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Extending Nearly-Linear Models

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Abstract

Nearly-Linear Models are a family of neighbourhood models, obtaining lower/upper probabilities from a given probability by a linear affine transformation with barriers. They include a number of known models as special cases, among them the Pari-Mutuel Model, the ε-contamination model, the Total Variation Model and the vacuous lower/upper probabilities. We classified Nearly-Linear models, investigating their consistency properties, in previous work. Here we focus on how to extend those Nearly-Linear Models that are coherent or at least avoid sure loss. We derive formulae for their natural extensions, interpret a specific model as a natural extension itself of a certain class of lower probabilities, and supply a risk measurement interpretation for one of the natural extensions we compute.

Keywords: Pari-Mutuel Model, Nearly-Linear Models, natural extension, coherent lower probabilities, risk measures, Value at Risk, Expected Shortfall.

1. Introduction

Within imprecise probabilities, special models play an important role, since they may be easier to understand for non-experts and often ensure simplified procedures for checking their consistency or making inferences with them. For instance, the *Pari-Mutuel Model* (PMM) obtains an upper probability

$$\overline{P}_{\mathrm{PMM}} = \min\{(1+\delta)P_0, 1\} \tag{1}$$

from a given probability P_0 and a parameter $\delta > 0$, and has been employed in betting, since it replaces a fair price (P_0) with a more realistic infimum selling price ($\overline{P}_{\mathrm{PMM}}$) [13, 15]. The upper probability $\overline{P}_{\mathrm{PMM}}$ is coherent and 2-alternating, and formulae for computing its natural extension are also known.

Recently, we explored a family of models generalising the PMM as well as other models [2, 3]. We termed them *Nearly-Linear (NL) Models*, since like $\overline{P}_{\mathrm{PMM}}$ in (1) they return an upper or a lower probability from a given P_0 by a linear affine transformation of P_0, with barriers to prevent obtaining values outside the interval $[0,1]$, as detailed in Section 2.1.

Each NL model is identified by two parameters. According to the different values the parameters may take, NL models are subdivided into three families, briefly recalled later on: the *Vertical Barrier* Models (VBM), the *Horizontal Barrier* Models (HBM), and the *Restricted Range* Models (RRM). As shown in [2, 3], VBMs correspond to lower and upper probabilities that are coherent. HBMs ensure coherence in certain cases and RRMs only in an extreme situation; both are generally only 2-coherent. On the other hand, these models can represent various kinds of beliefs, also some partly irrational and conflicting ones. With respect to this, NL models are somewhat similar to Neo-additive capacities, studied in [1, 5].

While the above mentioned aspects of NL models are investigated in detail in [2, 3], here we focus on extensions concerning these models. We concentrate on the *natural extension*, a well-known concept that is one of the pillars of the theory of Imprecise Probabilities in [15]. After recalling some preliminary material in Section 2, we consider the Vertical Barrier Model in Section 3. In 3.1, we show that the VBM itself may be viewed as a natural extension of less consistent (convex) models, while in Section 3.2 we study the natural extension of a VBM on gambles, generalising a known formula for the PMM. In Section 4 we determine when a HBM avoids sure loss and introduce formulae for computing its natural extension. We obtain simple expressions in the finite case, and a general formula for a coherent HBM. In Section 5 we present the results of an analogous investigation for the RRM. A hint about a risk measurement interpretation of VBMs is given in Section 6. Section 7 concludes the paper.

2. Preliminaries

Let $\mathscr{S}$ be a set of gambles (bounded random variables). A map $\underline{P} : \mathscr{S} \to \mathbb{R}$ is a *lower prevision* on $\mathscr{S}$ if (behaviourally) $\forall X \in \mathscr{S}$, $\underline{P}(X)$ is an agent's supremum buying price for X, while an *upper prevision* $\overline{P} : \mathscr{S} \to \mathbb{R}$ represents a collection of infimum selling prices for the gambles in $\mathscr{S}$ [15]. When $\mathscr{S}$ is made of (indicators of) events only, we speak of lower and upper probabilities of events, instead of previsions of their indicators.

It is possible to refer to lower or alternatively upper previsions only, by the conjugacy relation $\overline{P}(X) = -\underline{P}(-X)$, if $X \in \mathscr{S} \Rightarrow -X \in \mathscr{S}$. In the case of probabilities, conjugacy is written as

$$\overline{P}(A) = 1 - \underline{P}(\neg A),$$

assuming that $A \in \mathscr{S} \Rightarrow \neg A \in \mathscr{S}$.

Lower (and upper) previsions may satisfy different consistency requirements:

Definition 2.1 *[11, 15] Let $\underline{P} : \mathscr{S} \to \mathbb{R}$ be a given lower prevision and denote with $\mathbb{N}$ the set of natural numbers (including 0).*

(a) *$\underline{P}$ is a* coherent lower prevision *on $\mathscr{S}$ iff, $\forall n \in \mathbb{N}$, $\forall s_i \geq 0$, $\forall X_i \in \mathscr{S}$, $i = 0, 1, \ldots, n$, defining*

$$\underline{G} = \sum_{i=1}^{n} s_i\big(X_i - \underline{P}(X_i)\big) - s_0\big(X_0 - \underline{P}(X_0)\big),$$

it holds that $\sup \underline{G} \geq 0$.

(b) *$\underline{P}$ is a* convex lower prevision *on $\mathscr{S}$ iff (a) holds with the additional convexity constraint $\sum_{i=1}^{n} s_i = s_0 = 1$.*

$\underline{P}$ is centered convex or C-convex *iff it is convex, $\emptyset \in \mathscr{S}$ and $\underline{P}(\emptyset) = 0$.*

(c) *$\underline{P}$* avoids sure loss *on $\mathscr{S}$ iff (a) holds with $s_0 = 0$.*

(d) *$\underline{P}$ is* 2-coherent *on $\mathscr{S}$ iff, $\forall s_1 \geq 0$, $\forall s_0 \in \mathbb{R}$, $\forall X_0, X_1 \in \mathscr{S}$, defining $\underline{G}_2 = s_1\big(X_1 - \underline{P}(X_1)\big) - s_0\big(X_0 - \underline{P}(X_0)\big)$, it holds that* $\sup \underline{G}_2 \geq 0$.

When $\underline{P}$ avoids sure loss on $\mathscr{S}$, it is possible to apply a well-known procedure termed *natural extension* [14, 15] to find the least-committal coherent extension $\underline{E}$ of $\underline{P}$ on any $\mathscr{S}' \supset \mathscr{S}$. Least committal means that $\underline{E} = \underline{P}$ on $\mathscr{S}$ iff $\underline{P}$ is coherent on $\mathscr{S}$, that $\underline{E} \geq \underline{P}$ in general, and that if $\underline{Q}$ is coherent on $\mathscr{S}'$ and $\underline{Q} \geq \underline{P}$ on $\mathscr{S}$, then also $\underline{Q} \geq \underline{E}$ on $\mathscr{S}'$. If the starting point is an upper prevision $\overline{P}$, its natural extension $\overline{E}$ has a symmetric meaning.

In a general situation, when $\underline{P}$ (or $\overline{P}$) is defined on an arbitrary set of gambles $\mathscr{S}$, finding its natural extension may be operationally not easy at all. However, in the sequel we shall be concerned with some special situations that make this task simpler.

Precisely, let $\mathbb{P}$ be a partition of the sure event, $\mathscr{A}(\mathbb{P})$ the set of events logically dependent on $\mathbb{P}$ (the powerset of $\mathbb{P}$), $\mathscr{L}(\mathbb{P})$ the set of all gambles defined on $\mathbb{P}$. We shall consider lower probabilities $\underline{P}$ defined on $\mathscr{A}(\mathbb{P})$ that are coherent and 2-*monotone*, i.e. such that

$$\underline{P}(A \vee B) + \underline{P}(A \wedge B) \geq \underline{P}(A) + \underline{P}(B), \quad \forall A, B \in \mathscr{A}(\mathbb{P}).$$

(Correspondingly, $\overline{P}$ is 2-*alternating* if $\overline{P}(A \vee B) + \overline{P}(A \wedge B) \leq \overline{P}(A) + \overline{P}(B), \forall A, B \in \mathscr{A}(\mathbb{P})$.) Then, the natural extension of one such $\underline{P}$, or its conjugate $\overline{P}$, from $\mathscr{A}(\mathbb{P})$ to $\mathscr{L}(\mathbb{P})$ may be performed by making use of the Choquet integral (cf., e.g., [6, Chapter 4], [14, Appendix C]), which is helpful in obtaining the formulae that will be stated later on.

We shall also be concerned with *probability intervals*, which are lower and upper probability assignments on a *finite* partition $\mathbb{P}$. Denoting a probability interval with $I = [\underline{P}(\omega), \overline{P}(\omega)]_{\omega \in \mathbb{P}}$, $0 \leq \underline{P}(\omega) \leq \overline{P}(\omega) \leq 1, \forall \omega \in \mathbb{P}$, it is well-known, see e.g. [14, Section 7.1], that I avoids sure loss (on $\mathbb{P}$) iff $\sum_{\omega \in \mathbb{P}} \underline{P}(\omega) \leq 1 \leq \sum_{\omega \in \mathbb{P}} \overline{P}(\omega)$, and that if I avoids sure loss its natural extensions on $\mathscr{A}(\mathbb{P})$, $\underline{E}$ and its conjugate $\overline{E}$, are respectively 2-monotone and 2-alternating, and given by

$$\underline{E}(A) = \max\Big\{\sum_{\omega \Rightarrow A} \underline{P}(\omega), 1 - \sum_{\omega \Rightarrow \neg A} \overline{P}(\omega)\Big\} \qquad (2)$$

$$\overline{E}(A) = \min\Big\{\sum_{\omega \Rightarrow A} \overline{P}(\omega), 1 - \sum_{\omega \Rightarrow \neg A} \underline{P}(\omega)\Big\}. \qquad (3)$$

If $\underline{P}$ is a convex lower probability, it does not necessarily avoid sure loss: it does, iff $\underline{P}(\emptyset) \leq 0$ [10]. In any case, a convex probability is characterised as follows by an envelope theorem [10]:

Proposition 2.1 *$\underline{P} : \mathscr{S} \to \mathbb{R}$ is a convex lower probability on $\mathscr{S}$ iff there exist a non-empty set $\mathscr{M}$ of precise probabilities on $\mathscr{S}$ and a function $\alpha : \mathscr{M} \to \mathbb{R}$ such that*

$$\underline{P}(A) = \min\{P(A) + \alpha(P) : P \in \mathscr{M}\}, \quad \forall A \in \mathscr{S}.$$

2.1. Nearly-Linear Models

Denoting for the moment with $\mu : \mathscr{A}(\mathbb{P}) \to \mathbb{R}$ either a lower or an upper probability, we have

Definition 2.2 *[3] Given a probability P_0 on $\mathscr{A}(\mathbb{P})$, $a \in \mathbb{R}$ and $b > 0$, $\mu : \mathscr{A}(\mathbb{P}) \to \mathbb{R}$ is a* Nearly-Linear *imprecise probability iff $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$, and $\forall A \in \mathscr{A}(\mathbb{P}) \setminus \{\emptyset, \Omega\}$,*

$$\mu(A) = \min\{\max\{bP_0(A) + a, 0\}, 1\}.$$

In short, we write that μ is $NL(a,b)$. NL models are closed with respect to conjugacy [3]: if μ is $NL(a,b)$, then its conjugate $\mu^c(A) = 1 - \mu(\neg A)$ is also $NL(c,b)$, with

$$c = 1 - (a + b). \qquad (4)$$

Thus, every NL submodel is made up of a couple of conjugate imprecise probabilities. By convention we identify the lower probability $\underline{P}$ with the parameters (a,b), the upper probability $\overline{P}$ with (c,b). It can be shown that NL models can be partitioned into 3 submodels, as follows.

a) The Vertical Barrier Model

Definition 2.3 *A* Vertical Barrier Model (VBM) *is a NL model where $\underline{P}$ and its conjugate $\overline{P}$ are given by:*

$$\underline{P}(A) = \begin{cases} \max\{bP_0(A) + a, 0\} & \text{if } A \in \mathscr{A}(\mathbb{P}) \setminus \{\Omega\} \\ 1 & \text{if } A = \Omega \end{cases} \qquad (5)$$

$$\overline{P}(A) = \begin{cases} \min\{bP_0(A)+c,1\}, & \text{if } A \in \mathscr{A}(\mathbb{P}) \setminus \{\emptyset\} \\ 0 & \text{if } A = \emptyset \end{cases}$$

with c given by (4) *and a,b satisfying*

$$0 \leq a+b \leq 1, \quad a \leq 0. \tag{6}$$

In a VBM, $\underline{P}$ is coherent and 2-monotone, $\overline{P}$ is coherent and 2-alternating. Among VBMs, we find the following well-known neighbourhood models:

- If $a=0$, $0<b<1$ (hence $c=1-b>0$), the *ε-contamination model* (termed linear-vacuous mixture model in [15]):

$$\underline{P}_\varepsilon(A) = bP_0(A), \qquad \forall A \neq \Omega, \quad \underline{P}_\varepsilon(\Omega)=1,$$
$$\overline{P}_\varepsilon(A) = bP_0(A)+1-b, \quad \forall A \neq \emptyset, \quad \overline{P}_\varepsilon(\emptyset)=0.$$

- If $a+b=0$ (hence $c=1$), the *vacuous lower/upper probability model*:

$$\underline{P}_V(A)=0, \quad \forall A \neq \Omega, \quad \underline{P}_V(\Omega)=1,$$
$$\overline{P}_V(A)=1, \quad \forall A \neq \emptyset, \quad \overline{P}_V(\emptyset)=0.$$

  Note that we would obtain the same model also for $a+b<0$, thus the condition $a+b \geq 0$ that we require VBMs to satisfy is not restrictive.

- If $b=1$, $-1<a<0$ (hence $c=-a$), the *Total Variation Model* [7, 13]:

$$\underline{P}_{\mathrm{TVM}}(A) = \max\{P_0(A)-c,0\}, \quad \forall A \neq \Omega,$$
$$\overline{P}_{\mathrm{TVM}}(A) = \min\{P_0(A)+c,1\}, \quad \forall A \neq \emptyset,$$
$$\underline{P}_{\mathrm{TVM}}(\Omega)=1, \quad \overline{P}_{\mathrm{TVM}}(\emptyset)=0.$$

  $\underline{P}_{\mathrm{TVM}}$ ($\overline{P}_{\mathrm{TVM}}$) is the lower (upper) envelope of the probabilities whose total variation distance from P_0 does not exceed c ($\in]0,1[$). This model is strictly related with the PMM, cf. [13, Section 3.2].

- If $b=1+\delta>1$, $a=-\delta<0$, the *Pari-Mutuel Model* [8, 13, 15]

$$\underline{P}_{\mathrm{PMM}}(A) = \max\{(1+\delta)P_0(A)-\delta,0\},$$
$$\overline{P}_{\mathrm{PMM}}(A) = \min\{(1+\delta)P_0(A),1\}.$$

From a sellor's viewpoint, a VBM improves over a PMM, in the following sense: it is easy to see that $\overline{P}(A) \downarrow c \geq 0$ as $P_0(A) \downarrow 0$, and that $c=0$ in the PMM. Thus, $\overline{P}(A)$ does generally not tend to 0 with P_0, meaning that a VBM with $c>0$ ensures a minimum positive selling price for any event $A \neq \emptyset$, even those very unlikely (according to P_0).

b) The Horizontal Barrier Model

Definition 2.4 *A Horizontal Barrier Model (HBM) is a NL model where, $\forall A \in \mathscr{A}(\mathbb{P}) \setminus \{\emptyset,\Omega\}$,*

$$\underline{P}(A) = \min\{\max\{bP_0(A)+a,0\},1\},$$
$$\overline{P}(A) = \max\{\min\{bP_0(A)+c,1\},0\}, \tag{7}$$

with c given by (4) *and a,b satisfying the constraints $a+b>1$, $b+2a \leq 1$.*

It is easy to see that $a<0$, $c<0$, $b>1$ in a HBM. Further, in this model an agent acting as a buyer underestimates the riskiness of a transaction on a high P_0-probability event A (which s/he would buy at $\underline{P}(A)=1$ iff $P_0(A) \geq \frac{1-a}{b}$), while overestimating the risk of buying a low P_0-probability event B (which s/he would buy at $\underline{P}(B)=0$, i.e. for nothing, iff $P_0(B) \leq -\frac{a}{b}$). Despite its conveying an agent's somewhat contradictory attitudes towards risk evaluation, the HBM is not always incoherent. Although $\underline{P}$ and $\overline{P}$ in a HBM are generally only 2-coherent, it can be shown [3] that

Proposition 2.2 *$\overline{P}$ in a HBM is a coherent upper probability iff it is subadditive (i.e. iff $\overline{P}(A)+\overline{P}(B) \geq \overline{P}(A \vee B), \forall A,B \in \mathscr{A}(\mathbb{P})$). When $\overline{P}$ is coherent, it is 2-alternating too.*

There are also instances of HBMs where $\underline{P}=\overline{P}=P$, and P may or not be a precise probability.

c) The Restricted Range Model

Definition 2.5 *A Restricted Range Model (RRM) is a NL model where $\forall A \in \mathscr{A}(\mathbb{P}) \setminus \{\emptyset,\Omega\}$*

$$\underline{P}(A) = bP_0(A)+a, \quad \overline{P}(A) = bP_0(A)+c, \tag{8}$$

with c given by (4)*, and a,b satisfying*

$$a>0, \ b+2a \leq 1.$$

The name of this model arises from its property $\underline{P}(A) \in [a,a+b] \subset [0,1], \forall A \in \mathscr{A}(\mathbb{P}) \setminus \{\emptyset,\Omega\}$. It can be seen that again an agent using this model's $\underline{P}$ or $\overline{P}$ has conflicting attitudes towards risk, or towards high and low P_0-probability events. $\underline{P}$ and $\overline{P}$ are almost never coherent in the RRM: they are iff $\mathbb{P}$ is a partition of cardinality two, i.e. never in significant problems.

3. Vertical Barrier Models and Natural Extensions

With Vertical Barrier Models, we can discuss natural extensions under two different perspectives: next to considering the natural extension of a VBM to $\mathscr{L}(\mathbb{P})$, we may wonder whether the VBM itself is, or plays a role in, the natural extension of something else.

3.1. Vertical Barrier Models as Natural Extensions

We begin with this latter aspect, and consider the lower probability $\underline{Q}(A)$ which is the first term of the maximum defining $\underline{P}$ in Definition 2.3. It holds that

Proposition 3.1 *The lower probability*

$$\underline{Q}(A) = bP_0(A) + a, \quad \forall A \in \mathscr{A}(\mathbb{P}) \tag{9}$$

in Definition 2.3 of a VBM (where a, b satisfy the constraints (6)) avoids sure loss; $\underline{Q}$ is convex iff $b = 1$. Its natural extension on $\mathscr{A}(\mathbb{P})$ is precisely the lower probability $\underline{P}$ of the VBM it originates from.

Proof $\underline{Q}$ avoids sure loss because $\underline{Q} \leq \underline{P}$ by (5) (and (6), for $A = \Omega$), and $\underline{P} \leq P_0$, as is easily seen [2, Property (i') in Section 4]. Thus $\underline{Q} \leq P_0$ and the characterisation of avoiding sure loss in [15, Section 3.3.4 (a)] applies.

If $b = 1$, $\underline{Q}$ is convex, by Proposition 2.1 (with $\mathscr{M} = \{P_0\}$, $\alpha(P_0) = a$). If $b \neq 1$, $\underline{Q}$ is not convex, since then $\underline{Q}(\Omega) - \underline{Q}(\emptyset) = b$, contradicting [10, Proposition 3.4 (e)].

It remains to prove that $\underline{P}$ in (5) is the natural extension $\underline{E}_Q$ of $\underline{Q}$. By coherence (of $\underline{E}_Q$ and $\underline{P}$), $\underline{E}_Q(\Omega) = \underline{P}(\Omega) = 1$.

Now let $A \neq \Omega$, and recall that, since $\underline{P}$ is coherent and $\underline{P} \geq \underline{Q}$,

$$\underline{P} \geq \underline{E}_Q. \tag{10}$$

If $bP_0(A) + a > 0$, then $bP_0(A) + a = \underline{P}(A) \geq \underline{E}_Q(A) \geq \underline{Q}(A) = bP_0(A) + a$, with the first inequality following from (10). Then $\underline{P}(A) = \underline{E}_Q(A)$.

If $bP_0(A) + a \leq 0$, then $0 \leq \underline{E}_Q(A) \leq \underline{P}(A) = 0$, thus again $\underline{P}(A) = \underline{E}_Q(A)$. ∎

Thus, a VBM corrects the naive evaluation $\underline{Q}$ via natural extension, by introducing barriers to its values.

More generally, it can be shown that a generalisation of the VBM is the natural extension of a class of non-centered convex probabilities. The next result is useful for this.

Proposition 3.2 *Let I be a set of indexes and, for any $\alpha \in I$, let $P_\alpha : \mathscr{A}(\mathbb{P}) \to \mathbb{R}$ be a probability, $a_\alpha \leq 0$, such that $\inf_{\alpha \in I} a_\alpha \in \mathbb{R}$. Define $\underline{P}_\alpha = P_\alpha + a_\alpha$, $\underline{P} = \inf_{\alpha \in I} \underline{P}_\alpha$. Then $\underline{P}$ is convex and avoids sure loss. Letting $\underline{E}_\alpha$ be the natural extension of $\underline{P}_\alpha$, $\underline{E}_P$ the natural extension of $\underline{P}$, it holds that*

$$\underline{E}_P = \inf_{\alpha \in I} \underline{E}_\alpha. \tag{11}$$

By Proposition 3.2, $\underline{P}$ is a convex lower probability; since $\underline{P}(\emptyset) = \inf_{\alpha \in I} a_\alpha \leq 0$, $\underline{P}$ is also non-centered, outside the limiting situation $a_\alpha = 0, \forall \alpha \in I$. We may apply Proposition 3.1 to write explicitly $\underline{E}_\alpha$ in (11), since any $\underline{P}_\alpha$ is a lower probability of the type (9), with $b = 1$. We obtain that $\underline{E}_P(\Omega) = 1$, and, $\forall A \in \mathscr{A}(\mathbb{P}) \setminus \{\Omega\}$,

$$\underline{E}_P(A) = \inf_{\alpha \in I} \max\{P_\alpha(A) + a_\alpha, 0\}. \tag{12}$$

From (12) (and the coherence condition $\underline{E}_P(A) \geq 0$), $\underline{E}_P(A) = 0$ if $\exists \bar{\alpha} \in I : P_{\bar{\alpha}}(A) + a_{\bar{\alpha}} < 0$, while $\underline{E}_P(A) = \inf_{\alpha \in I}(P_\alpha(A) + a_\alpha)$ otherwise. Thus, we may rewrite (12) as follows:

$$\underline{E}_P(A) = \begin{cases} 1 & \text{if } A = \Omega \\ \max\{\inf_{\alpha \in I}(P_\alpha(A) + a_\alpha), 0\} & \text{otherwise.} \end{cases} \tag{13}$$

We conclude from (13) that the natural extension of the lower envelope $\underline{P}$ of a given set of lower probabilities $\underline{P}_\alpha$ that ensure the condition $a_\alpha \leq 0, \forall \alpha \in I$ (with $\inf_{\alpha \in I} a_\alpha > -\infty$) is formally analogue to a VBM (cf. (5)). It differs from it because it replaces the lower probability that avoids sure loss $bP_0(A) + a$ with $\inf_{\alpha \in I}(P_\alpha(A) + a_\alpha)$, a convex lower probability still avoiding sure loss.

3.2. The Natural Extension of a Vertical Barrier Model

Let us consider now the problem of extending a VBM (that is, extending its lower probability $\underline{P}$ or the conjugate upper probability $\overline{P}$) from $\mathscr{A}(\mathbb{P})$ to $\mathscr{L}(\mathbb{P})$. Take for instance $\underline{P}$: since it is coherent and 2-monotone, its natural extension $\underline{E}(X)$ on a gamble $X \in \mathscr{L}(\mathbb{P})$ is a Choquet integral and as such may be written in the form [14]

$$\underline{E}(X) = \inf X + \int_{\inf X}^{\sup X} \underline{P}(X > x)\, dx. \tag{14}$$

Equation (14) is a general formula for the natural extension of a 2-monotone coherent $\underline{P}$ to a gamble $X \in \mathscr{L}(\mathbb{P})$. In the case that $\underline{P}$ is the lower probability of a VBM, it may be further specialised, as stated in Proposition 3.3. The later Propositions 3.4 and 4.4 obtain similar results for the natural extension of $\overline{P}$ in a VBM and of $\underline{P}$ in a coherent HBM. The proofs of the three propositions follow a common pattern with minor modifications. We prove here only Proposition 3.3.

Proposition 3.3 *Let $\underline{P} : \mathscr{A}(\mathbb{P}) \to \mathbb{R}$ be the lower probability of a VBM. If $a < 0$, for any $X \in \mathscr{L}(\mathbb{P})$ define*

$$\tilde{x} = \sup\left\{x \in \mathbb{R} : P_0(X > x) \geq -\frac{a}{b}\right\},$$

$(\tilde{x} - X)^+ = \max\{\tilde{x} - X, 0\}$. Then

$$\underline{E}(X) = (a+b)\tilde{x} + \big(1 - (a+b)\big)\inf X - bE^{P_0}\big((\tilde{x} - X)^+\big), \tag{15}$$

where $E^{P_0}\big((\tilde{x} - X)^+\big)$ is the (precise) natural extension of P_0 to $(\tilde{x} - X)^+$.

If $a = 0$, the VBM is an ε-contamination (if $b \neq 0$) or vacuous (if $b = 0$) model and we get instead

$$\underline{E}(X) = (1-b)\inf X + bE^{P_0}(X), \tag{16}$$

with $E^{P_0}(X)$ (precise) natural extension of P_0 to X.

Proof We first write Equation (14) with $\underline{P}$ given by (5) (note that the events $(X > x)$ belong to $\mathscr{A}(\mathbb{P})$):

$$\underline{E}(X) = \inf X + \int_{\inf X}^{\sup X} \max\left\{bP_0(X > x) + a, 0\right\} dx. \quad (17)$$

In the second step, we split the integral in (17) into two, according to whether its argument is 0 or not. For this, we recognise that

$$bP_0(X > x) + a \geq 0 \quad \text{iff} \quad P_0(X > x) \geq -\frac{a}{b}.$$

If $a + b = 0$, $\underline{P}$ is the vacuous lower probability $\underline{P}_V$, whose natural extension $\underline{E}_V$ we already know, $\underline{E}_V(X) = \inf X, \forall X \in \mathscr{L}(\mathbb{P})$ [15].

If $a = 0$, $b > 0$, then $\tilde{x} = +\infty$. We shall discuss this subcase later, and for the moment *we suppose* $a < 0, a + b > 0$. Then, it holds that $0 < -\frac{a}{b} < 1$.

Given the preceding arguments, it is easy to realise that

$$\inf X \leq \tilde{x} \leq \sup X \quad (18)$$

and that (17) may be written as

$$\begin{aligned}\underline{E}(X) &= \inf X + \int_{\inf X}^{\tilde{x}} \big(bP_0(X > x) + a\big)\, dx \\ &= \inf X + b \int_{\inf X}^{\tilde{x}} P_0(X > x)\, dx + a\big(\tilde{x} - \inf X\big). \quad (19)\end{aligned}$$

To write the integral in (19) in a more convenient way, we recognise that, for $x < \tilde{x}$, and for any $\omega \in \mathbb{P}$,

$$\begin{aligned} X(\omega) > x \quad &\text{iff} \quad X(\omega) - x > 0 \\ &\text{iff} \quad \min\{X(\omega) - x, \tilde{x} - x\} > 0 \\ &\text{iff} \quad \min\{X(\omega) - \tilde{x} + \tilde{x} - x, \tilde{x} - x\} > 0 \\ &\text{iff} \quad \min\{X(\omega) - \tilde{x}, 0\} > x - \tilde{x}. \end{aligned}$$

Defining

$$Z = \min\{X - \tilde{x}, 0\}$$

and performing the substitution $z = x - \tilde{x}$, we get from (19)

$$\underline{E}(X) = \inf X + b \int_{\inf X - \tilde{x}}^{0} P_0(Z > z)\, dz + a\big(\tilde{x} - \inf X\big). \quad (20)$$

The next step consists in proving that

$$\inf Z = \inf X - \tilde{x}, \quad \sup Z = 0. \quad (21)$$

For this, recall (18) and distinguish the following cases:

- $\inf X < \tilde{x} < \sup X$.

  Then $\{\omega \in \mathbb{P} : X(\omega) < \tilde{x}\} \neq \emptyset$ and $\{\omega \in \mathbb{P} : X(\omega) > \tilde{x}\} \neq \emptyset$, $\sup Z = \max\{\sup_{X \leq \tilde{x}}(X - \tilde{x}), 0\} = 0$, $\inf Z = \min\{\inf_{X \leq \tilde{x}}(X - \tilde{x}), 0\} = \inf_{X \leq \tilde{x}}(X - \tilde{x}) = \inf X - \tilde{x}$.

- $\tilde{x} = \sup X$.

  Hence, $Z = X - \sup X$, and $\sup Z = 0$, $\inf Z = \inf X - \sup X = \inf X - \tilde{x}$.

- $\tilde{x} = \inf X$.

  Here $Z = 0$, and $\sup Z = 0 = \inf Z = \inf X - \tilde{x}$.

Now, since (21) holds, we may apply it and (14) to the integral in (20), replacing $\int_{\inf X - \tilde{x}}^{0} P_0(Z > z)\, dz = \int_{\inf Z}^{\sup Z} P_0(Z > z)\, dz$ with $E^{P_0}(Z) - \inf Z$ and getting

$$\begin{aligned}\underline{E}(X) &= \inf X + b\big(E^{P_0}(Z) - \inf Z\big) + a(\tilde{x} - \inf X) \\ &= \inf X + bE^{P_0}(Z) + (a+b)(\tilde{x} - \inf X), \quad (22)\end{aligned}$$

where $E^{P_0}(Z)$ is the natural extension of the precise probability P_0 to Z, which is its expectation [15, Section 3.2.2]. Since E^{P_0} is linear, and because $-Z = -\min\{X - \tilde{x}, 0\} = \max\{\tilde{x} - X, 0\} = (\tilde{x} - X)^+$, we obtain from (22)

$$\begin{aligned}\underline{E}(X) &= \inf X - b\underline{E}^{P_0}\big((\tilde{x} - X)^+\big) + (a+b)(\tilde{x} - \inf X) \\ &= (a+b)\tilde{x} + \big(1 - (a+b)\big)\inf X - b\underline{E}^{P_0}\big((\tilde{x} - X)^+\big).\end{aligned}$$

To complete the proof, we still have to consider the case $a = 0$, $b > 0$, where $\underline{P}$ is the lower probability of the ε-contamination model. Thus $\underline{P}$ is the restriction to events of the coherent and 2-monotone lower prevision

$$\underline{E}(X) = bE^{P_0}(X) + (1-b)\inf X \quad (23)$$

with $E^{P_0}(X)$ the linear natural extension of P_0 to X. Since the lower probability $\underline{P}$ is coherent and 2-monotone, its natural extension is its unique 2-monotone extension [14], and therefore coincides with $\underline{E}$ as given in (23). ■

Remark 3.1 *By Proposition 3.1, Proposition 3.3 also determines the natural extension on $\mathscr{L}(\mathbb{P})$ of the lower probability $\underline{Q}$ in Equation (9).*

If $a = -\delta < 0$, $b = 1 + \delta$, the VBM is a Pari-Mutuel Model, and $\underline{E}(X)$ in (15) boils down to

$$\underline{E}(X) = \tilde{x} - (1+\delta)E^{P_0}\big((\tilde{x} - X)^+\big)$$

which is in fact an expression for the natural extension of $\underline{P}_{\text{PMM}}$ that may be found in [15, p. 131].

In the special case $a = 0$, $b > 0$, the VBM is instead an ε-contamination model (defined on $\mathscr{A}(\mathbb{P})$). Here (16) states that its natural extension is again an ε-contamination model (defined on $\mathscr{L}(\mathbb{P})$). Putting $1 - b = \delta$, $\underline{E}(X)$ in (16) is rewritten in fact in the form $\underline{E}(X) = \delta \inf X + (1 - \delta)E^{P_0}(X)$, that appears in [15].

Lastly, it is interesting to compare the natural extension $\underline{E}(X)$ of a VBM with parameters a, b, given by (15), with that, $\underline{E}_k(X)$, of a VBM with parameters $a' = ka$, $b' = kb$, $k \in]0,1[$. Note that the choice of a', b' does not modify $\tilde{x}$, which remains the same. It can be shown that $\underline{E}_k$ is instead given by

$$\underline{E}_k(X) = k\underline{E}(X) + (1-k)\inf X,$$

i.e., $\underline{E}_k$ is a convex combination of the coherent prevision $\underline{E}(X)$ and $\inf X$. As $k \downarrow 0$, $\underline{E}_k \downarrow \inf X$; this is not surprising, since $k \downarrow 0$ means also $a' + b' \downarrow 0$, and $a' + b'$ is the maximum distance of $\underline{P}$ in a VBM(a', b') from the vacuous lower probability, whose natural extension on X is precisely $\inf X$.

It is also possible to derive a formula for the natural extension of the upper probability $\overline{P}$ of a VBM:

Proposition 3.4 *Let $\overline{P} : \mathscr{A}(\mathbb{P}) \to \mathbb{R}$ be the upper probability of a VBM. If $c < 1$, for any $X \in \mathscr{L}(\mathbb{P})$ define*

$$\hat{x} = \sup\left\{x \in \mathbb{R} : P_0(X > x) \geq 1 + \frac{a}{b}\right\}, \tag{24}$$

$(X - \hat{x})^+ = \max\{X - \hat{x}, 0\}$.

Then, if $c < 1$,

$$\overline{E}(X) = (1-c)\hat{x} + c \sup X + bE^{P_0}\left((X - \hat{x})^+\right), \tag{25}$$

where $E^{P_0}((X - \hat{x})^+)$ is the (precise) natural extension of P_0 to $(X - \hat{x})^+$.

If $c = 1$, $\overline{P} = \overline{P}_V$, the vacuous upper probability, whose natural extension is known to be $\overline{E}(X) = \sup X$.

4. Natural Extensions of Horizontal Barrier Models

Unlike the VBM, a HBM may be coherent (in rather special cases, as we recalled in Section 2.1 b)) or not. In general, it is only guaranteed to be 2-coherent, and in this case we might extend it with the 2-coherent natural extension studied in [12]. Rather than pursuing this avenue, in this paper we shall extend, when possible, the HBM with the usual (coherent) natural extension $\underline{E}$. For this, we need to know when a HBM (i.e., its $\underline{P}$ or $\overline{P}$) avoids sure loss, since this condition is necessary and sufficient for $\underline{E}(X)$ to be finite, $\forall X \in \mathscr{L}(\mathbb{P})$ [15].

In the case that partition $\mathbb{P}$ is finite, the following proposition answers this question and determines the natural extension $\overline{E}$ of $\overline{P}$ on $\mathscr{A}(\mathbb{P})$. Results are stated for upper probabilities, since for them most formulae with HBMs are more manageable. Since $\overline{P}$ is already defined on $\mathscr{A}(\mathbb{P})$, we stress that $\overline{E}$ is actually a least-committal correction of $\overline{P}$ rather than a real extension, in the case that $\overline{P}$ avoids sure loss but is not coherent on $\mathscr{A}(\mathbb{P})$.

Proposition 4.1 *Let $\overline{P} : \mathscr{A}(\mathbb{P}) \to \mathbb{R}$ be the upper probability of a HBM, with $\mathbb{P}$ finite.*

(a) $\overline{P}$ avoids sure loss on $\mathscr{A}(\mathbb{P})$ *iff*

$$\sum_{\omega \in \mathbb{P}} \overline{P}(\omega) \geq 1. \tag{26}$$

Assume that $\overline{P}$ avoids sure loss, and define, $\forall A \in \mathscr{A}(\mathbb{P})$,

$$\overline{E}(A) = \min\left\{\sum_{\omega \Rightarrow A} \overline{P}(\omega), 1\right\}. \tag{27}$$

Then

(b) $\overline{E}$ is a coherent and 2-alternating upper probability;

(c) $\overline{E}$ is the natural extension of $\overline{P}$ on $\mathscr{A}(\mathbb{P})$.

Proof (Outline) *Proof of (a).* If $\overline{P}$ avoids sure loss, by [15, Section 3.3.4 (a)] there is a probability P such that $P(A) \leq \overline{P}(A)$, $\forall A \in \mathscr{A}(\mathbb{P})$. Hence, $1 = \sum_{\omega \in \mathbb{P}} P(\omega) \leq \sum_{\omega \in \mathbb{P}} \overline{P}(\omega)$.

Conversely, let (26) hold. Then we may deduce that $\overline{P}$ avoids sure loss if we can establish that

(i) statement (b) holds, and

(ii) $\overline{E}(A) \leq \overline{P}(A)$, $\forall A \in \mathscr{A}(\mathbb{P})$.

In fact, there is then a probability P such that $P \leq \overline{E} \leq \overline{P}$, so $\overline{P}$ avoids sure loss, again by [15, Section 3.3.4 (a)].

We prove here only (i). For this, observe that if (26) holds, the probability interval $[0, \overline{P}(\omega)]_{\omega \in \mathbb{P}}$ avoids sure loss (while being not necessarily coherent). Then by (3) its natural extension on $\mathscr{A}(\mathbb{P})$ coincides with $\overline{E}$ in (27) and is 2-alternating.

Proof of (b). This is the proof of (i) above.

Proof of (c). It is achieved, by the properties of the natural extension, proving (ii) and that for any probability P, if $P \leq \overline{P}$, then $P \leq \overline{E}$. ■

Remark 4.1 *Interestingly, condition (26) characterises (together with $\overline{P}(\omega) \in [0,1]$, $\forall \omega \in \mathbb{P}$) a generic upper probability $\overline{P}$ that avoids sure loss* on $\mathbb{P}$ only *[15, Section 4.6.1]. Knowing additionally that $\overline{P}$ belongs to a HBM makes the same condition (26) equivalent to the fact that $\overline{P}$ avoids sure loss* on the broader environment $\mathscr{A}(\mathbb{P})$.

Moreover, the proof of (b) (i.e., of (i)) in Proposition 4.1 lets us deduce that, in a finite setting, a HBM $(\underline{P}, \overline{P})$ that avoids sure loss has the same natural extension of the probability interval $[0, \overline{P}(\omega)]_{\omega \in \mathbb{P}}$. In this sense, these two models are equivalent.

In general, however, NL models and (natural extensions of) probability intervals do not overlap. As proven in [3], a NL model is the natural extension on $\mathscr{A}(\mathbb{P})$ of a coherent probability interval in special instances only (including the PMM - which was already shown in [8] - and the ε-contamination model).

Proposition 4.1 displays the simple formula (27) for computing $\overline{E}$ on $\mathscr{A}(\mathbb{P})$, but is useful also for a second step, that of determining $\overline{E}(X)$ for any $X \in \mathscr{L}(\mathbb{P})$. In fact, $\overline{E}(X)$ may be thought of as the natural extension of $\overline{E}$ from $\mathscr{A}(\mathbb{P})$ to $\mathscr{A}(\mathbb{P}) \cup \{X\}$, which is given by a Choquet integral, since $\overline{E}$ is 2-alternating on $\mathscr{A}(\mathbb{P})$ by Proposition 4.1. The next proposition states the final result.

Proposition 4.2 *Let $(\underline{P}, \overline{P})$ be a HBM that avoids sure loss on $\mathscr{A}(\mathbb{P})$, $\mathbb{P} = \{\omega_1, \omega_2, \ldots, \omega_n\}$. Consider $X \in \mathscr{L}(\mathbb{P})$, taking $m \leq n$ distinct values $x_1 < x_2 < \ldots < x_m$.*

Define, for $j = 1,\dots,m$,

$$e_j = \bigvee_{\omega\in\mathbb{P}:\, X(\omega)=x_j} \omega$$

and let $k \in \{1,\dots,m\}$ be such that

$$\sum_{j=k}^{m} \overline{E}(e_j) \leq 1, \quad \sum_{j=k-1}^{m} \overline{E}(e_j) \geq 1, \tag{28}$$

with the convention $x_0 = 0$, $e_0 = \emptyset$ and where $\overline{E}$ is the natural extension of $\overline{P}$.

Then, we have

$$\overline{E}(X) = x_{k-1}\Big(1 - \sum_{j=k}^{m} \overline{E}(e_j)\Big) + \sum_{j=k}^{m} x_j \overline{E}(e_j). \tag{29}$$

Proof (Outline) The proof exploits the following form of the Choquet integral of the simple gamble X with respect to $\overline{P}$ [6, Equation (4.18)]:

$$(C)\int X\, d\overline{P} = \sum_{h=1}^{m} x_h \Big(\overline{P}(X \geq x_h) - \overline{P}(X \geq x_{h+1})\Big) \tag{30}$$

where by definition $(X \geq x_{m+1}) = \emptyset$. Equation (29) follows substituting $\overline{P}$, given by (7), in (30), after some manipulations (cf. also [9] for a similar technique). ■

Remark 4.2 *For some HBMs (they must satisfy the later Equation (31) for some $\bar{k} \in \{2,\dots,m\}$), distinct numbers may play the role of k in Equation (28). Yet, they return a unique $\overline{E}(X)$ from Equation (29).*

To see this, note that these numbers form an integer interval $J \subset \{1,\dots,m\}$. Take any two contiguous $\bar{k}-1$, $\bar{k} \in J$. From (28) with, alternatively, $k = \bar{k}$ and $k = \bar{k}-1$, we obtain the identity

$$\sum_{j=\bar{k}-1}^{m} \overline{E}(e_j) = 1. \tag{31}$$

Let us now compute $\overline{E}(X)$ with (29) in the two cases. With $k = \bar{k}-1$ and by (31), $\overline{E}(X)$ is given by

$$x_{\bar{k}-2}\Big(1 - \sum_{j=\bar{k}-1}^{m} \overline{E}(e_j)\Big) + \sum_{j=\bar{k}-1}^{m} x_j\overline{E}(e_j) = \sum_{j=\bar{k}-1}^{m} x_j\overline{E}(e_j).$$

With $k = \bar{k}$, and using again (31), $\overline{E}(X)$ takes the same value:

$$\begin{aligned} &x_{\bar{k}-1}\Big(1 - \sum_{j=\bar{k}}^{m} \overline{E}(e_j)\Big) + \sum_{j=\bar{k}}^{m} x_j\overline{E}(e_j) \\ &= x_{\bar{k}-1}\Big(1 - \sum_{j=\bar{k}-1}^{m} \overline{E}(e_j) + \overline{E}(x_{\bar{k}-1})\Big) + \sum_{j=\bar{k}}^{m} x_j\overline{E}(e_j) \\ &= \sum_{j=\bar{k}-1}^{m} x_j\overline{E}(e_j). \end{aligned}$$

By Proposition 4.2, in order to compute $\overline{E}$ we first have to group the atoms of $\mathbb{P}$ where $X(\omega)$ takes the same value (x_j on e_j) and to determine k. For this, we need to know every $\overline{E}(e_j)$, which is achieved easily by (27) of Proposition 4.1. If $m = n$, then obviously $e_j = \omega_j \in \mathbb{P}$, $\overline{E}(e_j) = \overline{P}(\omega_j)$.

The final formula (29) shows that the natural extension $\overline{E}$ is very similar to a classical expectation with the probability of e_j, i.e. that X takes the value x_j, replaced by its upper probability, and this for the highest $m-k+1$ values of X. The remaining values of X do not appear in the computation (29), but for x_{k-1}.

A special situation arises when $k = 1$: by (28) and Proposition 4.1 (a) applied to the partition $\mathbb{P}' = \{e_1, e_2, \dots, e_m\}$, it holds then that $\sum_{j=1}^{m} \overline{E}(e_j) = 1$, so that $\overline{E}$ is a precise probability on $\mathbb{P}'$ and $\overline{E}(X) = \sum_{j=1}^{m} x_j \overline{E}(e_j)$ is an expectation. However, this does not imply that the starting $\overline{P}$ in the HBM is a precise probability also outside $\mathbb{P}'$. For this, see the next example, where $\overline{P}$ is precise on $\mathbb{P}' = \mathbb{P}$, not even coherent as an upper probability elsewhere.

Example 4.1 *Given $\mathbb{P} = \{\omega_1, \omega_2, \omega_3\}$, $P_0(\omega_1) = 0.2$, $P_0(\omega_2) = P_0(\omega_3) = 0.4$, let $(\underline{P}, \overline{P})$ be a HBM with $b = 1.3$, $c = -0.1$. Then, $\overline{P}(\omega_1) = 1.3 \cdot 0.2 - 0.1 = 0.16$, $\overline{P}(\omega_2) = \overline{P}(\omega_3) = 0.42$ and $\overline{P}(\omega_1) + \overline{P}(\omega_2) + \overline{P}(\omega_3) = 1$, so $\overline{P}$ is a probability on $\mathbb{P}$. On the other hand, $\overline{P}(\omega_1 \vee \omega_2) = 0.68 > \overline{P}(\omega_1) + \overline{P}(\omega_2) = 0.58$ implies that $\overline{P}$ is not even coherent as an upper probability on $\mathscr{A}(\mathbb{P})$, being not subadditive.*

Yet, $\overline{E}(X)$ is computed as an expectation for $X \in \mathscr{L}(\mathbb{P})$. For instance, with $X(\omega_1) = -1$, $X(\omega_2) = 1$, $X(\omega_3) = 2$, from (29) $\overline{E}(X) = -1 \cdot 0.16 + 1 \cdot 0.42 + 2 \cdot 0.42 = 1.1$.

If $\overline{P}$ (and $\underline{P}$) in the HBM is coherent, we can still apply (29) to compute $\overline{E}(X)$: now $\overline{E}(e_j)$ is simply $\overline{P}(e_j)$.

Let us now turn to the general case that $\mathbb{P}$ may be infinite. We can still characterise the condition of avoiding sure loss for $\overline{P}$ as follows:

Proposition 4.3 *Let $(\underline{P}, \overline{P})$ be a HBM. $\overline{P} : \mathscr{A}(\mathbb{P}) \to \mathbb{R}$ (hence its conjugate $\underline{P}$) avoids sure loss on $\mathscr{A}(\mathbb{P})$ if and only if, for any $\mathbb{P}'$,* finite *partition coarser than $\mathbb{P}$, it holds that*

$$\sum_{e\in\mathbb{P}'} \overline{P}(e) \geq 1.$$

Since a coherent HBM is formed by $\overline{P}$, $\underline{P}$ that are, respectively, 2-alternating and 2-monotone, we can derive an expression for $\overline{E}(X)$ or $\underline{E}(X)$ by means of the Choquet integral. Unlike (29), the result applies no matter whether $\mathbb{P}$ is finite or not, and is stated in the next proposition for $\underline{P}$.

Proposition 4.4 *Let $(\underline{P}, \overline{P})$ be a HBM that is* coherent*. Let $X \in \mathscr{L}(\mathbb{P})$. Define*

$$\tilde{x}_u = \sup\Big\{x \in \mathbb{R} : P_0(X \leq x) \leq 1 + \frac{a}{b}\Big\}$$
$$\tilde{x}_l = \inf\Big\{x \in \mathbb{R} : P_0(X \leq x) \geq -\frac{c}{b}\Big\}$$

$$Z(\omega) = \max\left\{\min\{X(\omega) - \tilde{x_u}, 0\}, \tilde{x_l} - \tilde{x_u}\right\} \quad (\omega \in \mathbb{P}).$$

Then,

$$\underline{E}(X) = (a+b)\tilde{x_u} + \big(1-(a+b)\big)\tilde{x_l} + bE^{P_0}(Z), \quad (32)$$

where $E^{P_0}(Z)$ is the (precise) natural extension of P_0 to the gamble Z.

Note that the structure of (32) is similar to that of (15) and to the expression obtained for the PMM in [13, 15].

5. Natural Extensions of Restricted Range Models

An investigation similar to that performed in Section 4 for the HBM may be carried out for the RRM. This time more manageable formulae are obtained referring to lower probabilities. We gather the main results in the next proposition.

Proposition 5.1 *Let $\underline{P} : \mathscr{A}(\mathbb{P}) \to \mathbb{R}$ be the lower probability of a RRM.*

(a) *If $\mathbb{P}$ is infinite, $\underline{P}$ incurs sure loss.*

In the next items, $\mathbb{P}$ is finite *and made up of n elementary events.*

(b) *$\underline{P}$ avoids sure loss on $\mathscr{A}(\mathbb{P})$ iff $\sum_{\omega\in\mathbb{P}} \underline{P}(\omega) \leq 1$ iff $b+na \leq 1$.*

(c) *If $\underline{P}$ avoids sure loss on $\mathscr{A}(\mathbb{P})$, for any $A \in \mathscr{A}(\mathbb{P}) \setminus \{\Omega\}$, letting m_A be the number of atomic events of $\mathbb{P}$ implying A ($A = \bigvee_{i=1}^{m_A} \omega_{j_i}$), it is*

$$\underline{E}(A) = bP_0(A) + m_A a, \quad \underline{E}(\Omega) = 1. \quad (33)$$

$\underline{E}$ is 2-monotone on $\mathscr{A}(\mathbb{P})$.

(d) *If $\underline{P}$ avoids sure loss on $\mathscr{A}(\mathbb{P})$, for any $X \in \mathscr{L}(\mathbb{P})$, the natural extension $\underline{E}$ is given by*

$$\underline{E}(X) = (1-na-b)\min X + bE^{P_0}(X) + naE^{P_u}(X), \quad (34)$$

where $E^{P_0}(X), E^{P_u}(X)$ are the (usual) expectations of X referring to, respectively, P_0 and the probability P_u uniform on $\mathbb{P}$ ($P_u(\omega) = \frac{1}{n}$, $\forall\omega$).

Proposition 5.1 confirms, in its parts (a) and (b), that RRMs ensure weaker consistency properties than the other NL models, also as for the condition of avoiding sure loss. In particular, unlike HBMs (and of course VBMs) they cannot avoid sure loss if $\mathbb{P}$ is infinite.

When they avoid sure loss, their natural extensions on $\mathscr{A}(\mathbb{P})$ and on $\mathscr{L}(\mathbb{P})$ can be easily computed by means of, respectively, equations (33) and (34).

On $\mathscr{A}(\mathbb{P})$, $\underline{E}$ is actually a least-committal correction of $\underline{P}$ (and its conjugate $\overline{P}$). Note that $\underline{E}(\omega) = \underline{P}(\omega)$, so $\underline{P}$ is coherent on $\mathbb{P}$. Instead, $\overline{P}$ is coherent on $\mathbb{P}$ iff $n = 2$: this follows, recalling (33) at the second equality, (4) and (8) at the fourth, from $\overline{E}(\omega) = 1 - \underline{E}(\neg\omega) = 1 - bP_0(\neg\omega) - (n-1)a = 1 - b(1-P_0(\omega)) - (n-1)a = \overline{P}(\omega) - (n-2)a = \overline{P}(\omega)$ iff $n = 2$. In [3], we proved that $\overline{P}$ is coherent on $\mathscr{A}(\mathbb{P})$ iff $n = 2$; here we learn that not even the restriction of $\overline{P}$ on $\mathbb{P}$ is coherent for $n > 2$.

An interesting property, symmetric to that proven for HBMs in Remark 4.1, may be established observing that the probability interval $J = [\underline{P}(\omega), 1]_{\omega\in\mathbb{P}}$ avoids sure loss if $\underline{P}$ in the RRM does so (Proposition 5.1 (b)) and recalling (2), (33). The property states that a RRM $(\underline{P}, \overline{P})$ that avoids sure loss on $\mathscr{A}(\mathbb{P})$ and the probability interval J have the same natural extension $\underline{E}$ given by (33).

The natural extension $\underline{E}$ on $\mathscr{L}(\mathbb{P})$ is by (34) a convex linear combination of the vacuous lower prevision $\underline{P}_V(X) = \min X$ and the expectations $E^{P_0}(X), E^{P_u}(X)$.

6. An Interpretation in Terms of Risk Measures

Any upper prevision $\overline{P}$ induces a risk measure. Indeed, given a gamble Y, $\overline{P}(-Y)$ measures the riskiness of Y, that is, it represents the amount to be provided in order to manage possible losses from Y. Here, we shall refer the risk measures to $X = -Y$; this corresponds, when $Y \leq 0$, to thinking in terms of losses, a common practice for instance in insurance.

One may wonder whether the natural extension $\overline{E}$ of $\overline{P}$ in the VBM, given by (25) if $\underline{P}$ is non-vacuous, has an interpretation as a risk measure. For this, we remark that $\hat{x}$, given by (24), coincides with $\sup\{x \in \mathbb{R} : P_0(X \leq x) \leq -\frac{a}{b}\}$, which is a well-known risk measure [4], the *Value at Risk* of X at level $-\frac{a}{b}$, $VaR_{-\frac{a}{b}}(X)$.

As for $P_0((X-\hat{x})^+)$, it is the *Expected Shortfall* of X at level $-\frac{a}{b}$, denoted by $ES_{-\frac{a}{b}}(X)$ [4]. It measures how insufficient we expect *VaR* is in covering losses from X (the losses not covered by *VaR* are given by $(X-\hat{x})^+$). As a consequence, equation (25) can be equivalently written as follows:

$$\overline{E}(X) = (1-c)VaR_{-\frac{a}{b}}(X) + c\sup X + bES_{-\frac{a}{b}}(X). \quad (35)$$

If the VBM is a TVM (Section 2.1 b)), then $b = 1$, $-\frac{a}{b} = c$, and (35) specialises to

$$\overline{E}_{\text{TVM}}(X) = (1-c)VaR_c(X) + c\sup X + ES_c(X)$$

which is the *Total Variation risk measure* introduced in [13, Section 4].

In the case of the PMM, where $a = -\delta < 0$, $b = 1+\delta$, hence $c = 1-(a+b) = 0$ and $-\frac{a}{b} = \frac{\delta}{1+\delta}$, (35) boils down to

$$\overline{E}_{\text{PMM}}(X) = VaR_{\frac{\delta}{1+\delta}}(X) + (1+\delta)ES_{\frac{\delta}{1+\delta}}(X). \quad (36)$$

$\overline{E}_{\mathrm{PMM}}(X)$ is a known risk measure, termed Tail Value at Risk, *TailVaR* or $TVaR_{\frac{\delta}{1+\delta}}$; it corrects *VaR* adding to it a term proportional to the Expected Shortfall.

Passing from the PMM to a generic VBM, we see from (35) and (36) that the role of *VaR* gets weaker. In fact, *VaR* is replaced by a convex combination of *VaR* itself and of $\sup X$, $(1-c)VaR_{-\frac{a}{b}}(X)+c\sup X > VaR_{-\frac{a}{b}}(X)$, while the shortfall correction term is unchanged. Hence, $\overline{E}(X)$ corresponds to a more prudential risk measure than $\overline{E}_{\mathrm{PMM}}(X)$, since it requires a higher amount than $\overline{E}_{\mathrm{PMM}}(X)$ to cover risks arising from the same X. Recall also that $\sup X$ is the most prudential choice for a risk measure of X, that covering all losses that may arise from X. It is also remarkable that, replacing *VaR* with $(1-c)VaR$ and adding $c\sup X$ when passing from (36) to (35), we still obtain a coherent upper prevision, or equivalently a coherent risk measure, that may be viewed as a generalisation of *TailVaR*.

7. Conclusions

Inferences with Nearly-Linear Models that are coherent or at least avoid sure loss can be performed by means of the natural extension, using the formulae introduced in the previous sections. In the case of the VBM, its natural extension to gambles generalises that of the PMM, and has an interpretation in terms of risk measures. With the HBM, next to a similar procedure in the case it is coherent, simple formulae are available in a finite environment to correct the model to a coherent one if it avoids sure loss, and to extend it to gambles. Similar formulae let us compute the natural extensions of RRMs that avoid sure loss. As a task for future work, there remains to explore how to extend those NL models that do not avoid sure loss, but are only 2-coherent. The most natural approach seems to employ the results on the 2-coherent natural extension in [12]. Still regarding inferential problems, we would also like to study conditioning and the dilation effects with these models. Further features of NL models that might be investigated regard a deepening of their relationships with other models, such as distortion models, and a study (in the finite environment, when they are coherent) of the set of their extreme points.

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