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PENROSE TILING: A MATH-RELATED LEARNING INTERACTION BETWEEN HIGH SCHOOL STUDENTS AND A UNIVERSITY LAB

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Abstract The subject of this paper is the Penrose kite and dart tiling of the plane: an aperiodic tiling where, in his own words, "the tiles may be used to form an instructive game ... the virtue of the game lies in the very surprising variety which arises in the fitting together of pieces of only two kinds". We deal with a teaching experience in collaboration with a lab technician, a high school teacher and a graduating student: how to let high school students project and produce the kite and dart tiles using a Computer Numerical Control (CNC) machine. It is a project of work-related learning with 110 students coming from seven different high schools, in collaboration with the Model and Prototype Laboratory of the Department of Architecture at Roma Tre University. We discuss the effect of this activity on the learning process of the students, the methodology applied, the technical and organizational problems we faced in a Covid related period. Using the software GeoGebra, we will also show the geometric construction of Penrose tiles and how to produce an application to simulate the tessellation making.

1 Introduction

The use of problem solving [8], inspired by computer aided algorithms and visualization (for example: find a family of cones sharing the same given planar section), has become a common example of convergence [4] in mathematics education: a way to teach a subject by integrating knowledge, methods, and expertise from different disciplines well suited for cooperative or distance learning [2].

In this paper we deal with the problem of an aperiodic tiling of the plane, solved using pieces of two kinds only: the Penrose's kite and dart tiles. We start giving the geometric construction of the two tiles and learn how to communicate the corresponding instructions to a cutting machine, in order to produce many copies of the tiles. We then investigate few initial steps of possible tessellations and, finally, show how to simulate the tessellation on a computer using the software GeoGebra.

More precisely, we will discuss, in collaboration with a lab technician, a high school teacher and a graduating student, an original teaching experience: how to let high school students project and produce the kite and dart tiles using a numerical cutting machine. The activity followed a project of work-related learning with 110 students coming from seven different high schools, in collaboration with the Model and Prototype Laboratory of the Department of Architecture at Roma Tre University. In the following, we discuss in some details the effect of this activity on different contexts: the use of GeoGebra as a teaching tool, in a Covid related period, facilitated the geometric construction of Penrose tiles and allowed the construction of several applications to simulate the tessellation making and test different visualizations.

2 Penrose Tilings

Penrose's approach to aperiodic tiling was first introduced in his 1974 paper [5] using tiles of four different shapes, including rhombus, pentagons, and a five-pointed star [1]. Later ([3], [6]) he came up with a set of tiles of two kinds only which would cover the plane without overlapping or translational symmetry, provided a simple rule is applied.

The famous kite and dart tiles are two quadrilaterals (see Fig. 1) with internal angles which are multiple of 36° and sides which are in a golden ratio. The colored arcs describe a matching rule: any two tiles must join along sides with the same length matching arcs of the same color (see Fig.2). With such a rule the parallelogram is prohibited, thus avoiding translational invariance, and non-periodicity can be proven for any choice of the tiling [7].

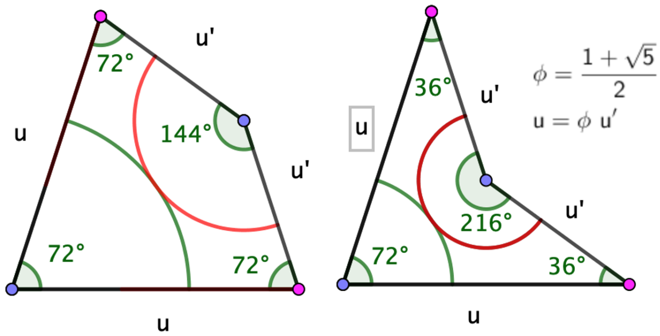


Figure 1. The Penrose kite and dart tiles. The long and short side of the two quadrilaterals are in a golden ratio and the internal angles are all multiple of 36°.

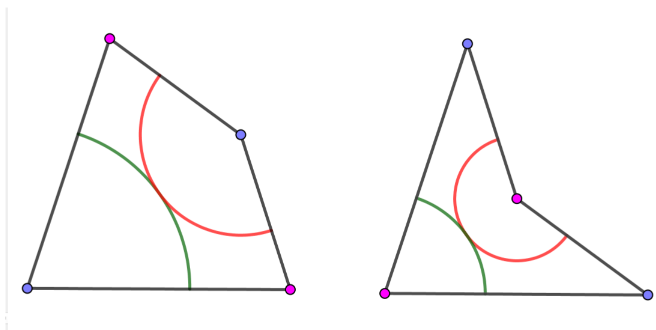


Figure 2. Kite and Dart tiles. The colored arcs provide the contact rule: join tiles matching arcs of the same color and sides of the same length. In particular, the parallelogram is excluded.

Given the tiles angles and the contact rule, there are seven vertex configurations (see Fig. 3) in a Penrose tiling: nevertheless, there is an infinite number of different non periodic tessellations.

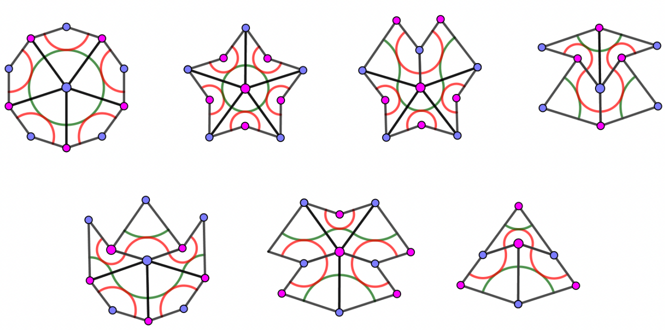


Figure 3. All the possible 7 vertex configurations of a Penrose tiling.

There is a lot of structure in a Penrose tiling, which allows the infinite and non-repetitive variety of designs together with the very rigid vertex constrains: the explanation is on a process called inflation which is out of the scope of our activity [7]. The amazing fact is that all finite configurations belong to one large finite region that is exactly duplicated infinitely many times on all patterns: without going too far, any region of size d would have an exact copy at a distance smaller than $2d$ [3]. Moreover, there are two infinite configurations which have pentagonal symmetry (see Fig. 4).

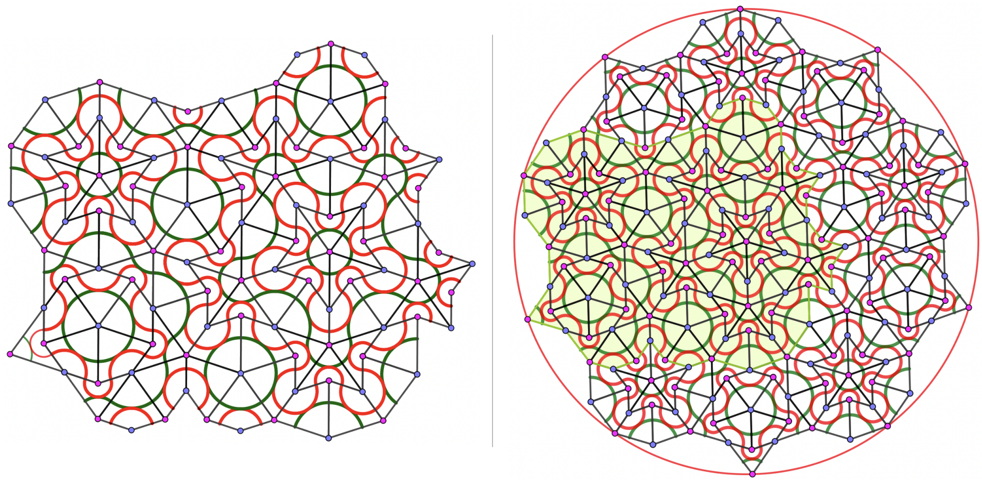


Figure 4. The apparently random configuration on the left can be recognized as a part of the pentagonal symmetric configuration on the right.

3 The PCTO Activity

The Paths for Transversal Competences and Orientation (Percorsi per le Competenze Trasversali e per l'Orientamento - PCTO) are one of the Italian responses to the recommendation of the European Council and Parliament on key competences for lifelong learning in 2006, and its subsequent modifications on 22 May 2018 [9].

The Italian ministry of education requires that these PCTO activities should involve students in their last three years of secondary school (in Italy, respectively 11^{th}, 12^{th} and 13^{th} grade) from 90 to 210 hours, according to the type of institution attended. Usually, they take the form of a project aimed at the solution of a preconstituted problem, a simulated formative enterprise or even the development of, or participation to, real entrepreneurial activities, almost always in collaboration with local cultural institutions, companies, research center, etc.

The specific PCTO on the Penrose tessellation, under the supervision of Prof. Falcolini, was organized in eight meetings, in the afternoon, for a total of 28 hours and was proposed to 110 students of seven schools in the Rome area from January to March 2022. At the end of the school year, in a final meeting, each school received a tessellation kit with one hundred pieces, designed and produced together with the students. Living in the SARS-CoV-2 era, most of the activities were carried out at a distance, through Microsoft Teams. While this impacted the social aspect of the activity, it also had the side benefit that every student could simultaneously try to reproduce what was shown and express immediately any doubt or problem encountered.

The organization of the eight meetings was the following: the first four meeting at a distance, using Teams,

1) Homage to Roger Penrose. Introduction to GeoGebra. Regular tilings of the plane.
2) Kite and dart tiles: geometric ruler-and-compass construction, properties, matching rule.
3) GeoGebra: examples with sliders and how to create a new tool. Homework: draw a circle with an inscribed regular polygon with n sides, for $n = 1, 2, \ldots, 60$; define a new tool for a chosen polygon and discuss how to tile the plane with it.
4) The new tools Kite and Dart using GeoGebra. How to communicate the instructions to a cutting machine.

The following two meetings took place at the Department of Architecture of Roma Tre University: all the students were staying in Aula Magna (with proper distances and safe face mask) while they visited the laboratory divided in small groups

5) Video on Penrose tiling. The seven vertex configurations. First visit to the Model and Prototype Laboratory: the use of a laser cutting machine, economy data for real tiles production.

6) Questions on special configurations, using real tiles. Second visit to the Model and Prototype Laboratory: compare the costs using a CNC milling machine. Project the tiles colors and design using GeoGebra.

   The last two meetings, after some new cases of positivity to Covid in one of the classes, were again carried out at a distance:

7) Tiles colors and design proposed by each school. Simplifications on GeoGebra new tools Kite and Dart.

8) A program to simulate the kite and dart tessellation [10] and the use of vertices constraint.

3.1 Some of the Activities

In the 2^{nd} meeting we discussed, using GeoGebra, the ruler-and-compass construction of the kite and dart tiles: combining the regular pentagon and golden ratio construction with the symmetry, it is possible to construct the two tiles and the arcs needed for the matching condition. Note that, on both tiles, also the marked (colored) arcs have radii which are in a golden ratio.

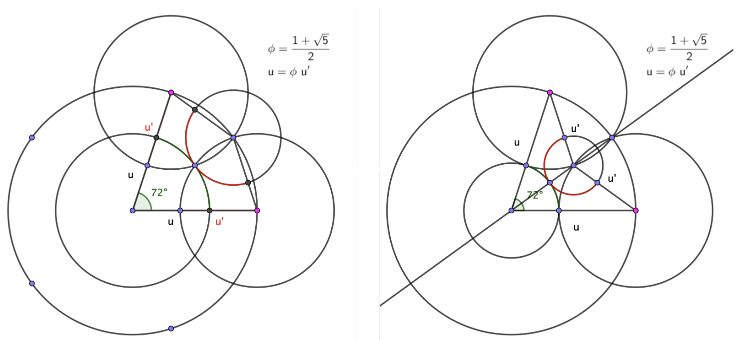


Figure 5. The ruler-and-compass construction of the kite and dart tiles, starting from a central angle of 72°.

In the 6^{th} meeting, our graduating student, A. D'Angeli, showed two of his original GeoGebra programs: the first reproducing Penrose tiling using colored circular sectors instead of arcs, and the second using directly the seven vertex configurations (see Fig. 3). In the first case, the sticking procedure is still based on the creation of tools that will connect the tiles by long or short side, while in the second case the program would fill at any step the chosen vertex using one of the seven vertex configurations (see Fig. 6).

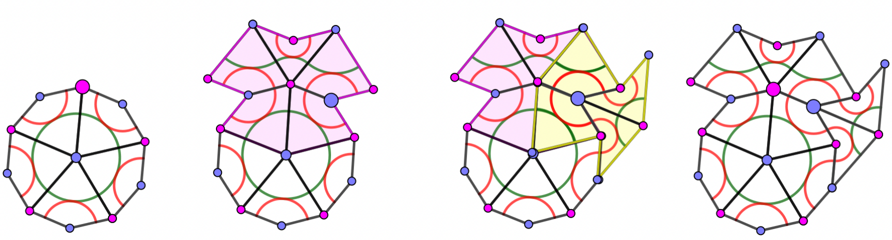


Figure 6. Vertex by vertex process: starting from the Sun module on the left, choose the large top vertex and follow the filling up of two adjacent vertices. Recognize in this picture the presence of four vertex configurations listed in Fig. 3.

The coloring process: color the kite first, with the circular sectors of the kite red and green; to color the remaining white background create a kite polygon and set the maximum opacity value using a cream color. After that create a tool that generates a dart of complementary coloring of the kite: the background in green or red and the circular sectors in cream. Be careful that another coloring of the kite tessellation is needed. In fact, it was needed a kite with a cream-colored background and the colors of the circular sectors reversed from the initial tool (see Fig. 7).

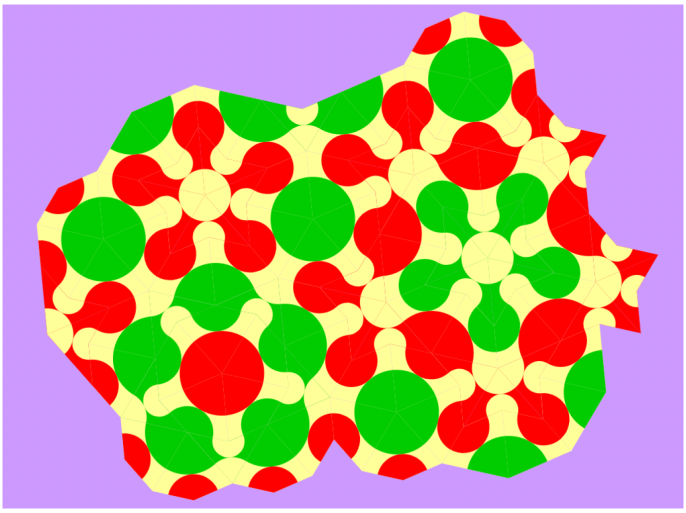


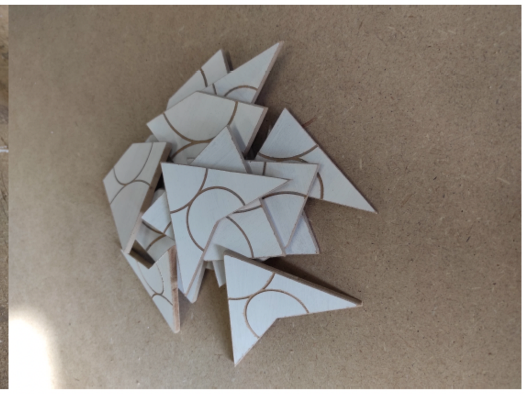
Figure 7. Penrose tessellation with three colors.

The second program should contain a well-chosen number of commands out of all the possible combinations of overlapping modules. We therefore presented it as a work in progress to be completed and improved by everyone using the GeoGebra?s Classroom function: with the educational purpose of giving students an outline of how to tile the plane through Penrose's modules. In addition, the program was designed as a preparatory step from virtual tiling to the concrete one.

Figure 8. Students in action at the 6^{th} meeting, in the Aula Magna of the Department of Architecture.

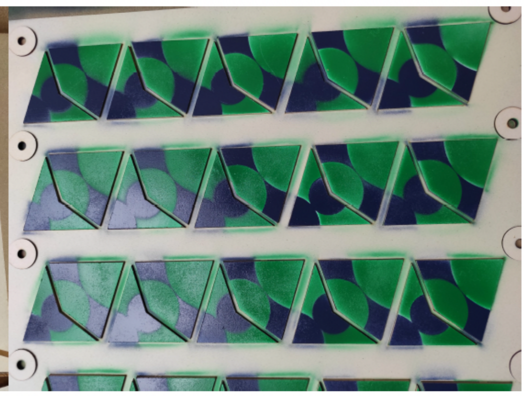
At the end, the author commented: *I displayed this program in the Aula Magna of Roma Tre's Architecture department at the end of the second live meeting. Exhibiting my work on a giant screen in front of almost a hundred students was a wonderful experience and, strangest of all, I felt at ease. I found it most natural explaining what I love. When your efforts are reciprocated by students with their interest and curiosity, teacher and student's work meet in a dimensionless* *environment, collapse to a point and then explode, giving birth to shared ideas: a world without discernment of who is teaching and who is learning.*

(A) Tiles to be sanded



(B) Sanded tiles

(C) Stencil for tiles' coloration



(D) Tiles colored with stencil

(E) A portion of the tessellated plane with the Penrose tiles with two colors

(F) The delivery of the tiles

Figure 9. The tiles production process and final delivery of the school kits.

These considerations emphasize the profound educational collaboration that took place between students and teachers during the eight meetings, especially the one in person. Students did not just think mathematics, they visualized it. Our teaching approach minimizes the difference between the visualization process and the thinking process. Therefore, a dormant curiosity is triggered in the student, prompting them to learn and ask questions. In our case, the next step was to touch the tiles and allow students to turn the virtual into the real through material experience compared to "touching" with the eyes.

The aesthetic part of the tiles was completely taken care of by the individual schools, thus both the color and arcs shape. In Fig. 8- 9 we show images that capture some moments of the PCTO and the tiles production process.

In the final 8th meeting we presented a downloadable program to simulate the kite and dart tessellation [10]. Most students started to use the program but many of them were also able to add some new tools or modify it.

3.2 A Teacher's Point of View

One of the participating classes, for a total of thirty students, was the 3DS from the high school "Socrate". This activity represented their first PCTO experience and almost every student had no previous experiences with GeoGebra. Due to this, the fact that the first meetings were devoted to the basic of GeoGebra proved to be very important: learning how to draw some simple pictures, how to use its basic instruments and, finally, to be able to reproduce the virtual construction of the tiles has been the right introduction to the software. Moreover, the students were then introduced to more advanced topics, like using a slider and dynamic constructions.

Since the first few meetings, to strengthen the students' control over GeoGebra, their teacher tried from time to time to assign small activities, related to the mathematical and physical subjects discussed during the normal class activity, that involved the use of the software. Drawing a parabola to discuss the number of its intersections with a line or studying the condition under which a ball is rolling without slipping on a horizontal surface (and simulating its motion) are all possible examples in which the use of GeoGebra might cast new insights and empower the students with a deeper understanding of the various subjects.

Two of the meetings took place at the Department of Architecture of Roma Tre University. As already said, the first one was devoted to the actual construction process of the tiles and the various possibilities to reduce the cost, in terms of the raw material, the power consumption and the time needed for the production. At the end, all students received a small sample of 2 kites and 2 darts: collaborating with their classes, they could start experimenting and exploring the possible ways to connect the two pieces.

In fact, in the following meeting, part of it was assigned to a guided laboratory in which the students, divided in small groups, investigated the properties of the Penrose tiling by construction. One of the most captivating problems was the following: "What is the minimum number of kites and darts necessary in order to reproduce all possible ways to build vertices in the tessellation?". Most of the students took quite seriously the challenge and some managed to obtain excellent results, as the one depicted in Fig. 10.

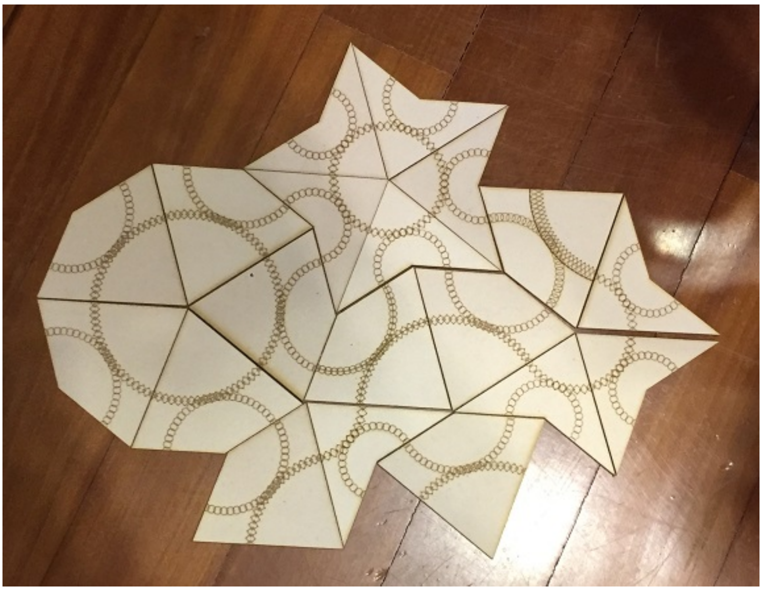


Figure 10. The work of one of the groups. It is possible to discern all the seven different *modules* or internal vertices. (Note the arcs on each tile, drawn with the laser, reproducing sequences of different shapes).

3.3 In the Laboratory

A peculiar feature of our PCTO activity has been the interaction with the Model and Prototype Laboratory of the Department of Architecture at Roma Tre University: translating the right commands for a cutting machine and trying optimizing the cost benefits analysis for a small-scale project and production of many tiles. Precisely this task has become the main subject of our full activity.

The ruler-and-compass construction of the kite and dart tiles (see Fig. 5) was the guide to convert the geometric exact algorithm into a cutting machine language: at first, we reproduced analogous commands using Autodesk AutoCAD routines.

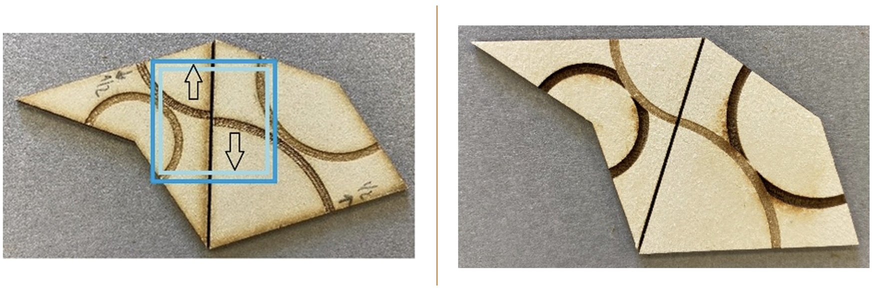


Figure 11. Working with arc thickness: the solution uses different line offsets.

Then we tried an automatic conversion, useful for several different realization of the tiles: from the swg to a dxf file, compatible with Autodesk Autocad, passing by the software Inkscape. Splines were set to low weight poly-lines with common unit 'pt', but we lose some information, like line colors and different arc level distinctions, in the process. A final clearing step, using a G-Code (a text file of machine commands, LaserCAD or DeltaCAM for CMC), fixed the chosen scale and gave to any line its correct priorities, special working functions (positioning of Tabs supports for milling) and parameters.

Our goal was to find an optimal choice between costs and benefits. A preliminary analysis fixed the nesting procedure: how to fit as many tiles as possible on a given panel to reduce waste. Nesting of pieces would depend on the chosen size of the tiles, on the machine and the materials used.

Our laser cutting machine has a wavelength of 10600 nm and an intensity of 130 Watt: the cutting speed was set to 1800 mm/min and has an estimated working cost of 60 Euro/h. The Computer Numerical Controlled (CNC) pantograph was set at a cutting speed of 1300 mm/min and has an estimated working cost of 80 Euro/h.

We used tiles with the long side of 7.5 cm. The material used was 2mm thick carton wood (around 6 Euro/mq) for the laser cutting machine and a 4 mm tick Medium Density Fiberboard (MDF, around 7 Euro/mq) for the CNC machine.

The laser has less waste during the cut (4 mm, only 5% of the tile size), no contact with machine tools and cut-ready production but is more time consuming.

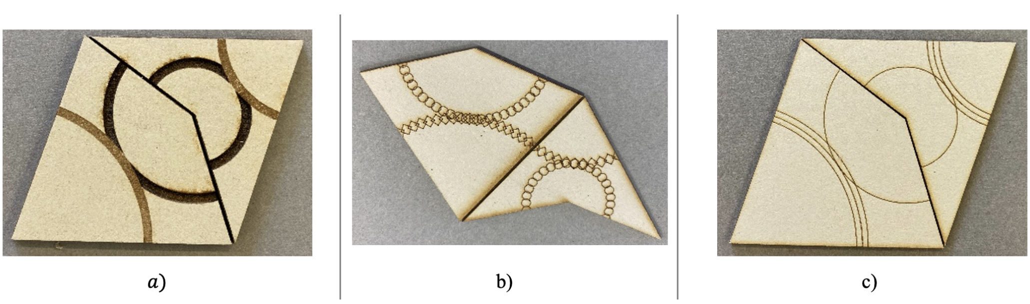


Figure 12. Different tiles production, using a laser cutter machine: a) the dark and brown colors depend on laser cutting speed and intensity; b) the arcs are sequences of different shapes; c) the difference is in the number of arcs.

The CNC machine works on stronger and more rigid materials, is less time consuming but the mills pulverize more material (12mm, with a mill diameter of 4 mm, around 15%), the panel must be secured during the cut and the pieces needed to be refined with sandpaper (see Fig. 9A).

Laser cutting cannot be used with MDF and several plastics both because of toxic fumes derived from the reaction between laser and the molecules of the materials, and the melting of some plastics due to the temperature of the laser. Thus, most of the tiles has been cut with the milling machine.

Moreover, our machines cannot directly color the two arcs drawn on the tiles (see Fig. 2): to distinguish them we tried several solutions, some of which were suggested by the students. At first, we used the laser cutting machine.

The first problem that occurred was related to line thickness: we had to carefully choose the direction of arc offsets to avoid discontinuities (see Fig. 11). Then we set machine parameters: using laser speed and intensity we got two engravings with different depth and shade (see Fig. 12a). The second problem was time: a 100 x70 cm panel took almost 7 hours of machine working time. Later we tried to draw thin arcs using a sequence of polygons and circles: the solution looks elegant and the working time much shorter (see Fig. 12b). During the 5^{th} meeting, a student proposed to use a different number of lines for the two arcs: simple and even more rapid, which means also cheaper.

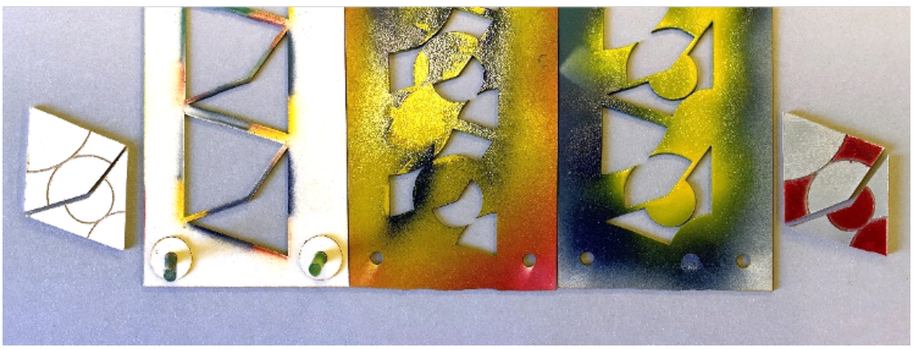
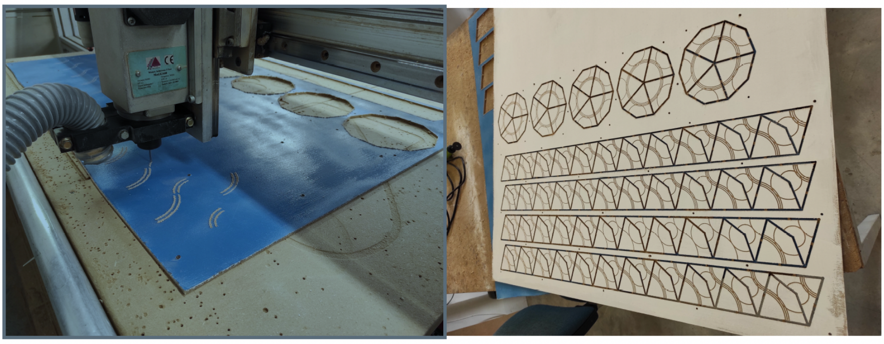


Figure 13. Coloring process. The tiles are first pos itioned and fixed on a laser cut stencil. Then, other two stencils are superimposed, one after the other, to spray the color paints.

After another trial, using dashed style, and a cost benefits analysis, we proposed a coloring process and the use of the CNC pantograph milling machine (see Fig. 14). First, we cut and sanded the 4mm MDF tiles and use a burin for thin arcs engraving, then we used a first laser-cut stencil to fix and control the tiles and two more stencils to color the tiles with a spray paint (see Fig. 13).



(A) Cut with pantograph CNC of a blue colored MDF's sheet

(B) MDF's sheet cut thanks to CAD project

Figure 14. The pantograph cutting machine using mills.

After the completion of the PCTO activity, however, the laser apparatus has been upgraded by changing the laser tube and using a different parameters? setting. With this configuration it is now sufficient one working hour to cut the tiles by laser cutting machine as opposed to the three working hours necessary using the CNC milling machine.

In conclusion, using laser or milling technology can be chosen considering the cost of the cutting job or the strength of the material. Carton wood has shown to be an optimal material to realize a puzzle game, whereas MDF could be a proper choice for wall or furniture coating.

4 Feedback and Outcomes

To assess the quality of the proposed activities, we prepared two questionnaires: the initial one was meant to evaluate the starting point of the students involved in our PCTO, while with the final one we tried to measure the impact on the students' knowledge and competencies collecting quantitative feedback regarding the entire activity.

Our questionnaires were implemented using Google Forms, allowing us an easy analysis of the results. From the initial one it emerged a general curiosity on mathematics (see Fig. 16a) but a lack of some specific information about Penrose, tiling and the software GeoGebra (see Fig. 15).

Do you know what is a *tiling* of the plane?	YES (33.3%)	NO (66.7%)	
Do you know the mathematical physicist Roger Penrose?	YES (32.4%)	NO (67.6%)	
Have you ever used the software GeoGebra?	YES (46.1%)	NO (53.9%)	
Can you draw a regular polygon using GeoGebra, varying the number n of its sides?	YES (7%)	NO (37.6%)	Perhaps (55.4%)

Figure 15. A table with some answers to selected questions from an initial questionnaire.

While the first observation was somewhat expected, as most of the students came from a scientific-oriented school, the second one gave a closer look at the fine structure of their previous knowledge. Note that, although the numerical data might suggest a correlation between the answers to the first two questions in Fig. 15, it is not so: many students knew about tiling but not who Penrose is, and vice versa.

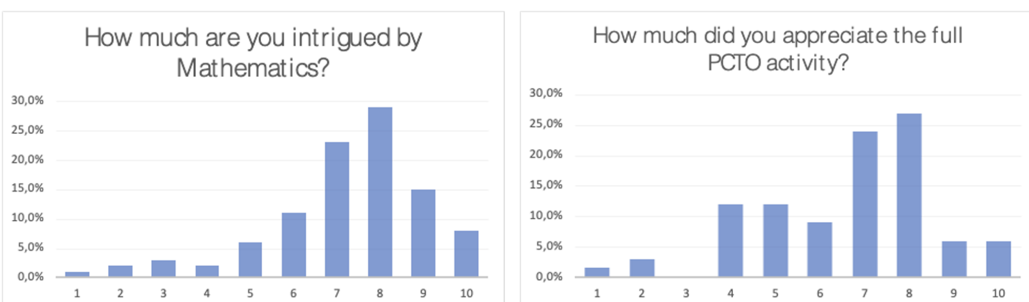


Figure 16. Comparison between a question from the initial questionnaire and a final feedback.

The answers collected at the end of the whole activity are quite encouraging. Apart from an adequate level of global satisfaction (Fig. 16b), the overall picture emerging is that most of the students mastered the basics of the Penrose kite and dart tiling: the vast majority know that the tiles may be constructed using only ruler and compass, although for a faster construction we presented in GeoGebra an alternative way not following a purist Euclidean approach.

Moreover, most of the students can identify the correct number of different modules that are possible in a Penrose tiling which we discussed in the 6^{th} meeting and the PCTO activity had, in most cases, a positive impact on their knowledge of the GeoGebra software. A critical point is represented by the third question presented in the table of Fig. 17: this is a rather technical question, not discussed in full detail during the meetings. While we cannot be satisfied with its results, we attribute them to the combined effect of the short time devoted to the specific subject and teaching method (at a distance) necessarily adopted due to SARS-CoV-2. We plan to investigate in greater detail these technical aspects in future edition of the activity, possibly in presence.

Are the Penrose kite and dart tiles constructible using only ruler and compass?	YES (75%)	NO (25%)		
How many different modules with a single internal vertex are there in a Penrose tiling?	5 (16.1%)	6 (11.3%)	**7 (67.8%)**	8 (4.8%)
What is the ratio among long and short side of a Penrose kite or dart tile?	correct (31%)	wrong (31%)	not answered (38%)	
How do you create a new tool in GeoGebra?	correct (74%)	wrong (9%)	not answered (17%)	

Figure 17. A table with some answers to selected questions from the final questionnaire.

As another qualitative outcome of our activity, every student in the school Socrate 3DS class had to produce a final report on the project, presenting the activities and expressing their positive and negative aspects. The reports sample was large enough to yield some interesting insights on the perception of the activity by the students. Many of them complained about the prolonged video sessions but most of them recognized a positive influence on their competences, as it is possible to appreciate from their words:

- "*Working in small groups proved to be effective. It made possible an exchange of ideas between us and, thanks to the individual contributions, led the group to create different types of modules*".
- "*Under the guidance of the professor and his tutors, I enjoyed discovering the GeoGebra software with all its tools. Thanks to this project, I've started using GeoGebra also on my own*".

From the teacher point of view, one of the most important results is a class of students proficient in GeoGebra. The possibility to use GeoGebra without explaining the basics during the everyday teaching activity and/or to assign open questions to be studied with it is a valuable opportunity. The very possibility to study a problem from another point of view leads the students to develop their scientific competences in a holistic way.

5 Conclusion

The Penrose kite and dart tiling of the plane has been the pivotal element in the PCTO activity presented. Exploring tilings in general and the Penrose one in particular, students have deepened their knowledge on the subject and have found themselves as key players of the whole learning project.

The interaction between a difficult subject, high school teachers and a university laboratory has been very fruitful and deep: the students mastered the Penrose's construction and put his ideas into practice by projecting and realizing the kite and dart tiles using a numerical cutting machine.

The use of the software GeoGebra, while necessary to overcome the difficulties related to the SARS-CoV-2, proved to be an excellent way to elicit interactive discussions during the meetings at a distance. While, in principle, it would have been possible to adopt a more "realistic" approach, i.e. to use real rulers and compasses if the meetings were in presence, we believe that experimenting with GeoGebra gave students insights that resulted priceless in the learning process.

This PCTO activity, while being at its first realization and despite the difficulties of the Covid era, proved to be quite successful. Students' feedback was good, and our evaluation of their knowledge retention was quite satisfying.

We therefore think that our experience, with some specific changes and personal improvements, could be useful for other similar activities in the future.

6 Acknowledgements

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