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NEW FIXED POINT THEOREMS IN OPERATOR VALUED EXTENDED HEXAGONAL b-METRIC SPACES

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Abstract In the current work, we broaden the class of C^*-algebra-valued hexagonal b-metric spaces and C^*-algebra-valued extended b-metric spaces by defining the class of C^*-algebra-valued extended hexagonal b-metric spaces and demonstrate a fixed point theorem with distinct contractive condition. In addition, an application is presented in the later part to demonstrate the existence and uniqueness of a particular type of operator equation in order to elucidate our results.

1 Introduction

The concept of Banach contraction is a basic outcome of the metric fixed point theory. It is a quite important and efficient tool in theoretical and applied sciences for solving the problems of Existence and uniqueness. In 2017, the conception of extended b-metric spaces was initiated by Tayyab Kamran et al. [10] as an extension of b-metric spaces [4]. Thereafter, the authors in [8] proposed the idea of extended hexagonal b-metric spaces by replacing the triangle inequality with hexagonal inequality. Recently, Asim et al. [1] developed a concept of C^*-algebra-valued extended b-metric spaces and Kalpana et al. [9] established a common fixed point theorem in the setting of C^*-algebra-valued hexagonal b-metric spaces. For further investigations on the concept of C^*-algebra, the readers can view [2, 3, 5, 6, 7, 11, 12, 13].

Deeply influenced by the above facts, we reveal the conception of C^*-algebra-valued extended hexagonal b-metric spaces and illustrate a fixed point theorem with distictive contractive condition. Eventually, an application is provided to guarantee the existence and uniqueness for the specific type of operator equation under the framework of C^*-algebra-valued extended hexagonal b-metric spaces.

2 Preliminaries

The conceptualization of extended b-metric spaces was commenced by Kamran et al. [10] that described in the following:

Definition 2.1. Given a nonempty set X and $E : X \times X \to [1, \infty)$, and $\tilde{d}_E : X \times X \to [0, \infty)$. If for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$
(1) $\tilde{d}_E(\mathfrak{a}, \mathfrak{b}) = 0 \Longleftrightarrow \mathfrak{a} = \mathfrak{b}$;
(2) $\tilde{d}_E(\mathfrak{a}, \mathfrak{b}) = \tilde{d}_E(\mathfrak{b}, \mathfrak{a})$;
(3) $\tilde{d}_E(\mathfrak{a}, \mathfrak{b}) \leq E(\mathfrak{a}, \mathfrak{b})[\tilde{d}_E(\mathfrak{a}, \mathfrak{c}) + \tilde{d}_E(\mathfrak{c}, \mathfrak{b})]$
then we say that the pair $(X, \tilde{d}_E)$ is an extended b-metric space.

Very recently, Kalpana et al. [8] generalized the above definition to the case of extended hexagonal b-metric spaces.

Definition 2.2. Let X be a non-empty set and $E : X \times X \to [1, \infty)$. A function $\tilde{d}_H : X \times X \to [0, \infty)$ is called an extended hexagonal b-metric if it satisfies:
(1) $\tilde{d}_H(\mathfrak{a}, \mathfrak{b}) = 0 \Leftrightarrow \mathfrak{a} = \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in X$;

(2) $\tilde{d}_H(\mathfrak{a},\mathfrak{b}) = \tilde{d}_H(\mathfrak{b},\mathfrak{a})$ for all $\mathfrak{a},\mathfrak{b} \in X$;
(3) $\tilde{d}_H(\mathfrak{a},\mathfrak{b}) \leq E(\mathfrak{a},\mathfrak{b})[\tilde{d}_H(\mathfrak{a},\mathfrak{c})+\tilde{d}_H(\mathfrak{c},\mathfrak{d})+\tilde{d}_H(\mathfrak{d},\mathfrak{e})+\tilde{d}_H(\mathfrak{e},\mathfrak{f})+\tilde{d}_H(\mathfrak{f},\mathfrak{b})]$ for all $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f} \in X$ and $\mathfrak{a} \neq \mathfrak{c},\ \mathfrak{c} \neq \mathfrak{d},\ \mathfrak{d} \neq \mathfrak{e},\ \mathfrak{e} \neq \mathfrak{f},\ \mathfrak{f} \neq \mathfrak{b}$;
The pair $(X, \tilde{d}_H)$ is called an extended hexagonal b-metric space.

We now discuss certain essential concepts and results in C^*-algebra.

Let $\mathbb{A}$ signifies the unital C^*-algebra and set $\mathbb{A}_h = \{\mathfrak{f} \in \mathbb{A} : \mathfrak{f} = \mathfrak{f}^*\}$. An element $\mathfrak{f} \in \mathbb{A}$ is said to be positive, if $\mathfrak{f} \in \mathbb{A}_h$ and $\sigma(\mathfrak{f}) \subseteq [0,\ \infty)$, where θ is a zero element in $\mathbb{A}$ and $\sigma(\mathfrak{f})$ is the spectrum of $\mathfrak{f}$, which is denoted by $\theta \preceq \mathfrak{f}$. The partial ordering on $\mathbb{A}_h$ given by $\mathfrak{f} \preceq \mathfrak{g}$ if and only if $\theta \preceq \mathfrak{g} - \mathfrak{f}$. The sets $\{\mathfrak{f} \in \mathbb{A} : \theta \preceq \mathfrak{f}\}$ and $\{\mathfrak{f} \in \mathbb{A} : \mathfrak{f}\mathfrak{g} = \mathfrak{g}\mathfrak{f},\ \forall g \in \mathbb{A}\}$ is represented as $\mathbb{A}_+$ and $\mathbb{A}^{'}$ as and $|\mathfrak{w}| = (\mathfrak{w}^*\mathfrak{w})^{\frac{1}{2}}$ respectively.
Very recently, Asim et al. [1] set up the idea of extended b-metric spaces to the C^*-algebra.

Definition 2.3. Let $X \neq \emptyset$ and $E : X \times X \to \mathbb{A}^I$. The mapping $\tilde{d}_E : X \times X \to \mathbb{A}$ is called a C^*-algebra-valued extended b-metric on X, if it satisfies the following (for all $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in X$):
(1) $\theta \preceq \tilde{d}_E(\mathfrak{a},\mathfrak{b})$ for all $\mathfrak{a},\mathfrak{b} \in X$ and $\tilde{d}_E(\mathfrak{a},\mathfrak{b}) = \theta$ iff $\mathfrak{a} = \mathfrak{b}$;
(2) $\tilde{d}_E(\mathfrak{a},\mathfrak{b}) = \tilde{d}_E(\mathfrak{b},\mathfrak{a})$ for all $\mathfrak{a},\mathfrak{b} \in X$;
(3) $\tilde{d}_E(\mathfrak{a},\mathfrak{b}) \preceq E(\mathfrak{a},\mathfrak{b})[\tilde{d}_E(\mathfrak{a},\mathfrak{c}) + \tilde{d}_E(\mathfrak{c},\mathfrak{b})]$.
The triplet $(X, \mathbb{A}, \tilde{d}_E)$ is called a C^*-algebra-valued extended b-metric space.

The definition of C^*-algebra-valued hexagonal b-metric spaces was defined in the following way by Kalpana et al. [9].

Definition 2.4. Let X be a nonempty set, and $A \in \mathbb{A}'$ such that $A \succeq I$. Suppose the mapping $\tilde{d}_H : X \times X \to \mathbb{A}$ satisfies:
(1) $\theta \preceq \tilde{d}_H(\mathfrak{a},\mathfrak{b})$ for all $\mathfrak{a},\mathfrak{b} \in X$ and $\tilde{d}_H(\mathfrak{a},\mathfrak{b}) = \theta \Leftrightarrow \mathfrak{a} = \mathfrak{b}$;
(2) $\tilde{d}_H(\mathfrak{a},\mathfrak{b}) = \tilde{d}_H(\mathfrak{b},\mathfrak{a})$ for all $\mathfrak{a},\mathfrak{b} \in X$;
(3) $\tilde{d}_H(\mathfrak{a},\mathfrak{b}) \preceq A[\tilde{d}_H(\mathfrak{a},\mathfrak{c})+\tilde{d}_H(\mathfrak{c},\mathfrak{d})+\tilde{d}_H(\mathfrak{d},\mathfrak{e})+\tilde{d}_H(\mathfrak{e},\mathfrak{f})+\tilde{d}_H(\mathfrak{f},\mathfrak{b})]$ for all $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f} \in X$ and $\mathfrak{a} \neq \mathfrak{c},\ \mathfrak{c} \neq \mathfrak{d},\ \mathfrak{d} \neq \mathfrak{e},\ \mathfrak{e} \neq \mathfrak{f},\ \mathfrak{f} \neq \mathfrak{b}$;
Then d is called a C^*-algebra-valued hexagonal b-metric on X and $(X, \mathbb{A}, \tilde{d}_H)$ is called a C^*-algebra-valued hexagonal b-metric space.

3 Main Results

Through this main section, we implement the idea of C^*-algebra valued extended hexagonal b-metric spaces as follows.
Hereafter $\mathbb{A}'_I$ signify the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}$ and $a \succeq I\}$ respectively.

Definition 3.1. Let X be a nonempty set and $E : X \times X \to \mathbb{A}'_I$. Suppose the mapping $\tilde{d}_H : X \times X \to \mathbb{A}$ satisfies:
(1) $\theta \preceq \tilde{d}_H(\mathfrak{a},\mathfrak{b})$ and $\tilde{d}_H(\mathfrak{a},\mathfrak{b}) = \theta \Leftrightarrow \mathfrak{a} = \mathfrak{b}$ for all $\mathfrak{a},\mathfrak{b} \in X$;
(2) $\tilde{d}_H(\mathfrak{a},\mathfrak{b}) = \tilde{d}_H(\mathfrak{b},\mathfrak{a})$ for all $\mathfrak{a},\mathfrak{b} \in X$;
(3) $\tilde{d}_H(\mathfrak{a},\mathfrak{b}) \preceq E(\mathfrak{a},\mathfrak{b})[\tilde{d}_H(\mathfrak{a},\mathfrak{c})+\tilde{d}_H(\mathfrak{c},\mathfrak{d})+\tilde{d}_H(\mathfrak{d},\mathfrak{e})+\tilde{d}_H(\mathfrak{e},\mathfrak{f})+\tilde{d}_H(\mathfrak{f},\mathfrak{b})]$ for all $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f} \in X$ and $\mathfrak{a} \neq \mathfrak{c},\ \mathfrak{c} \neq \mathfrak{d},\ \mathfrak{d} \neq \mathfrak{e},\ \mathfrak{e} \neq \mathfrak{f},\ \mathfrak{f} \neq \mathfrak{b}$;
The triplet $(X, \mathbb{A}, \tilde{d}_H)$ is called an C^*-algebra-valued extended hexagonal b-metric space.

Example 3.2. Let $X = \{1,2,3,4,5,6\}$ and $\mathbb{A} = \mathbb{R}^2$. If $\mathfrak{a},\mathfrak{b} \in \mathbb{A}$ with $\mathfrak{a} = (\mathfrak{a}_1,\mathfrak{a}_2)$, $\mathfrak{b} = (\mathfrak{b}_1,\mathfrak{b}_2)$, then the addition, multipilcation and scalar multipilcation can be defined as follows

$$\mathfrak{a}+\mathfrak{b} = (\mathfrak{a}_1+\mathfrak{b}_1,\ \mathfrak{a}_2+\mathfrak{b}_2),\ k\mathfrak{a} = (k\mathfrak{a}_1, k\mathfrak{a}_2),\ \mathfrak{a}\mathfrak{b} = (\mathfrak{a}_1\mathfrak{b}_1, \mathfrak{a}_2\mathfrak{b}_2).$$

Now, we define the metric $\tilde{d}_H : X \times X \to \mathbb{A}$ such that $\tilde{d}_H$ is symmetric and the control function $E : X \times X \to \mathbb{A}'_I$ as
$\tilde{d}_H(\mathfrak{e},\mathfrak{f}) = (0,0), \forall \mathfrak{e} = \mathfrak{f}, \tilde{d}_H(1,2) = (700,700)$,
$\tilde{d}_H(1,3) = \tilde{d}_H(1,4) = \tilde{d}_H(1,5) = \tilde{d}_H(2,3) = \tilde{d}_H(2,4) = \tilde{d}_H(2,5) = \tilde{d}_H(3,4) = \tilde{d}_H(3,5) = \tilde{d}_H(4,5) = (50,50), \tilde{d}_H(\mathfrak{e},6) = (150,150), \forall \mathfrak{e} = 2,3,4,5$ and the controlled function
$E(\mathfrak{e},\mathfrak{f}) = \mathfrak{e}+\mathfrak{f}, \forall \mathfrak{e},\mathfrak{f} \in X$.

It is easy to verify that $\tilde{d}_H$ is a C^*-algebra-valued extended hexagonal b-metric type space. Indeed, we have

$$\tilde{d}_H(1,2) = (700,700) \succ E(1,2)[\tilde{d}_H(1,3) + \tilde{d}_H(3,2)] = (300,300).$$

Therefore, $\tilde{d}_H$ is not a C^*-algebra-valued extended b-metric space.

Definition 3.3. A sequence $\{\mathfrak{e}_n\}$ in a C^*-algebra-valued extended hexagonal b-metric space $(X, \mathbb{A}, \tilde{d}_H)$ is said to be:
(i) convergent sequence if $\exists \mathfrak{e} \in X$ such that $\tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}) \to \theta$ $(n \to \infty)$ and we denote it by $\lim\limits_{n\to\infty} \mathfrak{e}_n = \mathfrak{e}$.
(ii) Cauchy sequence if $\tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}_m) \to \theta$ $(n, m \to \infty)$.

Definition 3.4. A C^*-algebra-valued extended hexagonal b-metric space $(X, \mathbb{A}, \tilde{d}_H)$ is said to be complete if every Cauchy sequence is convergent in X with respect to $\mathbb{A}$.

Theorem 3.5. *Let $(X, \mathbb{A}, \tilde{d}_H)$ be a complete C^*-algebra-valued extended hexagonal b-metric space and suppose $T : X \to X$ that meets the following criteria:*

$$\tilde{d}_H(T\mathfrak{e}, T\mathfrak{f}) \preceq G^*\, E(\mathfrak{e}, \mathfrak{f})\tilde{d}_H(\mathfrak{e}, \mathfrak{f})\, G \text{ for all } \mathfrak{e}, \mathfrak{f} \in X \tag{3.1}$$

where $G \in \mathbb{A}$ with $\|G\| < 1$. For $\mathfrak{e}_0 \in X$, choose $\mathfrak{e}_n = T^n\mathfrak{e}_0$. Assume that

$$\sup_{m\geq 1} \lim_{i\to\infty} \|E(\mathfrak{e}_i, \mathfrak{e}_{i+1})\| \|E(\mathfrak{e}_{i+1}, \mathfrak{e}_m)\| < \frac{1}{\|G\|^8} \tag{3.2}$$

and

$$\sup_{m\geq 1} \lim_{i\to\infty} \|E(\mathfrak{e}_{i+j}, \mathfrak{e}_{i+j+1})\| \|E(\mathfrak{e}_{i+1}, \mathfrak{e}_m)\| < \frac{1}{\|G\|^8}, \text{ for } j = 1, 2, 3. \tag{3.3}$$

Furthermore, presume that

$$\lim_{n,m\to\infty} \|E(\mathfrak{e}_n, \mathfrak{e}_m)\| < \frac{1}{\|G\|^2}, \text{ for each } \mathfrak{e} \in X. \tag{3.4}$$

Then, T has a unique fixed point in X.
Proof. *Let $\mathfrak{e}_0 \in X$ and set $\mathfrak{e}_{n+1} = T\mathfrak{e}_n = \ldots = T^{n+1}\mathfrak{e}_0, n = 1, 2, \ldots$. The element $\tilde{d}_H(\mathfrak{e}_1, \mathfrak{e}_0)$ in $\mathbb{A}$ is denoted by G_0. Then*

$$\begin{aligned} \tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}_{n+1}) &= \tilde{d}_H(T\mathfrak{e}_{n-1}, T\mathfrak{e}_n) \\ &\preceq G^*\, E(\mathfrak{e}_{n-1}, \mathfrak{e}_n)\, \tilde{d}_H(\mathfrak{e}_{n-1}, \mathfrak{e}_n) G \\ &\;\;\vdots \\ &\preceq (G^*)^n \prod_{k=1}^{n} E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\tilde{d}_H(\mathfrak{e}_0, \mathfrak{e}_1)G^n. \end{aligned} \tag{3.5}$$

Similarly, we get

$$\begin{aligned} \tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}_{n+2}) &\preceq (G^*)^n \prod_{k=1}^{n} E(\mathfrak{e}_{k-1}, \mathfrak{e}_{k+1})\tilde{d}_H(\mathfrak{e}_0, \mathfrak{e}_2)G^n, \\ \tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}_{n+3}) &\preceq (G^*)^n \prod_{k=1}^{n} E(\mathfrak{e}_{k-1}, \mathfrak{e}_{k+2})\tilde{d}_H(\mathfrak{e}_0, \mathfrak{e}_3)G^n \\ \text{and } \tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}_{n+4}) &\preceq (G^*)^n \prod_{k=1}^{n} E(\mathfrak{e}_{k-1}, \mathfrak{e}_{k+3})\tilde{d}_H(\mathfrak{e}_0, \mathfrak{e}_4)G^n. \end{aligned} \tag{3.6}$$

Now, we demonstrate that $\{\mathfrak{e}_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence i.e.,* $\lim_{n\to\infty}\tilde{d}_H(\mathfrak{e}_n,\mathfrak{e}_{n+p})=\theta$, *for* $p\in\mathbb{N}$. *For* $p=4m+1$, *where* $m\geq 1$, *we consider*

$$\begin{aligned}
\tilde{d}_H(\mathfrak{e}_n,\mathfrak{e}_{n+4m+1}) \preceq\ & E(\mathfrak{e}_n,\mathfrak{e}_{n+4m+1})[\tilde{d}_H(\mathfrak{e}_n,\mathfrak{e}_{n+1})+\tilde{d}_H(\mathfrak{e}_{n+1},\mathfrak{e}_{n+2})+\tilde{d}_H(\mathfrak{e}_{n+2},\mathfrak{e}_{n+3}) \\
& +\tilde{d}_H(\mathfrak{e}_{n+3},\mathfrak{e}_{n+4})] \\
& \vdots \\
& E(\mathfrak{e}_n,\mathfrak{e}_{n+4m+1})E(\mathfrak{e}_{n+4},\mathfrak{e}_{n+4m+1})\dots E(\mathfrak{e}_{n+4m-4},\mathfrak{e}_{n+4m+1}) \\
& [\tilde{d}_H(\mathfrak{e}_{n+4m-4},\mathfrak{e}_{n+4m-3})+\tilde{d}_H(\mathfrak{e}_{n+4m-3},\mathfrak{e}_{n+4m-2})+\tilde{d}_H(\mathfrak{e}_{n+4m-2},\mathfrak{e}_{n+4m-1}) \\
& +\tilde{d}_H(\mathfrak{e}_{n+4m-1},\mathfrak{e}_{n+4m})+\tilde{d}_H(\mathfrak{e}_{n+4m},\mathfrak{e}_{n+4m+1})] \\
=\ & \sum_{i=\frac{n}{4}}^{\frac{n+4m-4}{4}}\prod_{j=\frac{n}{4}}^{i}E(\mathfrak{e}_{4j},\mathfrak{e}_{n+4m+1})\Big[\tilde{d}_H(\mathfrak{e}_{4i},\mathfrak{e}_{4i+1})+\tilde{d}_H(\mathfrak{e}_{4i+1},\mathfrak{e}_{4i+2})+\tilde{d}_H(\mathfrak{e}_{4i+2},\mathfrak{e}_{4i+3}) \\
& +\tilde{d}_H(\mathfrak{e}_{4i+3},\mathfrak{e}_{4i+4})\Big]+\prod_{j=\frac{n}{4}}^{\frac{n+4m-4}{4}}E(\mathfrak{e}_{4j},\mathfrak{e}_{n+4m+1})\tilde{d}_H(\mathfrak{e}_{n+4m},\mathfrak{e}_{n+4m+1}) \\
\preceq\ & \sum_{i=\frac{n}{4}}^{\frac{n+4m-4}{4}}\prod_{j=\frac{n}{4}}^{i}E(\mathfrak{e}_{4j},\mathfrak{e}_{n+4m+1})\Big[(G^*)^{4i}\prod_{k=1}^{4i}E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\tilde{d}_H(\mathfrak{e}_0,\mathfrak{e}_1)G^{4i} \\
& +(G^*)^{4i+1}\prod_{k=1}^{4i+1}E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\tilde{d}_H(\mathfrak{e}_0,\mathfrak{e}_1)G^{4i+1} \\
& +(G^*)^{4i+2}\prod_{k=1}^{4i+2}E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\tilde{d}_H(\mathfrak{e}_0,\mathfrak{e}_1)G^{4i+2} \\
& +(G^*)^{4i+3}\prod_{k=1}^{4i+3}E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\tilde{d}_H(\mathfrak{e}_0,\mathfrak{e}_1)G^{4i+3}\Big] \\
& +\prod_{j=\frac{n}{4}}^{\frac{n+4m-4}{4}}E(\mathfrak{e}_{4j},\mathfrak{e}_{n+4m+1})(G^*)^{n+4m}\prod_{k=1}^{n+4m}E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\tilde{d}_H(\mathfrak{e}_0,\mathfrak{e}_1)G^{n+4m} \\
& \vdots \\
\preceq\ & \|G_0\|\sum_{i=\frac{n}{4}}^{\frac{n+4m-4}{4}}\prod_{j=\frac{n}{4}}^{i}\Bigg[\|E(\mathfrak{e}_{4j},\mathfrak{e}_{n+4m+1})\|\|\prod_{k=1}^{4i}E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\|\|G\|^{2(4i)} \\
& +\|E(\mathfrak{e}_{4j},\mathfrak{e}_{n+4m+1})\|\prod_{k=1}^{4i+1}\|E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\|\|G\|^{2(4i+1)} \\
& +\|E(\mathfrak{e}_{4j},\mathfrak{e}_{n+4m+1})\|\prod_{k=1}^{4i+2}\|E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\|\|G\|^{2(4i+2)} \\
& +\|E(\mathfrak{e}_{4j},\mathfrak{e}_{n+4m+1})\|\prod_{k=1}^{4i+3}\|E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\|\|G\|^{2(4i+3)}\Bigg]I \\
& +\|G_0\|\prod_{j=\frac{n}{4}}^{\frac{n+4m-4}{4}}\|E(\mathfrak{e}_{4j},\mathfrak{e}_{n+4m+1})\|\prod_{k=1}^{n+4m}\|E(\mathfrak{e}_{k-1},\mathfrak{e}_k)\|\|G\|^{2(n+4m)}I
\end{aligned}$$

where I is the unit element in $\mathbb{A}$. Let

$$a_i = \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i)} \|G_0\|, \tag{3.7}$$

$$b_i = \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i+1} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i+1)} \|G_0\|, \tag{3.8}$$

$$c_i = \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i+2} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i+2)} \|G_0\|, \tag{3.9}$$

and

$$d_i = \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i+3} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i+3)} \|G_0\|. \tag{3.10}$$

It is clear that $\sup_{m\geq 1} \lim_{i\to\infty} \left\| \frac{a_{i+1}}{a_i} \right\| = \|E(\mathfrak{e}_{4i+4}, \mathfrak{e}_{n+4m+1})\| \|E(\mathfrak{e}_{4i+3}, \mathfrak{e}_{4i+4})\| \|G\|^8 < 1$ *by the hypotheses of the theorem. In a similar manner, we can demonstrate that*

$$\sup_{m\geq 1} \lim_{i\to\infty} \left\| \frac{b_{i+1}}{b_i} \right\| < 1, \ \sup_{m\geq 1} \lim_{i\to\infty} \left\| \frac{c_{i+1}}{c_i} \right\| < 1 \text{ and } \sup_{m\geq 1} \lim_{i\to\infty} \left\| \frac{d_{i+1}}{d_i} \right\| < 1.$$

Therefore,

$$\sum_{i=\frac{n}{4}}^{+\infty} \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i)} \|G_0\| < +\infty,$$

$$\sum_{i=\frac{n}{4}}^{+\infty} \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i+1} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i+1)} \|G_0\| < +\infty,$$

$$\sum_{i=\frac{n}{4}}^{+\infty} \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i+2} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i+2)} \|G_0\| < +\infty$$

and

$$\sum_{i=\frac{n}{4}}^{+\infty} \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i+3} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i+3)} \|G_0\| < +\infty.$$

Consequently, we infer that

$$\left(\sum_{i=\frac{n}{4}}^{\frac{n+4m-4}{4}} \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i)} \|G_0\| \right) I,$$

$$\left(\sum_{i=\frac{n}{4}}^{\frac{n+4m-4}{4}} \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i+1} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i+1)} \|G_0\| \right) I,$$

$$\left(\sum_{i=\frac{n}{4}}^{\frac{n+4m-4}{4}} \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i+2} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i+2)} \|G_0\| \right) I$$

and

$$\left(\sum_{i=\frac{n}{4}}^{\frac{n+4m-4}{4}} \prod_{j=\frac{n}{4}}^{i} \|E(\mathfrak{e}_{4j}, \mathfrak{e}_{n+4m+1})\| \prod_{k=1}^{4i+3} \|E(\mathfrak{e}_{k-1}, \mathfrak{e}_k)\| \|G\|^{2(4i+3)} \|G_0\| \right) I$$

are Cauchy sequences in $\mathbb{A}$. Thereby, we obtain that $\tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}_{n+4m+1}) \to \theta$ as $n \to \infty$. By following the above steps, we can easily deduce that

$$\lim_{n\to\infty} \tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}_{n+4m+2}) = \lim_{n\to\infty} \tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}_{n+4m+3}) = \lim_{n\to\infty} \tilde{d}_H(\mathfrak{e}_n, \mathfrak{e}_{n+4m+4}) = \theta. \tag{3.11}$$

Therefore the sequence $\{\mathfrak{e}_n\}$ is Cauchy. As $(X, \tilde{d}_H)$ is complete, there exists $\mathfrak{e} \in X$ such that $\lim_{n\to\infty} \mathfrak{e}_n = \mathfrak{e}$. We will reveal that $\mathfrak{e}$ is a fixed point of T. Consider

$$\begin{aligned}
\tilde{d}_H(T\mathfrak{e}, \mathfrak{e}) &\preceq E(T\mathfrak{e}, \mathfrak{e})[\tilde{d}_H(T\mathfrak{e}, \mathfrak{e}_{n+1}) + \tilde{d}_H(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2}) + \tilde{d}_H(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+3}) \\
&\quad + \tilde{d}_H(\mathfrak{e}_{n+3}, \mathfrak{e}_{n+4}) + \tilde{d}_H(\mathfrak{e}_{n+4}, \mathfrak{e})] \\
&= E(T\mathfrak{e}, \mathfrak{e})[\tilde{d}_H(T\mathfrak{e}, T\mathfrak{e}_n) + \tilde{d}_H(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2}) + \tilde{d}_H(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+3}) \\
&\quad + \tilde{d}_H(\mathfrak{e}_{n+3}, \mathfrak{e}_{n+4}) + \tilde{d}_H(\mathfrak{e}_{n+4}, \mathfrak{e})] \\
&\preceq E(T\mathfrak{e}, \mathfrak{e})[G^* E(\mathfrak{e}, \mathfrak{e}_n)\tilde{d}_H(\mathfrak{e}, \mathfrak{e}_n)G + \tilde{d}_H(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2}) + \tilde{d}_H(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+3}) \\
&\quad + \tilde{d}_H(\mathfrak{e}_{n+3}, \mathfrak{e}_{n+4}) + \tilde{d}_H(\mathfrak{e}_{n+4}, \mathfrak{e})] \\
\Longleftrightarrow \|\tilde{d}_H(T\mathfrak{e}, \mathfrak{e})\| &\leq \|E(T\mathfrak{e}, \mathfrak{e})\| \big[\|G^2\|\|E(\mathfrak{e}, \mathfrak{e}_n)\|\|\tilde{d}_H(\mathfrak{e}, \mathfrak{e}_n)\| + \|\tilde{d}_H(\mathfrak{e}_{n+1}, \mathfrak{e}_{n+2})\| \\
&\quad + \|\tilde{d}_H(\mathfrak{e}_{n+2}, \mathfrak{e}_{n+3})\| + \|\tilde{d}_H(\mathfrak{e}_{n+3}, \mathfrak{e}_{n+4})\| + \|\tilde{d}_H(\mathfrak{e}_{n+4}, \mathfrak{e})\|\big]
\end{aligned}$$

which yields $\|\tilde{d}_H(T\mathfrak{e}, \mathfrak{e})\| \leq 0$ as $n \to \infty \Longleftrightarrow \tilde{d}_H(T\mathfrak{e}, \mathfrak{e}) \preceq \theta$ as $n \to \infty$ i.e., $\mathfrak{e}$ is a fixed point of T.
Unicity:
Let $\mathfrak{f}(\neq \mathfrak{e})$ be an another fixed point of T. As $\theta \preceq \tilde{d}_H(\mathfrak{e}, \mathfrak{f}) = \tilde{d}_H(T\mathfrak{e}, T\mathfrak{f}) \preceq G^* E(\mathfrak{e}, \mathfrak{f})\tilde{d}_H(\mathfrak{e}, \mathfrak{f})G$, we have

$$\begin{aligned}
0 \leq \|\tilde{d}_H(\mathfrak{e}, \mathfrak{f})\| &= \|\tilde{d}_H(T\mathfrak{e}, T\mathfrak{f})\| \\
&\leq \|G^* E(\mathfrak{e}, \mathfrak{f})\tilde{d}_H(\mathfrak{e}, \mathfrak{f})G\| \\
&\leq \|G^* G\|\|E(\mathfrak{e}, \mathfrak{f}\|\|\tilde{d}_H(\mathfrak{e}, \mathfrak{f})\| \\
&= \|G\|^2\|E(T^n\mathfrak{e}, T^m\mathfrak{f}\|\|\tilde{d}_H(\mathfrak{e}, \mathfrak{f})\|.
\end{aligned}$$

Taking limit $n \to \infty$ in the equation mentioned above and employing (3.4), we get $\|\tilde{d}_H(\mathfrak{e}, \mathfrak{f})\| < \|\tilde{d}_H(\mathfrak{e}, \mathfrak{f})\|$, which is impossible. Henceforth the fixed point $\mathfrak{e}$ is unique.

Example 3.6. Let $X = [0, 8]$ and $\mathbb{A} = M_2(\mathbb{R})$. Define partial ordering on $\mathbb{A}$ as

$$\begin{pmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 \\ \mathfrak{e}_3 & \mathfrak{e}_4 \end{pmatrix} \succeq \begin{pmatrix} \mathfrak{f}_1 & \mathfrak{f}_2 \\ \mathfrak{f}_3 & \mathfrak{f}_4 \end{pmatrix}$$
$$\Leftrightarrow \mathfrak{e}_i \geq \mathfrak{f}_i \text{ for } i = 1, 2, 3, 4.$$

For any $G \in \mathbb{A}$, its norm can be defined as, $\|G\| = \max_{1\leq i\leq 4} |a_i|$. Define $\tilde{d}_H : X \times X \to \mathbb{A}$ for all $\mathfrak{e}, \mathfrak{f} \in X$

$$\tilde{d}_H(\mathfrak{e}, \mathfrak{f}) = \begin{pmatrix} (\mathfrak{e} - \mathfrak{f})^6 & 0 \\ 0 & (\mathfrak{e} - \mathfrak{f})^6 \end{pmatrix}$$

with the controlled function

$$E(\mathfrak{e}, \mathfrak{f}) = \begin{cases} \begin{pmatrix} 2 + |\mathfrak{e} - \mathfrak{f}|^5 & 0 \\ 0 & 2 + |\mathfrak{e} - \mathfrak{f}|^5 \end{pmatrix}, & \text{if } \mathfrak{e} \neq \mathfrak{f} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } \mathfrak{e} = \mathfrak{f} \end{cases}$$

It is easy to verify that $(X, \mathbb{A}, \tilde{d}_H)$ is a complete C^*-algebra-valued extended hexagonal b-metric space. Define $T : X \to X$ by $T\mathfrak{e} = \frac{\mathfrak{e}}{4}$. We have

$$\begin{aligned}
\tilde{d}_H(T\mathfrak{e}, T\mathfrak{f}) &= \begin{pmatrix} \left(\frac{\mathfrak{e}}{4} - \frac{\mathfrak{f}}{4}\right)^6 & 0 \\ 0 & \left(\frac{\mathfrak{e}}{4} - \frac{\mathfrak{f}}{4}\right)^6 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4096}(\mathfrak{e} - \mathfrak{f})^6 & 0 \\ 0 & \frac{1}{4096}(\mathfrak{e} - \mathfrak{f})^6 \end{pmatrix} \\
&\preceq \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \left[2 + |\mathfrak{e} - \mathfrak{f}|^5\right](\mathfrak{e} - \mathfrak{f})^6 & 0 \\ 0 & \left[2 + |\mathfrak{e} - \mathfrak{f}|^5\right](\mathfrak{e} - \mathfrak{f})^6 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \\
&= G^* E(\mathfrak{e}, \mathfrak{f}) \tilde{d}_H(\mathfrak{e}, \mathfrak{f}) G
\end{aligned}$$

where $\|G\| = \frac{1}{4} < 1$. Notice that for each $\mathfrak{e} \in X$, $T^n\mathfrak{e} = \frac{\mathfrak{e}}{4^n}$. Thus

$$\sup_{m \geq 1} \lim_{i \to \infty} \|E(\mathfrak{e}_i, \mathfrak{e}_{i+1})\| \|E(\mathfrak{e}_{i+1}, \mathfrak{e}_m)\| = \sup_{m \geq 1} \left[4 + 2\left(\frac{\mathfrak{e}}{4^m}\right)^5\right] < 4^8 = \frac{1}{\|G\|^8}$$

and

$$\lim_{n \to \infty} \|E(\mathfrak{e}_n, \mathfrak{e}_m)\| = 2 < \infty.$$

As a result, all of the conditions of Theorem 3.5 are fulfilled. Accordingly T has a unique fixed point $(\mathfrak{e} = 0)$.

Corollary 3.7. *Let $(X, \mathbb{A}, \tilde{d}_H)$ be a complete C^*-algebra-valued hexagonal b-metric space and suppose $T : X \to X$ is a mapping satisfying the following condition:*

$$\tilde{d}_H(T\mathfrak{e}, T\mathfrak{f}) \preceq G^* F \tilde{d}_H(\mathfrak{e}, \mathfrak{f})\, G \text{ for all } \mathfrak{e}, \mathfrak{f} \in X \tag{3.12}$$

where $G \in \mathbb{A}, F \in \mathbb{A}'_I$ with $\|G\| < 1$ and $\|F\| > 1$. Then, T has a unique fixed point in X.
Proof. *The proof follows from Theorem 3.5 by defining $E : X \times X \to \mathbb{A}'_I$ via $E(\mathfrak{e}, \mathfrak{f}) = F$.*

4 Application

In this section, we show that a type of operator equation exists and is unique in the context of complete C^*-algebra-valued extended hexagonal b-metric spaces.

Example 4.1. Assume H is a Hilbert space, $L(H)$ is the set of linear bounded operators on H. Let $F_1, F_2, \ldots F_n, \ldots \in L(H)$ that satisfy $\sum_{n=1}^{\infty} \|F_n\|^6 < 1$ and $R \in L(H)_+$. Then the operator equation

$$C - \sum_{n=1}^{\infty} F_n^* C F_n = R$$

has a unique solution in $L(H)$.

Proof. Set $G = \left(\sum_{n=1}^{\infty} \|F_n\|\right)^6$, therefore it is obvious that $\|G\| < 1$ and $G > 0$. Now, select an operator $M \in L(H)$ that is positive. For $C, D \in L(H)$, set

$$\tilde{d}_H(C, D) = \|C - D\|^6 M.$$

Thereby $\tilde{d}_H$ is a C^*-algebra-valued extended hexagonal b-metric with a controlled function

$$E(C, D) = \begin{cases} I + \|C - D\|^5 M, \text{if } C \neq D \\ I, \text{if } C = D \end{cases}$$

As $L(H)$ is a Banach space, $(L(H), \tilde{d}_H)$ is a complete C^*-algebra-valued extended hexagonal b-metric space. Consider the map $T : L(H) \to L(H)$ defined by

$$TC = \sum_{n=1}^{\infty} F_n^* C F_n + R.$$

Then

$$\begin{aligned}
\tilde{d}_H(T(C), T(D)) &= \|T(C) - T(D)\|^6 M \\
&= \left\| \sum_{n=1}^{\infty} F_n^*(C - D)F_n \right\|^6 M \\
&\preceq \sum_{n=1}^{\infty} \|F_n\|^{12} \|C - D\|^6 M \\
&\prec \sum_{n=1}^{\infty} \|F_n\|^{12} \Big[I + \|C - D\|^5 M \Big] \|C - D\|^6 M \\
&= G^2\, E(C, D)\tilde{d}_H(C, D) \\
&= (GI)^* E(C, D)\tilde{d}_H(C, D)(GI).
\end{aligned}$$

Using Theorem 3.5, there exists a unique fixed point C in $L(H)$.

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