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Stochastic System Identification for Operational Modal Analysis: A Review

This paper reviews stochastic system identification methods that have been used to estimate the modal parameters of vibrating structures in operational conditions. It is found that many classical input-output methods have an output-only counterpart. For instance, the Complex Mode Indication Function (CMIF) can be applied both to Frequency Response Functions and output power and cross spectra. The Polyreference Time Domain (PTD) method applied to impulse responses is similar to the Instrumental Variable (IV) method applied to output covariances. The Eigensystem Realization Algorithm (ERA) is equivalent to stochastic subspace identification. [DOI: 10.1115/1.1410370]

1 Introduction

The application of system identification to vibrating structures yielded a new research domain in mechanical engineering, known as *experimental modal analysis*. In this case, a *modal model*, consisting of eigenfrequencies, damping ratios, mode shapes and modal participation factors, is identified from vibration data. Classically, one applies an artificial, measurable input to the system and one measures the output. From these measurements, the experimental model can be obtained by a variety of parameter estimation methods. However, cases exist where it is rather difficult to apply an artificial force and where one has to rely upon available ambient excitation sources. It is practically impossible to measure this ambient excitation and the outputs are the only information that can be passed to the system identification algorithms. Because in these cases the deterministic knowledge of the input is replaced by the assumption that the input is a realization of a stochastic process (white noise), one speaks of *stochastic system identification*. Specializing to the identification of vibrating structures the terms *output-only modal analysis* and *operational modal analysis* are used. A common problem of operational modal analysis methods is that if the white noise assumption is violated, for instance if the input contains in addition to white noise also some dominant frequency components, these frequency components cannot be separated from the eigenfrequencies of the system and will be identified as such.

The need to perform output-only modal analysis probably emerged first in civil engineering, where it is very difficult and expensive to excite constructions such as bridges and buildings with a hammer or shaker and to obtain artificially induced vibration levels that exceed the natural vibrations due to traffic or wind. Nevertheless, also in mechanical engineering, operational modal analysis proved to be very useful: for instance to obtain the modal parameters of a car during road testing or an aeroplane during flight tests.

Many textbooks exist that give an extensive overview of input-output modal parameter estimation methods [1–4]. We should also mention some recent efforts to compare input-output system identification methods for applications in structural dynamics [5,6]. Individual output-only modal parameter estimation methods are discussed in several papers, but an overview and comparison of different methods are missing. The present paper, which is based on [7], tries to fill that gap. Almost all methods discussed in this paper have successfully been applied to real-life vibration data. These applications are beyond the scope of this paper, but can, for instance, be found in [7–9].

2 Vibrating Structures: Models and Measurement Data

The aim of this section is to introduce some basic concepts that are most helpful in understanding the similarities and differences between stochastic system identification methods.

2.1 Models. One of the first steps of system identification is adopting a certain model structure. Afterwards the parameters of the chosen model are estimated from measurement data. A wide range of model structures is proposed in system identification literature, see for instance Ljung [10]. The general system identification procedure is to try out several model structures without bothering about the underlying physical input-output relations. The aim of this section is to discuss some models that can truly represent a vibrating structure excited by white noise. By consequence, these models are physically meaningful.

Physical Model. The dynamic behavior of a discrete mechanical system consisting of n_2 masses connected through springs and dampers is described by following matrix differential equation:

$$M\ddot{q}(t)+C_2\dot{q}(t)+Kq(t)=f(t) \tag{1}$$

where $M,C_2,K\in\mathbb{R}^{n_2\times n_2}$ are the mass, damping and stiffness matrices; $q(t)\in\mathbb{R}^{n_2}$ is the displacement vector at continuous time t. A dot over a time function denotes the derivative with respect to time. The vector $f(t)\in\mathbb{R}^{n_2}$ is the excitation force. For systems with distributed parameters (e.g., civil engineering structures), Eq. (1) is obtained as the Finite Element (FE) approximation of the system with only n_2 degrees of freedom (DOFs) left. Although the (nearly) physical model (1) is a good representation of a vibrating structure, it is not directly useful in an experimental modelling context. First, it is not possible (and also not necessary) to measure all DOFs of the FE model. Second, this equation is in continuous-time, whereas measurements are available as discrete time samples. And finally, there is some noise modelling needed: there may be other unknown excitation sources apart from $f(t)$ and measurement noise is always present in real life.

Stochastic State-Space Model. It can be shown that, by applying model reduction, sampling and modelling the noise, Eq. (1) can be converted to following *discrete-time stochastic state-space model* (see for instance [7,11] for a detailed derivation):

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$$x_{k+1}=Ax_k+w_k$$
$$y_k=Cx_k+\nu_k \tag{2}$$

where $y_k\in\mathbb{R}^l$ is the sampled output vector (the measurements); $x_k\in\mathbb{R}^n$ is the discrete state vector; $w_k\in\mathbb{R}^n$ is the process noise, typically due to disturbances and modelling inaccuracies, but here also due to the unknown excitation of the structure; $\nu_k\in\mathbb{R}^l$ is the measurement noise, typically due to sensor inaccuracy, but here also due to the unknown excitation of the structure; k is the time instant; l is the number of outputs; n is the system order ($n=2n_2$). The matrix $A\in\mathbb{R}^{n\times n}$ is the state transition matrix that completely characterizes the dynamics of the system by its eigenvalues; and $C\in\mathbb{R}^{l\times n}$ is the output matrix that specifies how the internal states are transformed to the outside world. The noise vectors are both unmeasurable vector signals assumed to be zero mean, white and with covariance matrices:

$$\mathbf{E}\left[\begin{pmatrix} w_p \\ \nu_p \end{pmatrix}(w_q^T \nu_q^T)\right]=\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}\delta_{pq} \tag{3}$$

where $\mathbf{E}$ is the expected value operator; δ_{pq} is the Kronecker delta.

It is precisely such a stochastic state-space model that will be identified when using so-called subspace identification methods (see Sections 4.2 and 5.1). In a second step, the modal parameters are obtained from the matrices A and C. The derivation starts with the eigenvalue decomposition of A:

$$A=\Psi\Lambda_d\Psi^{-1} \tag{4}$$

where $\Psi\in\mathbb{C}^{n\times n}$ is the eigenvector matrix and $\Lambda_d\in\mathbb{C}^{n\times n}$ is a diagonal matrix containing the discrete-time eigenvalues. The eigenfrequencies ω_i and damping ratios ξ_i are found from:

$$\mu_i=e^{\lambda_i\Delta t};\quad \lambda_i,\lambda_i^*=-\xi_i\omega_i\pm j\sqrt{1-\xi_i^2}\,\omega_i \tag{5}$$

where Δt is the sampling time. Finally, the mode shapes $V\in\mathbb{C}^{l\times n}$ are found as:

$$V=C\Psi \tag{6}$$

ARMA Model. The more classical system identification methods [10] identify models that do not contain the state. It can be shown [12,13] that the following so-called ARMA model is equivalent to the stochastic state-space model (2):

$$y_k+\alpha_1 y_{k-1}+\cdots+\alpha_p y_{k-p}=e_k+\gamma_1 e_{k-1}+\cdots+\gamma_p e_{k-p} \tag{7}$$

where, as before, y_k is the output vector and $e_k\in\mathbb{R}^l$ is a white noise vector sequence. The left-hand side is called the Auto-Regressive (AR) part and the right-hand side the Moving Average (MA) part, hence the name of the model. The matrices $\alpha_i\in\mathbb{R}^{l\times l}$ are the AR matrix parameters; matrices $\gamma_i\in\mathbb{R}^{l\times l}$ are the MA matrix parameters. Sometimes, as in the present case of multiple outputs, one speaks of ARMAV models as to stress their multivariable character. An ARMA model that is deduced from a state-space model has the same AR order as MA order. This is denoted as p in Eq. (7). The ARMA model order is related to the state-space model order as: $pl=n$. Since it is derived from a stochastic state-space model, also an ARMA model can truly represent a vibrating structure.

The modal parameters can be computed from the ARMA model by the eigenvalue decomposition of the companion matrix of the AR polynomial:

$$\begin{pmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I \\ -\alpha_p & -\alpha_{p-1} & \cdots & -\alpha_1 \end{pmatrix}\begin{pmatrix} V \\ V\Lambda_d \\ \cdots \\ V\Lambda_d^{p-2} \\ V\Lambda_d^{p-1} \end{pmatrix}=\begin{pmatrix} V \\ V\Lambda_d \\ \cdots \\ V\Lambda_d^{p-2} \\ V\Lambda_d^{p-1} \end{pmatrix}\Lambda_d \tag{8}$$

The observed mode shapes V are the first l rows of the eigenvector matrix. As in the state-space case, the eigenfrequencies and damping ratios can be computed from the discrete eigenvalues in Λ_d; see Eq. (5).

Frequency-Domain Model. Many identification methods identify frequency-domain models from samples of the Fourier transform of the measurement signals. Frequency-domain models are readily obtained by applying the z-transform to the discrete-time models (2), (7). For instance, the output power spectrum matrix of a state-space system can be written as [14]:

$$S_y(z)=C(zI-A)^{-1}G+R_0+G^T(z^{-1}I-A^T)^{-1}C^T\big|_{z=e^{j\omega\Delta t}} \tag{9}$$

where $S_y(z)\in\mathbb{C}^{l\times l}$ is the spectrum matrix containing the power and cross spectra between the outputs. The power spectra are real and located on the main diagonal. This expression can be evaluated for any number on the unit circle $z=e^{j\omega\Delta t}$ where ω [rad/s] can be any frequency of interest. Matrix $G\in\mathbb{R}^{n\times l}$ is the next state—output covariance matrix and $R_0\in\mathbb{R}^{l\times l}$ is the zero-lag output covariance matrix:

$$G=\mathbf{E}[x_{k+1}y_k^T],\quad R_0=\mathbf{E}[y_k y_k^T] \tag{10}$$

An in structural dynamics more common form of the output power spectrum matrix is obtained by applying the Laplace transform to a continuous-time model and introducing the modal parameters (see [7] for a detailed derivation):

$$S_y(s)=\left(\sum_{i=1}^{n}\frac{1}{s-\lambda_i}\{\nu_i\}\langle l_i^T\rangle\right)R_u\left(\sum_{i=1}^{n}\frac{1}{s^*-\lambda_i}\{l_i\}\langle\nu_i^T\rangle\right)\Bigg|_{s=j\omega} \tag{11}$$

where s is the Laplace variable; $\{\nu_i\}\in\mathbb{C}^l$ is the ith modal vector; $\langle l_i^T\rangle\in\mathbb{C}^m$ is the ith modal participation vector; m is the number of (white noise) inputs; $R_u\in\mathbb{R}^{m\times m}$ is the white noise input covariance matrix.

2.2 Measurement Data. In principle (output) data y_k are available as discrete samples of the time signal. The identification methods of Section 5 will be able to use directly these time signals. However, many system identification methods exist that require other types of data.

The identification methods of Section 4 require output covariances as primary data type. Output covariances are defined as:

$$R_i=\mathbf{E}[y_{k+i}y_k^T]=\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}y_{k+i}y_k^T \tag{12}$$

where the second equation follows from the ergodicity assumption. Of course, in reality, a finite number N of data is available and a covariance estimate is simply obtained by dropping the limit in (12).

The identification methods of Section 3 require a frequency domain representation of the output signals. The frequency-domain representation of stochastic signals is provided by the (power) spectrum $S_y\in\mathbb{C}^{l\times l}$, defined as the discrete-time Fourier transform of the covariance sequence:

$$S_y(e^{j\omega\Delta t})=\sum_{k=-\infty}^{\infty}R_k e^{-j\omega k\Delta t} \tag{13}$$

For details on methods to estimate covariances and spectra from measured time data, we refer to the extensive literature that exist on the subject; see for instance [15,16].

3 Frequency-Domain Spectrum-Driven Methods

The presentation order of the identification methods roughly corresponds to the historical application of stochastic system identification: from picking the peaks in spectrum plots (Section 3.1)

to time-domain subspace methods that make extensively used of concepts from numerical linear algebra (Section 5.1).

3.1 The Peak-Picking Method (PP). The simplest approach to estimate the modal parameters of a structure subjected to ambient loading is the so-called Peak-Picking (PP) method. The method is named after the key step of the method: the identification of the eigenfrequencies as the peaks of a spectrum plot. The method is for instance discussed in [15,17].

Under the conditions of low damping and well-separated eigenfrequencies, the spectrum (11) around an eigenfrequency ω_i can be approximated by:

$$S_y(j\omega_i) \approx \alpha_i \{\nu_i\} \langle \nu_i^H \rangle \tag{14}$$

where α_i is a scale factor depending on the damping ratio, the eigenfrequency, the modal participation factor and the input covariance matrix. Vector $\{\nu_i\}$ is the ith mode shape. From (14) it is clear that each column or row of the spectrum matrix at an eigenfrequency can be considered as an estimate of the mode shape at that frequency. Therefore it suffices to compute only 1 column or row of the spectrum matrix. In order to obtain damping ratios, it is often suggested to use the half-power bandwidth method, which is able to quantify the sharpness of a resonance peak. It is, however, widely accepted that this estimate is not a very accurate one.

Some refinements of the PP method exist. The coherence function between two channels tends to go to one at the resonance frequencies because of the high signal-to-noise ratio at these frequencies. Consequently, inspecting the coherence function can assist in selecting the eigenfrequencies. Also, the phase angles of the cross spectra are helpful: if real modes are expected, the phase angles should be either 0 or 180 deg at the resonance frequencies.

A violation of the basic assumptions (low damping and well-separated frequencies) leads to erroneous results. In fact the method identifies *operational deflection shapes* instead of mode shapes and for closely spaced modes such an operational deflection shape will be the superposition of multiple modes. Other disadvantages are that the selection of the eigenfrequencies can become a subjective task if the spectrum peaks are not very clear and that the eigenfrequencies have to be a subset of the discrete frequency values of the discrete Fourier transform.

Despite these drawbacks many civil engineering cases exist where the method is successfully applied; see for instance [18,19]. The popularity of the method is due to its implementation simplicity and its speed: the only algorithm that is needed is the Fast Fourier Transform (FFT) to convert time data to spectra. The computational speed can become irrelevant though, because of the large amount of user interaction needed to try to improve the estimation results.

3.2 The Complex Mode Indication Function (CMIF). A more advanced method consists of computing the Singular Value Decomposition (SVD) of the spectrum matrix:

$$S_y(j\omega) = U(j\omega)\Sigma(j\omega)U^H(j\omega) \tag{15}$$

where $U \in \mathbb{C}^{l \times l}$ is a complex unitary matrix containing the singular vectors as columns. The diagonal matrix $\Sigma \in \mathbb{R}^{l \times l}$ contains the real positive singular values in descending order. This ''method based upon the diagonalization of the spectral density matrix,'' as it was called, was already used in the beginning of the eighties to obtain the modes of a vibrating system subjected to natural excitation [20]. Some years later, the method was also applied to Frequency Response Functions (FRFs) and became known as the Complex Mode Indication Function (CMIF). As suggested by the name, the CMIF was originally intended as a tool to count the number of modes that is present in measurement data. As a useful byproduct, the CMIF also identifies the modal parameters from FRFs [21]. Recently, the spectrum-driven method received again attention as an alternative for the PP method in civil engineering applications [22]. In this paper, the old method was given a new name, the frequency-domain decomposition method.

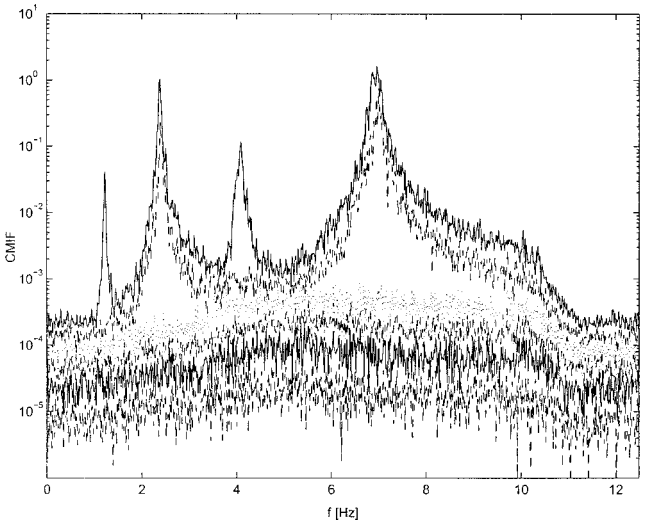

Fig. 1 The complex mode indication function (CMIF). The singular values of the spectrum matrix are plotted as a function of the frequency. Around 2.4 Hz and 7 Hz, two singular values are significant, indicating that there are two close modes.

The method is based on the fact that the transfer function or spectrum matrix evaluated at a certain frequency is only determined by a few modes. The number of significantly contributing modes determines the rank of the spectrum matrix. The SVD is typically used for estimating the rank of a matrix: the number of nonzero singular values equals the rank [23]. The singular values as a function of frequency $\Sigma(j\omega)$ is the actual CMIF; see Fig. 1 for an example. Therefore, plotting the CMIF yields the eigenfrequencies as local maxima. The CMIF is also able to detect closely spaced modes: more than one singular value will reach a local maximum around the close eigenfrequencies.

If only 1 mode is important at a certain eigenfrequency ω_i, the spectrum approximates a rank-one matrix and can be decomposed as (15):

$$S_y(j\omega_i) \approx \sigma_1(j\omega_i)\{u_1(j\omega_i)\}\langle u_1^H(j\omega_i)\rangle \tag{16}$$

By comparing (16) with (14), it is evident that the first singular vector at resonance is an estimate of the mode shape at that frequency. In case of mode multiplicity at a resonance frequency, every singular vector corresponding to a nonzero singular value is considered as a mode shape estimate.

In some sense, the CMIF method can be considered as an SVD extension of the PP method. The SVD is able to resolve mode multiplicity. The method can also be applied to a reduced spectrum matrix, where only the spectra between a chosen set of reference sensors and all outputs have to be computed. In this case, the maximum number of detectable multiple poles cannot exceed the smallest dimension of the reduced spectrum matrix.

Extensions of the CMIF method are possible that do estimate eigenfrequencies and damping ratios differently as in the PP method. After applying the SVD to the spectrum matrix, this matrix is in fact decomposed in single-DOF systems. To such a system, single-DOF modal parameter estimation methods could be applied, extensively documented in the modal analysis literature [1–4].

3.3 Maximum Likelihood Identification (ML). Contrary to the PP method or the CMIF which considers only one mode at a time, this method estimates the parameterized spectrum matrix as a whole. Maximum Likelihood (ML) identification is an optimization-based method that estimates the parameters of a model by minimizing an error norm. A discussion on the use of the ML estimator to identify parametric frequency-domain models

can be found in [24,25]. The ML method results in equations that are nonlinear in the unknown parameters. This requires an iterative procedure. Therefore it is no surprise that often mentioned drawbacks of ML estimators are the high computational load and the fact that they are not suited to handle large amounts of data. During the last years attention has been paid to the optimization of the ML method: the algorithm has been modified to keep the memory requirements as low as possible; and using an adapted parameterization and fast signal processing techniques, an important reduction of the computation time was possible. It has been shown that ML identification is a robust method to find the modal parameters of a structure from a large and noisy data set [26,27]. Originally intended for application to FRFs, the method was extended to use spectra as primary data, so that it also could be used in output-only cases [28].

4 Time-Domain Covariance-Driven Methods

It is known for some time that there exist similar mathematical expressions for impulse responses and output covariances (of a system excited by white noise) as a function of the system parameters; see for instance [12,15]. In modal analysis applications, this observation is used to feed classical impulse response based modal parameter estimation methods with output covariances instead [29]. Two such methods are discussed below.

4.1 The Instrumental Variable Method (IV). In this section, a method belonging to the class of so-called Instrumental Variable (IV) methods will be discussed. Although derived in a different way, the final equations of the IV method correspond to the Polyreference Time Domain (PTD) method after substituting impulse responses by output covariances. The PTD method is probably the most widely used traditional modal parameter estimation method. It contains the (Least Squares) Complex Exponential (LSCE) and the Ibrahim Time Domain (ITD) methods as special cases. For an overview, relations between these traditional (input-output) methods and the original references, see [1,3,30,31]. A more generic discussion and more references on IV methods can be found in [10].

An ARMA model (7) of suitable order can represent a vibrating structure. Unfortunately, the application of a classical *prediction error method* [10] to an ARMA model results in a highly nonlinear parameter estimation problem; see also Section 5.2. The nonlinearity is caused by the MA parameters. The advantage of the IV method is that it identifies only the AR parameters (and that this is achieved in a linear way), while the underlying model structure still is an ARMA model.

The idea of system identification is to ''fit'' a model to measured data y_k. A good parameter estimation method should extract the maximum information from the data, leaving residuals e_k that are uncorrelated with past data. This is formally written as:

$$\forall i>0 \ : \ \mathbf{E}[e_k y_{k-i}^T]=\mathbf{E}[e_k]\mathbf{E}[y_{k-i}^T]=0 \tag{17}$$

where the first equality says that e_k and y_{k-i} are uncorrelated; and the second equality follows from the zero-mean property of the noise sequence. If, on the contrary the residuals are correlated with past data, they still contain useful but unmodelled information and the model is not ideal. The derivation of the IV method starts by imposing conditions like (17) to the ARMA model (7) in order to get rid of the right-hand side (the MA part). The ''oldest'' noise term is e_{k-p}; so by post-multiplying the ARMA model by y_{k-p-i}^T (for $i>0$) and by taking the expectation we obtain:

$$\forall i>0 \ : \ \mathbf{E}[y_k y_{k-p-i}^T]+\alpha_1\mathbf{E}[y_{k-1}y_{k-p-i}^T]+\cdots + \alpha_p\mathbf{E}[y_{k-p}y_{k-p-i}^T]=0 \tag{18}$$

Because of stationarity and Eq. (12), we have: $\mathbf{E}[y_k y_{k-i}^T]=\mathbf{E}[y_{k+i}y_k^T]=R_i$; and the basic IV equation can be written in terms of the output covariances R_i:

$$\forall i>0 \ : \ R_{p+i}+\alpha_1 R_{p+i-1}+\cdots+\alpha_p R_i=0 \tag{19}$$

By replacing the output covariances by their estimates and writing down the equation for all available time lags i, the AR parameters $\alpha_1, \ldots, \alpha_p$ can be estimated by solving the resulting overdetermined set of equations in a least squares sense. Finally, the eigenvalues and the observed mode shapes are obtained from the eigenvalue decomposition of the companion matrix of the AR coefficients; see Eq. (8).

As in the previous methods of Section 3, it is also in the case of the IV method possible to reduce the dimensions of the involved matrices and the related computational effort by making use of a subset of reference sensors. Only the covariances between all outputs and this subset have to be computed. Details can be found in [7]. This corresponds to classical modal analysis, where the impulse response matrices are rectangular matrices having l rows (i.e., the number of outputs) and m columns (i.e., the number of inputs). In output-only cases, the impulse responses are substituted by output covariances and the inputs by the reference outputs; see also [8,29].

A typical problem of estimating a parametric model from data is the determination of the model order. A pth order ARMA model based on l outputs contains pl poles. Consequently, the ''expected'' number of poles covered by the data gives an indication of the model order. This expected number can be based on physical insight or counted as two times the number of peaks in the frequency-plot of a nonparametric spectrum estimate; see also the PP method, Section 3.1. A more accurate model order estimate is provided by the CMIF, a frequency-plot of the singular values of a nonparametric spectrum estimate (Section 3.2).

More formal procedures estimate models of different order and compare these models according to a quality criterion such as Akaike's Final Prediction Error or Rissanen's Minimum Description Length criterion [10].

However, in modal analysis one is usually not interested in a good model as such, but rather in the modal parameters extracted from that model. Practical experience with parametric models in modal analysis applications learned that it is better to over-specify the model order and to eliminate spurious numerical poles afterwards, so that only true physical system poles are left. The famous stabilization diagram [1,3] is a great tool to achieve this goal. The poles corresponding to a certain model order are compared to the poles of a one-order-lower model. If the eigenfrequency, the damping ratio and the related mode shape (or modal participation factor) differences are within preset limits, the pole is labeled as a stable one. The spurious numerical poles will not stabilize at all during this process and can be sorted out of the modal parameter data set more easily. Such a stabilization diagram is represented in Fig. 2.

Interesting to note and very relevant for civil engineering practice is that the IV method is robust against nonstationary inputs (i.e., white noise with time-varying covariances). This does not only follow from practical experience but has also been theoretically proven in [32].

4.2 Covariance-Driven Stochastic Subspace Identification (SSI-COV). Like the CMIF method can be considered as an SVD-enhanced PP method, covariance-driven subspace identification can—somewhat disrespectfully—be considered as an SVD-enhanced instrumental variable method. The covariance-driven Stochastic Subspace Identification method (SSI-COV) is addressing the so-called *stochastic realization problem*, i.e., the problem of identifying a stochastic state-space model (2) from output-only data.

Stochastic realization is closely related to deterministic (input-output) realization, that goes back to Ho and Kalman [33] and was

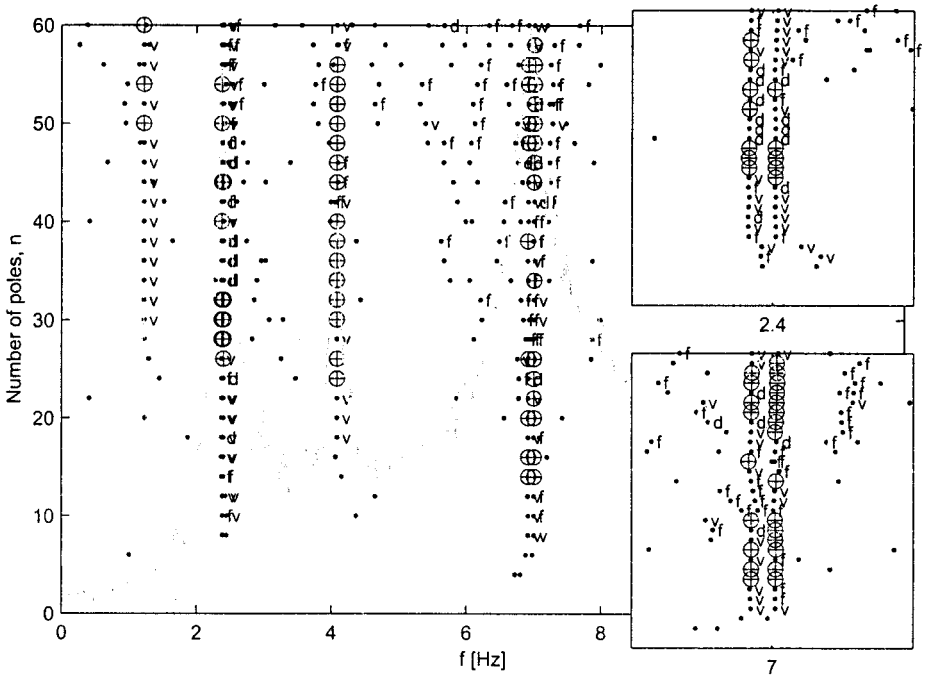

Fig. 2 Stabilization diagram obtained with the IV method. The used symbols are: "⊕" for a stable pole; ".v" for a pole with stable frequency and vector; ".d" for a pole with stable frequency and damping; ".f" for a pole with stable frequency and "." for a new pole. Two zooms are added that concentrate on the close modes around 2.4 and 7 Hz.

extended with the SVD to treat noisy data in [34] and [35]. The so-called Eigensystem Realization Algorithm (ERA), developed by Juang [36,37], is a modal analysis application of these deterministic realization algorithms. The stochastic (output-only) realization problem is solved in [12,38–40]. Application of stochastic realization to modal parameter estimation was reported by Benveniste and Fuchs [32]. They also proved that their algorithm is robust against nonstationary inputs (e.g., a white noise sequence with time-varying covariance).

Stochastic realization relies upon a fundamental property of stochastic state-space systems. It can be proven (see for instance [41,42]) that the output covariances R_i (12) can be decomposed as:

$$R_i = CA^{i-1}G \tag{20}$$

where G is defined in (10). The covariance sequence R_i can be estimated from measurement data; so it remains to be solved how to decompose the covariances as to find A, C, G. Hereto, a block Toeplitz matrix $T_{1|i} \in \mathbb{R}^{li \times li}$ is formed that consists of covariances:

$$T_{1|i} = \begin{pmatrix} R_i & R_{i-1} & \cdots & R_1 \\ R_{i+1} & R_i & \cdots & R_2 \\ \cdots & \cdots & \cdots & \cdots \\ R_{2i-1} & R_{2i-2} & \cdots & R_i \end{pmatrix}$$

$$= \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{i-1} \end{pmatrix} (A^{i-1}G \cdots AG \;\; G) \updownarrow n = O_i \Gamma_i \tag{21}$$

$$\underset{n}{\longleftrightarrow}$$

The second equality follows from applying property (20). The definitions of the *extended observability matrix* $O_i \in \mathbb{R}^{li \times n}$ and the *reversed extended stochastic controllability matrix* $\Gamma_i \in \mathbb{R}^{n \times li}$ are obvious from (21). For $li > n$, and in case of an observable and controllable system, the rank of the $li \times li$ Toeplitz matrix equals n, since it is the product of a matrix with n columns and a matrix with n rows.

The actual implementation of SSI-COV consists of estimating the covariances R_i, computing the SVD of $T_{1|i}$, truncate the SVD to the model order n, estimating O_i and Γ_i by splitting the SVD in two parts and finally estimating A, C, G from O_i and Γ_i. The modal parameters are found from A and C as indicated in (4) and (6). Implementation details can be found in the above-cited references. Again, it is possible to slightly reformulate the SSI-COV method, so that it only needs the covariances between all outputs and a set of references [7,8].

In theory, the system order n can be determined by inspecting the number of nonzero singular values of $T_{1|i}$ (21). In practice, however, the estimated covariance Toeplitz matrix is affected by "noise" leading to singular values that are all different from zero. As typical noise sources we have:

- Modelling inaccuracies. It is possible that the true system that generated the data cannot be modelled exactly as a stochastic state-space model. An attempt to model this system by a state-space model introduces an error in these cases.
- Measurement noise: introduced by the sensors and the electronics of the measurement hardware.
- Computational noise due to the finite precision of any computer.
- The finite number of data. The covariances have to be estimated, so that the factorisation property (20) does not hold exactly. As a consequence, the rank of the covariance Toeplitz matrix will not be exactly n.

Sometimes it is suggested to look at the "gap" between two successive singular values. The singular value where the maximal gap occurs yields the model order. This criterion should, however, not be applied too dogmatically, since in most practical cases there is no gap. Other statistical methods to determine the model order are discussed in [43].

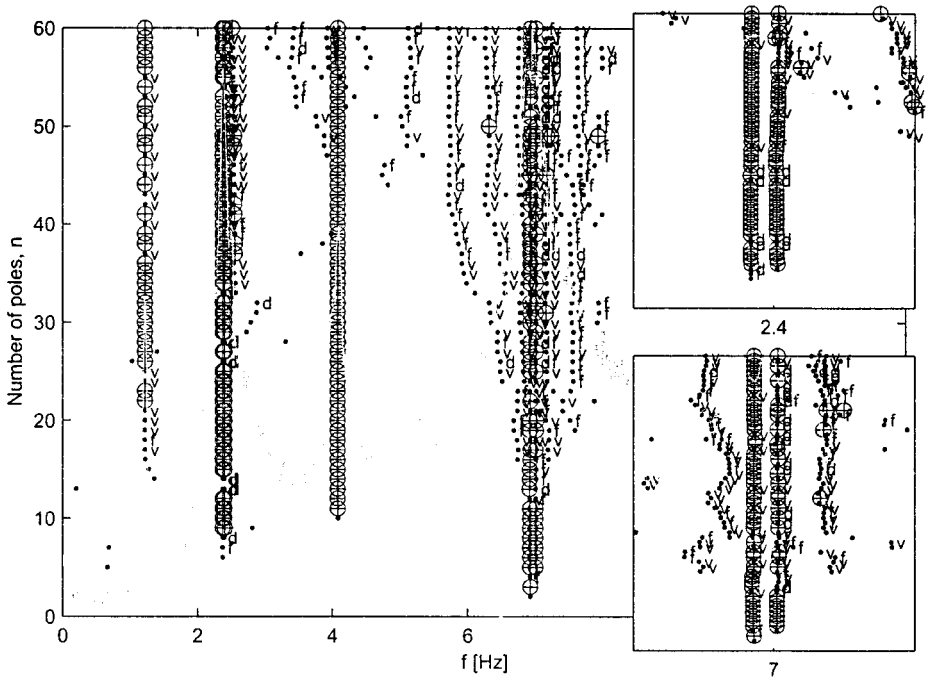

Fig. 3 Stabilization diagram obtained with the SSI-COV method. By comparing this diagram with the IV diagram (Fig. 2), it is clear that the IV method requires higher model orders to find stable poles.

However, to obtain a good model for modal analysis applications, it is probably a better idea to construct a stabilization diagram, by identifying a whole set of models with different order. The stabilization diagram was already introduced in Section 4.1. In case of the SSI-COV method, an efficient construction of the stabilization diagram is achieved by computing the SVD of the covariance Toeplitz matrix only once. Models of different order are then obtained by including a different number of singular values and vectors in the computation of O_i and Γ_i. A stabilization diagram obtained with the SSI-COV method is shown in Fig. 3.

5 Time-Domain Data-Driven Methods

5.1 Data-Driven Stochastic Subspace Identification (SSI-DATA). Recently, a lot of research effort in the system identification community was spent to subspace identification as evidenced by the book of Van Overschee and De Moor [42] and the second edition of Ljung's book [10]. Subspace methods identify state-space models from (input and) output data by applying robust numerical techniques such as *QR* factorization, SVD and least squares. As opposed to SSI-COV, the DATA-driven Stochastic Subspace Identification method (SSI-DATA) avoids the computation of covariances between the outputs. It is replaced by projecting the row space of future outputs into the row space of past outputs. In fact, the notions covariances and projections are closely related. They both are aimed to cancel out the (uncorrelated) noise. The first SSI-DATA algorithms can be found in [44]. A general overview of data-driven subspace identification (both deterministic and stochastic) is provided in [42].

Although somewhat more involved as compared to previous methods, it is also possible with SSI-DATA to reduce the dimensions of the matrices (and the memory requirements of the algorithm) by introducing the idea of the reference sensors. This is demonstrated in [7,11]. It is beyond the scope of this paper to explain the SSI-DATA method in detail. The interested reader is referred to the above-cited literature.

Covariance-Driven Versus Data-Driven Subspace Identification. At this point it is useful to discuss the similarities and differences between the SSI-COV (Section 4.2) and the SSI-DATA method (Section 5.1). First the similarities. Both methods start with a data reduction step. In the SSI-COV algorithm the raw time histories y_k, consisting of l channels and N data points (with $N \rightarrow \infty$), are converted to the covariances of the Toeplitz matrix (21). The number of elements is reduced from $l \times N$ to $li \times li$. In the SSI-DATA algorithm a similar reduction is obtained by projecting the row space of the future outputs into the row space of the past outputs. This projection is computed from the *QR* factorization of a big data Hankel matrix; see [7,11,42]. A significant data reduction is obtained because only a part of the R factor is needed in the sequel of the algorithm. Both methods then proceed with an SVD. This decomposition reveals the order of the system and the column space of O_i (21).

Several variants of stochastic subspace identification exist. They differ in the weighting of the data matrices before the application of the SVD. This weighting determines the state-space basis in which the identified model will be identified. Equivalent implementations exist for both SSI-COV and SSI-DATA. More details can be found in [40] and [42]. Well-known variants are Canonical Variate Analysis (CVA), Principal Components (PC) or Unweighted Principal Components (UPC). This last variant is sometimes also called Balanced Realization (BR).

There are also differences between the covariance-driven and data-driven approaches. In the SSI-COV method, the covariances can be computed in a very fast way by using the FFT algorithm [15,16]. The corresponding step in SSI-DATA is the relatively slow *QR* factorisation. Therefore SSI-COV is much faster than SSI-DATA. In favor of the data-driven method is that it is implemented as a numerically robust square root algorithm: the output data is not squared up as in the covariance-driven algorithm.

We should add that in practical applications of any of the variants of stochastic subspace identification, no accuracy differences can be observed when looking at the identified modal parameters.

5.2 Other Data-Driven Methods

The prediction Error Method Applied to an ARMA Model. Prediction Error Methods (PEM) can be considered as a general system identification framework [10]. These methods identify the parameters of a model by minimizing the so-called prediction errors. The straightforward application of PEM to estimate an ARMA model (7) from data results in a highly nonlinear optimization problem with related problems as: convergence not being guaranteed, local minima, sensitivity to initial values and a high computational load. In contrast to the nonlinear frequency-domain ML method (see Section 3.3), nonlinear time-domain methods (such as PEM applied to an ARMA model) never reached an acceptable level of robustness and applicability for real-life data [45–47]. Despite these drawbacks, some authors tried to apply the PEM to identify the modal parameters of civil engineering structures; see for instance [48,49].

The Prediction Error Method Applied to an AR Model. The nonlinearity of the PEM is caused by the MA part of the ARMA model (7). By omitting the moving-average part, an autoregressive model is obtained:

$$y_k + \alpha_1 y_{k-1} + \cdots + \alpha_p y_{k-p} = e_k \tag{22}$$

and the PEM reduces to a linear least squares problem, which is easily solved. Unfortunately, a pth-order AR model is not an equivalent representation of a vibrating structure with pl modes. The use of an AR model as a substitution of an ARMA model can only be justified if the AR model order goes to infinity: $p \to \infty$ [10]. In practice this means that many spurious poles will be introduced that need to be separated from the true system poles. The use of AR models for modal parameter estimation is, for instance, demonstrated in [50,51].

6 Experimental Comparison

In this section, the accuracy of the system identification results of the different methods in terms of the modal parameters are compared by means of a Monte-Carlo analysis consisting of 100 simulation runs.

The mast structure, shown in Fig. 4, is subjected to independent white noise inputs at all horizontal translation DOFs. The responses at 6 horizontal DOFs are simulated and afterwards contaminated by white measurement noise with $N/S = 10\%$ (N/S is the ratio of the rms values of the noise and output signal). The noisy outputs are then fed to 5 system identification methods: PP (Section 3.1), CMIF (Section 3.2), IV (Section 4.1), SSI-COV

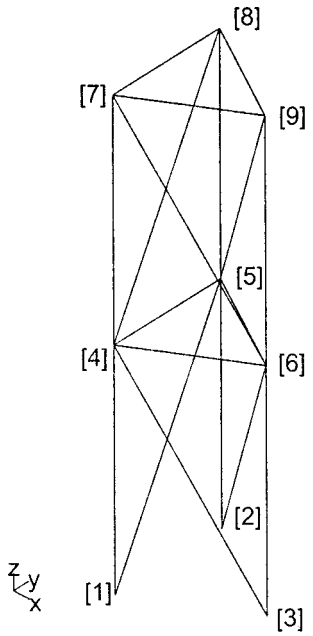

Fig. 4 FE model of the mast structure used in the Monte-Carlo analysis

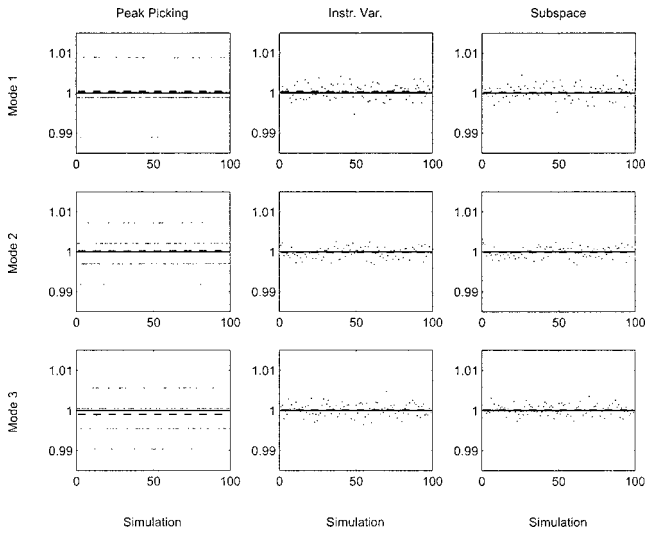

Fig. 5 Eigenfrequency estimation results from 100 Monte-Carlo simulations. The estimates are divided by the true values (a value of 1 on the graphs indicates a perfect estimate). These relative frequencies are shown as dots. The scatter of this quantity gives an idea about the variance of the estimate. The average estimate is also shown (as a dashed line). The deviation of this quantity from 1 (full line) corresponds to the bias of the estimate. The rows show the 3 modes; the columns represent the results of 3 identification methods: PP, IV and SSI-DATA.

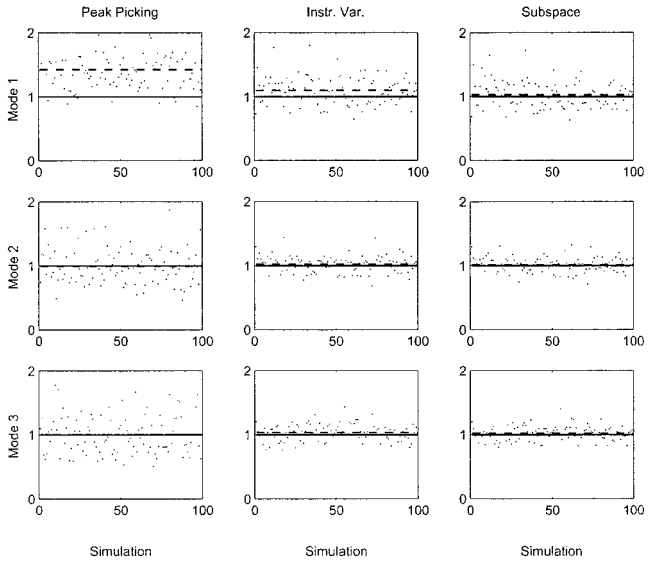

Fig. 6 Damping ratio estimation results from 100 Monte-Carlo simulations. The estimates are divided by the true values (a value of 1 on the graphs indicates a perfect estimate). These relative damping ratios are shown as dots. The scatter of this quantity gives an idea about the variance of the estimate. The average estimate is also shown (as a dashed line). The deviation of this quantity from 1 (full line) corresponds to the bias of the estimate. The rows show the 3 modes; the columns represent the results of 3 identification methods; PP, IV and SSI-DATA.

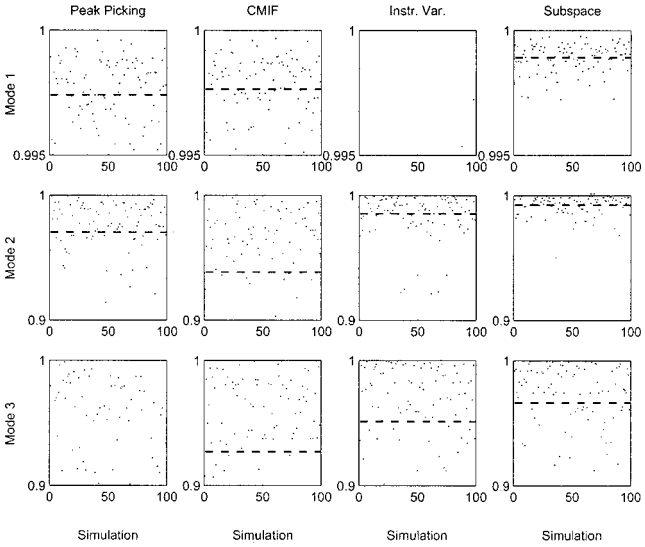

Fig. 7 Mode shape estimation results from 100 Monte-Carlo simulations. The correlation between the estimated and the true mode shapes are shown (as dots). The average correlation is also shown (as a dashed line). The rows show the 3 modes; the columns represent the results of 4 identification methods: PP, CMIF, IV and SSI-DATA. The scaling of the y-axis varies in vertical direction (to accommodate to the changing estimation quality of the different modes), but not in horizontal direction, allowing an easy comparison of the methods.

(Section 4.2) and SSI-DATA (Section 5.1). More details about the structure and the simulations can be found in [7].

Figures 1–3, which were already introduced previously, contain intermediate identification results of CMIF, IV and SSI-COV. As explained in Sections 4.1 and 4.2, no order selection criteria as such is applied. The stable poles are selected from the stabilization diagrams and they don't have to originate from one model. The modal parameter estimation results for the first three modes are represented in Figs. 5–7. In our discussion of the CMIF method, we did not include an alternative frequency or damping estimation procedures as compared to the PP method. The only difference is that the CMIF can detect closely spaced modes and finds the eigenfrequencies in a more objective way. Therefore the CMIF frequencies and damping ratios are not presented in the figures. The results of SSI-COV and SSI-DATA are so close to each other, that only SSI-DATA is presented.

The eigenfrequency estimates of the PP method can only take the discrete values determined by the frequency resolution of the spectrum (Fig. 5). All methods yield unbiased eigenfrequency estimates. Although still small, the standard deviation of the PP estimates is three times higher than for the other methods.

When looking at the damping estimates (Fig. 6), the high bias of the PP damping estimates is striking. It is rather a coincidence that mode 2 and 3 have unbiased damping estimates, as the situation changes when choosing different options for the non-parametric spectrum estimate (resolution, window, overlap, averages).

Regarding the mode shape estimates (Fig. 7), the IV estimates for the first mode are too bad to fit into the scales. Also the average correlation of the PP estimates of the third mode could not be represented. The subspace methods clearly outperform the others.

7 Conclusions

This paper reviewed stochastic system identification methods for operational modal analysis.

The basic peak-picking method (PP) finds the eigenfrequencies as the peaks of non-parametric spectrum estimates. This frequency selection procedure becomes a subjective task in case of noisy operational data, weakly excited modes and relatively close eigenfrequencies. The related half-power bandwidth damping estimation method is unreliable; and operational deflection shapes are identified instead of mode shapes.

The complex mode indication function (CMIF) is an SVD-extension of the PP method, allowing for a more objective selection of the eigenfrequencies and the identification of closely spaced modes. In seems however that the mode shape estimation quality depends on the selected singular vector around resonance (and that it is not always the vector at resonance that gives the best estimates).

The parametric methods (IV, SSI-COV, SSI-DATA) share the advantage that stabilization diagrams can be constructed by identifying parametric models of increasing order. These diagrams are very valuable in separating the true system poles from the spurious numerical poles.

The instrumental-variable method (IV) does not involve an SVD and consequently suffers from the lack of a noise-truncating mechanism. This is reflected in the fact that the mode shape estimates are less accurate than in the subspace methods and that higher order models are required to obtain good modal parameter estimates. A lot of additional poles are necessary for fitting the noise.

Both covariance- (SSI-COV) and data-driven subspace methods (SSI-DATA) seem to perform equally well concerning modal parameter estimation performance, although theoretically the numerical behavior of SSI-DATA should be better than that of SSI-COV since it avoids to square up the data. The SSI-COV method is considerably faster than the SSI-DATA method if its data-reduction step is carried out by the FFT, whereas SSI-DATA requires a slower *QR* factorization step. Evidently, because it only uses linear numerical algorithms, the SSI-DATA method is still much faster than non-linear prediction error methods that are sometimes proposed to estimate the modal parameters from operational data.

As was indicated in this paper, a lot of classical input-output methods carry over (after some modifications) to the output-only case. FRF-driven methods can be converted to spectrum-driven methods; impulse-response-driven methods are almost identical to output-covariance-driven methods and input-output data-driven methods are very similar to output-only data-driven methods.

References

[1] Heylen, W., Lammens, S., and Sas, P., 1995, *Modal Analysis Theory and Testing*, Department of Mechanical Engineering, Katholieke Universiteit Leuven, Leuven, Belgium.
[2] Maia, N. M. M., Silva, J. M. M., He, J., Lieven, N. A. J., Lin, R. M., Skingle, G. W., To, W.-M., and Urgueira, A. P. V., 1997, *Theoretical and Experimental Modal Analysis*, Research Studies Press, Taunton, Somerset, UK.
[3] Allemang, R. J., 1999, *Vibrations: Experimental Modal Analysis, Course Notes*, Seventh Edition, Structural Dynamics Research Laboratory, University of Cincinnati, OH. [http://www.sdrl.uc.edu/course_info.html].
[4] Ewins, D. J., 2000, *Modal Testing: Theory, Practice and Application*, Research Studies Press, Baldock, Hertfordshire, UK.
[5] Abdelghani, M., Verhaegen, M., Van Overschee, P., and De Moor, B., 1998, "Comparison study of subspace identification methods applied to flexible structures," Mech. Syst. Signal Process., **12**, No. 5, pp. 679–692.
[6] Petsounis, K. A., and Fassois, S. D., 2001, "Critical comparison and assessment of parametric time domain methods for the identification of vibrating structures." Mech. Syst. Signal Process., accepted for publication.
[7] Peeters, B., 2000, "System Identification and Damage Detection in Civil Engineering," PhD thesis, Department of Civil Engineering, Katholieke Universiteit Leuven, Belgium, [http://www.bwk.kuleuven.ac.be/bwm], Dec.
[8] Hermans, L., and Van der Auweraer, H., 1999, "Modal testing and analysis of structures under operational conditions: industrial applications," Mech. Syst. Signal Process., **13**, No. 2, pp. 193–216.
[9] Peeters, B., and De Roeck, G., 2001, "One-year monitoring of the Z24-Bridge: environmental effects versus damage events," Earthquake Eng. Struct. Dyn., **30**, No. 2, 149–171.
[10] Ljung, L., 1999, *System Identification: Theory for the User*, Second Edition, Prentice-Hall, Upper Saddle River, NJ.
[11] Peeters, B., and De Roeck, G., 1999, "Reference-based stochastic subspace identification for output-only modal analysis," Mech. Syst. Signal Process., **13**, No. 6, pp. 855–878.

[12] Akaike, H., 1974, ''Markovian representation of stochastic processes and its application to the analysis of autoregressive moving average processes,'' Annals of the Institute of Statistical Mathematics, **26**, pp. 363–387.
[13] Basseville, M., and Nikiforov I. V., 1993, *Detection of Abrupt Changes: Theory and Applications*, Prentice-Hall, Englewood Cliffs, NJ. [http://www.irisa.fr/sigma2/kniga].
[14] Caines, P., 1998, *Linear Stochastic Systems*, Wiley, New York.
[15] Bendat, J. S. and Piersol, A. G., 1993, *Engineering Applications of Correlation and Spectral Analysis*, Second Edition, Wiley, New York.
[16] Stoica, P. and Moses, R. L., 1997, *Introduction to Spectral Analysis*, Prentice-Hall, Englewood Cliffs, NJ.
[17] Felber, A. J., 1993, ''Development of a Hybrid Bridge Evaluation System,'' PhD thesis, University of British Columbia, Vancouver, Canada.
[18] Felber, A. J. and Cantieni, R., 1996, *Introduction of a New Ambient Vibration Testing System-Description of the System and Seven Bridge Tests*, Internal Report 156'521, EMPA, Dübendorf, Switzerland.
[19] Cunha, A., Caetano, E., Calçada, R., and Delgado, R., 1999, ''Modal identification and correlation with finite element parameters of Vasco da Gama Bridge,'' *Proceedings of IMAC 17*, pp. 705–711, Kissimmee, FL, Feb.
[20] Prevosto, M., 1982, ''Algorithmes d'Identification des Caractéristiques Vibratoires de Structures Mécaniques Complexes,'' PhD thesis, Université de Rennes I, France, Mar.
[21] Shih, C. Y., Tsuei, Y. G., Allemang, R. J., and Brown, D. L., 1988, ''Complex mode indication function and its application to spatial domain parameter estimation,'' Mech. Syst. Signal Process., **2**, No. 4, pp. 367–377.
[22] Brincker, R., Zhang, L., and Andersen, P., 2000, ''Modal identification from ambient responses using frequency domain decomposition,'' *Proceedings of IMAC 18, the International Modal Analysis Conference*, pp. 625–630, San Antonio, TX, Feb.
[23] Golub, G. H. and Van Loan, C. F., 1996, *Matrix Computations*, Third Edition, The Johns Hopkins University Press, Baltimore, MD.
[24] Schoukens J. and Pintelon R., 1991, *Identification of Linear Systems: a Practical Guideline to Accurate Modelling*, Pergamon Press, London, UK.
[25] Pintelon, R., Guillaume, P., Rolain, Y., Schoukens, J., and Van Hamme, H., 1994, ''Parametric identification of transfer functions in the frequency domain-a survey,'' IEEE Trans. Autom. Control, **AC-39**, No. 11, pp. 2245–2260.
[26] Guillaume P., Verboven P., and Vanlanduit S., 1998, ''Frequency-domain maximum likelihood identification of modal parameters with confidence intervals,'' *Proceedings of ISMA 23, Noise and Vibration Engineering*, K. U. Leuven, Belgium.
[27] Vanlanduit, S, Guillaume P., and Schoukens, J., 1998, ''High spatial resolution modal parameter estimation using a parametric MLE-like algorithm,'' *Proceedings of ISMA 23, Noise and Vibration Engineering*, K. U. Leuven, Belgium.
[28] Hermans, L., Guillaume, P., and Van der Auweraer, H., 1998, ''A frequency-domain maximum likelihood approach for the extraction of modal parameters from output-only data,'' *Proceedings of ISMA 23, Noise and Vibration Engineering*, K. U. Leuven, Belgium.
[29] James, G. H., Carne, T. G., and Lauffer, J. P., 1995, ''The natural excitation technique (NExT) for modal parameter extraction from operating structures,'' Int. J. Anal. Exp. Modal Anal., **10**, No. 4, pp. 260–277.
[30] Leuridan, J., 1984, ''Some Direct Parameter Model Identification Methods Applicable for Multiple Input Modal Analysis,'' PhD thesis, University of Cincinnati, OH.
[31] Allemang, R. J., Brown, D. L., and Fladung, W. A., 1994, ''Modal parameter estimation: a unified matrix polynomial approach,'' *Proceedings of IMAC 12, the International Modal Analysis Conference*, pp. 501–514, Honolulu, HI.
[32] Benveniste, A., and Fuchs, J.-J., 1985, ''Single sample modal identification of a nonstationary stochastic process,'' IEEE Trans. Autom. Control, **AC-30**, No. 1, pp. 66–74.
[33] Ho, B. L., and Kalman, R. E., 1966, ''Effective construction of linear state-variable models from input/output data,'' Regelungstechnik, **14**, pp. 545–548.
[34] Zeiger, H. P., and McEwen, A. J., 1974, ''Approximate linear realization of given dimension via Ho's algorithm,'' IEEE Trans. Autom. Control, **AC-19**, No. 2, pp. 53.
[35] Kung, S. Y., 1974, ''A new identification and model reduction algorithm via singular value decomposition.'' *Procedings of the 12th Asilomar Conference on Circuits, Systems and Computers*, pp. 705–714, Asilomar, CA, USA, Nov.
[36] Juang, J.-N., and Pappa, R. S., 1985, ''An eigensystem realization algorithm for modal parameter identification and model reduction,'' J. Guid. Control Dyn.
[37] Juang, J.-N., 1994, *Applied System Identification*, Prentice Hall, Englewood Cliffs, NJ.
[38] Desai, U. B., Pal, D., and Kirkpatrick, R. D., 1985, ''A realization approach to stochastic model reduction,'' Int. J. Control **42**, pp. 821–839.
[39] Aoki M., 1987, *State Space Modelling of Time Series*, Springer-Verlag, Berlin, Germany.
[40] Arun, K. S., and Kung, S. Y., 1990, ''Balanced approximation of stochastic systems,'' SIAM J. Matrix Anal. Appl., **11**, pp. 42–68.
[41] Akaike, H., 1974, ''Stochastic theory of minimal realization,'' IEEE Trans. Autom. Control, **19**, pp. 667–674.
[42] Van Overschee, P., and De Moor, B. 1996, *Subspace Identification for Linear Systems: Theory-Implementation-Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands.
[43] Camba-Mendez, G., and Kapetanios, G., 2001, ''Testing the rank of the Hankel covariance matrix: a statistical approach,'' IEEE Trans. Autom. Control, **AC-46**, No. 2, pp. 325–330.
[44] Van Overschee, P., and De Moor, B., 1993, ''Subspace algorithm for the stochastic identification problem,'' Automatica, **29**, No. 3, pp. 649–660.
[45] Viberg, M., 1995, ''Subspace-based methods for the identification of linear time-invariant systems,'' Automatica, **31**, No. 12, pp. 1835–1851.
[46] Kirkegaard P. H., and Andersen, P., 1997, ''State space identification of civil engineering structures from output measurements,'' Proceedings of IMAC 15, *The International Modal Analysis Conference*, pp. 889–895, Orlando, FL.
[47] Peeters, B., De Roeck, G., and Andersen, P., 1999, ''Stochastic system identification: uncertainty of the estimated modal parameters,'' *Proceedings of IMAC 17, The International Modal Analysis Conference*, pp. 231–237, Kissimmee, FL, Feb.
[48] Piombo, B., Ciorcelli, E., Garibaldi, L., and Fasana, A., 1993, ''Structures identification using ARMAV models,'' *Proceedings of IMAC 11, the International Modal Analysis Conference*, pp. 588–592, Orlando, FL.
[49] Andersen, P., 1997, ''Identification of Civil Engineering Structures Using Vector ARMA Models,'' PhD thesis, Department of Building Technology and Structural Engineering, Aalborg University, Denmark, May.
[50] Pandit, S. M., 1991, *Modal and Spectrum Analysis: Data Dependent Systems in State Space*, Wiley, New York.
[51] De, Roeck G., Claesen, W., and Van den Broeck, P., 1995, ''DDS-methodology applied to parameter identification of civil engineering structures,'' *Vibration and Noise '95*, pp. 341–353, Venice, Italy, Apr.