Designing Optimal Binary Rating Systems: Supplement

Nikhil Garg
Stanford University
nkgarg@stanford.edu

Ramesh Johari
Stanford University
rjohari@stanford.edu

February 18, 2019

Appendix A Mechanical Turk experiment, simulations, and results

In this section, we expand upon the results discussed in Section 5. We design and run an experiment that a real platform may run to design a rating system. We follow the general framework in Section 4. We first run an experiment to estimate a $\psi(\theta, y)$, the probability at which each item with quality θ receives a positive answer under different questions y. Then, we design $H(y)$, using our optimal β for various settings (different objectives w and matching rates g). Then, we simulate several markets (using the various matching rates g) and measure the performance of the different rating system designs H, as measured by various objective functions (2).

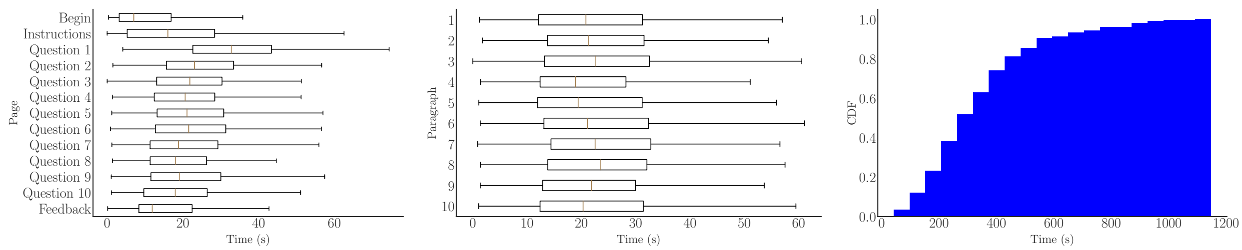
A.1 Experiment description

We now describe our Mechanical Turk experiment. We ask subjects to rate the English proficiency of ten paragraphs. These paragraphs are modified TOEFL (Test of English as a Foreign Language) essays with known scores as determined by experts (Educational Testing Service, 2005). Subjects were given six answer choices, drawn randomly from the following list: *Abysmal, Awful, Bad, Poor, Mediocre, Fair, Good, Great, Excellent, Phenomenal*, following the recommendation of Hicks et al. (2000). *Poor* and *Good* are always chosen, and the other four are sampled uniformly at random for each worker. One paragraph is shown per page; returning to modify a previous answer is not allowed; and paragraphs are presented in a random order. This data is used to calibrate a model of ψ for optimization, i.e. to simulate a system with a set of questions $\mathcal{Y}$, where each question y corresponds to a adjective, "Would you characterize the performance of this item as [adjective] or better?".[1]

Different experiment trials are described below. Pilots were primarily used to garner feedback regarding the experiment from workers (fair pay, time needed to complete, website/UI comments, etc). All trials yield qualitatively similar results in terms of both paragraph ratings and feedback rating distributions for various scales.

Pilot 1 30 workers. Similar conditions as final experiment (6 words sampled for paragraph ratings, all uniformly at random, 5 point scale feedback rating), with identical question phrasing, "How does the following rate on English proficiency and argument coherence?".

[1] The data from the experiment is also used for a separate paper, Garg and Johari (2018). In that work, we analyze the full multi-option question directly, but the main focus is reporting the results of a separate, live trial on a large online labor platform.



(a) Time spent per page.

(b) Time spent per paragraph.

(c) Distribution (CDF) of total time spent.

Figure 1: Additional information for MTurk experiment

Pilot 2 30 workers. 7 words sampled for paragraph ratings, 6 point scale feedback rating, with the following question phrasing: "How does the following person rate on English proficiency and argument coherence?".

Experiment 200 workers. 6 words sampled for paragraph ratings, with 2 fixed as described above, 5 point scale feedback rating. Question phrasing, "How does the following rate on English proficiency and argument coherence?".

We use paragraphs modified from a set published by the Educational Testing Service (Educational Testing Service, 2005). There are 10 paragraphs, 5 each on 2 different topics. For each topic, the paragraphs have 5 distinct expert scores. Paragraphs are shortened to just a few sentences, and the top rated paragraphs are improved and the worst ones are made worse, preserving the ranking according to the expert scores.

Figure 1a shows time spent on each page of the experiment, Figure 1b shows the time spent per paragraph, and Figure 1c shows the cumulative density function for time spent by workers. The paragraphs are presented to workers in a random order. No workers are excluded in our data and all workers were paid \$1.00, including the ones that spent 2-3 seconds per page. 7/60 workers in the pilots received a bonus of \$0.20 for providing feedback. The instructions advised workers to spend no more than a minute per question, though this was not enforced.

The instructions for the main experiment were as follows: "Please rate on English proficiency (grammar, spelling, sentence structure) and coherence of the argument, but not on whether you agree with the substance of the text." No additional context was provided.

A.2 Calculating optimal β and H

Figure 2 shows the empirical $\hat{\psi}(\theta, y)$ as measured through our experiment. The colors encode the true quality as rated by experts (light blue is best quality, dark blue is worst); recall there are 10 paragraphs with 5 distinct expert ratings (paragraphs 0 and 5 are rated the best, paragraphs 4 and 9 are rated the worst).

With the β calculated and visualized using the methods in Section 3, we now find the optimal H for various settings using the methods in Section 4. We view our set of paragraphs as *representative items* Θ from a larger universe of paragraphs. In particular, we view our worst quality paragraphs as in the 10th percentile of paragraphs, and our best items as in the 90th percentile. In other words, from the empirical $\hat{\psi}$, we carry out the methods in Section 4 using a ψ s.t. $\psi(.1, y) = (\hat{\psi}(4, y) + \hat{\psi}(9, y)/2$ (and similarly for $\psi(.3, y), \psi(.5, y), \psi(.7, y), \psi(.9, y)$, where e.g. $\hat{\psi}(4, y)$ is the empirical rate at which paragraph 4 received a positive rating on question y.

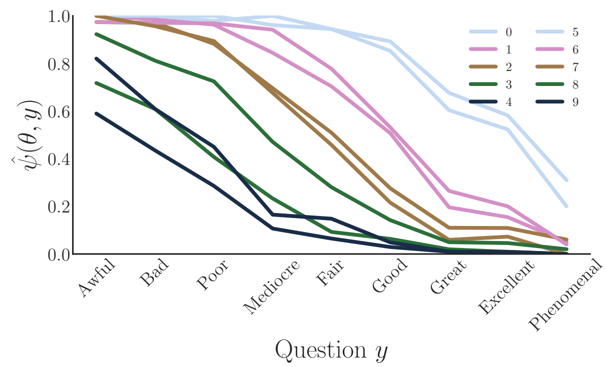
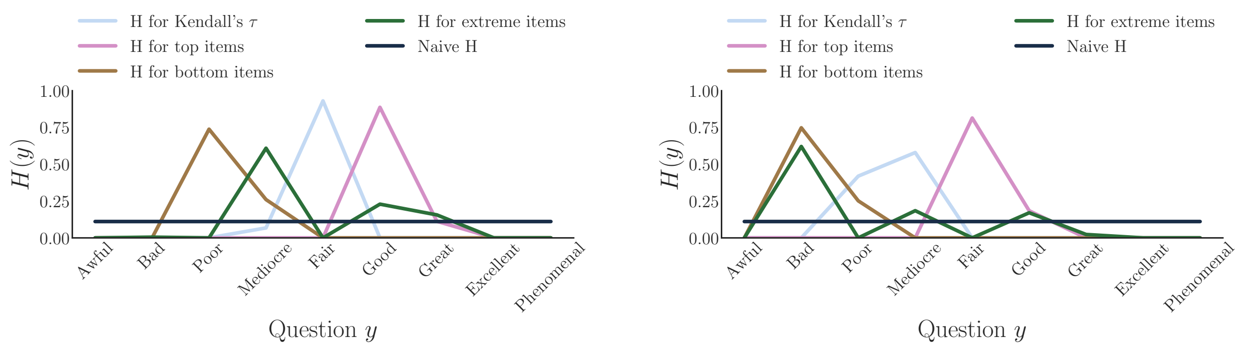


Figure 2: Paragraph rating distribution – for paragraph θ and rating word y, the empirical $\hat{\psi}(\theta, y)$ is shown. Colors encode the true quality as rated by experts (light blue is best quality, dark blue is worst).



(a) Uniform search, $g(\theta) = 1$.

(b) Linearly increasing search, $g(\theta) = \frac{1+10\theta}{11}$.

Figure 3: Optimal $H(y)$ varying by $w(\theta_1, \theta_2)$ using Mechanical Turk data

Then, we solve the optimization problem for H stated in Section 4. From the above discussion, we want to find an H such that the worst rated paragraphs in our experiment have a probability of receiving a positive rating that is approximately $\beta(.1)$.

Figure 3 shows the optimal H calculated for various platform settings. These distributions illustrate how often certain binary questions should be asked as it depends on the matching rates and platform objective. For example, as Figure 3a shows, when there is uniform matching and the platform cares about the entire ranking (i.e. has Kendall's τ or Spearman's ρ objective), it should ask most buyers to answer the question, "Would you rate this item as having 'Fair' quality or better?".

Several qualitative insights can be drawn from the optimal H. Most importantly, note that the optimal designs vary significantly with the platform objective and matching rates. In other words, given the same empirical data $\hat{\psi}$, the platform's design changes substantially based on its goals and how skewed matches are on the platform. Further, note that the differences in H follow from the differences in β that are illustrated in Figure 1: when the platform wants to accurately rank the best items, the questions that distinguish amongst the best (e.g., "Would you rate this item as having 'Good' quality?") are drawn more often.

A.3 Simulation description

Using the above data and subsequent designs, we simulate markets with a binary rating system as described in Section 3.1. Our simulations have the following characteristics.

- 500 items. Items have i.i.d. quality in $[0, 1]$. For item with quality θ, we model buyer rating data using the ψ collected from the experiment as follows. In particular, we presume the items are convex combinations of the representative items in our experiment – items with quality $\theta \in [.1, .3]$ are assumed to have rating probabilities $\psi(\theta, y) = \alpha\psi(.1, y) + (1 - \alpha)\psi(.3, y)$, where $\alpha = (\theta - .1)/.2$. Similarly for θ in other intervals. This process yields the $\tilde{\beta}$ shown in Figure 2b.

- In some simulations, all items enter the market at time $k = 0$ and do not leave. In the others, with *entry and exit*, each item independently leaves the market with probability .02 at the end of each time period, and a new item with quality drawn i.i.d. from $[0, 1]$ enters.

- There are 100 buyers, each of which matches to an item independently. In other words, matching is independent across items, and items can match more than once per time period.

- Matching is random with probability as a function of an item's *estimated* rank $\hat{\theta}$ according to score, rather than actual rank. In other words, the optimal systems were designed assuming item θ would match at rate $g(\theta)$; instead it matches according to $g(\hat{\theta})$, where $\hat{\theta}$ is the item's rank according to score. We use both $g = 1$ and *linear* search, $g(\hat{\theta}) = \frac{1+10\hat{\theta}}{11}$.

- $\mathcal{Y}$ is the set of 9 adjectives from our MTurk experiments.

- We test several possible H: *naive* with $H(y) = \frac{1}{|\mathcal{Y}|}$, and then the various optimal H calculated for the different sections, illustrated in Figure 3.

A.3.1 Simulation results

Figure 4 contains plots from a simulated system that has binary ratings. Figures 4a, 4b are with uniform search ($g = 1$), Figures 4c and 4d plot the objective prioritizing the worst items, and Figures 4e and 4f are with linearly increasing search. For each setting, we include both plots with and without birth/death.

Together, the results suggest that the asymptotic and rate-wise optimality of our calculated β hold even under deviations of the model, and that the real-world design approach outlined in Section 4 would provide substantial information benefits to platforms.

Several specific qualitative insights can be drawn from the figures, alongside those discussed in the main text.

1. From all the plots with uniform search, the H designed using our methods for the given setting outperforms other H designs, as expected, and the optimal β (for the given setting) significantly outperforms other designs both *asymptotically* and *rate-wise*.

2. Qualitatively, again with uniform search, heterogeneous item age also does not affect the results. In fact, it seems as if the optimal β and best possible H (given the data) as calculated from our methods outperforms other designs both *asymptotically* and *rate-wise*. Note that this is true even though items entering and leaving the market means that the system may not enter the asymptotics under which our theoretical results hold.

3. Figures 4c and 4d show the same system parameters as Figures 4a, 4b, i.e. uniform search. However, while 4a, 4b show Kendall's τ correlation over time, 4c and 4d show the objective prioritizing bottom items ($w = (1-\theta_1)(1-\theta_2)(\theta_1-\theta_2)$). Note that the β calculated for the actual objective outperforms that calculated for Kendall's τ, including asymptotically.

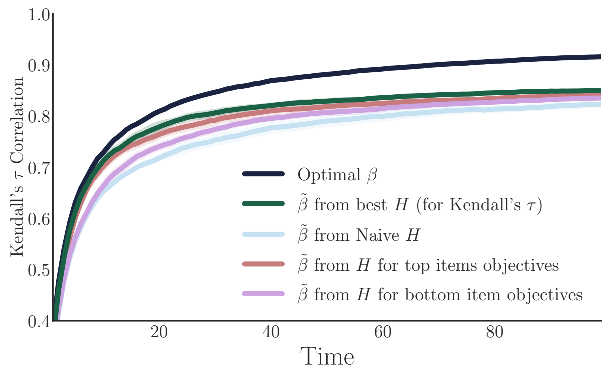
   Similarly, complementing the fact that H design changes significantly with the weight function, these plots show the value of designing while taking into account one's true objective value – the different designs perform differently. Mis-specifying one's objective (e.g. designing to differentiate the best items when one truly cares about the worst items) leads to a large gap in performance (e.g. see the gap between the dark green and red lines in 4c and 4d).

   Note that comparing the performance of β for the misspecified objective and H for the true objective is not a fair comparison: the former differentiates between all items (though potentially not in a rate-optimal way), while H is constrained by reality, i.e. ψ and $\mathcal{Y}$.

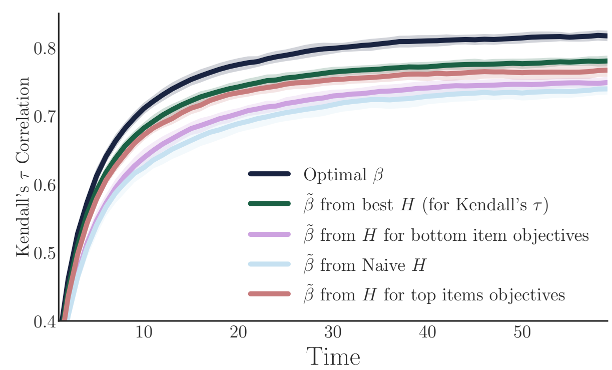
4. Now, consider Figures 4e and 4f, which plot the system with linearly increasing search. Note that, contrary to expectation, the optimal β for *uniform* search outperforms the β for the actual system simulated, with linear search! This pattern is especially true for small time k and with item birth/death.

   This inversion can be explained as follows. *Uniformization* occurs with heterogeneous age and matching according to observed quality: new items of high type are likely to be mis-ranked lower, while new items of low type are more likely to be mis-ranked higher. (We note that this may not matter in practice, where the search function itself is fit through data, which already captures this effect.) These errors are prominent at low time k and with item birth/death, i.e., in the latter our system never reaches the asymptotics at which the linear β is the optimal design.

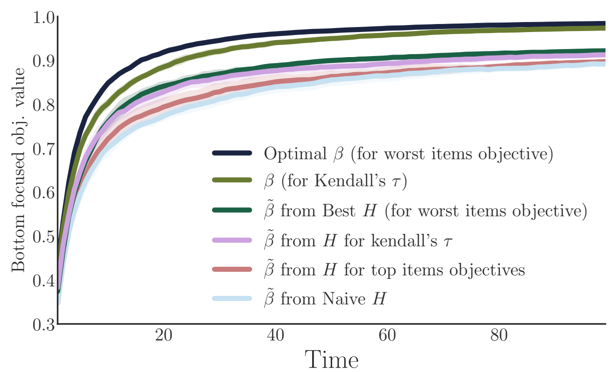
   This pattern can be seen more clearly by comparing the two β curves in Figures 4e, without item birth/death. At small k, when errors are common and so search is more effectively uniform, the β for uniform matching performs the best. However, as such errors subside over time, the performance of the β for linear search catches up and eventually surpasses that of uniform optimal β.



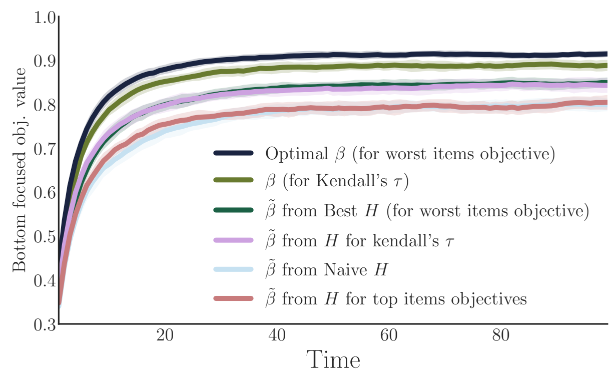
(a) Uniform matching, no birth/death



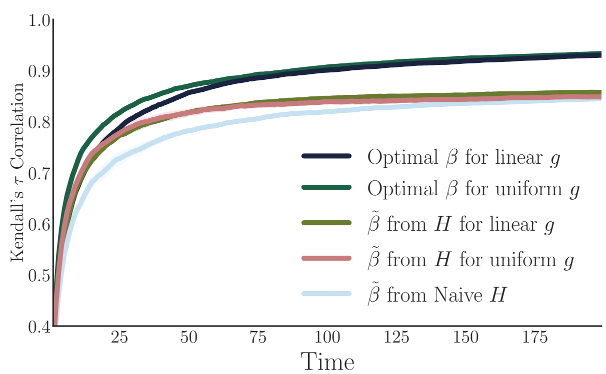
(b) Uniform matching, 2% probability of death per time



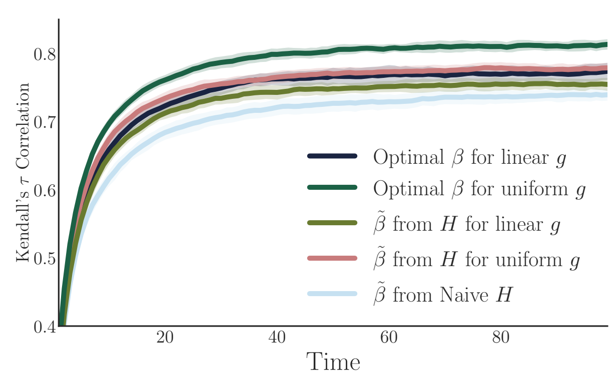
(c) Uniform matching, no birth/death, worst items weighted objective



(d) Uniform matching, 2% birth/death, worst items weighted objective



(e) Linear matching, no birth/death



(f) Linear matching, 2% probability of death per time

Figure 4: Simulations from data from Mechanical Turk experiment – Binary rating system

Appendix B Supplementary theoretical information and results

We now give some additional detail and develop additional results. Section B.1 contains the formal specification and update of our deterministic dynamical system. Section B.2 gives our algorithm, Nested Bisection, is far more detailed pseudo-code. Section B.3 formalizes our earlier qualitative discussion on how matching rates affects the function β. Section B.4 includes a convergence result for functions β_M as M increases. Finally, Section B.5 contains simple results on how one can learn $\psi(\theta, y)$ through experiments, even if one does not have a reference set of items Θ with known quality before one begins experiments.

B.1 Formal specification of system state update

Recall that $\mu_k(\Theta, X)$ is the mass of items with true quality $\theta \in \Theta \subseteq [0,1]$ and a reputation score $x \in X \subseteq [0,1]$ at time k. Let $E_k = \{\theta : n_k(\theta) = n_{k-1}(\theta) + 1\}$. These are the items who receive an additional rating at time k; for all $\theta \in E_k^c$, $n_k(\theta) = n_{k-1}(\theta)$. Our system is completely deterministic, and evolves according to the distributions of the individual seller dynamics.

For each $\theta \in E_k, x, x'$, define $\omega(\theta, x, x')$ as follows:

$$\omega(\theta, x, x') = \beta(\theta)\mathbb{I}\{n_k(\theta)x - n_{k-1}(\theta)x' = 1\} + (1 - \beta(\theta))\mathbb{I}\{n_k(\theta)x - n_{k-1}(\theta)x' = 0\}.$$

Then ω gives the probability of transition from x' to x when an item receives a rating. We then have:

$$\mu_{k+1}(\Theta, X) = \int_{E_k} \int_{x'=0}^{1} \int_{x \in X} \omega(\theta, x, x') dx \mu_k(dx', d\theta) + \int_{E_k^c} \int_{x \in X} \mu_k(dx, d\theta).$$

It is straightforward but tedious to check that the preceding dynamics are well defined, given our primitives.

B.2 Detailed algorithm

Here, we present the Nested Bisection algorithm, which is described at a high level and summarized in pseudo-code in the main text, in more detail.

ALGORITHM 1: Nested Bisection given in more detail

Data: Set size M, grid width δ, match function g /* Assume $\delta << \min_i t_i - t_{i-1}$ */

Result: β_M levels $\{t_0 \dots t_{M-1}\}$

Function *main* (M, δ, g)

- $t_0 = 0, t_{M-1} = 1$
- $\ell = 1 - \frac{1}{M-1}, u = 1 - \delta$
- **while** $u - \ell > \delta/2$ **do**
  - $j_{M-2} = \frac{r+\ell}{2}$
  - $\mathrm{rate_{last}} = -g_{M-2} \log(t_{M-2})$
  - $\{j_1 \dots j_{M-3}\} = \mathit{CalculateOtherLevels}(j_{M-2}, \mathrm{rate_{last}}, g)$
  - $\mathrm{rate_{first}} = -g_1 \log(1 - t_1)$
  - **if** $\mathit{rate}_{\mathit{first}} < \mathit{rate}_{\mathit{last}}$ **then** $\ell = j_{M-2}$
  - **else** $u = j_{M-2}$
- $\{t_1 \dots t_{M-2}\} = \mathit{CalculateOtherLevels}(u, g)$
- $t_{M-2} = u$
- return $\{t_i\}$

Function *PairwiseRate* $(t_{m-1}, t_m, g_m, g_{m-1})$

- return $-(g_{m-1} + g_m) \log \left[(1 - t_{m-1})^{\frac{g_{m-1}}{g_{m-1}+g_m}} (1 - t_m)^{\frac{g_m}{g_{m-1}+g_m}} + t_{m-1}^{\frac{g_{m-1}}{g_{m-1}+g_m}} t_m^{\frac{g_m}{g_{m-1}+g_m}} \right]$

Function *CalculateOtherLevels* $(j_{M-2}, \mathit{rate}_{\mathit{target}}, g)$

- /* Given target rate from current guess j_{M-2}, sequentially fix other levels. */
- **foreach** $m \in M - 3 \dots 1$ **do**
  - $j_m = \mathit{BisectNextLevel}(j_{m+1}, \mathrm{rate_{target}}, g_m, g_{m+1})$
- return $\{j_1 \dots j_{M-3}\}$

Function *BisectNextLevel* $(j_m, \mathit{rate}_{\mathit{target}}, g_{m-1}, g_m)$

- $\ell = 0, r = j_m - \delta$
- **while** $r - \ell > \delta/2$ **do**
  - $j_{m-1} = \frac{r+\ell}{2}$
  - $\mathrm{rate_m} = \mathit{PairwiseRate}(j_{m-1}, j_m, g_{m-1}, g_m)$
  - **if** $\mathit{rate}_m \le \mathit{rate}_{\mathit{target}}$ **then** $r = j_{m-1}$
  - **else** $\ell = j_{m-1}$
- return r

B.3 Formalization of effect of matching rates shifting

Matching concentrating at the top items moves mass of $\beta(\theta)$ away from high θ, and subsequently mass of $H(y)$ *away* from the questions that help distinguish the top items, as observed in Figures 1b and 3b above. Informally, this occurs because when matching concentrates, top items are accumulating many ratings more ratings comparatively, and so the amount of information needed per rating is comparatively less. We formalize this intuition in Lemma B.1 below.

The lemma states that if matching rates shift such that there is an index k above which matching rates increase and below which they decrease, then correspondingly the levels of β, (i.e. t_i) become closer together above k.

Lemma B.1. *Suppose $k, g, \tilde{g}$ such that $\forall j \in \{k+1 \dots M-1\}, g_j \geq \tilde{g}_j$, and $\forall j \in \{0 \dots k-1\}, g_j \leq \tilde{g}_j$, and $g_k = \tilde{g}_k$. Then, $t^*_k \geq \tilde{t}^*_k$.*

Proof. This proof is similar to that of Lemma 3.1, except that with the matching function changing the overall rate function can either increase, decrease, or stay the same. Suppose the overall rate function decreased or stayed the same when the matching function changed from $\tilde{g}$ to g. Then $g_{M-2} > \tilde{g}_{M-2}$ and the target rate is no larger, and so $t^*_{M-2} > \tilde{t}^*_{M-2}$. Then, $t^*_{M-3} > \tilde{t}^*_{M-3}$ (a smaller width is needed because the matching rates are higher and the rate is no larger, and the next value also increased). This shifting continues until $t^*_{k+1} > \tilde{t}^*_{k+1}$. Then, $t^*_k > \tilde{t}^*_k$.

Suppose instead that the overall rate function increased when the matching function changed from $\tilde{g}$ to g. Then $g_1 < \tilde{g}_1$ and the target rate is larger, and so $t^*_1 > \tilde{t}^*_1$. Then, $t^*_2 > \tilde{t}^*_2$ (a larger width is needed and the previous value also increased). This shifting continues until $t^*_{k-1} > \tilde{t}^*_{k-1}$. Then, $t^*_k > \tilde{t}^*_k$. $\square$

B.4 Limit of β as $M \to \infty$

Let β^w_M denote the optimal β with M intervals for weight function w, with intervals $\{S^{wM}_i\} = \{[s^{wM}_i, s^{wM}_{i+1})\}$ and levels $\boldsymbol{t^{wM}}$. Let $q_{wM}(\theta) = i/M$ when $\theta \in [s^{wM}_i, s^{wM}_{i+1})$, i.e. the *quantile* of interval item of type θ is in.

Then, we have the following convergence result for β_M.

Theorem B.1. *Let g be uniform. Suppose w such that q_{wM} converges uniformly. Then, $\forall C \in \mathbb{N}, \exists \beta^w$ s.t. $\beta^w_{C2^N+1} \to \beta^w$ uniformly as $N \to \infty$.*

The proof is technical and is below. We leverage the fact that, for g uniform, the levels of β_{2M} can be analytically written as a function of the levels of β_M. We believe (numerically observe) that this theorem holds for the entire sequence as opposed to the each such subsequence, and for general matching functions g. However, our proof technique does not carry over, and the proof would leverage more global properties of the optimal β_M.

Furthermore, the condition on w is light. For example, it holds for Kendall's τ, Spearman's ρ, and all other examples mentioned in this work.

This convergence result suggests that the choice of M when calculating a asymptotic and rate optimal β is not consequential. As M increases, the limiting value of W_k increases to 1 (i.e. the asymptotic value increases), but the optimal rate decreases to 0. As discussed above, with strictly increasing and continuous β, the asymptotic value is 1 but the large deviations rate does not exist, i.e. convergence is polynomial.

This result could potentially be strengthened as follows: first, show convergence on the entire sequence as opposed to these exponential subsequences, as conjectured; second, show desirable properties of the limiting function itself. It is conceivable but not necessarily true that the limiting function is “better” than other strictly continuous increasing functions in some rate sense, even though the comparison through large deviations rate is degenerate.

B.5 Learning $\psi(\theta, y)$ through experiments

Now, we show how a platform would run an experiment to decide to learn $\psi(\theta, y)$. In particular, one potential issue is that the platform does not have any items with know quality that it can use as representative items in its optimization. In this case, we show that it can use ratings within the experiment itself to identify these representative items. The results essentially follow from the law of large numbers.

We assume that $|\Theta| = L$ representative items $i \in \{1 \dots L\}$ are in the experiment, and each are matched N times. The experiment proceeds as follows: every time an item is matched, show the buyer a random question from $\mathcal{Y}$. For each word $y \in \mathcal{Y}$, track the empirical $\hat{\psi}(i, y)$, the proportion of times a positive response was given to question y. Alternatively, if $\mathcal{Y}$ is totally ordered (i.e. a positive rating for a given y also implies positive ratings would be given to all “easier” y'), and can be phrased as a multiple choice question, data collection can be faster: e.g., as we do in our experiments: $\mathcal{Y}$ consists of a set of totally ordered adjectives that can describe the item; the rater is asked to pick an adjective out of the set; this is interpreted as the item receiving a positive answer to the questions induced by the chosen answer and all worse adjectives, and a negative answer to all better adjectives.

First, suppose the platform approximately knows the quality θ_i of each item i, and θ_i are evenly distributed in $[0, 1]$. Suppose the items are ordered by index, i.e. $\theta_1 < \theta_2 < \dots < \theta_L$. Then let $\hat{\psi}(\theta, y) = \hat{\psi}(i, y)$ when $\theta \in [\theta_{i-1}, \theta_i]$. Call this procedure *KnownTypeExperiment.*

Lemma B.2. *Suppose $\psi(\theta, y)$ is Lipschitz continuous in θ. With KnownTypeExperiment, $\hat{\psi}(i, y) \to \psi(\theta_i, y) \forall y$ uniformly as $N \to \infty$. As $L \to \infty$, $\hat{\psi}(\theta, y) \to \psi(\theta, y) \forall \theta$ uniformly.*

Proof. The proof follows directly from the Strong Law of Large Numbers. As $N \to \infty$, $\forall i$, $\hat{\psi}(i, x) \to \psi(\theta_i, x)$ uniformly. Now, let $L \to \infty$. $\forall \epsilon, \exists L'$ s.t. $\forall L > L'$, $\forall \theta, \exists i$ s.t. $|\theta - \theta_i| < \epsilon$. $\psi(\theta, x)$ is Lipschitz in θ by assumption, and so $\hat{\psi}(\theta, x) \to \psi(\theta, x)$ uniformly. □

We now relax the assumption that the platform has an existing set of items with known qualities. Suppose instead the platform has many items L of unknown quality who are expected to match N times each over the experiment time period. For each item, the platform would again ask questions from $\mathcal{Y}$, drawn according to any distribution (with positive mass on each question). Then generate $\hat{\psi}(\theta, y)$ as follows: first, rank the items according to their ratings during the experiment itself. Then, for each y, $\hat{\psi}(\theta, y)$ is the empirical performance of the θth percentile item in the ranking, i.e. $\hat{\psi}(\theta, y) = \hat{\psi}(\theta_i, y)$ for $\theta \in \left[\frac{i-1}{L}, \frac{i}{L}\right]$. Call this procedure *UnknownTypeExperiment.*

Lemma B.3. *Suppose $\psi(\theta, y)$ is Lipschitz continuous in θ. With UnknownTypeExperiment, $\hat{\psi}(\theta, y) \to \psi(\theta, y) \forall y, \theta$ uniformly as $L, N \to \infty$.*

Proof. Fix L. Denote each item in the experiment as $i \in \{1 \dots L\}$ (with true quality $\theta_i \neq \theta_j$), and each item has N samples. Without loss of generality, assume the items are indexed according to their rank on the average of their scores on the samples, defined as the percentage of positive ratings received. $i = 1$ is then the worst item, and $i = L$ is the best item according to scores in the experiment.

For $\psi(\theta, x)$ increasing in θ, as $N \to \infty$, $Pr(\theta_i > \theta_j | i < j) \to 0$ almost surely by SLLN, and for a fixed L, $\{\theta_i\}$ this convergence is uniform. Furthermore, by SLLN, $\hat{\psi}(i, x) \to \psi(\theta_i, x)$ as $N \to \infty$. Recall $\hat{\psi}(\theta, x) = \hat{\psi}(i, x)$ for $\theta \in \left[\frac{i-1}{L}, \frac{i}{L}\right]$.

Now, let $L \to \infty$. $\forall \epsilon, \exists L'$ s.t. $\forall L > L'$, $\forall \theta, \exists i$ s.t. $|\theta - \theta_i| < \epsilon$. $\psi(\theta, x)$ is Lipschitz in θ by assumption, and so $\hat{\psi}(\theta, x) \to \psi(\theta, x)$ uniformly. $\square$

Appendix C Proofs

In this Appendix section, we prove our results.

Sections C.1-C.3 develop rate functions for P_k and W_k. While rates for P_k follow immediately from large deviation results, the rate function for W_k requires more effort as the quantity is an integral over a continuum of (θ_1, θ_2), each of which has a rate corresponding to that of $P_k(\theta_1, \theta_2)$.

Then in Section C.4 we prove Theorem 3.1 and Lemma 3.1.

Section C.5 then contains additional necessary lemmas required for the proof of the algorithm and convergence result, Theorem B.1. The main difficulty for the former is showing a Lipschitz constant in the resulting rate if a level t_i is shifted, which requires lower and upper bounds for t_1 and t_{M-2}, respectively. For the former, we need to relate the solutions of the sequence of optimization problems used to find β_M as M increases. It turns out that both properties follow by relating the levels of β_M to those of β_{2M-1}.

These additional lemmas are used to prove the algorithm approximation bound (Theorem 3.2) and the convergence result (Theorem B.1) in Section C.6 and C.7, respectively.

Finally in Section C.8 we prove the comments we make in the main text about Kendall's τ and Spearman's ρ rank correlations belonging in our class of objective functions, with asymptotic values of W_k maximized when $\boldsymbol{s}$ is equispaced in $[0, 1]$.

C.1 Rate functions for $P_k(\theta_1, \theta_2)$

Lemma C.1.

$$\lim_{k\to\infty} -\frac{1}{k} \log \left[\mu((x_k(\theta_1) - x_k(\theta_2)) \le 0 | \theta_1, \theta_2)\right] = \inf_{a\in\mathbb{R}} \{g(\theta_1) I(a|\theta_1) + g(\theta_2) I(a|\theta_2)\}$$

where $I(a|\ell) = \sup_z\{za - \Lambda(z|\theta)\}$, and $\Lambda(z|\theta)$ is the log moment generating function of a single sample from $x(\theta_1)$ and $g(\theta)$ is the sampling rate.

Proof. $\lim_{k\to\infty} -\frac{1}{k} \log \left[\mu((x_k(\theta_1) - x_k(\theta_2)) \le 0 | \theta_1, \theta_2)\right]$

$$= \lim_{k\to\infty} -\frac{1}{k} \log \left[\int_{a\in\mathbb{R}} \mu((x_k(\theta_1) = a|\theta_1) \mu(x_k(\theta_2) \ge a|\theta_2) da\right] \tag{1}$$

$$= \lim_{k\to\infty} -\frac{1}{k} \log \left[\int_{a\in\mathbb{R}} e^{-kg(\theta_1)I(a|\theta_1)} e^{-kg(\theta_2)I(a|\theta_2)} da\right] \tag{2}$$

$$= \inf_{a\in\mathbb{R}} \{g(\theta_1) I(a|\theta_1) + g(\theta_2) I(a|\theta_2)\} \qquad \text{Laplace principle} \tag{3}$$

$\square$

Where (2) is a basic result from large deviations, where $kg(\theta_i)$ is the number of samples item of quality θ_i has received.

Note that this lemma also appears in Glynn and Juneja (2004), which uses the Gartner-Ellis Theorem in the proof. Our proof is conceptually similar but instead uses Laplace's principle.

Recall that $\mathrm{KL}(a||b) = a \log \frac{b}{a} + (1-a) \log \frac{1-b}{1-a}$ is the Kullback-Leibler (KL) divergence between Bernoulli random variables with success probabilities a and b respectively. It is well known that for a Bernoulli random variable with success probability t,

$$I(a|t) = \mathrm{KL}(a||t)$$

Then, we have

Lemma C.2. *Let* $\theta_1 > \theta_2$ *and* $I(a|\theta) = KL(a||\beta(\theta))$*. Further, Let* $\overline{P}_k(\theta_1, \theta_2) = 1 - P_k(\theta_1, \theta_2)$*. Then,*

$$-\lim_{k\to\infty} \frac{1}{k} \log \overline{P}_k(\theta_1, \theta_2) = \inf_{a\in\mathbb{R}} \{g(\theta_1)I(a|\theta_1) + g(\theta_2)I(a|\theta_2)\}, \tag{4}$$

Proof. Follows directly from Lemma C.1.

$$\begin{aligned}
&-\lim_{k\to\infty} \frac{1}{k} \log \overline{P}_k(\theta_1, \theta_2|\beta) \\
&\quad= \lim_{k\to\infty} -\frac{1}{k} \log \left[1 + \mu_k(x_k(\theta_1) - x_k(\theta_2) < 0|\theta_1, \theta_2) - \mu_k(x_k(\theta_1) - x_k(\theta_2) > 0|\theta_1, \theta_2)\right] \\
&\quad= \lim_{k\to\infty} -\frac{1}{k} \log \left[2\mu_k(x_k(\theta_1) - x_k(\theta_2) < 0|\theta_1, \theta_2) + \mu_k(x_k(\theta_1) - x_k(\theta_2) = 0|\theta_1, \theta_2)\right] \\
&\quad= \lim_{k\to\infty} -\frac{1}{k} \log \left[\mu_k(x_k(\theta_1) - x_k(\theta_2) \le 0|\theta_1, \theta_2)\right] \\
&\quad= \inf_{a\in\mathbb{R}} \{g(\theta_1)I(a|\theta_1) + g(\theta_2)I(a|\theta_2)\} \qquad \text{Lemma } C.1
\end{aligned}$$

□

C.2 Laplace's principle with sequence of rate functions

In order to derive a rate function for $\overline{W}_k = (\lim_k W_k) - W_k$, we need to be able to relate its rate to that of $\overline{P}_k(\theta_1, \theta_2)$. The following theorem, related to Laplace's principle for large deviations allows us to do so.

Theorem C.1. *Suppose that* X *is compact with finite Lebesgue measure* $\mu(X) < \infty$*. Suppose that* $\varphi(x)$ *has an essential infimum* $\underline{\varphi}$ *on* X*, that* $\varphi_n(x)$ *has an essential infimum* $\underline{\varphi}_n$*, that both* φ *and all* φ_n *are nonnegative, and that* $\varphi_n \to \varphi$ *uniformly:*

$$\lim_{n\to\infty} \sup_{x\in X} |\varphi_n(x) - \varphi(x)| = 0.$$

Then:

$$\lim_{n\to\infty} \frac{1}{n} \log \int_X e^{-n\varphi_n(x)} dx = -\underline{\varphi}. \tag{5}$$

Proof. First, we note that for all x and n, $e^{-n\varphi_n(x)} \le e^{-n\underline{\varphi}_n}$. Therefore, letting $(*)$ denote the LHS of (5), we have:

$$(*) \le \lim_{n\to\infty} \frac{1}{n} \log \int_X e^{-n\underline{\varphi}_n} dx = -\underline{\varphi},$$

where the last limit follows from the fact that φ_n converges uniformly to φ, so that $\underline{\varphi}_n \to \underline{\varphi}$.

Next, for $\epsilon > 0$ let $A_n(\epsilon) = \{x : \varphi_n(x) \le \underline{\varphi}_n + \epsilon\}$ and let $A(\epsilon) = \{x : \varphi(x) \le \underline{\varphi} + \epsilon\}$. It follows (again by uniform convergence) that for all sufficiently large n, $A(\epsilon/2) \subseteq A_n(\epsilon)$, so that $\mu(A(\epsilon/2)) \le \mu(A_n(\epsilon))$ for all sufficiently large n. Further, $\mu(A(\epsilon/2)) > 0$, since $\underline{\varphi}$ is the essential infimum.

Since:

$$\int_X e^{-n\varphi_n(x)} dx \ge \mu(A_n(\epsilon)) e^{-n(\underline{\varphi}_n + \epsilon)},$$

it follows that:

$$(*) \geq -\underline{\varphi} - \epsilon + \lim_{n\to\infty} \frac{1}{n} \log \mu(A_n(\epsilon)).$$

To complete the proof, observe that since $\mu(A_n(\epsilon))$ is bounded below by a positive constant for all sufficiently large n, the last limit is zero. Therefore:

$$(*) \geq -\underline{\varphi} - \epsilon.$$

Since ϵ was arbitrary, this completes the proof. □

Remark C.1. *Let $X = [0,1] \times [0,1]$, $\varphi_n(\theta_1, \theta_2) = -\frac{1}{n} \log \overline{P}_n(\theta_1, \theta_2)$. Then, all the conditions for Theorem C.1 are met.*

C.3 Rate function for W_k

Our next lemma shows that we can obtain a nontrivial large deviations rate for W_k when β is a step-wise increasing function.

Recall $W_k = \int_{\theta_1 > \theta_2} w(\theta_1, \theta_2) P_k(\theta_1, \theta_2|\beta) d(\theta_1, \theta_2)$.

Let $\overline{P}_k(\theta_1, \theta_2) = 1 - P_k(\theta_1, \theta_2)$.

Further, let $\overline{W}_k = (\lim_k W_k) - W_k = \int_{\theta_1 > \theta_2} w(\theta_1, \theta_2) \overline{P}_k(\theta_1, \theta_2|\beta) d(\theta_1, \theta_2)$. (recall we assumed w integrates to 1 without loss of generality).

Lemma C.3. *Suppose β is piecewise constant with M levels $\{t_i\}$. Let $g_i \triangleq \inf_{\theta \in S_i} g(\theta) = g(s_i)$ Then,*

$$-\lim_{k\to\infty} \frac{1}{k} \log \overline{W}_k = \min_{0 \leq i \leq M-2} \inf_{a \in \mathbb{R}} \{g_{i+1} I(a|t_{i+1}) + g_i I(a|t_i)\} \triangleq r, \tag{6}$$

where $I(a|t) = KL(a||t)$ as defined in Lemma C.2.

Proof. When β is step-wise increasing with M levels $\{t_i\}$, then

$$\overline{W}_k = \sum_{0 \leq i < j < M} \int_{\theta_2 \in S_i, \theta_1 \in S_j} w(\theta_1, \theta_2) \overline{P}_k(\theta_1, \theta_2|\beta) d(\theta_1, \theta_2)$$

as $\overline{P}_k(\theta_1, \theta_2) = 0$ when $\beta(\theta_1) = \beta(\theta_2)$.

$-\lim_{k\to\infty}\frac{1}{k}\log\overline{W}_k$

$$= -\lim_{k\to\infty}\frac{1}{k}\log\int_{\theta_1>\theta_2} w(\theta_1,\theta_2)\overline{P}_k(\theta_1,\theta_2|\beta)d(\theta_1,\theta_2) \tag{7}$$

$$= -\lim_{k\to\infty}\frac{1}{k}\log\sum_{0\leq i<j<M}\int_{\theta_2\in S_i,\theta_1\in S_j} w(\theta_1,\theta_2)\overline{P}_k(\theta_1,\theta_2|\beta)d(\theta_1,\theta_2) \tag{8}$$

$$= -\max_{0\leq i<j<M}\left(\lim_{k\to\infty}\frac{1}{k}\left[\log\int_{\theta_2\in S_i,\theta_1\in S_j} w(\theta_1,\theta_2)\overline{P}_k(\theta_j,\theta_i|\beta)d(\theta_1,\theta_2)\right]\right) \tag{9}$$

$$= -\max_{0\leq i<j<M}\sup_{\theta_1\in S_j,\theta_2\in S_i}\left(\lim_{k\to\infty}\frac{1}{k}\left[\log w(\theta_1,\theta_2)\overline{P}_k(\theta_j,\theta_i|\beta)\right]\right) \tag{10}$$

$$= -\max_{0\leq i<j<M}\sup_{\theta_1\in S_j,\theta_2\in S_i}\left(\lim_{k\to\infty}\frac{1}{k}\log\overline{P}_k(\theta_j,\theta_i|\beta)\right) \tag{11}$$

$$= \min_{0\leq i<j<M}\inf_{\theta_1\in S_j,\theta_2\in S_i}\left(-\lim_{k\to\infty}\frac{1}{k}\log\overline{P}_k(\theta_j,\theta_i|\beta)\right) \tag{12}$$

$$= \min_{0\leq i<j<M}\inf_{\theta_1\in S_j,\theta_2\in S_i}\inf_{a\in\mathbb{R}}\{g(\theta_1)I(a|t_j)+g(\theta_2)I(a|t_i)\} \tag{13}$$

$$= \min_{0\leq i<j<M}\inf_{a\in\mathbb{R}}\{g_jI(a|t_j)+g_iI(a|t_i)\} \tag{14}$$

$$= \min_{0\leq i<M-1}\inf_{a\in\mathbb{R}}\{g_{i+1}I(a|t_{i+1})+g_iI(a|t_i)\} \tag{15}$$

The last line follows from adjacent t_i, t_{i+1} dominating the rate due to monotonicity properties.

Line (10) follows from Theorem C.1.

Line (9) follows from: $\forall a_i^\epsilon \geq 0$, $\limsup_{\epsilon\to 0}\left[\epsilon\log\left(\sum_i^N a_i^\epsilon\right)\right] = \max_i^N \limsup_{\epsilon\to 0}\epsilon\log(a_i^\epsilon)$, which is a finite case version (with fewer assumptions) of Theorem C.1. See, e.g., Lemma 1.2.15 in (Dembo and Zeitouni, 2010) for a proof of this property.

□

Lemma C.4. *$\beta(\theta)$ is piecewise constant $\iff \exists c(\beta)>0$ s.t. $-\lim_{k\to\infty}\frac{1}{k}\log(\overline{W}_k)=c(\beta)$.*

Proof. $\implies$ follows directly from Lemma C.3: $\inf_{a\in\mathbb{R}}\{g_{i+1}I(a|t_{i+1})+g_iI(a|t_i)\}>0$ when $t_i\neq t_{i+1}$, which holds when β is piece-wise constant with the appropriate number of levels.

$\impliedby$ Consider β that is not piece-wise constant. Recall that we further assume that β is non-decreasing, and discontinuous only on a measure 0 set. Following algebra steps similar to those in Lemma C.3, but for general β:

$$-\lim_{k\to\infty}\frac{1}{k}\log\overline{W}_k = -\lim_{k\to\infty}\frac{1}{k}\log\int_{\theta_1>\theta_2} w(\theta_1,\theta_2)\overline{P}_k(\theta_1,\theta_2|\beta)d(\theta_1,\theta_2) \tag{16}$$

$$= \inf_{\theta_1>\theta_2}\left(-\lim_{k\to\infty}\frac{1}{k}\log\overline{P}_k(\theta_j,\theta_i|\beta)\right) \tag{17}$$

$$= 0 \tag{18}$$

Where the last line follows from β continuous at some θ_1, and so $\lim_{\theta_2\to\theta_1}\overline{P}_k(\theta_1,\theta_2|\beta)=1$.

Intuitively, what goes wrong with continuous β is that $\overline{P}_k(\theta_1,\theta_2|\beta)$ does not converge uniformly:

$$\forall\epsilon,k,\exists\theta_2\neq\theta_1\ \ \overline{P}_k(\theta_1,\theta_2)>\epsilon$$

i.e. close by items are very hard to distinguish from one another. Then, because the large deviations rate of $\overline{W}_k$ is dominated by the worst rates under the integral, we don't get a positive rate.

□

C.4 Proofs of Lemma 3.1 and Theorem 3.1

Remark C.2. *The KL divergence for two Bernoulli random variables is continuous and strictly convex, with minima at $a = b$, when $a, b \in (0, 1)$. Note that* $\inf_a\{g_i KL(a||t_i) + g(i+1)KL(a||t_{i+1})\}$, *for all feasible g, is also continuous and strictly convex in t_i, t_{i+1}, with minima at $t_i = t_{i+1}$.*

One consequence of the above fact is that fixing either t_i or t_{i+1} and moving the other farther away monotonically increases KL, while moving it closer decreases KL.

C.4.1 Proof of Theorem 3.1

Proof. We use the same notation as the proof for Lemma C.3.

Part 1.

$$\lim_{k\to\infty} W_k = \lim_{k\to\infty} \sum_{0\le i<j<M} \left[\int_{\theta_2\in S_i,\theta_1\in S_j} w(\theta_1,\theta_2)P_k(\theta_1,\theta_2|\beta)d(\theta_1,\theta_2)\right] \tag{19}$$

$$= \sum_{0\le i<j<M} \int_{\theta_2\in S_i,\theta_1\in S_j} w(\theta_1,\theta_2)d(\theta_1,\theta_2) \tag{20}$$

(20) follows from bounded convergence and $P_k(\theta_1, \theta_2|\beta) \to 1$ for $\theta_1 \in S_j, t_2 \notin S_j$. Thus choosing $\boldsymbol{s}$ to maximize (20) maximizes the asymptotic value of W_k.

Part 2. Follows directly from Lemma C.3. □

C.4.2 Proof of Lemma 3.1

Proof. Recall $r(\boldsymbol{t}) \triangleq -\lim_{k\to\infty}\frac{1}{k}\log(W - W_k) = \min_{0\le i\le M-2}\inf_{a\in\mathbb{R}}\{g_{i+1}\mathrm{KL}(a||t_{i+1}) + g_i\mathrm{KL}(a||t_i)\}$. We show the following:

$r(\mathbf{t}) =$

$$\min\left(\log(1-t_1)^{-g_1},\right.$$
$$\log\left[(1-t_{i-1})^{\frac{g_{i-1}}{g_{i-1}+g_i}}(1-t_i)^{\frac{g_i}{g_{i-1}+g_i}} + t_{i-1}^{\frac{g_{i-1}}{g_{i-1}+g_i}}t_i^{\frac{g_i}{g_{i-1}+g_i}}\right]^{-g_{i-1}-g_i} \quad \text{for } 1 < i < M-1,$$
$$\left.\log(t_{M-2})^{-g_{M-2}}\right)$$

and $\boldsymbol{t}^*$ maximizes $r_w(\boldsymbol{t}) \iff$ all the terms inside the minimization $r_w(\boldsymbol{t}^*)$ are equal. Further, the optimal levels $\boldsymbol{t}^*$ are unique. The result immediately follows, that $\{t_i\}$ is the unique solution that equalizes the rates inside the minimization, by noting that the optimal r has $t_0 = 0, t_{M-1} = 1$.

We first prove the alternative form for r. Note that $\{g_{i-1}\mathrm{KL}(a||t_{i-1}) + g_i\mathrm{KL}(a||t_i)\}$ is convex in a, and so we can find an analytic form for the infinum over a.

Let $a_i = \arg\inf_{a\in[t_{i-1},t_i]}\{g_{i-1}\mathrm{KL}(a||t_{i-1}) + g_i\mathrm{KL}(a||t_i)\}$

$$\implies \nabla_{a_i}\left[g_{i-1}\mathrm{KL}(a_i||t_{i-1}) + g_i\mathrm{KL}(a_i||t_i)\right] = 0$$

$$\implies \nabla_{a_i}\left[g_{i-1}\left(a_i\log\frac{a_i}{t_{i-1}} + (1-a_i)\log\frac{1-a_i}{1-t_{i-1}}\right) + g_i\left(a_i\log\frac{a_i}{t_i} + (1-a_i)\log\frac{1-a_i}{1-t_i}\right)\right] = 0$$

$$\implies g_{i-1}\left(\log\frac{a_i}{t_{i-1}} - \log\frac{1-a_i}{1-t_{i-1}}\right) + g_i\left(\log\frac{a_i}{t_i} - \log\frac{1-a_i}{1-t_i}\right) = 0$$

$$\implies \log\left(\frac{a_i}{1-a_i}\right)^{g_{i-1}+g_i} = \log\left(\frac{t_{i-1}}{1-t_{i-1}}\right)^{g_{i-1}} + \log\left(\frac{t_i}{1-t_i}\right)^{g_i}$$

$$\implies \frac{a_i}{1-a_i} = \left[\left(\frac{t_{i-1}}{1-t_{i-1}}\right)^{g_{i-1}}\left(\frac{t_i}{1-t_i}\right)^{g_i}\right]^{\frac{1}{g_{i-1}+g_i}}$$

$$\implies a_i = \frac{c}{1+c}, \text{ where } c = \left[\left(\frac{t_{i-1}}{1-t_{i-1}}\right)^{g_{i-1}}\left(\frac{t_i}{1-t_i}\right)^{g_i}\right]^{\frac{1}{g_{i-1}+g_i}}$$

Then,
$g_{i-1}\mathrm{KL}(a_i||t_{i-1}) + g_i\mathrm{KL}(a_i||t_i)$

$$= g_{i-1}a\log\frac{a}{t_{i-1}} + g_i a\log\frac{a}{t_i} + g_{i-1}(1-a)\log\frac{1-a}{1-t_{i-1}} + g_i(1-a)\log\frac{1-a}{1-t_i}$$

$$= a\left[(g_{i-1}+g_i)\log\frac{a}{1-a} + g_{i-1}\log\frac{1-t_{i-1}}{t_{i-1}} + g_i\log\frac{1-t_i}{t_i}\right] + \log(1-a)^{g_{i-1}+g_i} - \log(1-t_{i-1})^{g_{i-1}}(1-t_i)^{g_i}$$

$$= (g_{i-1}+g_i)\log(1-a) - \log(1-t_{i-1})^{g_{i-1}}(1-t_i)^{g_i} \tag{21}$$

$$= -(g_{i-1}+g_i)\log\left[\left[1 + \left[\left(\frac{t_{i-1}}{1-t_{i-1}}\right)^{g_{i-1}}\left(\frac{t_i}{1-t_i}\right)^{g_i}\right]^{\frac{1}{g_{i-1}+g_i}}\right](1-t_{i-1})^{\frac{g_{i-1}}{g_{i-1}+g_i}}(1-t_i)^{\frac{g_i}{g_{i-1}+g_i}}\right]$$

$$= -(g_{i-1}+g_i)\log\left[(1-t_{i-1})^{\frac{g_{i-1}}{g_{i-1}+g_i}}(1-t_i)^{\frac{g_i}{g_{i-1}+g_i}} + t_{i-1}^{\frac{g_{i-1}}{g_{i-1}+g_i}}t_i^{\frac{g_i}{g_{i-1}+g_i}}\right] \tag{22}$$

Where line (21) uses $\frac{a}{1-a} = c$ and $(g_{i-1}+g_i)\log c = \log\left[\left(\frac{t_{i-1}}{1-t_{i-1}}\right)^{g_{i-1}}\left(\frac{t_i}{1-t_i}\right)^{g_i}\right]$. Note that the first and last rates emerge, respectively, by plugging in $t_0 = 0, t_M = 1$, which holds trivially at the optimum from monotonicity.

We note that a similar derivation, of the large deviation rate for two binomial distributions with different probability of successes and match rates, appears in Glynn and Juneja (2004). In that work, the authors seek to optimize the g in order to identify the single best item out of a set of possible items, and a concave program emerges. In this work, because we optimize the probability of successes and care about retrieving a ranking of the items, no such concave or convex program emerges.

Now we show that $\boldsymbol{t}^*$ maximizes $r_w(\boldsymbol{t}) \iff$ all the terms inside the minimization $r_w(\boldsymbol{t})$ are equal.

equalizes $\implies$ optimal. Let $r(i)$ be the ith term in the minimization, starting at $i = 1$. Note that (holding the other fixed) increasing t_i increases the ith term monotonically and decreases the $(i+1)$th term monotonically. Suppose β s.t. $r(i) = r(j)\forall i, j$. To increase the minimization term, one must increase $r(i), \forall i$. To increase $r(1)$, t_1 must increase, regardless of what the other levels are. Then, to increase $r(2)$, t_2 must increase … to increase $r(M-2)$, t_{M-2} must increase. However, to increase $r(M-1)$, t_{M-2} must decrease, and we have a contradiction. Thus, one cannot increase all terms simultaneously.

equalizes $\Longleftarrow$ optimal. Suppose $\mathbf{t}$ maximizes $r(\mathbf{t})$ but the terms inside the minimization are not equal. Then $\exists i$ s.t. $r(i) = \min_j r(j)$ and either $r(i) \neq r(i-1)$ or $r(i) \neq r(i+1)$. $r(i)$ can be increased without lowering the overall rate. This method can be repeated $\forall i : r(i) = \min_j r(j)$ and so $\mathbf{t}$ would not be optimal, a contradiction.

Uniqueness follows from the overall rate unique determining t_1, t_{M-2} and so iteratively uniquely determining the rest. $\square$

C.5 Additional necessary lemmas

Now, we begin the set-up that will lead to a proof for Theorem 3.2. It turns out that proving the theorem requires, in the process, essentially proving our convergence result with $M \to \infty$, Theorem B.1. For Theorem 3.2, we need a lower bound for t_1 as a function of M. This seems hard to do in general. Luckily, in our case, there is a property for how $\mathbf{t}^*$ changes when M is doubled. Using this property, we can derive that $t_1^* \geq \mathcal{O}(M^{-3})$.

Recall that step-wise increasing β with M intervals $S_i = [s_i, s_{i+1})$ has levels $\{t_i\}_{i=0}^{M-1}$, where $t_0 = 1, t_{M-1} = 1$, and $s_0 \triangleq 0$, $s_M \triangleq 1$.

Furthermore, we use the following notation for the large deviation rate

$$r_i = -(g_{i-1} + g_i) \log \left[(1 - t_{i-1})^{\frac{g_{i-1}}{g_{i-1}+g_i}} (1 - t_i)^{\frac{g_i}{g_{i-1}+g_i}} + t_{i-1}^{\frac{g_{i-1}}{g_{i-1}+g_i}} t_i^{\frac{g_i}{g_{i-1}+g_i}} \right] \tag{23}$$

for $i \in \{1 \dots M-1\}$, which implies $r_1 = -g_1 \log(1 - t_1)$ and $r_{M-1} = -g_{M-2} \log(t_{M-2})$.

We further use r^{M-1} to be the rate achieved by the optimal β_M with M intervals.

Lemma C.5. *Suppose g uniform, i.e. $g_i = 1, \forall i$ and that β_M has values $\{t_i\}_{i=0}^{M-1}$. Then β_{2M-1} has values $\{t_i'\}_{i=0}^{2M-2}$, where $t_{2i}' = t_i, \forall i \in \{0 \dots M-1\}$, $t_1' = \frac{1}{2}\left(1 - \sqrt{1 - t_1}\right)$ and $t_{2M-3}' = \frac{1}{2}\left(1 + \sqrt{t_{M-2}}\right)$.*

Proof. We first set the values $t_{2i}' = t_i$ and then optimally choose the remaining values t_k', k odd. Then, we show that the resulting large deviation rates between all adjacent pairs are equal. Then, by the proof of Lemma 3.1, which showed that equalizing the rates between adjacent intervals is a sufficient condition for optimality, β_{2M-1} has the levels $\{t_i'\}_{i=0}^{2M-2}$.

Let r' denote rates between adjacent t' as r does for t. Supposing $t_2' = t_1$, we find t_1' such that $r_1' = r_2'$ and $t_1' < t_2'$.

$$\begin{aligned}
-\log(1 - t_1') &= -2 \log \left[\sqrt{(1 - t_1')(1 - t_2')} + \sqrt{t_1' t_2'} \right] \\
\implies 1 - t_1' &= (1 - t_1')(1 - t_2') + t_1' t_2' + 2\sqrt{(1 - t_1')(1 - t_2') t_1' t_2'} \\
\implies t_1' &= \frac{1}{2}\left(1 - \sqrt{1 - t_2'}\right) = \frac{1}{2}\left(1 - \sqrt{1 - t_1}\right)
\end{aligned}$$

Similarly, $r_{2M-3}' = r_{2M-2}'$ when $t_{2M-3}' = \frac{1}{2}\left(1 + \sqrt{t_{M-2}}\right)$. It follows that $r_1' = r_2' = r_{2M-3}' = r_{2M-2}'$ by choosing such t_1', t_{2M-3}'.

Next, we find $t'_k \in (t'_{k-1}, t'_{k+1})$ for $k \in \{3, 5, \dots 2M-5\}$ such that the rates $r'_k = r'_{k+1}$.

$$-2\log\left[\sqrt{(1-t'_k)(1-t'_{k-1})} + \sqrt{t'_k t'_{k-1}}\right] = -2\log\left[\sqrt{(1-t'_k)(1-t'_{k+1})} + \sqrt{t'_k t'_{k+1}}\right]$$

$$\implies t'_k = \frac{c}{1+c}, \text{ where } c = \left[\frac{\sqrt{1-t'_{k+1}} - \sqrt{1-t'_{k-1}}}{\sqrt{t'_{k-1}} - \sqrt{t'_{k+1}}}\right]^2$$

Now, we show that $r'_k = r'_j, \forall j, k$ by showing that the difference between each rate r_i and its analogous rate r'_{2i} is constant. $r_k = r_j, \forall j, k$ by assumption and so $r'_k = r'_j, \forall j, k$ follows.

$r_{M-1} = -\log t_{M-2}$ and $r'_{2M-2} = -\log \frac{1}{2}(1 + \sqrt{t_{M-2}})$. Thus if $r_i = -\log x$ for some x, then $r_{2i} = -\log \frac{1}{2}(1 + \sqrt{x})$ would imply that all the rates are equal. Thus, it is sufficient to show that

$$\left[\sqrt{(1-t'_{2i-1})(1-t'_{2i})} + \sqrt{t'_{2i-1}t'_{2i}}\right]^2 = \frac{1}{2}\left[1 + \sqrt{(1-t_{i-1})(1-t_i)} + \sqrt{t_{i-1}t_i}\right] \tag{24}$$

$$\equiv \left[\sqrt{\left(1 - \frac{c}{1+c}\right)(1-t_i)} + \sqrt{\frac{c}{1+c}t_i}\right]^2 = \frac{1}{2}\left[1 + \sqrt{(1-t_{i-1})(1-t_i)} + \sqrt{t_{i-1}t_i}\right] \tag{25}$$

$$\text{where } c = \left[\frac{\sqrt{1-t_i} - \sqrt{1-t_{i-1}}}{\sqrt{t_{i-1}} - \sqrt{t_i}}\right]^2$$

The proof for (25) is algebraically tedious and is shown in Remark C.3 below.

Then, by the proof of Lemma 3.1, which shows that equalizing the rates inside the minimization terms implies an optimal $\{t_i\}$, β_{2M-1} has the levels $\{t'_i\}_{i=0}^{2M-2}$.

□

Remark C.3.

$$\left[\sqrt{\left(1 - \frac{c}{1+c}\right)(1-t_i)} + \sqrt{\frac{c}{1+c}t_i}\right]^2 = \frac{1}{2}\left[1 + \sqrt{(1-t_{i-1})(1-t_i)} + \sqrt{t_{i-1}t_i}\right]$$

$$\textit{where } c = \left[\frac{\sqrt{1-t_i} - \sqrt{1-t_{i-1}}}{\sqrt{t_{i-1}} - \sqrt{t_i}}\right]^2$$

Proof. Let $x = \sqrt{t_i}, y = \sqrt{1-t_i}, z = \sqrt{t_{i-1}}$, and $w = \sqrt{1-t_{i-1}}$. Note that $x > z, w > y, y = 1 - x^2, w = 1 - z^2$. Then,

$$\frac{c}{c+1} = \frac{(y-w)^2}{2 - 2xz - 2yw}, \text{ and } \frac{1}{c+1} = \frac{(x-z)^2}{2 - 2xz - 2yw}$$

(To show the above two equalities, factor out $\frac{1}{(x-z)^2}$ from numerator and denominator, and substitute $y = 1 - x^2, w = 1 - z^2$).

Now, the left hand side:

$$\begin{aligned}
&\left[\sqrt{\left(1-\frac{c}{1+c}\right)(1-t_i)}+\sqrt{\frac{c}{1+c}t_i}\right]^2 \\
=&\frac{1}{2-2xz-2yw}\left[\sqrt{(x-z)^2y^2}+\sqrt{(y-w)^2x^2}\right]^2 \\
=&\frac{(x-z)^2y^2+(y-w)^2x^2+2xy(x-z)(w-y)}{2-2xz-2yw} && \sqrt{(y-w)^2}=w-y,\sqrt{(x-z)^2}=x-z \\
=&\frac{z^2y^2+w^2x^2-2wxyz}{2-2xz-2yw}
\end{aligned}$$

The right hand side:

$$\begin{aligned}
&\frac{1}{2}\left[1+\sqrt{(1-t_{i-1})(1-t_i)}+\sqrt{t_{i-1}t_i}\right] \\
&=\frac{1}{2}\left[1+(wy+xz)\right]
\end{aligned}$$

Multiplying both sides by $2-2xz-2yw$, we have:

$$\begin{aligned}
\left[\sqrt{\left(1-\frac{c}{1+c}\right)(1-t_i)}+\sqrt{\frac{c}{1+c}t_i}\right]^2 &= \frac{1}{2}\left[1+\sqrt{(1-t_{i-1})(1-t_i)}+\sqrt{t_{i-1}t_i}\right] \\
\equiv z^2y^2+w^2x^2-2wxyz &= 1-(wy+xz)^2 \\
\equiv z^2(1-x^2)+(1-z^2)x^2-2wxyz &= 1-w^2y^2-x^2z^2-2wxyz \\
\equiv z^2-2x^2z^2+x^2 &= 1-(1-z^2)(1-x^2)-x^2z^2 \\
\equiv 0 &= 0
\end{aligned}$$

□

Corollary C.1. *Suppose g uniform, i.e. $g_i = 1, \forall i$. $\forall \epsilon > 0, \exists M$ s.t. $\forall M' \geq M$, $r^{M'} < \epsilon$.*

Proof. Let $M = 2^N, M' = 2^{N+1} - 1$, for some N. We show that $r^{M'} \leq \frac{1}{2} r^M$. The corollary follows by noting that $r^{K'} < r^K \, \forall K' > K$ and that $r^K < \infty, \forall K$.

$$\begin{aligned}
r^M - r^{M'} &= -\log t^M_{M-2} + \log t^{M'}_{M'-2} \\
&= -\log t^M_{M-2} + \log\left[\frac{1}{2}+\frac{1}{2}\sqrt{t^M_{M-2}}\right] && \text{Lemma } C.5 \\
&= \log\left[\frac{1}{2}\frac{1}{t^M_{M-2}}+\frac{1}{2}\frac{1}{\sqrt{t^M_{M-2}}}\right] \\
&\geq -\frac{1}{2}\log t^M_{M-2} && \sqrt{t^M_{M-2}} \geq t^M_{M-2} \\
\implies r^{M'} &\leq \frac{1}{2} r^M
\end{aligned}$$

□

Corollary C.2. *Suppose g uniform, i.e. $g_i = 1, \forall i$. $\forall \delta > 0, \exists N$ s.t. $\forall M \geq N$, $\max_k t_k^M - t_{k-1}^M < \delta$.*

Proof. This corollary follows directly from Corollary C.1. If the rates are upper bounded, then so are the level differences.

We first find where the rate is minimized given a width between levels of δ

$$\begin{aligned} x_m &= \arg\min_x -2\log\left[\sqrt{(1-x-\delta)(1-x)} + \sqrt{x(x+\delta)}\right] \\ &= \frac{1}{2} - \frac{1}{2}\delta \end{aligned}$$

Then given an upper bound of ϵ on the rate, there is a bound on δ determined by the largest possible difference at levels symmetric around $\frac{1}{2}$.

$$\begin{aligned} r^L &= -2\log\left[2\sqrt{(\frac{1}{2}-\delta)(\frac{1}{2}+\delta)}\right] \\ &= -\log\left[1-4\delta^2\right] \\ &\geq \epsilon \text{ when } \delta > \frac{1}{2}\sqrt{1-e^{-\epsilon}} \end{aligned}$$

□

Lemma C.6. *Suppose g is non-decreasing in θ. Then, $t_{M-2} \geq 1 - \frac{1}{M-1}$.*

Proof. Note that, with uniform matching, $\forall x \in (0,1], y \in [0, 1-x]$ the rate with values $t_{i-1} = y, t_i = y + x$ is no more than the last with $t_{M-2} = 1 - x$. With width x, in other words, the extreme points have a larger rates than the middle points. For $i \notin \{1, M-1\}$:

$$\begin{aligned} r_i &= \inf_a \{g_{i-1}\mathrm{KL}(a||t_{i-1}) + g_i \mathrm{KL}(a||t_i)\} \\ &= \inf_a \{\mathrm{KL}(a||y) + \mathrm{KL}(a||y+x)\} && \text{uniform matching} \\ &= -2\log\left[(1-y)^{\frac{1}{2}}(1-y-x)^{\frac{1}{2}} + y^{\frac{1}{2}}(y+x)^{\frac{1}{2}}\right] && (26) \\ &= -\log\left[(1-y)(1-y-x) + y(y+x) + 2\left[(1-y)(1-y-x)y(y+x)\right]^{1/2}\right] \\ &\leq -\log(1-x) \end{aligned}$$

Where line (26) follows from line (22).

By the proof of Lemma 3.1, the optimal levels equalize the rates between each level. Then, when g is non-decreasing, $g_{M-2} \geq g_\ell, \forall \ell \in \{1 \ldots M-3\}$. Then, at the same level differences, the rate corresponding to the last level is no smaller. Thus, to equalize the rates, the last width must be no larger than any other width. Thus, $t_{M-2} \geq 1 - \frac{1}{L}$. □

Lemma C.7. *With uniform matching ($g_i = 1$), $r^{2^{N+1}-1} \geq \frac{1}{5} r^{2^N}$.*

Proof. Let $K = 2^N, K' = 2^{N+1} - 1$. Note that $t^K_{K-1} \geq \frac{1}{2}$ by Lemma C.6.

$$
\begin{aligned}
r^K - r^{K'} &= -\log t^K_{K-2} + \log t^{K'}_{K'-2} \\
&= -\log t^K_{K-2} + \log\left[\frac{1}{2} + \frac{1}{2}\sqrt{t^K_{K-2}}\right] && \text{Lemma } C.5 \\
&= \log\left[\frac{1}{2}\frac{1}{t^K_{K-2}} + \frac{1}{2}\frac{1}{\sqrt{t^K_{K-2}}}\right] \\
&\leq \log\left(t^K_{K-2}\right)^{-\frac{4}{5}} && \frac{1}{2}\left(t^K_{K-2}\right)^{-1} + \frac{1}{2}\left(t^K_{K-2}\right)^{-\frac{1}{2}} \leq \left(t^K_{K-2}\right)^{-\frac{4}{5}} \text{ when } t^K_{K-2} \in \left[\frac{1}{2}, 1\right] \\
\implies r^{K'} &\geq \frac{1}{5} r^K
\end{aligned}
$$

□

Lemma C.8. *With uniform matching ($g_i = 1$), $\exists C > 0$ s.t. $\forall M, t^M_1 \geq CM^{-3}$.*

Proof. By Lemma C.7, $\exists C_2 > 0$ s.t. $r^M \geq C_2 5^{-\lceil \log_2 M \rceil}$. Then

$$
\begin{aligned}
-\log(1 - t^M_1) &= r^M \\
&\geq C_2 5^{-\lceil \log_2 M \rceil} \\
\implies t^M_1 &\geq 1 - \exp\left[-C_2 5^{-\lceil \log_2 M \rceil}\right] \\
&\geq 1 - \exp\left[-C_3 M^{-\frac{1}{\log_5 2}}\right] \\
&\geq \frac{e-1}{e} C_3 M^{-\frac{1}{\log_5 2}} && e^{-x} \leq 1 - \frac{e-1}{e}x \text{ for } x \in [0, 1] \\
\implies \exists C > 0 \text{ s.t. } t^M_1 &\geq CM^{-3}
\end{aligned}
$$

□

Corollary C.3. *With monotonically non-decreasing g, $\exists C > 0$ s.t. $\forall M, t^M_1 \geq CM^{-3}$.*

Proof. The result follows from noting that t^M_1 with uniform matching lower bounds the first value with any other monotonically non-decreasing g, which is a direct application of Lemma B.1 – scale g such that $g_1 = 1$. Then, $g_j \geq 1, j > 1$ and $g_0 \leq 1$. Then, the condition of the lemma holds. □

Lemma C.9. *The run-time of NestedBisection is $O(M \log^2 \frac{1}{\delta})$, where δ is the bisection grid width and M is the number of intervals.*

Proof. The outer bisection, in *main*, runs at most $\log_2 \frac{2}{\delta} + 1$ iterations. Each outer iteration calls *BisectNextLevel* $M - 3$ times, and the inner bisection in each call runs for at most $\log_2 \frac{2}{\delta}$ iterations. Thus the run-time of algorithm is $O(M \log^2 \frac{1}{\delta})$. □

C.6 Proof for Theorem 3.2

Finally, we are ready to prove Theorem 3.2. It follows from formalizing the relationship between δ, the bisection grid width, and ϵ, the additive approximation error in the rate function.

Proof. Recall M is the number of intervals (levels) in β. We use j, t, t^* to denote the levels in a certain iteration, the returned levels, and the optimal levels, respectively. We use $r(\cdot)$ to denote the individual rates between returned levels, i.e. $r(1) = -g_1 \log(1-t_1)$, $r(m) = \{g_{m-1}\mathrm{KL}(a_m||t_{m-1}) + g_m \mathrm{KL}(a_m||t_m)\}, m \in \{2 \dots M-2\}, r(M-1) = -g_{M-2}\log(t_{M-2})$, and use r^* to denote the optimal rate.

By Lemma C.6, $t^*_{M-2} \geq 1 - \frac{1}{M-1}$. By assumption, $t^*_{M-2} < 1-\delta$. Thus, $t^*_{M-2} \in [1-\frac{1}{M-1}, 1-\delta]$, the starting interval for the outer bisection.

First, suppose the outer bisection terminates such that $t_{M-2} \leq t^*_{M-2} + \delta$. We prove that this case always occurs below.

In this case, $r^* - r(M-1)$ is at most $-g_{M-2}\log(t^*_{M-2}) + g_{M-2}\log(t^*_{M-2}+\delta) = g_{M-2}\log\left(\frac{t^*_{M-2}+\delta}{t^*_{M-2}}\right)$. For all $m \in \{M-2 \dots 2\}$, in the final *CalculateOtherLevels* call the algorithm will use bisection to match the corresponding rate with this last rate, $r(M-1) = -g_{M-2}\log(t_{M-2})$, setting t_{m-2} to the smallest value such that $r(m) \leq r(M-1)$ (i.e. the right end of the final interval is chosen).

Then, $\forall m \in \{M-2 \dots 2\}$, $r(m) \in [r(M-1) - \epsilon(\delta), r(M-1)]$, where $\epsilon(\delta)$ is an upper bound on the change in the rate functions with a shift of δ in one of the parameters.

For now, assume $r(1) = -g_1 \log(t_1) \geq r(M-1)$. We prove that this occurs below. Then,

$$\begin{aligned} r(m) &\geq r(M-1) - \epsilon(\delta) && \forall m \in \{1 \dots M\} \\ &\geq -g_{M-2}\log\left(t^*_{M-2}+\delta\right) - \epsilon(\delta) \end{aligned}$$

Now we characterize $\epsilon(\delta)$ in the region $[t^*_1 + \delta, t^*_{M-2}+\delta]$. In particular, we want to bound the rate loss from the other levels $r(m), m > 1$ after the $g_{M-2}\log\left(\frac{t^*_{M-2}+\delta}{t^*_{M-2}}\right)$ loss in in $r(M-1)$. Note that the only source of error is a level shifting right by δ. $r_j(\cdot)$ denotes individual rates between levels j in an intermediary iteration. Let a'_i be the minimum point inside the rate infimum after the shift by δ.

$$\begin{aligned} \epsilon(\delta) &= \sup_{t_{i-1},t_i} \left[g_{i-1}\mathrm{KL}(a_i||t_{i-1}) + g_i\mathrm{KL}(a_i||t_i) - g_{i-1}\mathrm{KL}(a'_i||t_{i-1}+\delta) - g_i\mathrm{KL}(a'_i||t_i)\right] \\ &\leq \sup_{t_{i-1},t_i} \left[g_{i-1}\mathrm{KL}(a'_i||t_{i-1}) + g_i\mathrm{KL}(a'_i||t_i) - g_{i-1}\mathrm{KL}(a'_i||t_{i-1}+\delta) + g_i\mathrm{KL}(a'_i||t_i)\right] && a_i \text{ is inf point} \\ &= \sup_{t_{i-1},t_i} g_{i-1}\left[a'_i \log\frac{t_{i-1}+\delta}{t_{i-1}} + (1-a'_i)\log\frac{1-t_{i-1}-\delta}{1-t_{i-1}}\right] \\ &\leq \sup_{t_{i-1},t_i} g_{i-1}\left[a'_i\log\frac{t_{i-1}+\delta}{t_{i-1}}\right] && \text{2nd term negative} \\ &\leq g_{M-2}\left[\log\frac{t^*_1+\delta}{t^*_1}\right] && t_j \geq t^*_1, g_j \leq g_{M-2} \\ \implies r(m) &\geq r^* - g_{M-2}\log\left(\frac{t^*_{M-2}+\delta}{t^*_{M-2}}\right) - g_{M-2}\left[\log\frac{t^*_1+\delta}{t^*_1}\right] \\ &\geq r^* - g_{M-2}\frac{\delta}{t^*_{M-2}} - g_{M-2}\frac{\delta}{t^*_1} && \log(1+x) \leq x \\ &\geq r^* - \delta g_{M-2}\left[\frac{M-1}{M-2} + \frac{1}{t^*_1}\right] && t^*_{M-2} \geq 1 - \frac{1}{M-1} \end{aligned}$$

By Corollary C.3, $\exists C > 0$ s.t. $t^*_1 \geq CM^{-3} \implies r(m) \geq r^* - \delta g_{M-2}\left[\frac{M-1}{M-2} + CM^3\right]$. Then, let $\delta = \frac{\epsilon}{g_{M-2}\left[\frac{M-1}{M-2}+CM^3\right]}$. Supposing the algorithm terminates in such an iteration, it finds an ϵ-optimal

β in time $O\left(M \log^2 \frac{g_{M-2}\left[\frac{M-1}{M-2}+CM^3\right]}{\epsilon}\right) = O\left(M \log^2 \frac{M}{\epsilon}\right)$.

Next, we show that the algorithm only terminates the outer bisection when $u \leq t^*_{M-2} + \delta$. The claim follows from $\ell \leq t^*_{M-2}$ being an algorithm invariant. The initial $\ell = 1 - \frac{1}{M-1} \leq t^*_{M-2}$ by Lemma C.6. ℓ can only be set to be $> t^*_{M-2}$ if in the current iteration, $j_{M-2} > t^*_{M-2}$ and $r_j(1) < r_j(M-1)$. However, if $j_{M-2} \geq t^*_{M-2}$, then $r_j(1) \geq r_j(M-1)$ ($j_m \geq t^*_m \forall m$), following from a shifting argument like that given in Lemma 3.1 and that the inner bisection is such that $r_j(m) \leq r_j(M-1), m \in \{2 \dots M-2\}$, i.e. all the values $t_m > t^*_m$. Thus, $\ell \leq t^*_{M-2}$ is an algorithm invariant and $u > t^*_{M-2} + \delta \implies u - \ell > \delta$.

Finally, we show that $r(1) \geq r(M-1)$ at the returned $\{t_i\}$. By assumption, in the initial iteration, $u \geq t^*_{M-2}$, and recall that the returned $\{t_i\}$ such that $t_{M-2} = u$ from the final iteration. As shown in the previous paragraph, $j_{M-2} \geq t^*_{M-2} \implies r_j(1) \geq r_j(M-1)$. Thus, if the algorithm terminates in the first iteration, then $r(1) \geq r(M-1)$. In any subsequent iteration, u is changed only if $r_j(1) \geq r_j(M-1)$ at its new value. Thus, $r_j(1) \geq r_j(M-1)$ is an algorithm invariant, and $r(1) \geq r(M-1)$.

The algorithm terminates in finite time. Thus, it terminates when $t_{M-2} = u \leq t^*_{M-2} + \delta$ and finds a (ϵ, M, g)-optimal β in time $O\left(M \log^2 \frac{M}{\epsilon}\right)$.

□

In Theorem 3.2, there is an guarantee of an additive error away from the optimal rate. To instead have a multiplicative error bound for uniform matching, one can use the lower bound on the optimal rate from Lemma C.7, $\exists C > 0$ s.t. $r^* \geq CM^{-3}$. Then, for uniform matching, the algorithm returns a $(1-\epsilon)$ multiplicative approximation in time $O\left(M \log^2 \frac{M}{\epsilon}\right)$.

C.7 Proof of Theorem B.1

Let β^w_M denote the optimal β with M intervals for weight function w, with intervals $\boldsymbol{s^{wM}}$ and levels $\boldsymbol{t^{wM}}$. Let $q_{wM}(\theta) = i/M$ when $\theta \in [s^{wM}_i, s^{wM}_{i+1})$, i.e. the *quantile* of interval item of type θ is in. Then we have the following convergence result for β_M.

Theorem B.1. *Let g be uniform. Suppose w such that q_{wM} converges uniformly. Then, $\forall C \in \mathbb{N}, \exists \beta^w$ s.t. $\beta^w_{C2^N+1} \to \beta^w$ uniformly as $N \to \infty$.*

Proof. Note that the condition on q implies that $\exists \overline{M}$ s.t. $\forall M > \overline{M}, \forall \theta, \exists x_\theta$ such that $\theta \in \left[s^M_{\lfloor x_\theta M \rfloor}, s^M_{\lceil x_\theta M \rceil}\right)$.

Let $M' = 2M-1, M'' = 4M-3, M^q = 2^q M - 2^q + 1$. $\theta \in \left[s^M_{\lfloor x_\theta M \rfloor}, s^M_{\lceil x_\theta M \rceil}\right) \implies \beta_M(\theta) = t^M_{\lfloor x_\theta M \rfloor} \in \left[t^M_{\lfloor x_\theta M \rfloor - 1}, t^M_{\lfloor x_\theta M \rfloor + 1}\right]$. Then,

$$\begin{aligned}
\beta_{M'}(\theta) &= t^{M'}_{\lfloor x_\theta M' \rfloor} \\
&= t^{M'}_{\lfloor x_\theta (2M-1) \rfloor} \\
&\in \left[t^{M'}_{2\lfloor x_\theta M \rfloor - 2}, t^{M'}_{2\lfloor x_\theta M \rfloor + 2}\right] \\
&\subset \left[t^{M}_{\lfloor x_\theta M \rfloor - 1}, t^{M}_{\lfloor x_\theta M \rfloor + 1}\right] && \text{Lemma } C.5
\end{aligned}$$

And, for general q,

$$\begin{aligned}
\beta_{M^q}(\theta) &= t^{M^q}_{\lfloor x_\theta(2^q M - 2^q + 1)\rfloor} \\
&\in \left[t^{M^q}_{\lfloor x_\theta 2^q M\rfloor - 2^q}, t^{M^q}_{\lfloor x_\theta(2^q M)\rfloor + 1}\right] \\
&\subset \left[t^{M^q}_{2^q\lfloor x_\theta M\rfloor - 2^q}, t^{M^q}_{2^q\lfloor x_\theta M\rfloor + 1}\right] \\
&\subset \left[t^{M}_{\lfloor x_\theta M\rfloor - 1}, t^{M}_{\lfloor x_\theta M\rfloor + 1}\right] && \text{Lemma } C.5
\end{aligned}$$

Then, $\forall N' > 1, \theta$: $\beta_{2^{N'}M - 2^{N'} + 1}(\theta) \in \left[t^{M}_{\lfloor x_\theta M\rfloor - 1}, t^{M}_{\lfloor x_\theta M\rfloor + 1}\right]$ and

$$|\beta_{2^{N'}M - 2^{N'} + 1}(\theta) - \beta_M(\theta)| \le t^{M}_{\lfloor x_\theta M\rfloor + 1} - t^{M}_{\lfloor x_\theta M\rfloor - 1}$$

By Corollary C.2, $\forall \delta > 0, \exists K$ s.t. $\forall K' > K$, $t^{K'}_{\lfloor x_\theta K'\rfloor + 1} - t^{K'}_{\lfloor x_\theta K'\rfloor - 1} < 2\delta$.
By the Cauchy criterion, $\exists \beta$ s.t. $\beta_{(C-1)2^N+1} \to \beta$ uniformly.
By change of variables, $\exists \beta$ s.t. $\beta_{C2^N+1} \to \beta$ uniformly. □

Corollary C.4. *For Kendall's tau and Spearman's rho correlation measures, $\exists \beta$ s.t. $\beta_{2^N} \to \beta$ uniformly as $N \to \infty$.*

Proof. For Kendall's tau and Spearman's rho, $\{s_i\}$ is spaced such that $\forall i, j, s_i - s_{i-1} = s_j - s_{j-1}$. Thus, $x_\theta = \theta$ meets the criterion. □

C.8 Kendall's tau and Spearman's rho related proofs

Definition C.1 (see e.g. Embrechts et al. (2003); Nelsen (2007))**.** *The population version of Kendall-tau correlation between item true quality and rating scores is proportional to*

$$W_k^\tau \triangleq 2\int_{\theta_1 > \theta_2} P_k(\theta_1, \theta_2) d\theta_1 d\theta_2$$

Similarly, given items with qualities $\theta_1, \theta_2, \theta_3$, the population version of Spearman's rho correlation between item true quality and rating scores is

$$W_k^\rho \triangleq 6\int_{\theta_1 > \theta_2, \theta_3} P_k(\theta_1, \theta_3) d\theta_1 d\theta_2 d\theta_3$$

Lemma C.10. *Spearman's ρ can also be written as being proportional to $\int_{\theta_1>\theta_2}(\theta_1 - \theta_2) P_k(\theta_1, \theta_2) d\theta_1 d\theta_2$, i.e. with $w(\theta_1, \theta_2) = (\theta_1 - \theta_2)$.*

Proof. Recall $P_k(\theta_1, \theta_3) =$
$Pr((\theta_1 - \theta_2)(x_1^k - x_3^k) > 0)$

$$\begin{aligned}
&= \int_{\theta_1 > \theta_2, \theta_3} Pr(x_1^k - x_3^k > 0) d\theta_1 d\theta_2 d\theta_3 + \int_{\theta_1 < \theta_2, \theta_3} Pr(x_1^k - x_3^k < 0) d\theta_1 d\theta_2 d\theta_3 \\
&= \int_{\theta_1, \theta_3} Pr(x_1^k - x_3^k > 0)\left[\int_{\theta_2 = 0}^{\theta_1} d\theta_2\right] d\theta_1 d\theta_3 + \int_{\theta_1, \theta_3} Pr(x_1^k - x_3^k < 0)\left[\int_{\theta_2 = \theta_1}^{1} d\theta_2\right] d\theta_1 d\theta_3 \\
&= \int_{\theta_1, \theta_3}\left[Pr(x_1^k - x_3^k > 0)\theta_1] + Pr(x_1^k - x_3^k < 0)(1 - \theta_1)\right] d\theta_1 d\theta_3 \\
&= \int_{\theta_1, \theta_3}\left[Pr(x_1^k - x_3^k < 0) + \theta_1\left[Pr(x_1^k - x_3^k > 0) - Pr(x_1^k - x_3^k < 0)\right]\right] d\theta_1 d\theta_3
\end{aligned}$$

Similarly,
$Pr((\theta_1-\theta_2)(x_1^k-x_3^k)<0)=$

$$
\begin{aligned}
&= \int_{\theta_1,\theta_3}\left[Pr(x_1^k-x_3^k>0)+\theta_1\left[Pr(x_1^k-x_3^k<0)-Pr(x_1^k-x_3^k>0)\right]\right]d\theta_1 d\theta_3 \\
&= \int_{\theta_1,\theta_3}\left[Pr(x_3^k-x_1^k>0)+\theta_3\left[Pr(x_3^k-x_1^k<0)-Pr(x_3^k-x_1^k>0)\right]\right]d\theta_1 d\theta_3
\end{aligned}
$$

Where the second equality follows from θ_1, θ_3 interchangeable. Then

$$
\begin{aligned}
W_k^\rho &= 3\int_{\theta_1,\theta_2}(\theta_1-\theta_2)P_k(\theta_1,\theta_2)d\theta_1 d\theta_2 \\
&= \int_{\theta_1>\theta_2} 6(\theta_1-\theta_2)P_k(\theta_1,\theta_2)d\theta_1 d\theta_2
\end{aligned}
$$

□

Note that Spearman's ρ is similar to Kendall's τ with an additional weighting for how far apart the two values that are flipped are.

Lemma C.11. *When w is constant, i.e. for Kendall's τ rank correlation, the intervals $\boldsymbol{s}$ that maximize* (20),

$$\sum_{0\le i<j<M}\int_{\theta_2\in S_i,\theta_1\in S_j} w(\theta_1,\theta_2)d(\theta_1,\theta_2) = \sum_{0\le i<j<M}(s_{i+1}-s_i)(s_{j+1}-s_j) \tag{27}$$

, are $\{s_i=\frac{i}{M}\}_{i=0}^M$.

Lemma C.12. *When w is $(\theta_1-\theta_2)$, i.e. for Spearman's ρ rank correlation, the intervals $\boldsymbol{s}$ that maximize* (20),

$$\sum_{0\le i<j<M}\int_{\theta_2\in S_i,\theta_1\in S_j} w(\theta_1,\theta_2)d(\theta_1,\theta_2) \tag{28}$$

are $\{s_i=\frac{i}{M}\}_{i=0}^M$, i.e. the same as those for Kendall's τ.

Proof.

$$
\begin{aligned}
\sum_{0\le i<j<M}\int_{\theta_2\in S_i,\theta_1\in S_j} w(\theta_1,\theta_2)d(\theta_1,\theta_2) &= \sum_{0\le i<j<M}\int_{\theta_2\in S_i,\theta_1\in S_j}(\theta_1-\theta_2)d(\theta_1,\theta_2) \\
&= \sum_{0<i<j\le M}\left(\frac{s_j+s_{j-1}}{2}-\frac{s_i+s_{i-1}}{2}\right)(s_i-s_{i-1})(s_j-s_{j-1})
\end{aligned}
$$

Finding an asymptotically optimal $\{s_i\}$ then is a constrained third order polynomial maximization problem with M variables. The maximum is achieved at $\{s_i=\frac{i}{M}\}_{i=0}^{i=M}$, as for Kendall's tau correlation. □

References

Amir Dembo and Ofer Zeitouni. *Large Deviations Techniques and Applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010. ISBN 978-3-642-03310-0 978-3-642-03311-7. DOI: 10.1007/978-3-642-03311-7.

Educational Testing Service. TOEFL iBT Writing Sample Responses, 2005. URL `http://toefl.uobabylon.edu.iq/papers/ibt_2014_12148630.pdf`.

Paul Embrechts, Filip Lindskog, and Alexander Mcneil. Chapter 8 - Modelling Dependence with Copulas and Applications to Risk Management. In Svetlozar T. Rachev, editor, *Handbook of Heavy Tailed Distributions in Finance*, volume 1 of *Handbooks in Finance*, pages 329–384. North-Holland, Amsterdam, 2003. DOI: 10.1016/B978-044450896-6.50010-8.

Nikhil Garg and Ramesh Johari. Designing Informative Rating Systems for Online Platforms: Evidence from Two Experiments. October 2018. URL `https://arxiv.org/abs/1810.13028v1`.

Peter Glynn and Sandeep Juneja. A large deviations perspective on ordinal optimization. In *Simulation Conference, 2004. Proceedings of the 2004 Winter*, volume 1. IEEE, 2004.

Fred Hicks, Lee Valentine, John Morrow, and Ian McDonald. Choosing Natural Adjective Ladders, 2000. URL `http://www.mcdonald.me.uk/storytelling/lichert_article.htm`.

Roger B. Nelsen. *An Introduction to Copulas*. Springer Science & Business Media, June 2007. ISBN 978-0-387-28678-5.