

The Virasoro algebra and its representations in physics

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Abstract

In this report for the course “Lie algebras and quantum groups” at KTH I discuss the origin of the Virasoro algebra, give the physical motivation for studying its unitary irreducible highest weight representations, and examine the necessary and sufficient conditions for such representations to exist.

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1 Introduction

The Virasoro algebra, defined by the basis elements $\{L_n, \hat{c}\}_{n \in \mathbb{Z}}$ with commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{\hat{c}}{12}m(m^2 - 1)\delta_{m+n,0}, \quad [L_m, \hat{c}] = 0, \quad (1)$$

is an infinite-dimensional Lie algebra with many applications in physics. It often appears in problems with conformal symmetry and where the essential space-time is one- or two-dimensional and space is periodic, i.e. compactified to a circle. An example of such a setting is string theory where the string worldsheet is two-dimensional and cylindrical in the case of closed strings.

In the following sections we will see how the Virasoro algebra appears as a central extension of the Witt algebra and study the conditions for highest weight representations to be unitary and irreducible. When such conditions are satisfied the representations are seen to give rise to physically acceptable states.

2 The Virasoro algebra as a central extension

The Virasoro algebra is actually the unique central extension of the Lie algebra of the group $\text{Diff}(S^1)$ of diffeomorphisms (smooth 1-to-1 maps) of the circle S^1 . This can be observed as follows. Since $\text{Lie}(\text{Diff}(S^1)) = \mathcal{X}(S^1)$, the algebra of smooth vector fields on the circle, we can write an arbitrary element $X \in \mathcal{X}(S^1)$ as

$$X = f(z)\frac{d}{dz} = \sum_{n \in \mathbb{Z}} a_n e^{in\phi} i \frac{d}{d\phi}, \quad (2)$$

using the representation $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and a Fourier expansion of f . A basis in $\mathcal{X}(S^1)$ is then $\{L_n\}_{n \in \mathbb{Z}}$, where $L_n := ie^{in\phi} \frac{d}{d\phi}$. These satisfy the algebra

$$[L_m, L_n] = (m - n)L_{m+n}, \quad (3)$$

which is also called the Witt algebra.

To obtain the central extensions of the Witt algebra (3), we add a number of central basis elements $\{\hat{c}_k\}$ to the algebra and postulate the relations

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + C_{mn}^k \hat{c}_k, \\ [L_m, \hat{c}_k] &= [\hat{c}_j, \hat{c}_k] = 0. \end{aligned} \quad (4)$$

Using the properties of the Lie bracket and the freedom to absorb terms in the structure constants C_{mn}^k by a change of basis, one finds (see [1] or [2]) that we can take

$$\begin{aligned} C_{mn}^k \hat{c}_k &= \hat{c}(m)\delta_{m+n,0}, \\ \hat{c}(m) &= \frac{1}{12}m(m^2 - 1)\hat{c}. \end{aligned} \quad (5)$$

Thus, we are left with only one central element \hat{c} , and with the standard normalization chosen in (5) we end up with the Virasoro algebra (1). In [1] it is also shown that this central extension is nontrivial, i.e. it cannot be transformed back into the Witt algebra by a change of basis. The Virasoro algebra is thus the unique central extension of the Witt algebra.

3 Representations of the Virasoro algebra

3.1 Conditions for representations in physics

We now turn our attention to the representation theory of the Virasoro algebra. As can be seen from physical applications (important examples are found in e.g. [2]), the eigenvalues of the operator L_0 often correspond to energies or masses of a physical system. Thus, we would like the eigenvalues of L_0 to be non-negative, or at least bounded from below, in a physically relevant representation. Motivated by this we consider *highest weight* representations of the Virasoro algebra, i.e. we assume we have a representation in a vector space V containing a highest weight vector $v \neq 0$, satisfying

$$\begin{aligned} L_0 v &= h v, \\ L_n v &= 0 \quad \forall n > 0, \end{aligned} \tag{6}$$

for some $h \in \mathbb{C}$. Assuming the representation is also *irreducible* we have that, since \hat{c} is a central element (Casimir operator),

$$\hat{c} w = c w \quad \forall w \in V, \tag{7}$$

where $c \in \mathbb{C}$ is called the *central* or *conformal charge*.

In the highest weight representation the operators L_n and L_{-n} for $n > 0$ act as annihilation and creation operators respectively. They lower resp. raise the eigenvalue of L_0 as can be seen by

$$[L_0, L_n] = -n L_n \quad \Rightarrow \quad L_0 L_n v = (h - n) L_n v. \tag{8}$$

Using the irreducibility of the representation and the Poincaré-Birkhoff-Witt theorem (see e.g. [3]) we have that every state vector $w \in V$ is on the form

$$\begin{aligned} w &= L_{-j}^{n_j} \dots L_{-2}^{n_2} L_{-1}^{n_1} v, \quad n_i \geq 0 \\ L_0 w &= (h + N) w, \end{aligned} \tag{9}$$

where $N = n_1 + 2n_2 + \dots + jn_j$ is called the *level* of the state w . Since the level is always non-negative, we indeed have the physically plausible situation that eigenvalues of L_0 are bounded from below.

In physical applications we would like the representation space V to have an inner product $\langle \cdot, \cdot \rangle$. We restrict our attention to *unitary* representations, i.e.

$$L_n^\dagger = L_{-n} \quad \forall n \in \mathbb{Z}. \tag{10}$$

This condition allows us to calculate all the inner products between the states in V in terms of h , c and $|v|^2 := \langle v, v \rangle$. E.g. for $n > 0$,

$$\begin{aligned} \langle L_{-n} v, L_{-n} v \rangle &= \langle v, L_n L_{-n} v \rangle - \langle v, L_{-n} L_n v \rangle = \langle v, [L_n, L_{-n}] v \rangle \\ &= \left(2nh + \frac{c}{12} n(n^2 - 1) \right) |v|^2. \end{aligned} \tag{11}$$

Also included in the condition of unitarity is the requirement that the inner product is positive definite (i.e. that it really *is* an inner product in the mathematical sense). A state with negative norm is called a *ghost* state and the existence of such a state in V means the representation is not unitary. The existence of a state with zero norm still violates the unitarity, but in this case it is not quite as bad since we can quote out the subspace of such states to get a positive definite space.

3.2 Restrictions on h and c

The numbers h and c actually characterize the unitary highest weight representations uniquely, as is shown in [4]. Our main task now is to determine which values of h and c that allow the corresponding irreducible highest weight representation to be unitary. First, we note that (10) implies that L_0 is Hermitian, so $h \in \mathbb{R}$. Furthermore, considering $n = 1$ and then $n \gg 0$ in (11), we have that

$$|L_{-n}v|^2 \geq 0 \quad \forall n > 0 \quad (12)$$

implies

$$h \geq 0 \quad \text{and} \quad c \geq 0. \quad (13)$$

One can proceed with restrictions on h and c by considering all states (9) on a specific level $N \in \mathbb{N}$. Since states on different levels have different eigenvalues of L_0 they are orthogonal. Hence, for the inner product to be positive semidefinite it is necessary to require that all the matrices

$$M_N(c, h) := \left[\langle \Psi_i, \Psi_j \rangle \right]_{1 \leq i, j \leq \pi(N)}, \quad N = 0, 1, 2, \dots \quad (14)$$

are positive semidefinite, where $\Psi_i = L_{-N}^{n_N} \dots L_{-2}^{n_2} L_{-1}^{n_1} v$ are the states on the level $N = \sum_{k=1}^N kn_k$ and $\pi(N)$ is the number of states on that level. π is given as the partition function

$$\sum_{N=0}^{\infty} \pi(N) q^N = \prod_{n=1}^{\infty} (1 - q^n)^{-1}. \quad (15)$$

The first few levels can be calculated explicitly to give further restrictions on the allowed values of c and h . As an example (see [2]) the requirements

$$\det M_N(c, h) \geq 0 \quad \text{for} \quad N = 0, 1, 2, \quad (16)$$

with $\Psi_1 = L_{-1}^2 v$, $\Psi_2 = L_{-2} v$ for the level $N = 2$, adds the restriction

$$\frac{1}{2}c + (c - 5)h + 8h^2 \geq 0. \quad (17)$$

Before giving more drastic restrictions, we consider the special case $c = 0$. For this case there is a simple argument (first given in [5]) which shows that the representation must be trivial. For an arbitrary level N , the matrix

$$M := \begin{bmatrix} \langle L_{-2N}v, L_{-2N}v \rangle & \langle L_{-N}^2 v, L_{-2N}v \rangle \\ \langle L_{-2N}v, L_{-N}^2 v \rangle & \langle L_{-N}^2 v, L_{-N}^2 v \rangle \end{bmatrix} \quad (18)$$

must be positive semidefinite. Some straightforward calculations show that

$$\det M = 4N^2 h^2 (8h - 5N), \quad (19)$$

where we from now on choose the normalization $|v|^2 = 1$. For large N this expression becomes negative unless $h = 0$. Unitarity thus requires $h = c = 0$ and by (11) and (6) we find that $L_n v = 0 \quad \forall n \in \mathbb{Z}$, i.e. we end up with the trivial one-dimensional representation. Since $\hat{c} = c = 0$ corresponds to the Witt

algebra (3) we also conclude that the only irreducible unitary highest weight representation of the Witt algebra is the trivial representation.¹

Continuing our analysis of which values on c and h that are allowed for the representation to be unitary, we now state a theorem given by Friedan, Qiu and Shenker ([6]):

Theorem 1 *In order for there to be a unitary highest weight representation of the Virasoro algebra corresponding to given values of c and h , it is necessary that either*

$$c \geq 1 \quad \text{and} \quad h \geq 0, \quad (20)$$

or

$$c = 1 - \frac{6}{(m+2)(m+3)} \quad (21)$$

and

$$h = h_{p,q}(c) := \frac{((m+3)p - (m+2)q)^2 - 1}{4(m+2)(m+3)}, \quad (22)$$

where $m = 0, 1, 2, \dots$, $p = 1, 2, \dots, m+1$, $q = 1, 2, \dots, p$.

The proof of this theorem relies on a general formula for the determinant of the matrices (14) called the *Kac determinant formula*:

$$\det M_N(c, h) = C_N \prod_{k=1}^N \eta_k(c, h)^{\pi(N-k)}, \quad (23)$$

where

$$\eta_k(c, h) := \prod_{\substack{p, q \in \mathbb{Z}^+ \\ pq=k}} (h - h_{p,q}(c)) \quad (24)$$

and C_N is some positive constant for each level N . Here, the m in (22) is regarded as a function of c using (21). The determinant formula was given in a slightly different form by Kac in [7] and was proved by Feigin and Fuchs in [8]. An almost complete proof is also given in [4].

The restriction on the values of h and c stated in Theorem 1 can be found by considering the curves $h_{p,q}(c)$, $0 < c < 1$, $p, q \in \mathbb{Z}^+$. Using the Kac determinant formula it is found that $\det M_N(c, h) < 0$ for some N whenever (c, h) is outside $h_{p,q}$ and $c < 1$. This corresponds to an odd number of ghosts and thus cannot yield a unitary representation. Further analysis, involving the study of subrepresentations, shows that only the points listed in the theorem can be completely ghost-free. This proof was first given in [6], with details added in [9]. A similar proof can be found in [10].

3.3 Existence of allowed representations

So far we have only considered the conditions necessary for a representation of the Virasoro algebra to be unitary. It remains to be discussed whether the restrictions on h and c given in Theorem 1 are *sufficient* to yield unitary representations.

¹This is not in conflict with the defining representation (2) of the Witt algebra, since that is not a highest weight representation.

It is rather straightforward to show that the first possibility in Theorem 1, i.e. $h \geq 0$, $c \geq 1$, does indeed yield a positive semidefinite inner product. Since, for a fixed N , the matrices $M_N(c, h)$ depend continuously on h and c , we only need to consider the case of a fixed $c > 1$ and varying $h > 0$. For such values of c , $h_{p,q}(c)$ is either non-real or negative, so that $\det M_N(c, h) \neq 0$ in the entire region. For h sufficiently large the matrix $M_N(c, h)$ is positive definite, as is explicitly shown in [10]. Hence, by continuity we have that $M_N(c, h)$ is positive semidefinite for all $h \geq 0$, $c \geq 1$.

The remaining unitary representations allowed by Theorem 1 can all be constructed using the fact that to each affine Kac-Moody algebra (see [2]) there is an associated Virasoro algebra. This is called Sugawara's construction and we only give a quick outline of it here.

In short, an affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated with a finite-dimensional simple Lie algebra \mathfrak{g} is an algebra spanned by $\{T_a^m, \hat{k}\}_{a=1, \dots, \dim \mathfrak{g}, m \in \mathbb{Z}}$, where \hat{k} is central and

$$[T_a^m, T_b^n] = \lambda_{ab}^c T_c^{m+n} + \hat{k} m \delta_{a,b} \delta^{m+n,0}. \quad (25)$$

The structure constants λ_{ab}^c are those of the subalgebra $\mathfrak{g} \cong \text{Span}_{\mathbb{C}}\{T_a^0\}$. From $\hat{\mathfrak{g}}$ one constructs an associated Virasoro algebra as

$$L_n^{\mathfrak{g}} := \frac{1}{2k + Q_{\psi}} \sum_{m \in \mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} \circ T_a^{m+n} T_a^{-m} \circ, \quad (26)$$

where $\circ \cdot \circ$ denotes a normal ordering employed to handle the infinite sum, k is the value of \hat{k} in an irreducible representation, and Q_{ψ} is the value of the quadratic Casimir operator in the adjoint representation of \mathfrak{g} (see [2], [11], [12] or [13]). A given unitary representation of the Kac-Moody algebra $\hat{\mathfrak{g}}$ then naturally transforms into a unitary representation of the Virasoro algebra. The value of c in such a representation can be shown to be²

$$c_{\mathfrak{g}} = \frac{2k \dim \mathfrak{g}}{2k + Q_{\psi}}. \quad (27)$$

The conditions for a representation of a Kac-Moody algebra to be unitary are stated in e.g. [12]. By considering a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and forming a new Virasoro algebra $K_m := L_m^{\mathfrak{g}} - L_m^{\mathfrak{h}}$ one finds that for certain algebras \mathfrak{g} the resulting central charge $c_K = c_{\mathfrak{g}} - c_{\mathfrak{h}}$ attains all the values (21) and the values of h in (22) give rise to unitary representations. Thus, the conditions given in Theorem 1 are sufficient to allow unitary representations. Details of this construction are given in [2], [11] and [12].

4 Summary

We have seen that the Virasoro algebra has a rather geometric origin as the unique central extension of the Witt algebra, i.e. the smooth vector fields on a circle. Motivated by physical considerations, we found that the representations of the Virasoro algebra of interest in most physical applications are the unitary

²In the case that \mathfrak{g} is semi-simple one instead gets a sum of of the c -numbers $c_{\mathfrak{g}_i}$ corresponding to each simple factor \mathfrak{g}_i .

irreducible highest weight representations. These are completely characterized by the central charge c and the L_0 -eigenvalue h for the highest weight vector.

We concluded that a necessary and sufficient condition for an irreducible highest weight representation of the Virasoro algebra to be unitary is that the values c and h satisfy

$$c \geq 1 \quad \text{and} \quad h \geq 0,$$

or

$$c = 1 - \frac{6}{(m+2)(m+3)} \quad \text{and} \quad h = \frac{((m+3)p - (m+2)q)^2 - 1}{4(m+2)(m+3)}$$

$$\text{for } m = 0, 1, 2, \dots, \quad p = 1, 2, \dots, m+1, \quad q = 1, 2, \dots, p.$$

This includes the case $c = h = 0$, for which the representation is trivial.

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