Appendix A. Technicalities

A.1. Proofs of Section 2

PROOF OF THEOREM 1. The equivalence $\text{cl}\,\text{posi}(\mathscr{G}) = \mathsf{Cn}_{\mathfrak{T}}(\mathscr{G})$ holds since in the derivation tree of a sequent $\mathscr{G} \triangleright g$, applications of the closure rule can be lifted up to the root and joint in a single inference step. The other equivalence traces back to Walley (1991). □

A.2. Proofs of Section 4

PROOF OF PROPOSITION 4. As for $\mathfrak{T}$, the equivalence $\text{cl}\,\text{posi}(\mathscr{G}) = \mathsf{Cn}_{\mathfrak{T}^\star}(\mathscr{G})$ holds since in the derivation tree of a sequent $\mathscr{G} \triangleright g$, applications of the closure rule can be lifted up to the root and joint in a single inference step.

Next, for the equivalence between items (1)-(3), first of all, notice that $\text{posi}(\mathscr{G} \cup \Sigma^{\geq}) \subseteq \text{cl}\,\text{posi}(\mathscr{G} \cup \Sigma^{\geq}) \subseteq \mathsf{Cn}_{\mathfrak{T}^\star}(\mathscr{G})$. Hence, (3) implies (2) implies (1). For the remaining implications we reason as follows. Assume that (2) holds, and assume $f + \delta \in \text{posi}(\mathscr{G} \cup \Sigma^{\geq})$, for every $\delta > 0$. This means $f \in \text{cl}(\text{posi}(\mathscr{G} \cup \Sigma^{\geq}))$. Suppose $f \in \Sigma^{<}$, then since $\Sigma^{<}$ is open, $f + \delta \in \Sigma^{<}$ for some $\delta > 0$, contradicting P-coherence of $\text{posi}(\mathscr{G} \cup \Sigma^{\geq})$. We therefore conclude that $f \notin \Sigma^{<}$, and that (3) holds. Now, assume (2) does not hold, i.e. $f \in \Sigma^{<}$ and $f \in \text{posi}(\mathscr{G} \cup \Sigma^{\geq})$. Hence, $-f$ is in the interior of $\Sigma^{\geq}$, meaning that for some $\delta > 0$, $-f - \delta = g \in \Sigma^{\geq}$. From this we get that $-1 = \frac{g+f}{\delta} \in \text{posi}(\mathscr{G} \cup \Sigma^{\geq})$: (1) does not hold. □

PROOF OF PROPOSITION 5. Since (1) always implies (2), we need to verify the other direction. Assume $\mathsf{Cn}_{\mathfrak{T}^\star}(\mathscr{G})$ is not P-coherent. By Proposition 4, $-1 \in \mathsf{Cn}_{\mathfrak{T}^\star}(\mathscr{G})$, and thus $-\varepsilon \in \mathsf{Cn}_{\mathfrak{T}^\star}(\mathscr{G})$, for every $\varepsilon \geq 0$. Let $f \in \mathscr{L}_R$. If (*) holds there is $\varepsilon > 0$ such that $f + \varepsilon \in \Sigma^{\geq} \subseteq \mathsf{Cn}_{\mathfrak{T}^\star}(\mathscr{G})$. Hence, by closure under linear combinations, $f + \varepsilon + (-\varepsilon) = f \in \mathsf{Cn}_{\mathfrak{T}^\star}(\mathscr{G})$.

Assume (b) holds. It is enough to prove that $-B \subseteq \text{posi}(B \cup \{-1\})$. Fix $b \in B$. By hypothesis $-\frac{1}{\varepsilon}b - \sum_{i=1}^{\ell} \lambda_i b_i = -1$, with $\lambda_i \geq 0$ and $\varepsilon > 0$, for some $\{b_1, \ldots, b_\ell\} \subseteq B$. Hence $-b = \varepsilon(-1 + \sum_{i=1}^{\ell} \lambda_i b_i) \in \text{posi}(B \cup \{-1\})$. We thus conclude that $\mathsf{Cn}_{\mathfrak{T}^\star}(\mathscr{G})$ is logical inconsistent. □

In the next Proposition we explicit another property of $\Sigma^{\geq}$ and verify that implies (*).

Proposition 1 *Assume that $\Sigma^{\geq}$ contains a basis B of $\mathscr{L}_R$ and for every $b \in B$ there is a finite $\{b_1, \ldots, b_\ell\} \subset B$ such that $b + \sum_{i=1}^{\ell} \lambda_i b_i = \varepsilon > 0$, with $\lambda_i \geq 0$. Then condition (*) holds*

Proof It is enough to check that, for $b \in B$, there is $\varepsilon > 0$ such that $-b + \varepsilon \in \Sigma^{\geq}$. But this is immediate since by hypothesis we know there is a finite $\{b_1, \ldots, b_\ell\} \subset B$ such that $b + \sum_{i=1}^{\ell} \lambda_i b_i = \varepsilon > 0$, with $\lambda_i \geq 0$. Hence $-b + \varepsilon = \sum_{i=1}^{\ell} \lambda_i b_i$, which is in $\Sigma^{\geq}$ since the latter is a cone that includes B. ■

PROOF OF PROPOSITION 8. This follows by Proposition 1 and the fact that Bernstein's polynomials form a partition of unity. □

PROOF OF PROPOSITION 9. Assume that a polynomial $f(\theta)$ belongs to (15). If there exists a monomial of $f(\theta)$ of degree ℓ less than d, then we can multiply it for the Bernstein partition of unity of degree $d - \ell$. The resulting polynomial will then belong to (17). The opposite direction of the proof is obvious. □

References

Peter Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman & Hall/CRC Monographs on Statistics & Applied Probability. Taylor & Francis, 1991. ISBN 9780412286605.