Noise Robust Core-stable Coalitions of Hedonic Games Supplementary Material

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1. n agents 2 support partial information noise model

In a two support noise model we have $\mathcal{N}_{sp} = \{1, \alpha\}$ with $\alpha > 1$, such that for any coalition $S \subseteq N$, $\mathbb{P}[\alpha(S) = \alpha] = p = 1 - \mathbb{P}[\alpha(S) = 1]$. We derive the agreement probability, $f_T(p, \alpha)$ in the following lemma. Note that this lemma serves as the base case in the Mathematical induction based proof of the Theorem 11 in the main paper.

Lemma 1 *Let $\tilde{\pi}$ be $\tilde{\epsilon}$-PAC stable partition of noisy game $(N, \tilde{\boldsymbol{v}})$, and let $\tilde{\pi}$ be a ϵ-PAC stable outcome of the noise-free game $(N, \boldsymbol{v})$, where ϵ is identified in Theorem 5 of the paper. Then the agreement probability $f_T(p, \alpha)$ is given by*

$$f_T(p, \alpha) = \begin{cases} 1, & if\ \tilde{\pi}(i) = T,\ \forall\ i \in T \\ p + (1-p)^{|\mathcal{R}(T)|+1-|\mathcal{I}(\alpha,T)|}, & otherwise \end{cases}$$

where $\mathcal{I}(\alpha, T) = \left\{ \tilde{\pi}(i) \in \mathcal{R}(T) \;\middle|\; \frac{\tilde{v}_i(\tilde{\pi}(i))}{\tilde{v}_i(T)} \geq \alpha \right\}$.

Proof Recall from Theorem 5 in main paper we have the following

$$\mathbb{P}_{T \sim \tilde{\mathcal{D}}}[\cup_{i \in T}\ v_i(\tilde{\pi}(i)) \geq v_i(T)] \geq (1 - \tilde{\epsilon}) f_T(\boldsymbol{p}, \boldsymbol{\alpha}).$$

Also, recall that the agreement event is defined as

$$M(\tilde{\pi}, T) \coloneqq \{(\{\alpha(\tilde{\pi}(i))\}_{\tilde{\pi}(i) \in \mathcal{R}(T)}, \alpha(T)) : \cap_{i \in T}\{v_i(\tilde{\pi}(i)) \geq v_i(T)\ \cap\ \alpha(\tilde{\pi}(i)) v_i(\tilde{\pi}(i)) \geq \alpha(T) v_i(T)\}\},$$

and $f_T(p, \alpha) = \mathbb{P}[M(\tilde{\pi}, T)]$ is the probability of agreement event. Moreover,

$$\mathcal{R}(T) \coloneqq \{\tilde{\pi}(i) \mid i \in T\}; \quad \mathcal{I}(\alpha, T) = \left\{ \tilde{\pi}(i) \in \mathcal{R}(T) \;\middle|\; \frac{\tilde{v}_i(\tilde{\pi}(i))}{\tilde{v}_i(T)} \geq \alpha \right\}.$$

To find the agreement probability, $f_T(\boldsymbol{p}, \boldsymbol{\alpha})$ we consider two cases $\mathcal{I}(\alpha, T) = \emptyset$, and $\mathcal{I}(\alpha, T) \neq \emptyset$. For these cases we identify the possible noise values $\{\alpha(\tilde{\pi}(i))\}_{\tilde{\pi}(i) \in \mathcal{R}(T)}$, $\alpha(T)$ that are element of $M(\tilde{\pi}, T)$.

- **Case 01:** $[\mathcal{I}(\alpha,T)=\emptyset]$. In this case, we have following elements in $M(\tilde{\pi},T)$.
  - $\alpha(\tilde{\pi}(i))=1,\ \forall\ \tilde{\pi}(i)\in\mathcal{R}(T)$ and $\alpha(T)=1$. The probability of such choice of α's is
$$(1-p)^{|\mathcal{R}(T)|+1}. \tag{1}$$
  - $\alpha(\tilde{\pi}(i))=\alpha$ for **exactly one** $\tilde{\pi}(i)\in\mathcal{R}(T)$, and $\alpha(\tilde{\pi}(i))=1$ for remaining coalitions in $\mathcal{R}(T)$, and $\alpha(T)=\alpha$. Probability of such choice of α's is $(p\times(1-p)^{|\mathcal{R}(T)|-1})\times p$. And there are $\binom{|\mathcal{R}(T)|}{1}$ ways of selecting **exactly one** coalition $\tilde{\pi}(i)\in\mathcal{R}(T)$. Thus, the probability of above α's is $\binom{|\mathcal{R}(T)|}{1}p(1-p)^{|\mathcal{R}(T)|-1}p$.
    In general, for any $k\in\{0,1,\dots,|\mathcal{R}(T)|\}$ coalitions $\tilde{\pi}(i)\in\mathcal{R}(T)$, take $\alpha(\tilde{\pi}(i))=\alpha$. Moreover, $\alpha(\tilde{\pi}(i))=1$ for remaining $|\mathcal{R}(T)|-k$ coalitions and take $\alpha(T)=\alpha$. Further, we have $\binom{|\mathcal{R}(T)|}{k}$ similar choices. So, the probability of the above choice of α's is
$$\begin{aligned}\sum_{k=0}^{|\mathcal{R}(T)|}\left\{\binom{|\mathcal{R}(T)|}{k}p^k(1-p)^{|\mathcal{R}(T)|-k}\right\}\times p &= p\times\left(\sum_{k=0}^{|\mathcal{R}(T)|}\binom{|\mathcal{R}(T)|}{k}p^k(1-p)^{|\mathcal{R}(T)|-k}\right)\\ &= p.\end{aligned} \tag{2}$$
    This is because for any coalition S, we have $\mathbb{P}[\alpha(S)=\alpha]=p=1-\mathbb{P}[\alpha(S)=1]$ and the fact that binomial probabilities summed up to 1.
- **Case 02:** $[\mathcal{I}(\alpha,T)\neq\emptyset]$. Then, in addition to the above possible cases, we will have a few other cases, which are:
  - $\alpha(\tilde{\pi}(i))=\alpha$ for **exactly one** $\tilde{\pi}(i)\in\mathcal{I}(\alpha,T)$, $\alpha(\tilde{\pi}(i))=1$ for remaining coalitions in $\mathcal{R}(T)$ and $\alpha(T)=1$. Probability of such choice of α's is $p(1-p)^{|\mathcal{R}(T)|-1}(1-p)=p(1-p)^{|\mathcal{R}(T)|}$. And there are $\binom{|\mathcal{I}(\alpha,T)|}{1}$ ways of choosing **exactly one** coalition $\tilde{\pi}(i)\in\mathcal{I}(\alpha,T)$. Thus the overall probability is $\binom{|\mathcal{I}(\alpha,T)|}{1}p(1-p)^{|\mathcal{R}(T)|}$.
    In general, we have $\alpha(\tilde{\pi}(i))=\alpha$ for any $k\in\{1,2,\dots,|\mathcal{I}(\alpha,T)|\}$ coalitions $\tilde{\pi}(i)\in\mathcal{I}(\alpha,T)$. Moreover, $\alpha(\tilde{\pi}(i))=1$ for remaining $|\mathcal{R}(T)|-k$ coalitions, and $\alpha(T)=1$. Probability of such choice of α's is $p^k(1-p)^{|\mathcal{R}(T)|-|\mathcal{I}(\alpha,T)|}(1-p)$. And there are $\binom{|\mathcal{I}(\alpha,T)|}{k}$ ways of selecting k coalitions $\tilde{\pi}(i)\in\mathcal{I}(\alpha,T)$. Thus the overall probability is
$$\sum_{k=1}^{|\mathcal{I}(\alpha,T)|}\binom{|\mathcal{I}(\alpha,T)|}{k}p^k(1-p)^{|\mathcal{R}(T)|-|\mathcal{I}(\alpha,T)|}(1-p). \tag{3}$$

The probability of event $M(\tilde{\pi},T)$, i.e., $\mathbb{P}[M(\tilde{\pi},T)]$ is obtained by adding probabilities given in Equations (1), (2) and (3).

$$\begin{aligned}\mathbb{P}[M(\tilde{\pi},T)] &= (1-p)^{|\mathcal{R}(T)|+1}+p+\sum_{k=1}^{|\mathcal{I}(\alpha,T)|}\binom{|\mathcal{I}(\alpha,T)|}{k}p^k(1-p)^{|\mathcal{R}(T)|-|\mathcal{I}(\alpha,T)|}(1-p)\\ &= (1-p)^{|\mathcal{R}(T)|+1}+p+(1-p)^{|\mathcal{R}(T)|-|\mathcal{I}(\alpha,T)|+1}\left[1-(1-p)^{|\mathcal{I}(\alpha,T)|}\right]\end{aligned}$$

$$= \quad p + (1-p)^{|\mathcal{R}(T)|-|\mathcal{I}(\alpha,T)|+1}.$$

This ends the proof. ■

If $\tilde{\pi}(i) \neq T$ for at least one $i \in T$, then $f_T(p,\alpha) = 1,\ \forall\ \alpha$ if and only if $p = 0$ or $p = 1$. That is, if the value of all the coalitions are retained, or if values of all of them are inflated by α, then for all $i \in T$, and for all $\tilde{\pi}(i) \in \mathcal{R}(T)$, one has $\tilde{\pi}(i) \succeq_i T$, and $\tilde{\pi}(i) \succeq'_i T$. Thus, $\tilde{\pi}$ is ϵ-PAC stable outcome of *unknown noise-free* game and hence $\tilde{\pi}$ is noise-robust.

Corollary 2 *When $\tilde{\pi} = N$, i.e., the grand coalition is $\tilde{\epsilon}$-PAC stable outcome in the noisy game, then $\mathcal{R}(T) = \{N\}$ for any coalition T. Thus, $\mathcal{I}(\alpha,T) = \emptyset$, or $\mathcal{I}(\alpha,T) = \{N\}$. Therefore, $f_T(p,\alpha)$ simplifies to*

$$f_T(p,\alpha) = \begin{cases} 1, & if\ \mathcal{I}(\alpha,T) = \{N\} \\ (1-p)^2 + p, & if\ \mathcal{I}(\alpha,T) = \emptyset. \end{cases} \tag{4}$$

2. n agents 2 support partial information noisy games without core

Suppose $\tilde{\pi}$ is not $\tilde{\epsilon}$-PAC stable partition fo the noisy game $(N, \tilde{\boldsymbol{v}})$. Moreover, let the noise support be $\mathcal{N}_{sp} = \{1, \alpha\}$, the following lemma provides the expression of $h_T(p,\alpha)$. Note that this lemma serves as the base case for the Mathematical induction based proof of Theorem 15 in the main paper.

Lemma 3 *Suppose $\tilde{\pi}$ is not a $\tilde{\epsilon}$-PAC stable outcome of the noisy game $(N, \tilde{\boldsymbol{v}})$, then the agreement probability $h_T(p,\alpha)$ for noise support $\mathcal{N}_{sp} \in \{1,\alpha\}$ is given by*

$$h_T(p,\alpha) = \begin{cases} 1, & if\ \ \tilde{\pi}(i) = T,\ \forall\ i \in T \\ (1-p) + p^{|\mathcal{R}(T)|+1-|\mathcal{J}(\alpha,T)|}, & otherwise, \end{cases} \tag{5}$$

where $\mathcal{J}(\alpha,T) \coloneqq \left\{ \tilde{\pi}(i) \in \mathcal{R}(T) \;\middle|\; \frac{\tilde{v}_i(\tilde{\pi}(i))}{\tilde{v}_i(T)} \geq \frac{1}{\alpha} \right\}$.

Proof From Theorem 13 of the main paper, we have the following

$$\mathbb{P}[\cup_{i\in T} v_i(\tilde{\pi}(i)) \geq v_i(T)] \geq (1-\tilde{\epsilon}) h_T(\boldsymbol{p}, \boldsymbol{\alpha}).$$

To get $h_T(p,\alpha) \coloneqq \mathbb{P}[F(T,\tilde{\pi})]$ we consider two cases viz. $\mathcal{J}(\alpha,T) = \emptyset$, and $\mathcal{J}(\alpha,T) \neq \emptyset$. For these cases, we identify the possible noise values elements of $F(T,\tilde{\pi})$.

- **Case 01:** $[\mathcal{J}(\alpha,T) = \emptyset]$. In this case, we have the following possibilities:
  - $\alpha(\tilde{\pi}(i)) = \alpha,\ \forall\ \tilde{\pi}(i) \in \mathcal{R}(T)$, and $\alpha(T) = \alpha$. Probability of such a choice of α's is
$$p^{|\mathcal{R}(T)|+1}. \tag{6}$$

- $\alpha(\tilde{\pi}(i)) = 1$ for $k \in \{0, 1, \ldots, |\mathcal{R}(T)|\}$ coalitions $\tilde{\pi}(i) \in \mathcal{R}(T)$, and $\alpha(\tilde{\pi}(i)) = \alpha$ for remaining $|\mathcal{R}(T)| - k$ coalitions. Moreover, $\alpha(T) = 1$. Probability of such choice of α's is $(1-p)^k p^{|\mathcal{R}(T)|-k}(1-p)$. Further, there are $\binom{|\mathcal{R}(T)|}{k}$ ways of selecting k coalitions $\tilde{\pi}(i)$ from $\mathcal{R}(T)$. Thus, the overall probability is

$$\sum_{k=0}^{|\mathcal{R}(T)|} \binom{|\mathcal{R}(T)|}{k} (1-p)^k p^{|\mathcal{R}(T)|-k}(1-p) = 1 - p. \tag{7}$$

- **Case 02:** $[\mathcal{J}(\alpha, T) \neq \emptyset]$. In addition to the above possible cases, we have a few other cases:
  - $\alpha(\tilde{\pi}(i)) = 1$ for any $k \in \{1, 2, \ldots, |\mathcal{J}(\alpha, T)|\}$ coalitions $\tilde{\pi}(i) \in \mathcal{J}(\alpha, T)$. Moreover, $\alpha(\tilde{\pi}(i)) = \alpha$ for remaining coalitions in $\mathcal{R}(T)$. Also, $\alpha(T) = \alpha$. Probability of such choice of α's is $(1-p)^k p^{|\mathcal{R}(T)|-k} p = (1-p)^k p^{|\mathcal{R}(T)|-k+1}$. And there are $\binom{|\mathcal{J}(\alpha,T)|}{k}$ ways of selecting k coalitions $\tilde{\pi}(i) \in \mathcal{J}(\alpha, T)$. Thus the overall probability is

$$\sum_{k=1}^{|\mathcal{J}(\alpha,T)|} \binom{|\mathcal{J}(\alpha, T)|}{k} (1-p)^k p^{|\mathcal{R}(T)|-k+1}. \tag{8}$$

The probability $\mathbb{P}[F(T, \tilde{\pi})]$ is obtained by adding probabilities given in Equations (6), (7) and (8).

$$\begin{aligned}
\mathbb{P}[F(T, \tilde{\pi})] &= p^{|\mathcal{R}(T)|+1} + (1-p) + \sum_{k=1}^{|\mathcal{J}(\alpha,T)|} \binom{|\mathcal{J}(\alpha, T)|}{k} (1-p)^k p^{|\mathcal{R}(T)|-k+1} \\
&= p^{|\mathcal{R}(T)|+1} + (1-p) + p^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha,T)|+1} \left[1 - p^{|\mathcal{J}(\alpha,T)|} \right] \\
&= (1-p) + p^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha,T)|+1}.
\end{aligned}$$

This ends the proof. ■

If $\tilde{\pi}(i) \neq T$ for at least one $i \in T$, then $h_T(p, \alpha) = 1, \ \forall \ \alpha$ if $p = 0$ or $p = 1$. That is, if the value of all coalitions are retained, or if value of all of them are inflated by α, then coalition $T \succeq_i \tilde{\pi}(i)$, and $T \succeq_i' \tilde{\pi}(i)$ for all $i \in T$. Thus, neither noise-free nor noisy game will have $\tilde{\pi}$ as PAC stable outcome. Moreover, if we allow $h_T(p, \alpha) = \eta$ for some user-given satisfaction η, we get a noise set in accordance to the Remark 14 in the main paper. In this case, the noise set also depends on $|\mathcal{R}(T)|$, and $|\mathcal{J}(\alpha, T)|$ for coalition T. Hence, the partition is η noise-robust non core-stable for the noise set $I^\star(T, \eta)$.

3. Proof of Theorem 15 of main paper

Theorem: For n agent noisy hedonic game $(N, \tilde{\boldsymbol{v}})$ with $\mathcal{N}_{sp} = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, the agreement probability $h_T(\mathbf{p}, \boldsymbol{\alpha})$ is given by:

$$h_T(\mathbf{p}, \boldsymbol{\alpha}) = \begin{cases} 1, & if\ \tilde{\pi}(i) = T,\ \forall\ i \in T, \\ \sum_{r,s\in[l]:\alpha_r>\alpha_s} p_r^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_r,\alpha_s,T)|+1} \times \{(p_s + p_r)^{|\mathcal{J}(\alpha_r,\alpha_s,T)|} - p_r^{|\mathcal{J}(\alpha_r,\alpha_s,T)|}\} & \\ + \sum_{a=1}^{l} p_a \left(\sum_{b=a}^{l} p_b\right)^{|\mathcal{R}(T)|}, & otherwise. \end{cases}$$

Proof We will prove this via Mathematical induction on the noise support $l \geq 2$. Clearly, this is true for $l = 2$ (from Lemma 3 above). Let us assume that it is true for $l = k$, i.e.; there are sets

$$\mathcal{J}(\alpha_r, \alpha_s, T) = \left\{ \tilde{\pi}(i) \in \mathcal{R}(T) \;\middle|\; \frac{\tilde{v}_i(\tilde{\pi}(i))}{\tilde{v}_i(T)} \geq \frac{\alpha_s}{\alpha_r} \right\},$$

such that the support $\alpha(S) = \{\alpha_1, \ldots, \alpha_k\},\ \forall\ S \subseteq N$ where $\alpha_s < \alpha_r,\ \forall\ 1 \leq s < r \leq k$. For this k we have $f_T(p_j, \alpha_j : j \in [k]) =: h_T(\boldsymbol{p}, \boldsymbol{\alpha})$ (by assumption)

$$h_T(\boldsymbol{p}, \boldsymbol{\alpha}) = \sum_{a=1}^{k} p_a \left(\sum_{b=a}^{k} p_b\right)^{|\mathcal{R}(T)|} + \sum_{r,s\in[k]:\alpha_r>\alpha_s} p_r^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_r,\alpha_s,T)|+1}((p_r+p_s)^{|\mathcal{J}(\alpha_r,\alpha_s,T)|} - p_r^{|\mathcal{J}(\alpha_r,\alpha_s,T)|}).$$

We will now show that this is true for $l = k+1$. To this end define $\mathcal{J}(\alpha_{k+1}, \alpha_s, T)$ for all $s \in [k]$ such that $\alpha_{k+1} > \alpha_s$

$$\mathcal{J}(\alpha_{k+1}, \alpha_s, T) = \left\{ \tilde{\pi}(i) \in \mathcal{R}(T) \;\middle|\; \frac{\tilde{v}_i(\tilde{\pi}(i))}{\tilde{v}_i(T)} \geq \frac{\alpha_s}{\alpha_{k+1}} \right\}.$$

Now, there are two cases, $\mathcal{J}(\alpha_{k+1}, \alpha_s, T) = \emptyset,\ \forall\ \alpha_s,\ s \in [k]$, or $\mathcal{J}(\alpha_{k+1}, \alpha_s, T) \neq \emptyset$ for at least for one $s \in [k]$.

Case 01: $[\mathcal{J}(\alpha_{k+1}, \alpha_s, T) = \emptyset,\ \forall\ \alpha_s,\ s \in [k]]$. Apart from the existing $\{\alpha(\tilde{\pi}(i)\}_{\tilde{\pi}(i)\in\mathcal{R}(T)}$ and $\alpha(T)$ for k support case, with this extra $k+1$, it will also have $\alpha(T) = \alpha_{k+1}$ and $\alpha(\tilde{\pi}(i)) = \alpha_{k+1},\ \forall\ \tilde{\pi}(i) \in \mathcal{R}(T)$. The probability of such extra α's is $p_{k+1}\left(\sum_{b=k+1}^{k+1} p_b\right)^{|\mathcal{R}(T)|}$. Therefore, the overall probability is

$$\sum_{a=1}^{k} p_a \left(\sum_{b=a}^{k} p_b\right)^{|\mathcal{R}(T)|} + p_{k+1} \left(\sum_{b=k+1}^{k+1} p_b\right)^{|\mathcal{R}(T)|} = \sum_{a=1}^{k+1} p_a \left(\sum_{b=a}^{k+1} p_b\right)^{|\mathcal{R}(T)|}.$$

Case 02: $[\mathcal{J}(\alpha_{k+1}, \alpha_s, T) \neq \emptyset$ for at least one $s \in [k]]$. In this case, apart from all $\alpha(T)$ and $\{\alpha(\tilde{\pi}(i))\}_{\tilde{\pi}(i)\in\mathcal{J}(\alpha_r,\alpha_s,T)}$, we have $\{\alpha(\tilde{\pi}(i))\}_{\forall\ \tilde{\pi}(i)\in\mathcal{J}(\alpha_{k+1},\alpha_r,T)}, \alpha(T)$. For this set, the possible pairs are such that $\alpha(\tilde{\pi}(i)) = \alpha_r,\ \ \forall\ \tilde{\pi}(i) \in \mathcal{R}(T) \setminus \mathcal{J}(\alpha_{k+1}, \alpha_r, T)$, and $\alpha(T) = \alpha_{k+1}$. Thus, their combined probability is $p_{k+1}^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_{k+1},\alpha_s,T)|+1}((p_{k+1} - p_s)^{|\mathcal{J}(\alpha_{k+1},\alpha_r,T)|} - p_{k+1}^{|\mathcal{J}(\alpha_{k+1},\alpha_r,T)|})$. Hence for $k+1$ support, the probability is

$$\sum_{r,s\in[k]:\alpha_r>\alpha_s} p_r^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_r,\alpha_r,T)|+1}((p_r + p_s)^{|\mathcal{J}(\alpha_r,\alpha_s,T)|} - p_r^{|\mathcal{J}(\alpha_r,\alpha_s,T)|})$$

$$+\, p_{k+1}^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_{k+1},\alpha_s,T)|+1}((p_{k+1}+p_s)^{|\mathcal{I}(\alpha_{k+1},\alpha_s,T)|}-p_{k+1}^{|\mathcal{J}(\alpha_{k+1},\alpha_s,T)|}).$$

From case 01 and case 02 with $k+1$ support, we have

$$h_T(p_j,\alpha_j;j\in[k+1]) = \sum_{r,s\in[k+1]:\alpha_r>\alpha_s} p_r^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_r,\alpha_s,T)|+1}\left((p_r+p_s)^{|\mathcal{J}(\alpha_r,\alpha_s,T)|}-p_r^{|\mathcal{J}(\alpha_r,\alpha_s,T)|}\right) + \sum_{a=1}^{k+1} p_a \left(\sum_{b=a}^{k+1} p_b\right)^{|\mathcal{R}(T)|}.$$

Furthermore, it is true for $k+1$ support. Thus, from the principle of Mathematical induction, this is true for any $l \geq 2$. ■

4. 2 agent 2 support model

In this Section, we will provide further details about the 2 agents' full information noisy game with 2 support of the noise distribution. First, we consider the following noisy game.

$$\tilde{v}_1(12) > \tilde{v}_1(1); \;\; \tilde{v}_2(12) > \tilde{v}_2(2). \qquad \text{(game 1)}$$

We also consider the other possible noisy games with 2 agents in later subsections.

4.1. Proof of Lemma 18 of main paper

Lemma: For noisy game 1 with complete information on $\tilde{\boldsymbol{v}}$ and $\mathcal{N}_{sp} = \{1,\alpha\}$ we have

$$\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1] = \begin{cases} 1-p(1-p^2), & if\ \alpha \geq \overline{r} \\ 1-p(1-p), & if\ \underline{r} \leq \alpha < \overline{r} \\ 1, & if\ \alpha < \underline{r}, \end{cases} \tag{9}$$

where $\overline{r} = \max\left\{\frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}, \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)}\right\}$, and $\underline{r} = \min\left\{\frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}, \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)}\right\}$.

Also, this prediction probability $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1]$ is convex in p. So, while the minimal value for $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1]$ occurs for noise probabilities around $p = 0.5$ (depending on $\alpha, \overline{r}$ and $\underline{r}$), the maximal value of it is 1 at $p = 0$ and $p = 1$.

Proof For noisy game 1, we have $\tilde{\pi} = N$. Now, consider the noise support $\mathcal{N}_{sp} = \{1, \alpha\}$, where $\alpha > 1$ such that $\mathbb{P}[\alpha(S) = \alpha] = p = 1 - \mathbb{P}[\alpha(S) = 1]$, for some fixed and unknown p. Given noisy game 1, there are 8 possible combinations of α's (because each coalition has two options). We will now enumerate all such possibilities:

1. $\alpha(1) = 1; \alpha(2) = 1;\ \alpha(12) = 1$. The probability of such alpha is $(1-p)^3$. Thus, the noise-free values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \tilde{v}_2(2), v_1(12) = \tilde{v}_1(12)\ and\ v_2(12) = \tilde{v}_2(12)$. Therefore, The noise-free game is:

$$v_1(12) > v_1(1); \;\; v_2(12) > v_2(2).$$

   From this game we have $\pi = \tilde{\pi} = N$.

2. $\alpha(1) = 1; \alpha(2) = 1;\ \alpha(12) = \alpha$ Probability of such alpha's is $p(1-p)^2$. Thus the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \tilde{v}_2(2), v_1(12) = \frac{\tilde{v}_1(12)}{\alpha}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha}$. Therefore, the actual preferences will depend on the relative values of α and $\tilde{\boldsymbol{v}}$. If α and $\tilde{\boldsymbol{v}}$'s are such that $\frac{\tilde{v}_1(12)}{\alpha} > \tilde{v}_1(1)$ and $\frac{\tilde{v}_2(12)}{\alpha} > \tilde{v}_2(2)$, then $\pi = N$, otherwise $\pi = \{\{1\},\{2\}\}$.

3. $\alpha(1) = 1; \alpha(2) = \alpha;\ \alpha(12) = 1$. The probability of such alpha is $p(1-p)^2$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha}, v_1(12) = \tilde{v}_1(12)$ *and* $v_2(12) = \tilde{v}_2(12)$. Since, $\tilde{v}_2(12) > \tilde{v}_2(2) > \frac{\tilde{v}_2(2)}{\alpha}$. The noise-free game is:

$$v_1(12) > v_1(1);\ \ v_2(12) > v_2(2).$$

So, we have $\pi = \tilde{\pi} = N$.

4. $\alpha(1) = \alpha;\ \alpha(2) = 1;\ \alpha(12) = 1$. Probability of such alpha's is $p(1-p)^2$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha};\ \ v_2(2) = \tilde{v}_2(2), v_1(12) = \tilde{v}_1(12)$ *and* $v_2(12) = \tilde{v}_2(12)$. Since, $\tilde{v}_1(12) > \tilde{v}_1(1) > \frac{\tilde{v}_1(1)}{\alpha}$. Therefore The noise-free game is:

$$v_1(12) > v_1(1);\ \ v_2(12) > v_2(2).$$

From this game we have $\pi = \tilde{\pi} = N$.

5. $\alpha(1) = 1; \alpha(2) = \alpha;\ \alpha(12) = \alpha$. The probability of this alpha is $p^2(1-p)$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha}, v_1(12) = \frac{\tilde{v}_1(12)}{\alpha}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha}$. The actual preferences will depend on the relative values of α and $\tilde{\boldsymbol{v}}$. If α and $\tilde{\boldsymbol{v}}$'s are such that $\frac{\tilde{v}_1(12)}{\alpha} > \tilde{v}_1(1)$, then $\pi = N$, otherwise $\pi = \{\{1\},\{2\}\}$.

6. $\alpha(1) = \alpha; \alpha(2) = 1;\ \alpha(12) = \alpha$. The probability of such alpha is $p^2(1-p)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha},\ \ v_2(2) = \tilde{v}_2(2);\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha}$. The actual preferences will depend on the relative values of α and $\tilde{\boldsymbol{v}}$. If α and $\tilde{\boldsymbol{v}}$'s are such that $\frac{\tilde{v}_2(12)}{\alpha} > \tilde{v}_2(2)$, then $\pi = N$ otherwise $\pi = \{\{1\},\{2\}\}$.

7. $\alpha(1) = \alpha; \alpha(2) = \alpha;\ \alpha(12) = 1$. Probability of such alpha's is $p^2(1-p)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha};\ v_1(12) = \tilde{v}_1(12)$ *and* $v_2(12) = \tilde{v}_2(12)$. Since, $\tilde{v}_1(12) > \tilde{v}_1(1) > \frac{\tilde{v}_1(1)}{\alpha}$. and, $\tilde{v}_2(12) > \tilde{v}_2(2) > \frac{\tilde{v}_2(2)}{\alpha}$. The noise-free game is:

$$v_1(12) > v_1(1);\ \ v_2(12) > v_2(2).$$

From this game we have $\pi = \tilde{\pi} = N$.

8. $\alpha(1) = \alpha; \alpha(2) = \alpha;\ \alpha(12) = \alpha$. The probability of such alpha is p^3. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha}$. Therefore, the noise-free game is:

$$v_1(12) > v_1(1);\ \ v_2(12) > v_2(2).$$

From this game it is clear that $\pi = \tilde{\pi} = N$.

Recall, $\overline{r} = \max\left\{\frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}, \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)}\right\}$, and $\underline{r} = \min\left\{\frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}, \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)}\right\}$. Out of 8 cases there are 5 cases (case 1,3,4,7,8) in which the grand coalition $\pi = \tilde{\pi} = N$ is formed in noise-free game. In these conditions, the relative value of $\tilde{v}_1(\cdot), \tilde{v}_2(\cdot)$ should satisfy $\alpha \geq \overline{r}$, and this constitute the first expression $p^3 + p^2(1-p) + 2p(1-p)^2 + (1-p)^3$ of $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1]$. Apart from this, if the inflation interval is $\underline{r} \leq \alpha < \overline{r}$, then $\pi = \tilde{\pi} = N$ is also possible from case (6) with probability $p^2(1-p)$. Thus, $p^2(1-p)$ will be added to the above prediction probability. So, we have $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1]$ corresponding to it. Moreover, finally, if $\alpha < \underline{r}$, all cases are allowable, and hence the grand coalition will always form in the noise-free game. Thus,

$$\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1] = \begin{cases} p^3 + p^2(1-p) + 2p(1-p)^2 + (1-p)^3, & if\ \alpha \geq \overline{r} \\ p^3 + 2p^2(1-p) + 2p(1-p)^2 + (1-p)^3, & if\ \underline{r} \leq \alpha < \overline{r} \\ 1, & if\ \alpha < \underline{r}. \end{cases} \tag{10}$$

Simplifying these polynomials, we have

$$\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1] = \begin{cases} 1 - p(1-p^2), & if\ \alpha \geq \overline{r} \\ 1 - p(1-p), & if\ \underline{r} \leq \alpha < \overline{r} \\ 1, & if\ \alpha < \underline{r}. \end{cases} \tag{11}$$

This ends the proof. ■

If we allow some user given satisfaction ζ on the prediction probability, i.e., $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1] = \zeta$, we get the following noise interval

$$I^\star(\zeta = 0.9) = \begin{cases} [0, 0.101] \cup [0.946, 1], & if\ \alpha \geq \overline{r}; \\ [0, 0.113] \cup [0.887, 1], & if\ \underline{r} \leq \alpha < \overline{r} \\ 1, & if\ \alpha < \underline{r}. \end{cases} \tag{12}$$

4.2. Details of the other 2 agent noisy games

Here we will give the prediction probabilities for other possible noisy games with 2 agents and 2 noise support.

4.2.1. Both agents prefer staying alone in noisy game

As opposed to the noisy game 1, in noisy game 2 both agents prefer to stay alone. The noisy preferences of agents are as follows:

$$\tilde{v}_1(1) > \tilde{v}_1(12);\ \ \tilde{v}_2(2) > \tilde{v}_2(12). \tag{game 2}$$

Clearly $\tilde{\pi} = \{\{1\},\{2\}\} \neq N$ is the core-stable outcome. The following lemma provides prediction probability, $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 2]$ for noisy game 2.

Lemma 4 *For noisy game 2 with full information of $\tilde{\boldsymbol{v}}$'s, the prediction probability that unknown noise-free game has $\pi = \tilde{\pi}$ as a core-stable outcome is*

$$\mathbb{P}[\pi = \tilde{\pi} \mid game\ 2] = \begin{cases} 1 - p^2(1-p), & if\ \frac{1}{\alpha} < \underline{r} \\ 1, & if\ \frac{1}{\alpha} \geq \underline{r}. \end{cases} \tag{13}$$

Moreover, the minimal and maximal values of above prediction probability are 0.85 *(when* $p = 2/3$*), and* 1*, respectively.*

Similar to game 1, the probability of formation of partition $\pi = \{\{1\}, \{2\}\}$ in an *unknown* noise-free game is always more than 0.85. So, the safety value is 0.85. The prediction probability is 1 when $\frac{1}{\alpha} \geq \underline{r}$ for any noise probability p. Moreover, for some user-given satisfaction ζ, we obtain the corresponding p by setting $\mathbb{P}[\pi = \{\{1\}, \{2\}\} \mid game\ 2] = \zeta$. In particular, we have

$$I^\star(\zeta = 0.9) = \begin{cases} [0, 0.413] \cup [0.867, 1], & if\ \frac{1}{\alpha} < \underline{r} \\ [0, 1], & \frac{1}{\alpha} \geq \underline{r}. \end{cases} \tag{14}$$

It is easy to see that the allowable p is larger than the interval given in Equation (12) for game 1. So, the partition $\tilde{\pi} = \{\{1\}, \{2\}\}$ is noise robust for larger number of inflation probabilities p. Again the noise set will shrink if we increase the satisfaction ζ.

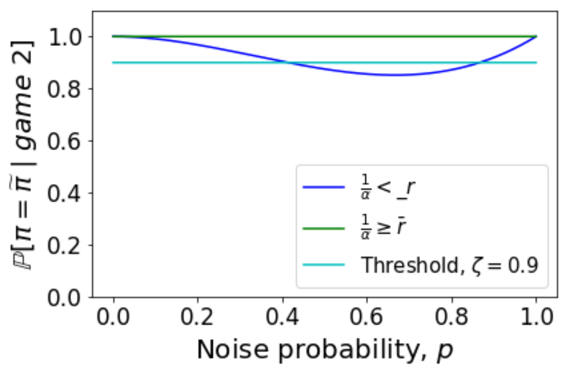

Figure 1: The prediction probability $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 2]$. For $\zeta = 0.9$, the noise regimes are given in Equation (14).

4.2.2. Agent 1 prefers to stay alone and agent 2 prefers grand coalition in noisy game

Now, we consider a noisy game where agent 1 prefers to stay alone, whereas agent 2 prefers the grand coalition. In particular, the preferences in the noisy game are

$$\tilde{v}_1(1) > \tilde{v}_1(12); \quad \tilde{v}_2(12) > \tilde{v}_2(2). \tag{game 3}$$

Again $\tilde{\pi} = \{\{1\}, \{2\}\} \neq N$ is noisy core-stable outcome. The prediction probability, $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 3]$ is given in the Lemma below.

Lemma 5 *For noisy game 3 with full information of* $\tilde{\boldsymbol{v}}$*'s, the prediction probability that unknown noise-free game has* $\pi = \tilde{\pi}$ *as a core-stable outcome is given by:*

$$\mathbb{P}[\pi = \tilde{\pi} \mid game\ 3] = \begin{cases} 1 - p(1-p), & if\ \frac{1}{\alpha} < \frac{\tilde{v}_1(12)}{\tilde{v}_1(1)} \\ 1, & if\ \frac{1}{\alpha} \geq \frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}. \end{cases} \tag{15}$$

Moreover, the minimal and maximal values of above prediction probability are 0.75 *(when* $p = 0.5$*), and* 1*, respectively.*

Similar to game 1 and game 2 the probability of formation of partition $\pi = \{\{1\},\{2\}\}$ in an *unknown* noise-free game is always more than 0.75 that is the safety value for game 3. The prediction probability is 1 when $\frac{1}{\alpha} \geq \frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}$ for any noise probability p. Moreover, for some user-given satisfaction, ζ we obtain the corresponding p by setting $\mathbb{P}[\pi = \{\{1\},\{2\}\} \mid game\ 3] = \zeta$. In particular,

$$I^\star(\zeta = 0.9) = \begin{cases} [0, 0.113] \cup [0.887, 1], & if\ \frac{1}{\alpha} < \frac{\tilde{v}_1(12)}{\tilde{v}_1(1)} \\ [0, 1], & if\ \frac{1}{\alpha} \geq \frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}. \end{cases} \tag{16}$$

The following figure shows the prediction probabilities for game 3.

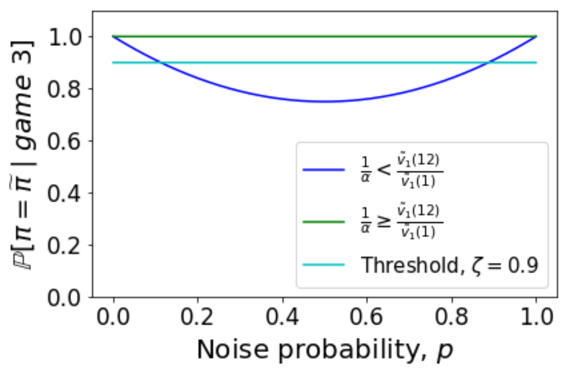

Figure 2: The prediction probability $\mathbb{P}[\tilde{\pi} = \pi \mid game\ 3]$. For $\zeta = 0.9$, we obtain the noise regimes as given in Equation (16).

4.2.3. Agent 1 prefers grand coalition and agent 2 prefers to stay alone

Finally, consider a noisy game symmetric to game 3. Here agent 1 prefers a grand coalition, and agent 2 prefers to stay alone. In particular, we have the following preferences.

$$\tilde{v}_1(12) > \tilde{v}_1(1); \ \ \tilde{v}_2(2) > \tilde{v}_2(12). \tag{game 4}$$

Again $\tilde{\pi} = \{\{1\},\{2\}\} \neq N$ is a noisy core-stable outcome. In the following lemma, we find the prediction probability when noisy game 4 is considered.

Lemma 6 *For noisy game 4 with full information of* $\tilde{\mathbf{v}}$*'s, the prediction probability that noise-free game has* $\pi = \tilde{\pi}$ *as as core-stable outcome is given by:*

$$\mathbb{P}[\pi = \tilde{\pi} \mid game\ 4] = \begin{cases} 1 - p(1-p), & if\ \frac{1}{\alpha} < \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)} \\ 1, & if\ \frac{1}{\alpha} \geq \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)}. \end{cases} \tag{17}$$

So, the minimal and maximal values of above prediction probability are 0.75 *(when* $p = 0.5$*) and* 1 *respectively.*

In this case also, the noise regime can be obtained using $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 4] = \zeta$. In particular,

$$I^\star(\zeta = 0.9) = \begin{cases} [0, 0.113] \cup [0.887, 1], & if\ \frac{1}{\alpha} < \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)} \\ [0, 1], & if\ \frac{1}{\alpha} \geq \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)}. \end{cases} \quad (18)$$

Figure 3 shows the prediction probabilities for game 4.

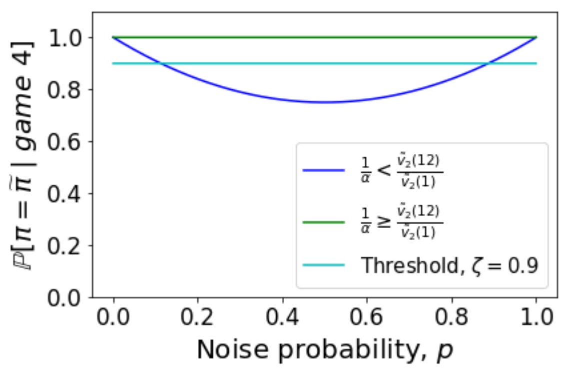

Figure 3: The prediction probability $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 4]$. For $\zeta = 0.9$, we obtain the noise regimes as given in Equation (18).

5. 2 agents 3 support noise model

In this section, we consider two player noisy hedonic game with three support noise model, i.e., $\mathcal{N}_{sp} = \{1, \alpha_1, \alpha_2\}$, with $\alpha_1 > 1$, and $\alpha_2 < 1$. Note that $\alpha_1, \alpha_2 > 0$. Let $\mathbb{P}[\alpha(S) = \alpha_1] = p_1$; $\mathbb{P}[\alpha(S) = \alpha_2] = p_2$; *and* $\mathbb{P}[\alpha(S) = 1] = 1 - p_1 - p_2$. That is the value of each coalition is either inflated with probability p_1, or deflated with probability p_2 or retained with probability $1 - p_1 - p_2$. The following lemma provides the prediction probability for game 1.

5.1. Proof of Lemma 20 of main paper

Lemma: For the 3 support noise model the prediction probability $\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1]$ is

$$\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1] = \begin{cases} g(p_1, p_2), & if\ \ \alpha_1 \geq \overline{r}\ ;\ \frac{1}{\alpha_2} \geq \overline{r}\ ;\ \frac{\alpha_1}{\alpha_2} \geq \overline{r} \\ 1, & if\ \ \alpha_1 < \underline{r}\ ;\ \frac{1}{\alpha_2} < \underline{r}\ ;\ \frac{\alpha_1}{\alpha_2} < \underline{r} \end{cases} \quad (19)$$

where $g(p_1, p_2) = p_1^3 + p_2^3 + 2(p_1(1-p_1-p_2)^2 + p_2^2(1-p_1-p_2) + p_1p_2(1-p_1-p_2) + p_1p_2^2) + p_1^2p_2 + p_1^2(1-p_1-p_2) + p_2(1-p_1-p_2)^2 + (1-p_1-p_2)^3$.

Proof For game 1, with $l = 3$ support of noise there are 27 possible cases for α's. Since there are 3 coalitions, each coalition's value can either be retained, inflated by α_1, or deflated by α_2. We will now enumerate all of them:

1. $\alpha(1) = 1; \alpha(2) = 1;\ \alpha(12) = 1$ Probability of such alpha's is $(1 - p_1 - p_2)^3$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \tilde{v}_2(2), v_1(12) = \tilde{v}_1(12)$ *and* $v_2(12) = \tilde{v}_2(12)$. The noise-free game is: $v_1(12) > v_1(1);\quad v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

2. $\alpha(1) = 1; \alpha(2) = 1;\ \alpha(12) = \alpha_1$. Probability of such alpha's is $p_1(1 - p_1 - p_2)^2$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \tilde{v}_2(2), v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_1}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_1}$. The noise-free game preferences are unclear; they will depend on the relative values of α_1 and $\tilde{v}$. If α_1 and $\tilde{v}$'s are such that $\frac{\tilde{v}_1(12)}{\alpha_1} > \tilde{v}_1(1)$ and $\frac{\tilde{v}_2(12)}{\alpha_1} > \tilde{v}_2(2)$ then $\pi = N$, otherwise $\pi = \{\{1\}, \{2\}\}$.

3. $\alpha(1) = 1; \alpha(2) = \alpha_1;\ \alpha(12) = 1$. Probability of such alpha's is $p_1(1 - p_1 - p_2)^2$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_1}, v_1(12) = \tilde{v}_1(12)$ *and* $v_2(12) = \tilde{v}_2(12)$. Since $\tilde{v}_2(12) > \tilde{v}_2(2) > \frac{\tilde{v}_2(2)}{\alpha_1}$. The noise-free game is: $v_1(12) > v_1(1);\quad v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

4. $\alpha(1) = \alpha_1;\ \alpha(2) = 1;\ \alpha(12) = 1$. Probability of such alpha's is $p_1(1 - p_1 - p_2)^2$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_1};\quad v_2(2) = \tilde{v}_2(2), v_1(12) = \tilde{v}_1(12)$ *and* $v_2(12) = \tilde{v}_2(12)$. Since, $\tilde{v}_1(12) > \tilde{v}_1(1) > \frac{\tilde{v}_1(1)}{\alpha_1}$. The noise-free game is: $v_1(12) > v_1(1)$; $v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

5. $\alpha(1) = 1; \alpha(2) = \alpha_1;\ \alpha(12) = \alpha_1$. Probability of such alpha's is $p_1^2(1 - p_1 - p_2)$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_1}, v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_1}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_1}$. The noise-free game preferences are unclear; they will depend on the relative values of α_1 and $\tilde{v}$. If α_1 and $\tilde{v}$'s are such that $\frac{\tilde{v}_1(12)}{\alpha_1} > \tilde{v}_1(1)$, then $\pi = N$, otherwise $\pi = \{\{1\}, \{2\}\}$.

6. $\alpha(1) = \alpha_1; \alpha(2) = 1;\ \alpha(12) = \alpha_1$. Probability of such alpha's is $p_1^2(1 - p_1 - p_2)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_1},\ \ v_2(2) = \tilde{v}_2(2);\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_1}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_1}$. The noise-free game preferences are unclear; they will depend on the relative values of α_1 and $\tilde{v}$. If α_1 and $\tilde{v}$'s are such that $\frac{\tilde{v}_2(12)}{\alpha_1} > \tilde{v}_2(2)$, then $\pi = N$, otherwise $\pi = \{\{1\}, \{2\}\}$.

7. $\alpha(1) = \alpha_1; \alpha(2) = \alpha_1;\ \alpha(12) = 1$. Probability of such alpha's is $p_1^2(1 - p_1 - p_2)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_1},\quad v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_1};\ v_1(12) = \tilde{v}_1(12)$ *and* $v_2(12) = \tilde{v}_2(12)$. Since, $\tilde{v}_1(12) > \tilde{v}_1(1) > \frac{\tilde{v}_1(1)}{\alpha_1}$. and, $\tilde{v}_2(12) > \tilde{v}_2(2) > \frac{\tilde{v}_2(2)}{\alpha_1}$. The noise-free game is: $v_1(12) > v_1(1);\quad v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

8. $\alpha(1) = \alpha_1; \alpha(2) = \alpha_1;\ \alpha(12) = \alpha_1$. The probability of such alpha is p_1^3. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_1},\quad v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_1};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_1}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_1}$. The noise-free game is: $v_1(12) > v_1(1);\quad v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

9. $\alpha(1) = 1; \alpha(2) = 1;\ \alpha(12) = \alpha_2$. Probability of such alpha's is $p_2(1 - p_1 - p_2)^2$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \tilde{v}_2(2), v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_2}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_2}$. Since $\alpha_2 < 1$, thus $\frac{\tilde{v}_1(12)}{\alpha_2} > \tilde{v}_1(12) > \tilde{v}_1(1) = v_1(1)$. Similarly, $\frac{\tilde{v}_2(12)}{\alpha_2} > \tilde{v}_2(12) > \tilde{v}_2(2) = v_2(2)$. The noise-free game is: $v_1(12) > v_1(1);\quad v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

10. $\alpha(1) = 1; \alpha(2) = \alpha_2;\ \alpha(12) = 1$. Probability of these alpha's is $p_2(1-p_1-p_2)^2$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_2}, v_1(12) = \tilde{v}_1(12)\ and\ v_2(12) = \tilde{v}_2(12)$. The noise-free game preferences are unclear; they will depend on the relative values of α_2 and $\tilde{v}$. If α_2 and $\tilde{v}$'s are such that $\tilde{v}_2(12) > \frac{\tilde{v}_2(2)}{\alpha_2}$ then $\pi = N$, otherwise $\pi = \{\{1\}, \{2\}\}$.

11. $\alpha(1) = \alpha_2;\ \alpha(2) = 1;\ \alpha(12) = 1$. Probability of such alpha's is $p_2(1-p_1-p_2)^2$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_2};\ \ v_2(2) = \tilde{v}_2(2), v_1(12) = \tilde{v}_1(12)\ and\ v_2(12) = \tilde{v}_2(12)$. The noise-free game preferences are unclear; they will depend on the relative values of α_2 and $\tilde{v}$. If α_2 and $\tilde{v}$'s are such that $\tilde{v}_1(12) > \frac{\tilde{v}_1(1)}{\alpha_2}$ then $\pi = N$, otherwise $\pi = \{\{1\}, \{2\}\}$.

12. $\alpha(1) = 1; \alpha(2) = \alpha_2;\ \alpha(12) = \alpha_2$. probability of such alpha's is $p_2^2(1-p_1-p_2)$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1);\ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_2}, v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_2}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_2}$ Since $\alpha_2 < 1$, thus $\frac{\tilde{v}_1(12)}{\alpha_2} > \tilde{v}_1(12) > \tilde{v}_1(1) = v_1(1)$, and $\frac{\tilde{v}_2(12)}{\alpha_2} > \frac{\tilde{v}_2(2)}{\alpha_2}$. The noise-free game is: $v_1(12) > v_1(1);\ \ v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

13. $\alpha(1) = \alpha_2; \alpha(2) = 1;\ \alpha(12) = \alpha_2$. Probability of such alpha's is $p_2^2(1-p_1-p_2)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_2},\ \ v_2(2) = \tilde{v}_2(2);\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_2}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_2}$. Since $\alpha_2 < 1$ thus $\frac{\tilde{v}_2(12)}{\alpha_2} > \tilde{v}_2(12) > \tilde{v}_2(2) = v_2(2)$, and $\frac{\tilde{v}_1(12)}{\alpha_2} > \frac{\tilde{v}_1(1)}{\alpha_2}$. The noise-free game is: $v_1(12) > v_1(1);\ \ v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

14. $\alpha(1) = \alpha_2; \alpha(2) = \alpha_2;\ \alpha(12) = 1$. Probability of such alpha's is $p_2^2(1-p_1-p_2)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_2},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_2};\ v_1(12) = \tilde{v}_1(12)\ and\ v_2(12) = \tilde{v}_2(12)$. The noise-free game preferences are unclear; they will depend on the relative values of α_2 and $\tilde{v}$. If α_2 and $\tilde{v}$'s are such that $\tilde{v}_1(12) > \frac{\tilde{v}_1(1)}{\alpha_2}$ and $\tilde{v}_1(12) > \frac{\tilde{v}_2(2)}{\alpha_2}$ then $\pi = N$, otherwise $\pi = \{\{1\}, \{2\}\}$.

15. $\alpha(1) = 1; \alpha(2) = \alpha_1;\ \alpha(12) = \alpha_2$. Probability of such alpha's is $p_1p_2(1-p_1-p_2)$. Thus, the actual values are $v_1(1) = \tilde{v}_1(1),\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_1};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_2}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_2}$. Since $\frac{\tilde{v}_1(12)}{\alpha_2} > \tilde{v}_1(12) > \tilde{v}_1(1)$ and $\frac{\tilde{v}_2(12)}{\alpha_2} > \tilde{v}_2(12) > \tilde{v}_2(2) > \frac{\tilde{v}_2(2)}{\alpha_1}$. The noise-free game is: $v_1(12) > v_1(1);\ \ v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

16. $\alpha(1) = 1; \alpha(2) = \alpha_2;\ \alpha(12) = \alpha_1$. Probability of such alpha's is $p_1p_2(1-p_1-p_2)$. Thus, actual values are $v_1(1) = \tilde{v}_1(1),\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_2};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_1}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_1}$. The noise-free game preferences are unclear; it will depend on the relative values α_1, α_2 and $\tilde{v}$. If α_1, α_2 and $\tilde{v}$'s are such that $\frac{\tilde{v}_1(12)}{\alpha_1} > \tilde{v}_1(1)$ and $\frac{\tilde{v}_2(12)}{\alpha_1} > \frac{\tilde{v}_2(2)}{\alpha_2}$ then $\pi = N$, otherwise $\pi = \{\{1\}, \{2\}\}$.

17. $\alpha(1) = \alpha_1; \alpha(2) = 1;\ \alpha(12) = \alpha_2$. Probability of such alpha's is $p_1p_2(1-p_1-p_2)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_1},\ \ v_2(2) = \tilde{v}_2(2);\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_2}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_2}$. Since $\frac{\tilde{v}_1(12)}{\alpha_2} > \tilde{v}_1(12) > \tilde{v}_1(1)$ and $\frac{\tilde{v}_2(12)}{\alpha_2} > \tilde{v}_2(12) > \tilde{v}_2(2)$. The noise-free game is: $v_1(12) > v_1(1);\ \ v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

18. $\alpha(1) = \alpha_2; \alpha(2) = 1;\ \alpha(12) = \alpha_1$. Probability of such alpha's is $p_1p_2(1-p_1-p_2)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_2},\ \ v_2(2) = \tilde{v}_2(2);\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_1}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_1}$.

The noise-free game preferences are unclear; it will depend on the relative values of α_1, α_2, and $\tilde{v}$. If α_1, α_2 and $\tilde{v}$'s are such that $\frac{\tilde{v}_1(12)}{\alpha_1} > \frac{\tilde{v}_1(1)}{\alpha_2}$ and $\frac{\tilde{v}_2(12)}{\alpha_1} > \tilde{v}_2(2)$ then $\pi = N$, otherwise $\pi = \{\{1\},\{2\}\}$.

19. $\alpha(1) = \alpha_1; \alpha(2) = \alpha_2;\ \alpha(12) = 1$. Probability of such alpha's is $p_1p_2(1-p_1-p_2)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_1},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_2};\ v_1(12) = \tilde{v}_1(12)\ and\ v_2(12) = \tilde{v}_2(12)$. The noise-free game preferences are unclear; it will depend on the relative values of α_1, α_2, and $\tilde{v}$. If α_1, α_2 and $\tilde{v}$'s are such that $\tilde{v}_2(12) > \frac{\tilde{v}_2(2)}{\alpha_2}$ then $\pi = N$ otherwise $\pi = \{\{1\},\{2\}\}$.

20. $\alpha(1) = \alpha_2; \alpha(2) = \alpha_1;\ \alpha(12) = 1$. Probability of such alpha's is $p_1p_2(1-p_1-p_2)$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_2},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_1};\ v_1(12) = \tilde{v}_1(12)\ and\ v_2(12) = \tilde{v}_2(12)$. The noise-free game preferences are unclear; it will depend on the relative values of α_1, α_2, and $\tilde{v}$. If α_1, α_2 and $\tilde{v}$'s are such that $\tilde{v}_1(12) > \frac{\tilde{v}_1(1)}{\alpha_2}$ then $\pi = N$, otherwise $\pi = \{\{1\},\{2\}\}$

21. $\alpha(1) = \alpha_1; \alpha(2) = \alpha_1;\ \alpha(12) = \alpha_2$. Probability of such alpha's is $p_1^2p_2$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_1},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_1};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_2}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_2}$. Since, $\frac{\tilde{v}_1(12)}{\alpha_2} > \tilde{v}_1(12) > \tilde{v}_1(1) > \frac{\tilde{v}_1(1)}{\alpha_1}$, and $\frac{\tilde{v}_2(12)}{\alpha_2} > \tilde{v}_2(12) > \tilde{v}_2(2) > \frac{\tilde{v}_2(2)}{\alpha_1}$. The noise-free game is: $v_1(12) > v_1(1);\ \ v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

22. $\alpha(1) = \alpha_1; \alpha(2) = \alpha_2;\ \alpha(12) = \alpha_1$. Probability of such alpha's is $p_1^2p_2$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_1},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_2};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_1}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_1}$. The noise-free game preferences are unclear; it will depend on the relative values of α_1, α_2, and $\tilde{v}$. If α_1, α_2 and $\tilde{v}$'s are such that $\frac{\tilde{v}_2(12)}{\alpha_1} > \frac{\tilde{v}_2(2)}{\alpha_2}$ then $\pi = N$ otherwise $\pi = \{\{1\},\{2\}\}$.

23. $\alpha(1) = \alpha_1; \alpha(2) = \alpha_2;\ \alpha(12) = \alpha_2$. Probability of such alpha's is $p_1p_2^2$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_1},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_2};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_2}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_2}$. Since, $\frac{\tilde{v}_1(12)}{\alpha_2} > \tilde{v}_1(12) > \tilde{v}_1(1) > \frac{\tilde{v}_1(1)}{\alpha_1}$. The noise-free game is: $v_1(12) > v_1(1)$; $v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

24. $\alpha(1) = \alpha_2; \alpha(2) = \alpha_1;\ \alpha(12) = \alpha_1$. Probability of such alpha's is $p_1^2p_2$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_2},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_1};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_1}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_1}$ Clearly, the preferences in the noise-free game are not clear; it will depend on the relative values of α_1, α_2 and $\tilde{v}$. If α_1, α_2 and $\tilde{v}$'s are such that $\frac{\tilde{v}_1(12)}{\alpha_1} > \frac{\tilde{v}_1(1)}{\alpha_2}$ then $\pi = N$ otherwise $\pi = \{\{1\},\{2\}\}$.

25. $\alpha(1) = \alpha_2; \alpha(2) = \alpha_1;\ \alpha(12) = \alpha_2$. Probability of such alpha's is $p_1p_2^2$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_2},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_1};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_2}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_2}$. Since, $\frac{\tilde{v}_2(12)}{\alpha_2} > \tilde{v}_2(12) > \tilde{v}_2(2) > \frac{\tilde{v}_2(2)}{\alpha_1}$. The noise-free game is: $v_1(12) > v_1(1)$; $v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

26. $\alpha(1) = \alpha_2; \alpha(2) = \alpha_2;\ \alpha(12) = \alpha_1$. Probability of such alpha's is $p_1p_2^2$. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_2},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_2};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_1}\ and\ v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_1}$. The noise-free game preferences are unclear; it will depend on the relative values α_1,

α_2 and $\tilde{v}$. If α_1, α_2 and $\tilde{v}$'s are such that $\frac{\tilde{v}_1(12)}{\alpha_1} > \frac{\tilde{v}_1(1)}{\alpha_2}$ and $\frac{\tilde{v}_2(12)}{\alpha_1} > \frac{\tilde{v}_2(2)}{\alpha_2}$ then $\pi = N$ otherwise $\pi = \{\{1\},\{2\}\}$.

27. $\alpha(1) = \alpha_2; \alpha(2) = \alpha_2;\ \alpha(12) = \alpha_2$. The probability of such alpha is p_2^3. Thus, the actual values are $v_1(1) = \frac{\tilde{v}_1(1)}{\alpha_2},\ \ v_2(2) = \frac{\tilde{v}_2(2)}{\alpha_2};\ v_1(12) = \frac{\tilde{v}_1(12)}{\alpha_2}$ *and* $v_2(12) = \frac{\tilde{v}_2(12)}{\alpha_2}$. The noise-free game is: $v_1(12) > v_1(1);\ \ v_2(12) > v_2(2)$. So, $\pi = \tilde{\pi}$ in this case.

Since $\overline{r} = \max\left\{\frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}, \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)}\right\}$, and $\underline{r} = \min\left\{\frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}, \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)}\right\}$. From above cases, we see that in 14 out of 27 cases (case 1,3,4,7,8,9,12,13,15,17,21,23,25,27) we have $\pi = \tilde{\pi} = N$ in noise-free game. In these cases, the relative value of $\tilde{v}_1(\cdot), \tilde{v}_2(\cdot)$ should satisfy $\alpha_1 \geq \overline{r}, \frac{1}{\alpha_2} \geq \overline{r}, \frac{\alpha_1}{\alpha_2} \geq \overline{r}$. The prediction probability in this case is given below as $g(p_1, p_2)$. Whereas if we allow for the cases, say $\alpha_1 < \underline{r}$; $\frac{1}{\alpha_2} < \underline{r}$; $\frac{\alpha_1}{\alpha_2} < \underline{r}$, then the prediction probability is 1. So, these are the two extreme cases. However, if we take any other range of α's, the prediction probability will be more than $g(p_1, p_2)$ and less than 1. Thus,

$$\mathbb{P}[\pi = \tilde{\pi} \mid game\ 1] = \begin{cases} g(p_1, p_2), & if\ \ \alpha_1 \geq \overline{r}\ ;\ \frac{1}{\alpha_2} \geq \overline{r}\ ;\ \frac{\alpha_1}{\alpha_2} \geq \overline{r} \\ 1, & if\ \ \alpha_1 < \underline{r}\ ;\ \frac{1}{\alpha_2} < \underline{r}\ ;\ \frac{\alpha_1}{\alpha_2} < \underline{r}, \end{cases} \tag{20}$$

where $g(p_1, p_2) = p_1^3 + p_2^3 + 2(p_1(1-p_1-p_2)^2 + p_2^2(1-p_1-p_2) + p_1p_2(1-p_1-p_2) + p_1p_2^2) + p_1^2p_2 + p_1^2(1-p_1-p_2) + p_2(1-p_1-p_2)^2 + (1-p_1-p_2)^3$. ■

5.2. Safety value via global minima for 2 agents and 3 support noise model

Here we will show that the above prediction probability given in Equation (20) can be non-convex in p_1, p_2. So, the global minima are difficult to hope for.

Note that $\frac{\partial g(p_1,p_2)}{\partial p_1} = 3p_1^2 - (p_2-1)^2$ and $\frac{\partial g(p_1,p_2)}{\partial p_2} = -2p_1(p_2-1) - 3p_2^2 + 6p_2 - 2$. Hence, we have $\frac{\partial^2 g(p_1,p_2)}{\partial^2 p_1} = 6p_1$, $\frac{\partial^2 g(p_1,p_2)}{\partial p_1 p_2} = \frac{\partial^2 g(p_1,p_2)}{\partial p_2 p_1} = -2(p_2-1)$, and $\frac{\partial^2 g(p_1,p_2)}{\partial p_2^2} = -2p_1 - 6p_2 + 6$. Thus, the Hessian of $g(p_1, p_2)$ is

$$H(g(p_1, p_2)) = \begin{bmatrix} 6p_1 & -2(p_2-1) \\ -2(p_2-1) & -2p_1 - 6p_2 + 6 \end{bmatrix}.$$

For $p_1 = 0.3$ and $p_2 = 0.5$, we have

$$H(g(p_1, p_2)) = \begin{bmatrix} 0.18 & 1 \\ 1 & 2.4 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 2.78$, and $\lambda_2 = -0.20$. So, $g(p_1, p_2)$ is not a convex function. Therefore, finding the global minima is difficult.

Though the above prediction probability is non-convex, one can get the noise set such that the prediction probability is more than a given satisfaction ζ. Similar to the 2 support cases, where the prediction probability was a convex function, but the noise regimes were disjoint intervals, in 3 support cases also, we get disjoint sets. However, computing the exact safety value is problematic because it is the global minima of the non-convex prediction probability function. Note that the safety value is a fundamental limit such that below a user-given satisfaction ζ, the partition is noise robust in the entire noise probability simplex.

As earlier, in the noise regimes where the prediction probability is more than ζ, a partition $\tilde{\pi}$ that is core-stable in a noisy game will remain core-stable in a noise-free game.