Chapter 4. Plane sections of a cone

In this chapter I will discuss what the intersection of a plane with a right circular cone looks like. A number of topics discussed will be of use in drawing figures associated with such intersections.

I begin with a few elementary topics which arise. Many of these are elementary and perhaps tedious, but if you try to draw the various figures accurately you will need them (or possibly simple variations).

1. Solving quadratic equations numerically

The first topic involves solving quadratic equations in the presence of limited precision in calculations. Suppose we know that

$$ax^2 + bx + c = 0$$

has two real roots, which we want to compute. The quadratic formula gives

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \left(\frac{b}{2a}\right)\left(-1 \pm \sqrt{1 - \frac{4ac}{b^2}}\right)$$

If $4ac/b^2$ is small then one of these involves the subtraction of two positive numbers about the same size, and a large rounding error will result. This is called a **cancellation** error. To avoid it, we first observe that since

$$ax^2 + bx + c = a(x^2 + (b/a)x + (c/a)) = a(x - \lambda_1)(x - \lambda_2)$$

the product of the two roots is c/a. Therefore we set in sequence

$$
\begin{aligned}
B &= b/2a \\
D &= 4ac/b^2 \\
  &= c/aB^2 \\
\ell &= -1 - \sqrt{1 - D} \\
\lambda_1 &= B\ell \\
\lambda_2 &= BD/\ell
\end{aligned}
$$

2. Planes and projections

I recall here some elementary formulas involving projections in 3D. The basic formula everything else depends on solves the following problem: *Given vectors u and v in 3D, what is the perpendicular projection of u onto the line along v?* The answer is

$$u_0 = \left(\frac{u \bullet v}{v \bullet v}\right) v$$

as is usually explained when dot products are introduced.

Suppose now we are given a plane

$$Ax + By + Cz = D \ .$$

Recall that the geometrical significance of the coefficient vector $\alpha = (A, B, C)$ is that it is perpendicular to the plane. A unique multiple of α will actually lie on the plane. It is $\alpha_0 = t(A, B, C)$ where
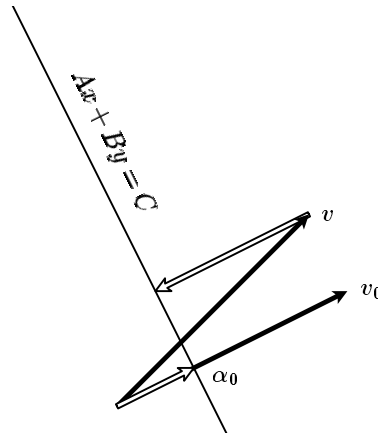
$$A(tA) + B(tB) + C(tC) = D, \quad t = \frac{D}{A^2 + B^2 + C^2} = \frac{D}{\alpha \bullet \alpha} \ .$$

Suppose $v = (x, y, z)$ given. What is the perpendicular projection $v_\perp$ of v onto the plane? Let v_0 be the projection of the vector v onto the line along α. Then the picture shows that

$$v_\perp = v - (v_0 - \alpha_0) = v - \frac{v \bullet \alpha - D}{\alpha \bullet \alpha} \, \alpha \ .$$
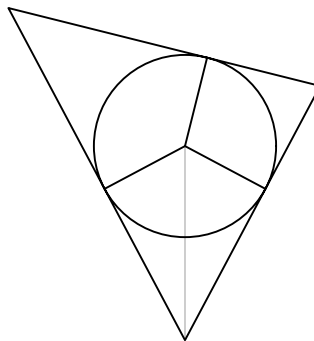
The distance from v to the plane is therefore

$$\frac{|v \bullet \alpha - D|}{\|\alpha\|} \ .$$

3. Angular bisectors

A circle is said to be **inscribed** in a triangle if it lies completely inside the triangle and touches all three sides.

In this section we want to find (1) the centre of this circle, (2) its radius, and (3) the points of contact with the sides. We will assume that the triangle is oriented so that the origin is one vertex and the positive y-axis bisects one of its angles. Let θ be half this vertex angle. We shall also assume that the side opposite the origin is given by an equation $Ax + By = C$.

The centre of the circle lies on the y-axis. Let it be $P = (0, y)$. Its distance from the left and right sides is thus guaranteed to be the same. This common distance is $y \sin \theta$. Its distance from the other side is

$$\frac{|By - C|}{\sqrt{A^2 + B^2}}$$

so we must solve the equation

$$y \sin \theta = \frac{|By - C|}{\sqrt{A^2 + B^2}} \ .$$

We do this by squaring both sides and rearranging to get

$$y^2 \sin^2 \theta (A^2 + B^2) = (By - C)^2$$

leading to the quadratic equation

$$\left(B^2 - (A^2 + B^2) \sin^2 \theta\right)y^2 - 2BCy + C^2 = 0 \ .$$

Exercise 3.1. *The number y is one of the two roots of this equation. What is the other root? Which is the larger root? What does it mean if one root is negative?*

The coordinates of the two contact points on the left and right sides are $(\pm y \sin \theta \cos \theta, y - y \sin^2 \theta)$. According to the discussion of projections, the third contact point is

$$(0, y) - \left(\frac{By - C}{A^2 + B^2}\right)(A, B) \ .$$

4. Drawing cones

We start with a standard cone whose vertex angle is 2θ.
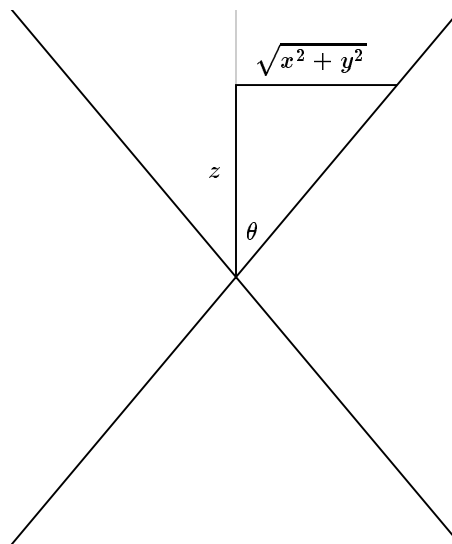
Its equation is

$$x^2 + y^2 = z^2 \tan^2 \theta \ .$$

We shall want to draw it in projection after a rigid transformation has been applied to it. There are several ways to draw a cone in 3D, but in these notes I shall draw only its outline, which consists of a pair of straight lines. The first step in drawing the transformed cone is to find the transformed axis. This is simple enough to calculate. By means of a 2D rotation, we can in effect assume it to be the y-axis. The only extra piece of information needed to calculate the cone's outline is the angle its axis makes with respect to the (x, y) plane. Call it α. Here is the picture we get for different values of α, in effect rotating the top of the cone towards us.
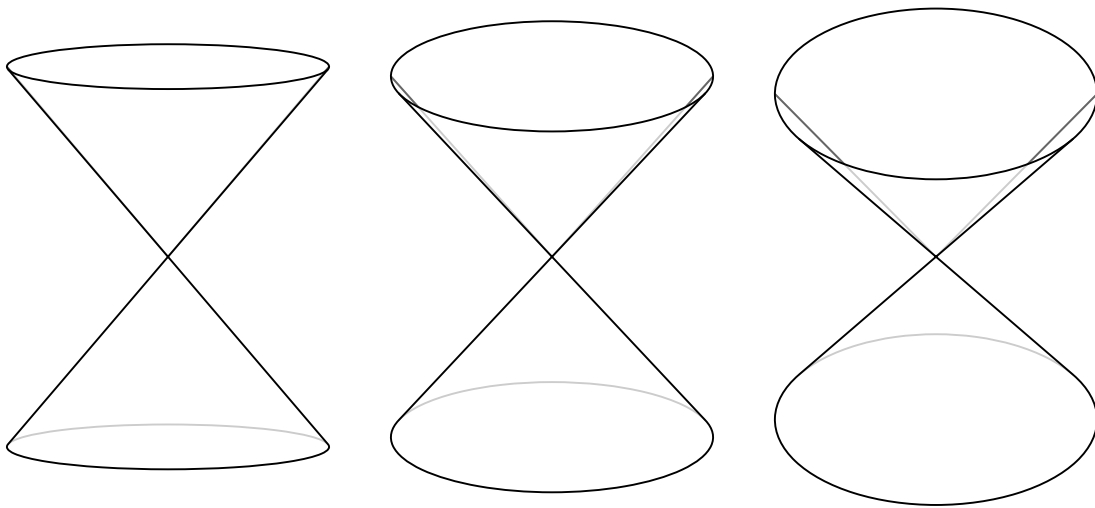
The point is that the front of the cone obscures its middle, even in projection. In effect we confront the following problem: *Given an ellipse of axes a and b with centre at $(x, 0)$, what are the tangents from the origin to this ellipse?* This is easily answered by scaling the ellipse vertically so as to turn it into a circle. The angle φ has

$$\sin \varphi = a/x$$

where $r = a/x$.
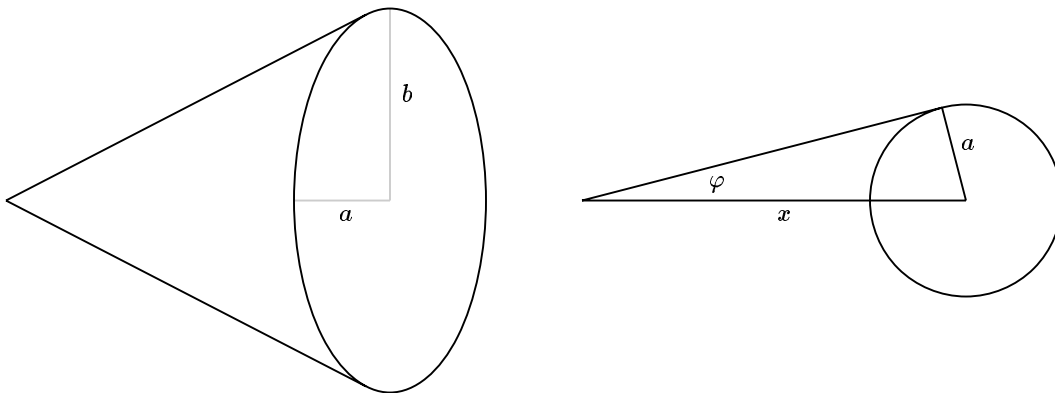
Note that this same angle φ is that made by the radius to the point of contact with the vertical, and is thus essentially the angular parameter of the corresponding point on the original ellipse The exact point of contact on the original ellipse is therefore

$$(x - a \sin \varphi, b \cos \varphi) \, .$$

5. Conic intersections

The intersection of any cone and a plane is always an ellipse, a parabola, or an hyperbola. It is simple to parametrize it, and not too difficult to tell exactly what its location and dimensions are (when the cone is right-circular). Namely, we can essentially parametrize it as we do a circle. Recall that as t varies from 0 to 2π the point $(\cos t, \sin t)$ runs around the unit circle in the (x, y) plane. Therefore the point

$$(\tau \cos t, \tau \sin t, 1)$$

runs around the intersection of our cone with the plane $z = 1$, where $\tau = \tan \theta$. If the intersection is an ellipse, there will be for every point on this circle a unique point on the same line through the origin and also on the plane, of the form

$$\kappa(\tau \cos t, \tau \sin t, 1) \ .$$

If the plane is $Ax + By + Cz = D$ then we must have

$$\kappa(A\tau \cos t + B\tau \sin t + C) = D, \quad \kappa = \frac{D}{A\tau \cos t + B\tau \sin t + C} \ .$$

If the intersection is not an ellipse then this parametrization will fail when the denominator is 0. This makes sense, since a circle cannot possibly parametrize an hyperbola or parabola. It is instructive to look carefully at the denominator to see exactly when it vanishes. If

$$A\tau \cos t + B\tau \sin t + C = 0$$

this means that the point $(\tau \cos t, \tau \sin t)$ lies on the planar line

$$Ax + By + C = 0 \ .$$
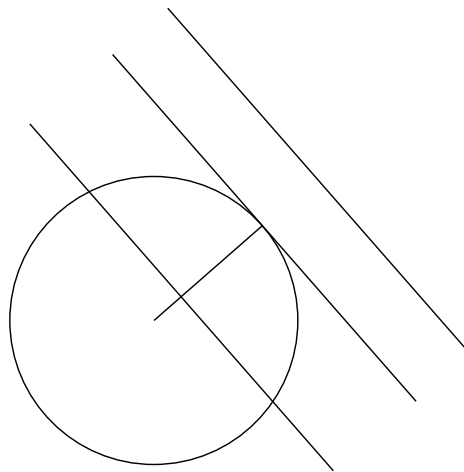
There are three possibilities as to exactly what the intersection of a circle and a straight line can be. (1) There is no intersection. (2) The line is tangent to the circle, and there is one point of intersection. (3) There are two points of

intersection. How do we tell which case occurs? Recall that the geometrical significance of the coefficients A, B, C is that the direction perpendicular to the line is (A, B) and the signed distance from the origin to the line is

$$\frac{-C}{\sqrt{A^2 + B^2}} \ .$$

The line will therefore intersect the circle of radius τ only when

$$\frac{|C|}{\sqrt{A^2 + B^2}} \leq \tau$$

or

$$C^2 \leq \tau^2(A^2 + B^2) \ .$$

It will intersect in exactly one point when this is an exact equality. Furthermore we can see how to find the intersection geometrically. The two points of intersection must be symmetrically placed with respect to the perpendicular direction (A, B). Multiplying (A, B) by τ/C, we may as well assume that the direction vector (A, B) just touches the circle. The angle between this direction and the angle α towards the intersection points satisfies

$$\cos \alpha = \frac{-C}{\sqrt{A^2 + B^2}} \ .$$

We can now see that if the intersection is an ellipse we parametrize it by the whole circle; if it is a parabola we parametrize it by the circle minus a single point; and if an hyperbola we parametrize each piece of the hyperbola by one of the arcs we get by removing the two points of intersection with the line $Ax + By + C = 0$.

6. Focal points

Certain points called the **focal points** play an important role in the theory of conic sections. It was only with Kepler's discovery of his first law of planetary motion that they became of general interest, but they were in fact discovered first for purely mathematical reasons by the Greeks, and in effect allowed the Greeks to formulate the algebraic equations describing conic sections long before coordinate systems were introduced in full generality (by Descartes in the 17th century).

We shall look only at ellipses. Here the fundamental fact is that one can locate two points F_1 and F_2 inside the ellipse with the following simple property:

- *If a line is drawn from one focus to the periphery of the ellipse and then to the other focus, the total length of the path is independent of which point on the periphery we move to.*
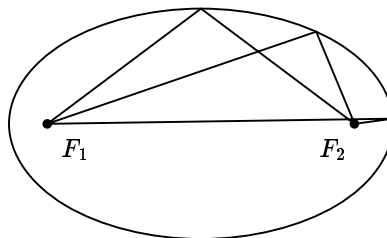
Before we begin to think about why this is true, we must locate the points F_1 and F_2. We can do this if we apply the characteristic property to just two paths. If a and b are the semi-major axes of the ellipse, then its equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \ .$$

If $F_1 = (-f, 0)$ then $F_2 = (f, 0)$. We first consider the path from F_2 to the right hand end extremity of the ellipse, then back to F_1. Its length is $(a - f) + a + f = 2a$. On the other and we consider the path from F_2 to the top point $(0, b)$ and then to F_1. Its length must also be $2a$, and by Pythagoras' Theorem

$$f = \sqrt{a^2 - b^2} \ .$$

We shall see two proofs of the assertion above about the equality of path lengths. The first is algebraic. Choose a point $P = (x, y)$ on the ellipse. We can calculate the distances from F_1 to P and F_2 to P and add them them up to see whether or not we get $2a$. I leave the details as an exercise—at one point a bit of tricky algebra is required. This proof is straightforward, but not very illuminating.
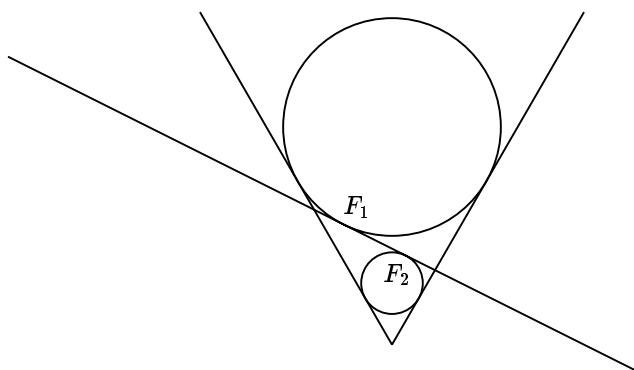
7. Dandelin's spheres

The proof of the focal property of ellipses that is in many ways the most satisfying was not discovered until the early nineteenth century, although the property itself had been known for more than 2000 years before that. The discoverer was a Belgian mathematician named Dandelin, who is relatively unknown. His proof starts with the construction of the ellipse as a conic section. So we choose a right circular cone and slice it with a plane which is horizontal enough to produce an ellipse. Dandelin's proof then inserts into the cone, above and below the plane, the largest spheres which will fit and at the same time touch the plane. We may choose our coordinates so that the axis of the cone is the z-axis, and the equation of the plane is

$$Ax + Cz = D \ .$$

Exercise 7.1. *If the original equation is $Ax + By + Cz = D$, what change of coordinates do we have to make to get the new $B = 0$? What is the value of the new A?*

If we then look sideways at the configuration (projecting along the y-axis), we see this picture:


Each sphere will touch the plane at a single point. *The points of contact of the spheres with the plane are defined in Dandelin's proof to be the focal points of the ellipse.* Of course they have to be inside the cone and hence inside the ellipse. It is more or less evident from a symmetry argument that the curve of intersection has an axis of symmetry and that the focal points lie on it.

Here is the three-dimensional view of Dandelin's construction of the foci:
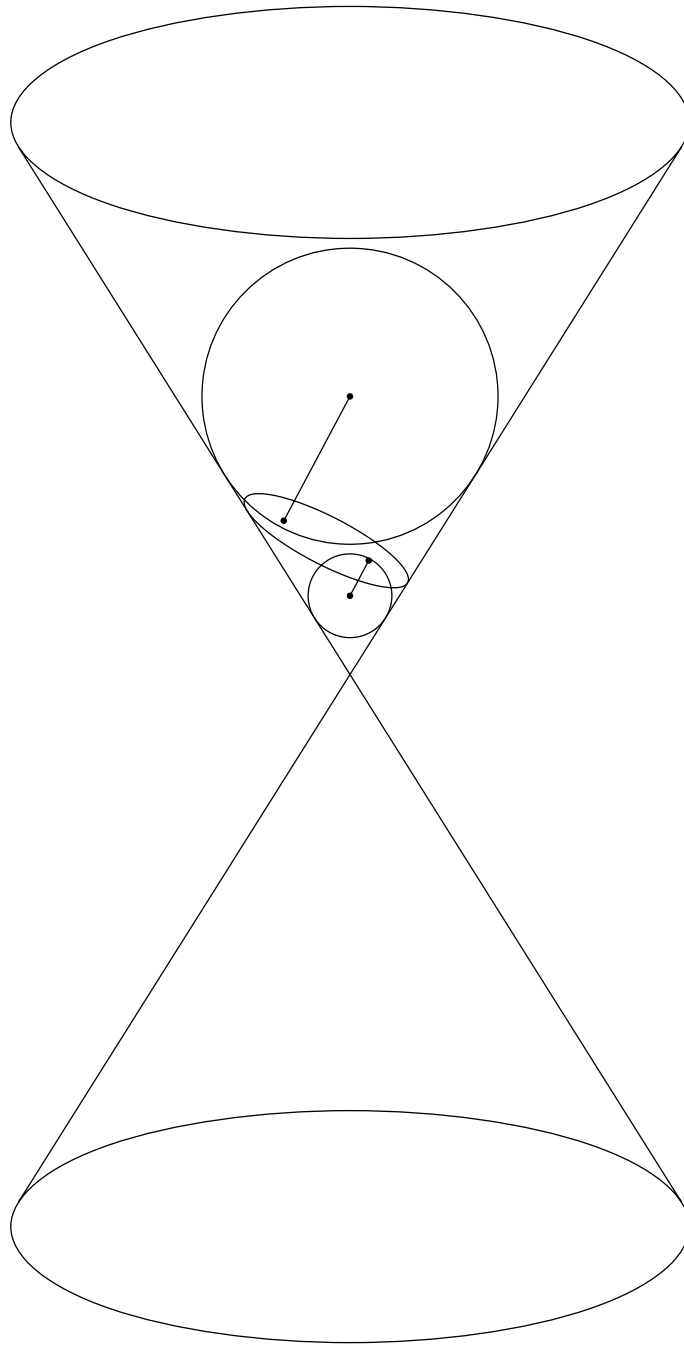
And here is the diagram to be used in the proof of the focal property:
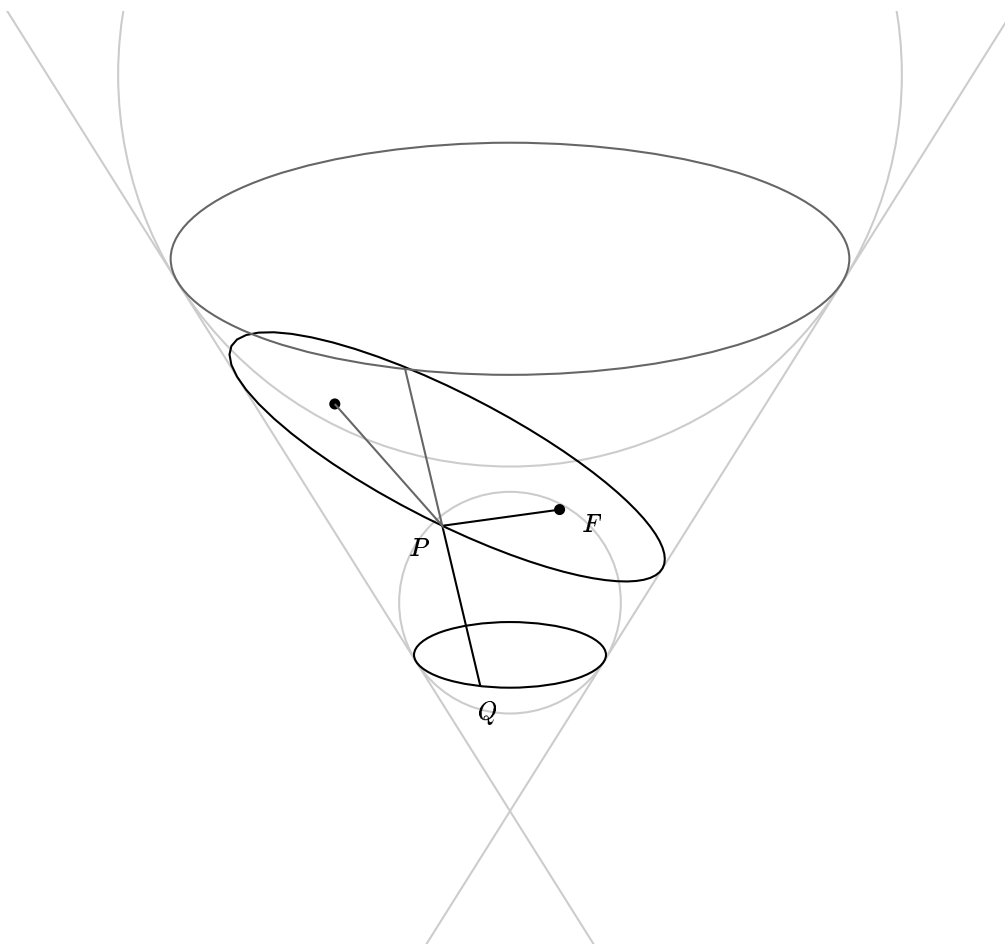
The proof now runs like this: Pick one of the spheres, say the smaller one. Let F be the focus touching that sphere. and let P be a point on the intersection curve. Draw a line segment from F to P, and and another along the surface of the cone from P to the circle of contact with the sphere, say hitting at Q. The two lines PF and PQ are both tangent to the sphere, and must be the same length. Therefore the distance from F to P is the same as the distance from P to the circle of contact. Repeat this argument for the other sphere. We see that the total distance from F_1 to P to F_2 is the same as the distance along the cone between the two circles of contact, and therefore does not depend on P.

Exercise 7.2. *Prove that two tangents from a point to a sphere have the same length.*

8. The directrix

If E is an ellipse with semi-major axes a and b, the **eccentricity** of the curve is defined to be the ratio

$$e = \frac{\sqrt{a^2 - b^2}}{a} \ .$$

This is also the ratio of the focal distance f to a. If $e = 0$ the curve is a circle, and the closer e gets to 1 the flatter the ellipse is. The eccentricity can also be defined for other conic sections.

For each conic section and choice of focus F there exists a unique line ℓ, called the **directrix** determined by these data, with the following property: if P is any point on the curve then
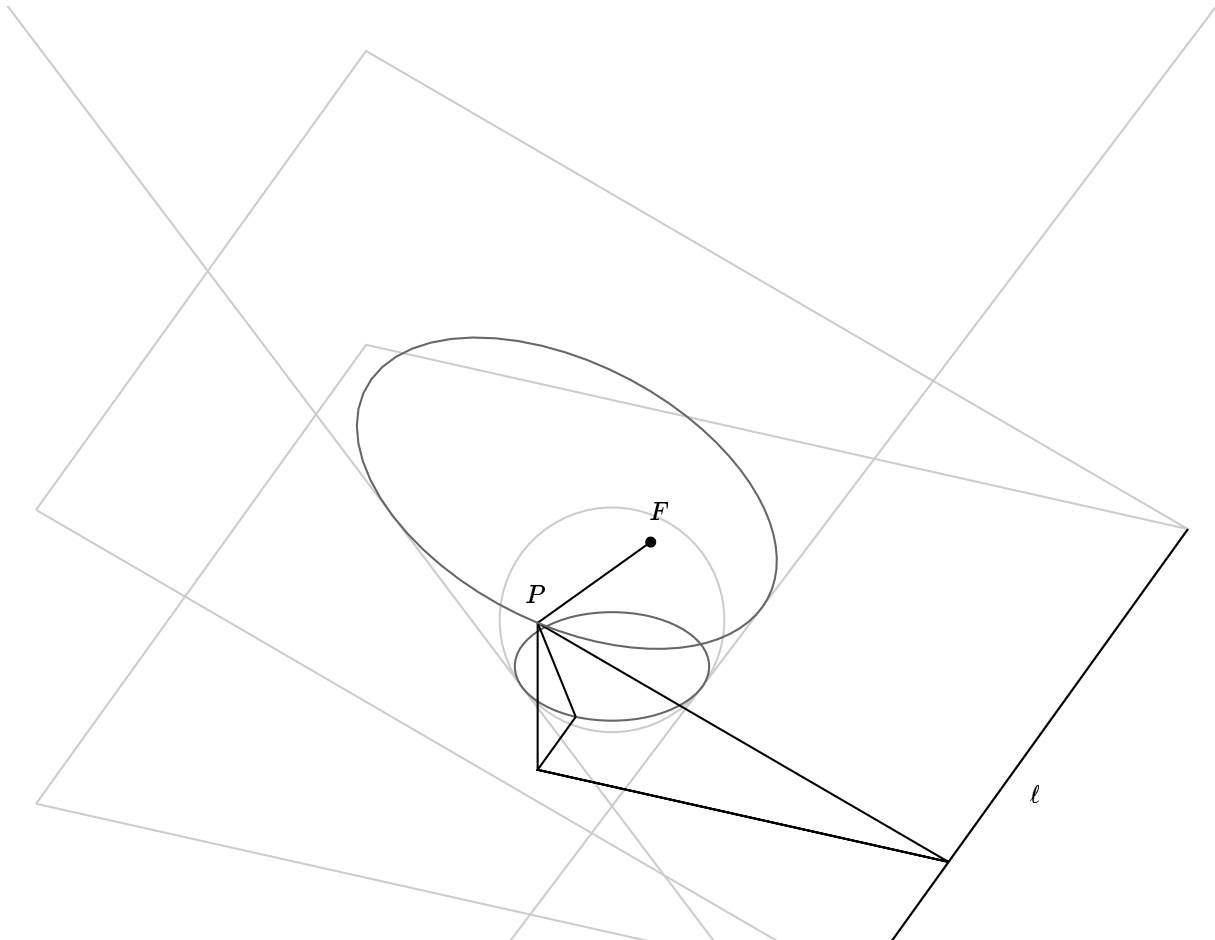
$$\frac{d(P, F)}{d(P, \ell)} = e \ .$$

It is simple enough to figure out where the directrix of an ellipse has to be if it is to have this property, and it is straightforward to calculate that this line does indeed have this property.

Exercise 8.1. *For the ellipse*

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

tell where its directrix has to be. Prove the focal-directrix property for it.

The calculation is not too instructive. More satisfying is that we can use Dandelin's construction to give a geometrical proof of it.

Exercise 8.2. *Use the picture above to explain why $d(P, F)/d(P, \ell)$ is independent of P.*