcal{E})}(\alpha) = \beta$. For an example of this, let $E = \mathcal{E}(\alpha,\beta)$ where $\beta \geq \alpha^{++}$, and let U be the associated ultrafilter, that is, $x \in U$ if and only if $\alpha \in i^E(x)$. Then $U = \mathcal{E}(\alpha,\beta')$ for some $\beta' < \beta$. In fact $\mathcal{E}(\alpha,\beta') = \mathcal{E}_{\gamma'}$ where (since $\mathcal{E}(\alpha,\beta')$ is a measure) $\gamma' = \alpha^{++L[i^U(\mathcal{E})]}$. There are β' many ordinals $\gamma'' < \gamma'$ such that $\mathcal{E}_{\gamma''}$ is a full extender, so we must have $\beta' \leq \alpha^{++L[i^U(\mathcal{E})]}$. Now $\{\,\nu : o(\nu) > \nu^{++}\,\} \in U$, so $o^{i^U(\mathcal{E})}(\alpha) > \alpha^{++L[i^U(\mathcal{E})]}$. Thus $o^{i^U(\mathcal{E})}(\alpha) > \alpha^{++L[i^U(\mathcal{E})]} \geq \beta'$.

Because of this, it is not strictly true that comparisons of these models use only linear iterations; however the tree iterations which they do use have a particularly simple form: there is a single trunk with no side branches of length more than one, and furthermore each extender used in the iteration tree is a member of (though not necessarily on the extender sequence of) the model to which it is applied. These simple trees can be modified to obtain linear iterations (cf. [2, 23]) or they can be handled directly without using the stronger techniques required for larger extenders (*c.f.* [58]). Schindler [49] has extended such linearization techniques to work for cardinals below the sharp for a class of strong cardinals, and it is not known how much further they can be stretched.

Before we can state a covering lemma for models with extenders, we need to develop some notation for dealing with indiscernibles for extenders. Consider for contrast the more familiar case of indiscernibles for measures. If U is an ultrafilter on κ in some model M, and $i = i^U : M \to \mathrm{Ult}(M,U)$ is the canonical embedding, then κ is an indiscernible for $i(U)$ in the sense that

$$\forall x \in i``\mathcal{P}(\kappa)\ \big(\kappa \in x \iff i^{-1}(x) \in U \iff x \in i(U)\big),$$

and κ generates $\mathrm{Ult}(M,U)$ in the sense that

$$\mathrm{Ult}(M,U) = \{\, i^U(f)(\kappa) : f \in M \,\}.$$

Now let E be a (κ,λ)-extender, and let $i = i^E : M \to \mathrm{Ult}(M,E)$ be the canonical embedding. In this case the role previously played by the ordinal κ is played by the interval $[\kappa,\lambda)$: if we write E_a for the ultrafilter corresponding to $a \in [\lambda]^{<\omega}$ then

$$\forall x \in i``\mathcal{P}(\kappa^{|a|})\ \big(a \in x \iff i^{-1}(x) \in E_a \iff x \in i(E_a)\big),$$

and

$$\mathrm{Ult}(M,E) = \{\, i(f)(a) : f \in M \wedge a \in [\kappa,\lambda)^{<\omega} \,\}.$$

Thus a plays the role of an indiscernible for $i(E_a) = i(E)_{i(a)}$. Since $i(a) = i``a$ it will be sufficient to consider individual ordinals in the interval $[\kappa, \lambda)$.

In the example above we regard the critical point κ of i as a *principal indiscernible* for $i(E)$, and we will call an ordinal $\alpha \in [\kappa, \lambda)$ the *indiscernible for* $i(E)_{i(\alpha)}$ *belonging to* κ. In order to extend these concepts to an iterated ultrapower, we will write a system of indiscernibles for a model $M = L[\mathcal{E}]$ as a pair $(\mathcal{C}, b)$ of functions, where $\mathcal{C}_\gamma$ is the set of principal indiscernibles for the extender $\mathcal{E}_\gamma$ and $b(\gamma, \alpha, \xi)$ is the indiscernible (if there is one) for $(\mathcal{E}(\gamma))_\xi$ which belongs to α. Here is the precise definition:

4.49 Definition. If $i_{0,\theta}\colon M_0 \to M_\theta = M$ is an iterated ultrapower then the system $(\vec{\mathcal{C}}, b)$ of indiscernibles for M_θ generated by $i_{0,\theta}$ is defined as follows:

1. $\alpha \in \mathcal{C}_\gamma$ if and only if there are $\nu < \nu' \leq \theta$ such that $\alpha = \operatorname{crit}(i_{\nu,\nu'})$ and $\mathcal{E}_\gamma = \mathcal{E}_\gamma^{M_{\nu'}} = i_{\nu,\nu'}(E_\nu)$ where E_ν is the extender such that $M_{\nu+1} = \operatorname{Ult}(M_\nu, E_\nu)$.

2. If $\alpha \in \mathcal{C}(\gamma)$, with ν and ν' as in clause 1, then $b(\gamma, \alpha, \eta)$ is defined if and only if $\eta \in i``[\alpha, \lambda)$ where E_ν is a (α, λ)-extender. In this case $b(\gamma, \alpha, \eta) = i^{-1}_{\nu,\nu'}(\eta)$.

In order to obtain an abstract definition of a system of indiscernibles for a model $M = L[\mathcal{E}]$, without any assumption that the system came from an iterated ultrapower, we replace clause 4.15(2) of the definition 4.15 of a system of indiscernibles for sequences of measures with clause $(2')$ below.

$2'$ For any function $f \in M$ there is a finite sequence $\vec{a}$ of ordinals such that if $\alpha \in \mathcal{C}(\gamma)$, with $\vec{a} \cap [\alpha, \gamma) = \varnothing$, and $\bar{b} = b(\gamma, \alpha, b)$, then $\bar{b} \in x \iff x \cap V_\gamma \in (\mathcal{E}(\gamma))_b$.

Some obvious changes need to be made to the definition of a h-coherent system of indiscernibles, and the definition of $X = h``(\rho; \mathcal{C})$ needs to be modified to $h``(\rho; \mathcal{C}, b)$.

4.50 Theorem (Covering for short extenders). *Assume that $n < \omega$ and that there is no inner model M such that $\{\, \alpha < \kappa : o^M(\alpha) = \alpha^{+n} \,\}$ is unbounded in κ for any cardinal κ. Let κ be a cardinal of K, set $\lambda = \kappa^{+n}$, and suppose $X = Y \cap K_\lambda$ where $Y \prec H(\lambda^+)$ and ${}^{\operatorname{cf}(\kappa)}Y \subseteq Y$. Then there is a pair $(\mathcal{C}, b)$, a function $h \in K$, and an ordinal $\rho < \kappa$ such that*

1. *The pair $(\mathcal{C}, b)$ is a h-coherent system of indiscernibles for K.*

2. $\operatorname{dom}(\mathcal{C}) \cup \operatorname{dom}(b) \subseteq X$, *and* $\operatorname{ran}(b) \cup \bigcup \operatorname{ran}(\mathcal{C}) \subseteq X$.

3. *For all $\nu \in X - \rho$, one of the following four conditions hold:*

   (a) $\nu \in h``(X \cap \nu)$.

(b) *$\nu = s^{\mathcal{C}}(\gamma, \xi)$ for some $\xi \in X \cap \nu$ and $\gamma \in h``(X \cap \nu)$.*

(c) *$\nu = a^{\mathcal{C},X}(\gamma, \xi)$ for some $\xi \in X \cap \nu$ and $\gamma \in h``(X \cap \nu)$. Furthermore, this clause never holds if ν is a limit point of X.*

(d) *$\nu = b(\gamma, \alpha, a)$ for some $\alpha \in X \cap \nu$ and $\gamma, a \in h``\alpha$.*

4. *If X' is another set satisfying the hypothesis of the theorem, and $\mathcal{C}'$, b', X' satisfy clauses (1-4), then there is a finite set $\vec{d}$ of ordinals such that if $\xi, \gamma \in X \cap X'$ with $[\xi, \gamma) \cap \vec{d} = \varnothing$, then*

$$\begin{gathered} s^{\mathcal{C}}(\gamma, \xi) \leq s^{\mathcal{C}'}(\gamma, \xi) \\ a^{\mathcal{C},X}(\gamma, \xi) \leq a^{\mathcal{C}',X'}(\gamma, \xi) \\ b(\gamma, \alpha, \nu) = b'(\gamma, \alpha, \nu). \end{gathered}$$

In particular, the left side of the above relations is defined whenever the right side is defined.

Longer extenders require the second, and more interesting, modification to the covering lemma which was alluded to in observation 7: an extender can not necessarily be reconstructed from its countable sequences of indiscernibles. Again we contrast indiscernibles for extenders with those for measures. Suppose that M is a model of set theory and $C \subseteq \kappa$ is a ω-sequence of indiscernibles for M, in the sense that $U = \{\, x \subseteq \kappa : C - x \text{ is finite} \,\}$ is a normal M-ultrafilter on κ. Then the added hypothesis ${}^{\omega}M \subseteq M$ implies that $C \in M$, so that $U \in M$ and hence $U \in K^M$.

Now if (C, b) is similarly a system of indiscernibles for a M-extender E, then the situation is more complicated. Again, C is an ω-sequence of indiscernibles which generates the normal measure $U = E_\kappa$ associated with E. In fact all of the ultrafilters E_a are members of M, since E_a is generated by the ω-sequence $\langle\, b(\nu, \alpha, a) : \nu \in C \,\rangle)$. It is not clear, however, that these ultrafilters E_a can be reassembled in M to obtain the extender E, and in fact Gitik showed in [21] (see chapter [15]) that this reassembly is not always possible. He also showed that it is possible under the stronger hypothesis of theorem 4.50, namely that $\{\, \alpha < \kappa : o(\alpha) > \alpha^{+n} \,\}$ is bounded in κ for some $n < \omega$. In addition he discovered a game which does provide the desired reassembly, provided that it is applied to a sequence (C, b) of indiscernibles such that $\mathrm{ot}(C)$ has uncountable cofinality. Hence a version of the covering lemma can be obtained for these longer extenders by systematically replacing ω with ω_1 [23]:

4.51 Theorem (Covering up to $0^{\P}$). *Assume that $0^{\P}$ does not exist. Let κ be a cardinal of K with $\mathrm{cf}(\kappa) > \omega$, and set $\lambda = o(\kappa)^{+K}$. Then if $\kappa \not\subseteq X = Y \cap K_\lambda$, where $Y \prec H(\lambda)$ and ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$, then there is a pair $(\mathcal{C}, b)$ such that the conclusion of theorem 4.50 holds, except that clause 3c is modified* *as follows, where we write $a_\iota^{\mathcal{C},X}(\gamma,\xi)$ for the ιth accumulation point for E_γ above ξ:*

(3c′) $\nu = a_\iota^{\mathcal{C},X}(\gamma,\xi)$ *for some* $\xi \in X \cap \nu$*, some* $\gamma \in h\,``(X \cap \nu)$*, and some* $\iota < \omega_1$*.*

Furthermore clause (3c′) *does not hold for any limit point* ν *of* X *with* $\mathrm{cf}(\nu) > \omega$*.*

For details, see [23], which shows that these results are strong enough to give the correct lower bound for the consistency strength of a failure of the singular cardinal hypothesis at a cardinal of cofinality greater than ω.

Up to a Woodin Cardinal

The following form of the weak covering lemma is proved for countably closed cardinals in chapter [46]. This proof originally appeared in [43], and the general case is proved in [42].

4.52 Theorem. *Suppose that there is no inner model with a Woodin cardinal, and that the Steel core model K exists. Then $(\lambda^+)^K = \lambda^+$ for every singular cardinal λ.*

By "the Steel core model K exists" we mean that Steel's construction of the core model up to a Woodin cardinal, described in chapter [46], succeeds in constructing a class model K satisfying the weak covering lemma. It is known that this follows from the assumption that there is a class of subtle cardinals.

The proof involves several technical difficulties which either do not occur or are easily dealt with below $0^\P$, but it closely parallels the earlier proofs. Like the first part of the proof of theorem 4.19 it gives, for any suitable covering set X, a mouse $\widetilde{M}$, a system $\widetilde{\mathcal{C}}$ of indiscernibles for $\widetilde{M}$, and an ordinal $\rho < \kappa$ such that $X = h^{\widetilde{M}}\,``(X \cap \rho; \widetilde{\mathcal{C}})$. However the system $\widetilde{\mathcal{C}}$ of indiscernibles comes from an iteration tree, not a linear iteration, and no known analysis of such indiscernibles yields any useful information. The proof of lemma 4.52 sidesteps this problem: like the proof of the covering lemma 4.5 for sequences of measures, it relies on the observation that there are no measures, and hence no indiscernibles, in the interval $(\lambda, \lambda^{+K}]$.

Theorem 4.52 is actually weaker than it appears at first: its hypothesis that the Steel core model exists has no parallel in the covering lemmas for smaller cardinals. Recall that the proof of the full covering lemma for sequences of measures involved first proving the weak covering lemma for the model K^{c} constructed using countably complete measures, and then defining the true core model K as a elementary submodel of K^{c}. There are at least two problems in extending this procedure past $0^\P$. The most important of these is the fact that countable completeness is not, so far as is known, sufficient to ensure iterability of extender sequences significantly beyond a strong cardinal. Steel, in his construction of K^{c}, replaces countable completeness with a stronger notions, which he calls *countable certification.* However the proof of the weak covering lemma, as given for sequences of measures, does not work for K^{c} as defined from countably certified measures. Instead Steel defines K^{c} by using a measurable cardinal in V, which provides the certification needed to prove that K^{c} satisfies a form of the weak covering lemma which is slightly weaker than the countably closed weak covering property, definition 3.46, but is sufficiently strong to support the definition of K and the proof of the full covering lemma. Further work by Steel, Jensen and others has weakened the strength required to a subtle cardinal; however there is no clear strategy for obtaining the weak covering property with any weaker assumptions. In contrast, Mitchell and Schindler [44] have obtained a model which is iterable and (in what appear to be the appropriate senses) universal with no large cardinal assumptions.

The second problem involves the proof that the iteration from the basic construction in the proof of the core model drops immediately, that is, that $1 \in D$. In the case of the covering lemma for extenders below $0^{\P}$ this argument splits. The argument used to show that the weak covering lemma holds for countably complete cardinals λ is similar to that for sequences of measures but requires an extra assumption that $o(\alpha) < \lambda$ for all $\alpha < \lambda$. The proof of the full covering lemma, on the other hand, uses a different proof relying on the weak covering lemma; it does not show that the iteration drops, but instead shows that even when the iteration does not drop there is still a Skolem function $g^X \in K$ (derived, for a suitable set X, from an extender $\mathcal{E}_\gamma$ of length κ and critical point less than $\inf(\kappa - X)$) such that $X = g^X``(X \cap \rho^X; \mathcal{C}^X)$ for some $\rho^X < \lambda$. Beyond $0^{\P}$ the notion of $o(\alpha)$ is not meaningful, so only the second argument, which requires the weak covering lemma, is usable.

A few other results are known which use the ideas of the covering lemma. One of these is theorem 1.16, asserting that that any Jónsson cardinal κ is Ramsey in K. This proof avoids a measurable cardinal at κ, since if κ were measurable then it would be Ramsey, and it avoids smaller measurable cardinals by selecting a set of indiscernibles witnessing that κ is Ramsey which contains only nonmeasurable cardinals. Others such results demonstrate that certain properties of the smaller core models extend to larger cardinals: Schindler proves in [50] that if M is a model which contains all of its countable subsets then the core model K^M defined inside M is an iterated ultrapower of K, and Gitik, Schindler and Shelah proves in [24] that if $\kappa > \omega_2$ is a cardinal in K then any sound mouse M extending $K \| \kappa$ and projecting to κ is an initial segment of K.

Beyond a Woodin Cardinal

As was pointed out earlier, not even the weak covering lemma is valid for a model containing a Woodin cardinal δ: Woodin has defined a notion of forcing, the nonstationary tower forcing [62], such that the cardinal δ is still Woodin in the generic extension $L[\mathcal{E}][G]$ but there are cofinally many singular cardinals $\lambda < \delta$ such that $\lambda^{+L[\mathcal{E}]} < \lambda^{+L[\mathcal{E}][G]}$. Indeed it seems likely that every sufficiently large successor cardinal less than δ is collapsed by this forcing.

It is possible that this situation is analogous to that of Prikry forcing at a measurable cardinal, in that one could hope for an analogue of the Dodd-Jensen lemma stating that any failure of the weak covering lemma is achieved by some variant of stationary tower forcing. Some very weak results in this direction are proved in [41], but there are many more questions than theorems. One difficulty is that the stationary tower forcing, unlike Prikry forcing, has a number of variants; furthermore there are other forcings, notably Woodin's "all sets generic" forcing, which require a Woodin cardinal in the universe and which may be relevant to this question.

More promising developments deal with core models which do not contain Woodin cardinals, but which are large in the sense that they have inner models with Woodin cardinals. The best result so far is due to Schimmerling and Woodin, in [47]:

4.53 Theorem. *Suppose that $\mathcal{E}$ is a good extender sequence and the model $W = L[\mathcal{E}, x]$ is sufficiently iterable. Then either there is an amenable ultrafilter U on W with* $\operatorname{crit}(W) > \operatorname{rank}(x)$ *such that* $\operatorname{Ult}(W, U)$ *is well-founded, or else W has the weak covering property above* $\operatorname{rank}(x)$.

Using this and unpublished work of Woodin they prove the following theorem on the existence of a core model:

4.54 Theorem. *Suppose that there is an inaccessible cardinal and $L(\mathbb{R})$ does not satisfy the axiom of determinacy in any set generic extension. Then one of the following is true:*

1. *The Steel core model exists, and satisfies the weak covering lemma.*
2. *There is a set x, and a good sequence $\mathcal{E}$ satisfying* $\operatorname{crit}(\mathcal{E}_\gamma) > \operatorname{rank}(x)$ *for all $\gamma \in \operatorname{dom}(\mathcal{E})$, such that the model $W = L[\mathcal{E}, x]$ is iterable, does not contain any measurable cardinals above* $\operatorname{rank}(x)$, *and has the weak covering property at all cardinals larger than* $\operatorname{rank}(x)$.

Notice that a model $L[\mathcal{E}']$ will satisfy clause (2) whenever $\mathcal{E}'$ is not a proper class, even though $L[\mathcal{E}']$ may contain many Woodin cardinals; however in order to satisfy clause (2) we must take $x = \mathcal{E}'$ and let the sequence $\mathcal{E}$ of the lemma be empty.

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