A SUPPLEMENTAL DOCUMENT

A.1 Additional Numerical Results

We provide additional results for the disjunctive version of the problem that is introduced in Section 5. We consider the class of problems $B_{\mathrm{LB}}(R, K, p, \Delta)$ described in [20], where $L = 1$ and the probability that user 1 finds page j attractive is given as

$$p_{1,j} = \begin{cases} p & \text{if } j \leq K \\ p - \Delta & \text{otherwise.} \end{cases}$$

Similar to [20], we set $p = 0.2$ and vary other parameters, namely R, K, and Δ. We run both CTS and CUCB for 100000 rounds in all problem instances, and report their regrets averaged over 20 runs in Table 1.

Table 1: Average regret and its standard deviation for CTS and CUCB for the class of problems $B_{\mathrm{LB}}(R, K, 0.2, \Delta)$.

			CTS		**CUCB**	
R	K	Δ	**Regret**	**Std. Dev.**	**Regret**	**Std. Dev.**
16	2	0.15	155.4	14.1	1284.1	52.4
16	4	0.15	103.2	9.0	998.9	33.2
16	8	0.15	52.1	9.8	549.5	16.8
32	2	0.15	321.4	18.9	2718.8	61.2
32	4	0.15	252.2	17.0	2227.0	55.4
32	8	0.15	155.4	25.7	1531.0	21.9
16	2	0.075	276.9	50.7	2057.6	79.6
16	4	0.075	205.4	25.7	1496.5	65.2
16	8	0.075	113.1	40.4	719.4	53.7

We observe that CTS outperforms CUCB in all problem instances in terms of the average regret. Next, we compare the performance of CTS with CascadeKL-UCB proposed in [20] using the results reported in Table 1 in [20]. We observe that CTS outperforms CascadeKL-UCB in all problem instances as well. As a final remark, we see that for both CTS and CUCB, the regret increases as the number of pages (R) increases, decreases as the number of recommended items (K) increases, and increases as Δ decreases.

A.2 Additional Facts

We introduce Fact 2 in order to bound the expression in Lemma 1 and Fact 3 in order to bound the expression in Lemma 3.

Fact 2. *(Multiplicative Chernoff Bound ([21] and [13])) Let $X_1, \ldots, X_n$ be Bernoulli random variables taking values in $\{0, 1\}$ such that $\mathbb{E}[X_t | X_1, \ldots, X_{t-1}] \geq \mu$ for all $t \leq n$, and $Y = X_1 + \ldots + X_n$. Then, for all $\delta \in (0, 1)$,*

$$\Pr[Y \leq (1 - \delta) n\mu] \leq e^{-\frac{\delta^2 n\mu}{2}}.$$

Fact 3. *(Results from Lemma 7 in [12]) Given $Z \subseteq \tilde{S}^*$, let τ_j be the round at which $\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)) \wedge \neg\mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))$ occurs for the jth time, and let $\tau_0 = 0$. If $\forall i \in Z, N_i(\tau_j + 1) \geq q$ and $0 < \varepsilon \leq 1/\sqrt{e}$, then*

$$\mathbb{E}\left[\sum_{t=\tau_j+1}^{\tau_{j+1}} \mathbb{I}\{\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)), \mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))\}\right] \leq \prod_{i \in Z} B_q - 1 \tag{16}$$

where B_q is given as

$$B_q = \begin{cases} \min\left\{\frac{4}{\varepsilon^2}, 1 + 6\alpha_1' \frac{1}{\varepsilon^2} e^{-\frac{\varepsilon^2}{2} q} + \frac{2}{e^{\frac{1}{8}\varepsilon^2 q} - 2}\right\} & \text{if } q > \frac{8}{\varepsilon^2} \\ \frac{4}{\varepsilon^2} & \text{otherwise} \end{cases}$$

and α'_1 is a problem independent constant.

Moreover,

$$\sum_{q=0}^{T}\left(\prod_{i\in Z} B_q - 1\right) \leq 13\alpha'_2\left(\frac{2^{2|Z|+3}\log\frac{|Z|}{\varepsilon^2}}{\varepsilon^{2|Z|+2}}\right) \tag{17}$$

where α'_2 is a problem independent constant.

A.3 Proof of Lemma 1

The proof is similar to the proof of Lemma 3 in [12]. However, additional steps are required to take probabilistic triggering into account. Consider a base arm $i \in [m]$. Let τ^i_w be the round for which base arm i is in the triggering set of the selected super arm for the wth time. Hence, we have $i \in \tilde{S}(\tau^i_w)$ for all $w > 0$. Also let $\tau^i_0 = 0$. Then, we have:

$$\begin{aligned}
&\mathbb{E}[|\{t : 1 \leq t \leq T, i \in \tilde{S}(t), |\hat{\mu}_i(t) - \mu_i| > \varepsilon \vee \mathcal{B}_{i,2}(t)\}|] \\
&\quad = \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{I}\{i \in \tilde{S}(t), |\hat{\mu}_i(t) - \mu_i| > \varepsilon \vee \mathcal{B}_{i,2}(t)\}\right] \\
&\quad \leq \mathbb{E}\left[\sum_{w=0}^{T}\sum_{t=\tau^i_w+1}^{\tau^i_{w+1}} \mathbb{I}\{i \in \tilde{S}(t), |\hat{\mu}_i(t) - \mu_i| > \varepsilon \vee \mathcal{B}_{i,2}(t)\}\right] \\
&\quad = \sum_{w=0}^{T} \mathbb{E}[\mathbb{I}\{i \in \tilde{S}(\tau^i_{w+1}), |\hat{\mu}_i(\tau^i_{w+1}) - \mu_i| > \varepsilon \vee \mathcal{B}_{i,2}(\tau^i_{w+1})\}] \\
&\quad = \sum_{w=0}^{T} \mathbb{E}[\mathbb{I}\{|\hat{\mu}_i(\tau^i_{w+1}) - \mu_i| > \varepsilon \vee \mathcal{B}_{i,2}(\tau^i_{w+1})\}] \\
&\quad \leq 1 + \sum_{w=1}^{T} \Pr[|\hat{\mu}_i(\tau^i_{w+1}) - \mu_i| > \varepsilon \vee \mathcal{B}_{i,2}(\tau^i_{w+1})] \\
&\quad = 1 + \sum_{w=1}^{T} \Pr[|\hat{\mu}_i(\tau^i_{w+1}) - \mu_i| > \varepsilon \wedge \neg\mathcal{B}_{i,2}(\tau^i_{w+1})] + \sum_{w=1}^{T} \Pr[\mathcal{B}_{i,2}(\tau^i_{w+1})] \\
&\quad = 1 + \sum_{w=1}^{T} \Pr[|\hat{\mu}_i(\tau^i_{w+1}) - \mu_i| > \varepsilon, N_i(\tau^i_{w+1}) > (1-\rho)wp_i] + \sum_{w=1}^{T} \Pr[N_i(\tau^i_{w+1}) \leq (1-\rho)wp_i] \\
&\quad \leq 1 + \sum_{w=1}^{T} \Pr[|\hat{\mu}_i(\tau^i_{w+1}) - \mu_i| > \varepsilon, N_i(\tau^i_{w+1}) > (1-\rho)wp^*] + \sum_{w=1}^{T} \Pr[N_i(\tau^i_{w+1}) \leq (1-\rho)wp_i] \\
&\quad \leq 1 + \sum_{w=1}^{T} 2e^{-2(1-\rho)wp^*\varepsilon^2} + \mathbb{I}\{p^* < 1\} \cdot \sum_{w=1}^{T} e^{-\frac{\rho^2 wp^*}{2}} \qquad (18) \\
&\quad \leq 1 + \frac{1}{(1-\rho)p^*\varepsilon^2} + \frac{2\mathbb{I}\{p^* < 1\}}{\rho^2 p^*}
\end{aligned}$$

where the second term in (18) is obtained by observing that

$$\begin{aligned}
&\Pr[|\hat{\mu}_i(\tau^i_{w+1}) - \mu_i| > \varepsilon, N_i(\tau^i_{w+1}) > (1-\rho)wp^*] \\
&\leq \sum_{k=\lceil(1-\rho)wp^*\rceil}^{\infty} \Pr[|\hat{\mu}_i(\tau^i_{w+1}) - \mu_i| > \varepsilon | N_i(\tau^i_{w+1}) = k] \Pr[N_i(\tau^i_{w+1}) = k]
\end{aligned}$$

and applying Hoeffding's inequality, and the third term in (18) is obtained by using Fact 2.

A.4 Proof of Lemma 2

The proof is similar to the proof of Lemma 1 in [12]. Let $\boldsymbol{\theta'} := (\boldsymbol{\theta'}_{\tilde{S}^*}, \boldsymbol{\theta}_{\tilde{S}^{*c}}(t))$ be such that

$$\|\boldsymbol{\theta'}_{\tilde{S}^*} - \boldsymbol{\mu}_{\tilde{S}^*}\|_\infty \leq \varepsilon \ . \tag{19}$$

Claim 1: For all S' such that $\tilde{S}' \cap \tilde{S}^* = \emptyset$, $S' \neq \text{Oracle}(\boldsymbol{\theta'})$.

Claim 1 holds since

$$\begin{aligned}
r(S', \boldsymbol{\theta'}) &= r(S', \boldsymbol{\theta}(t)) && (20)\\
&\leq r(S(t), \boldsymbol{\theta}(t)) && (21)\\
&\leq r(S(t), \boldsymbol{\mu}) + B\left(\frac{\Delta_{S(t)}}{B} - (\tilde{k}^{*2}+1)\varepsilon\right) && (22)\\
&= r(S(t), \boldsymbol{\mu}) + \Delta_{S(t)} - B(\tilde{k}^{*2}+1)\varepsilon \\
&= r(S^*, \boldsymbol{\mu}) - B(\tilde{k}^{*2}+1)\varepsilon && (23)\\
&< r(S^*, \boldsymbol{\mu}) - B\tilde{k}^*\varepsilon \\
&\leq r(S^*, \boldsymbol{\theta'}) && (24)
\end{aligned}$$

where (20) follows from Assumption 3 since $\boldsymbol{\theta'}$ and $\boldsymbol{\theta}(t)$ only differ on arms in $\tilde{S}^*$ and $\tilde{S}' \cap \tilde{S}^* = \emptyset$, (21) holds since $S(t) \in \text{OPT}(\boldsymbol{\theta}(t))$, (22) is by $\neg\mathcal{D}(t)$ and Assumption 3, (23) is by the definition of $\Delta_{S(t)}$, and (24) is again by Assumption 3.

Next, we consider two cases:

Case 1a: $\tilde{S}^* \subseteq \tilde{\text{Oracle}}(\boldsymbol{\theta'})$ for all $\boldsymbol{\theta'} = (\boldsymbol{\theta'}_{\tilde{S}^*}, \boldsymbol{\theta}_{\tilde{S}^{*c}}(t))$ that satisfies (19).

Case 1b: There exists $\boldsymbol{\theta'} = (\boldsymbol{\theta'}_{\tilde{S}^*}, \boldsymbol{\theta}_{\tilde{S}^{*c}}(t))$ that satisfies (19) for which $\tilde{S}^* \not\subseteq \tilde{\text{Oracle}}(\boldsymbol{\theta'})$. For this $\boldsymbol{\theta'}$, let $S_1 = \text{Oracle}(\boldsymbol{\theta'})$ and $Z_1 = \tilde{S}_1 \cap \tilde{S}^*$. Together with Claim 1, for this case, we have $Z_1 \neq \tilde{S}^*$ and $Z_1 \neq \emptyset$.

Note that Case 1a and Case 1b are complements of each other.

When Case 1a is true, for any given $\boldsymbol{\theta'}$, with an abuse of notation, let $S_0 := \text{Oracle}(\boldsymbol{\theta'})$. Then, we have $r(S_0, \boldsymbol{\theta'}) \geq r(S^*, \boldsymbol{\theta'}) \geq r(S^*, \boldsymbol{\mu}) - B\tilde{k}^*\varepsilon$. If $S_0 \notin \text{OPT}$, then we have $r(S^*, \boldsymbol{\mu}) = r(S_0, \boldsymbol{\mu}) + \Delta_{S_0}$. Combining the two results above, we obtain $r(S_0, \boldsymbol{\theta'}) \geq r(S_0, \boldsymbol{\mu}) + \Delta_{S_0} - B\tilde{k}^*\varepsilon$. By Assumption 3, this implies that $\|\boldsymbol{\theta'}_{\tilde{S}_0} - \boldsymbol{\mu}_{\tilde{S}_0}\|_1 \geq \frac{\Delta_{S_0}}{B} - \tilde{k}^*\varepsilon > \frac{\Delta_{S_0}}{B} - (\tilde{k}^{*2}+1)\varepsilon$. Thus, from the discussion above, we conclude that either $S_0 \in \text{OPT}$ or $\|\boldsymbol{\theta'}_{\tilde{S}_0} - \boldsymbol{\mu}_{\tilde{S}_0}\|_1 > \frac{\Delta_{S_0}}{B} - (\tilde{k}^{*2}+1)\varepsilon$. This means $\mathcal{E}_{\tilde{S}^*,1}(\boldsymbol{\theta'}) = \mathcal{E}_{\tilde{S}^*,1}(\boldsymbol{\theta}(t))$ holds. Hence, if Case 1a is true, then Lemma 2 holds for $Z = \tilde{S}^*$.

In Case 1b, we also have $r(S_1, \boldsymbol{\theta'}) \geq r(S^*, \boldsymbol{\theta'}) \geq r(S^*, \boldsymbol{\mu}) - B\tilde{k}^*\varepsilon$. Consider any $\boldsymbol{\theta''} = (\boldsymbol{\theta''}_{Z_1}, \boldsymbol{\theta}_{Z_1^c}(t))$ such that

$$\|\boldsymbol{\theta''}_{Z_1} - \boldsymbol{\mu}_{Z_1}\|_\infty \leq \varepsilon. \tag{25}$$

We see that

$$\begin{aligned}
\|\boldsymbol{\theta''}_{\tilde{S}_1} - \boldsymbol{\theta'}_{\tilde{S}_1}\|_1 &= \sum_{i \in \tilde{S}_1 \cap \tilde{S}^*} |\theta''_i - \theta'_i| + \sum_{i \in \tilde{S}_1 \cap \tilde{S}^{*c}} |\theta''_i - \theta'_i| \\
&\leq \sum_{i \in Z_1} (|\theta''_i - \mu_i| + |\mu_i - \theta'_i|) \\
&\leq 2(\tilde{k}^* - 1)\varepsilon
\end{aligned}$$

hence $r(S_1, \boldsymbol{\theta''}) \geq r(S_1, \boldsymbol{\theta'}) - 2B(\tilde{k}^* - 1)\varepsilon \geq r(S^*, \boldsymbol{\mu}) - B\tilde{k}^*\varepsilon - 2B(\tilde{k}^* - 1)\varepsilon = r(S^*, \boldsymbol{\mu}) - B(3\tilde{k}^* - 2)\varepsilon$.

Claim 2: For all S' such that $\tilde{S}' \cap Z_1 = \emptyset$, $S' \neq \text{Oracle}(\boldsymbol{\theta''})$.

Similar to Claim 1, Claim 2 holds since

$$r(S', \boldsymbol{\theta''}) = r(S', \boldsymbol{\theta}(t))$$

$$
\begin{aligned}
&\le r(S(t), \boldsymbol{\theta}(t)) \\
&\le r(S(t), \boldsymbol{\mu}) + B\left(\frac{\Delta_{S(t)}}{B} - (\tilde{k}^{*2}+1)\varepsilon\right) \\
&= r(S(t), \boldsymbol{\mu}) + \Delta_{S(t)} - B(\tilde{k}^{*2}+1)\varepsilon \\
&= r(S^*, \boldsymbol{\mu}) - B(\tilde{k}^{*2}+1)\varepsilon \\
&< r(S^*, \boldsymbol{\mu}) - B(3\tilde{k}^{*}-2)\varepsilon \\
&\le r(S_1, \boldsymbol{\theta''}).
\end{aligned}
$$

Claim 2 implies that when Case 1b holds, we have $\tilde{\text{Oracle}}(\boldsymbol{\theta''}) \cap Z_1 \neq \emptyset$. Hence, we consider two cases again for $\text{Oracle}(\boldsymbol{\theta''})$:

Case 2a: $Z_1 \subseteq \tilde{\text{Oracle}}(\boldsymbol{\theta''})$ for all $\boldsymbol{\theta''} = (\boldsymbol{\theta''}_{Z_1}, \boldsymbol{\theta}_{Z_1^c}(t))$ that satisfies (25).

Case 2b: There exists $\boldsymbol{\theta''} = (\boldsymbol{\theta''}_{Z_1}, \boldsymbol{\theta}_{Z_1^c}(t))$ that satisfies (25) for which $Z_1 \not\subseteq \tilde{\text{Oracle}}(\boldsymbol{\theta''})$. For this $\boldsymbol{\theta''}$ let $S_2 = \text{Oracle}(\boldsymbol{\theta''})$ and $Z_2 = \tilde{S}_2 \cap Z_1$. Together with Claim 2, for this case, we have $Z_2 \neq Z_1$ and $Z_2 \neq \emptyset$.

Similar to Case 1a, when Case 2a is true, then Lemma 2 holds for $Z = Z_1$. Thus, we can keep repeating the same arguments iteratively, and the size of Z_i will decrease by at least 1 at each iteration. After at most $\tilde{k}^* - 1$ iterations, Case $(\cdot)$b will not be possible. In order to see this, suppose that we come to a point where $|Z_i| = 1$. As in all iterations, either Case$(i+1)$a or Case$(i+1)$b must hold. However, when Case$(i+1)$b holds, Claim $i+1$, which follows from Case(i)b, implies that there exists a $Z_{i+1} \subseteq Z_i$ such that $Z_{i+1} \neq \emptyset$ and $Z_{i+1} \neq Z_i$, which is not possible when $|Z_i| = 1$. Therefore, we conclude that some Case $(i+1)$a must hold, where $Z_i \subseteq \tilde{S}^*$, $Z_i \neq \emptyset$, and $\mathcal{E}_{Z_i,1}(\boldsymbol{\theta}(t))$ occurs.

Finally, we need to show that Claim $i+1$ holds for all iterations. We focus on the claim

$$
r(S^*, \boldsymbol{\mu}) - B(\tilde{k}^{*2}+1)\varepsilon < r(S^*, \boldsymbol{\mu}) - B(\tilde{k}^* + 2\sum_{k=1}^{i}(\tilde{k}^* - k))\varepsilon
$$

as repeating other arguments for all iterations is straightforward. The given inequality is true as $\tilde{k}^* + 2\sum_{k=1}^{i}(\tilde{k}^* - k) \le \tilde{k}^* + 2\sum_{k=1}^{\tilde{k}^*-1}(\tilde{k}^* - k) = \tilde{k}^{*2} < \tilde{k}^{*2} + 1$. Note that, when checking Claim $i+1$, we know that i previous iterations have passed, hence $\tilde{k}^*$ must be larger than $i+1$.

A.5 Proof of Lemma 3

Given Z, we re-index the base arms in Z such that z_i represents ith base arm in Z. We also introduce a counter $c(t)$, and let $c(1) = 1$. If at round t, $\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)) \wedge \neg\mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))$ occurs and a feedback for $z_{c(t)}$ is observed, i.e., $z_{c(t)} \in S'(t)$, the counter is updated with probability $p^*/p^{S(t)}_{z_{c(t)}}$ in the following way:

$$
c(t+1) = \begin{cases} c(t)+1 & \text{if } c(t) < |Z| \\ 1 & \text{if } c(t) = |Z| \end{cases}
$$

If the counter is not updated at round t, $c(t+1) = c(t)$. Note that when $\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)) \wedge \neg\mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))$ occurs, $z_{c(t)} \in Z \subseteq \tilde{\text{Oracle}}(\boldsymbol{\theta}(t)) = \tilde{S}(t)$, hence we always have $0 < p^*/p^{S(t)}_{z_{c(t)}} \le 1$. Moreover, the probability that the counter is updated, i.e., $c(t+1) \neq c(t)$, given $\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)) \wedge \neg\mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))$ occurs is constant and equal to p^* for all rounds t for which $\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)) \wedge \neg\mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))$ occurs. To see this, consider a parameter vector $\boldsymbol{\theta}$ such that $\mathcal{E}_{Z,1}(\boldsymbol{\theta}) \wedge \neg\mathcal{E}_{Z,2}(\boldsymbol{\theta})$ holds and let $S = \text{Oracle}(\boldsymbol{\theta})$, then $\Pr[c(t+1) \neq c(t) | \boldsymbol{\theta}(t) = \boldsymbol{\theta}] = \Pr[z_{c(t)} \in S'(t) | S(t) = S] \cdot (p^*/p^S_{z_{c(t)}}) = p^S_{z_{c(t)}} \cdot (p^*/p^S_{z_{c(t)}}) = p^*$.

Let τ_j be the round at which $\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)) \wedge \neg\mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))$ occurs for the jth time, and let $\tau_0 := 0$. Then, the counter is updated only at rounds τ_j with probability p^*. Let $\eta_{q,k}$ be the round τ_j such that $c(\tau_j + 1) = k+1$ and $c(\tau_j) = k$ holds for the $(q+1)$th time. Let $\eta_{0,0} = 0$ and $\eta_{q,|Z|} = \eta_{q+1,0}$. We know that $0 = \eta_{0,0} < \eta_{0,1} < \ldots < \eta_{0,|Z|} = \eta_{1,0} < \eta_{1,1} < \ldots$.

We use two important observations to continue with proof. Firstly, due to the way the counter is updated, for $t \ge \eta_{q,0} + 1$ we have $N_i(t) \ge q$, $\forall i \in Z$. Secondly, for non-negative integers j_1 and j_2, $\Pr[\eta_{q,k+1} = \tau_{j_1+j_2+1} | \eta_{q,k} =$

$\tau_{j_1}] = p^*(1-p^*)^{j_2}$. This holds since for the given event to hold, the counter must not be updated at rounds $\tau_{j_1+1}, \tau_{j_1+2}, ..., \tau_{j_1+j_2}$, each of which happens with probability $1-p^*$, and must be updated at round $\tau_{j_1+j_2+1}$ which happens with probability p^*.

Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{t=\eta_{q,k}+1}^{\eta_{q,k+1}} \mathbb{I}\{\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)), \mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))\}\right] \\
&= \sum_{j_1=0}^{\infty} \Pr[\eta_{q,k} = \tau_{j_1}] \sum_{j_2=0}^{\infty} \Pr[\eta_{q,k+1} = \tau_{j_1+j_2+1} | \eta_{q,k} = \tau_{j_1}] \\
&\quad \times \sum_{j=j_1}^{j_1+j_2} \mathbb{E}\left[\sum_{t=\tau_j+1}^{\tau_{j+1}} \mathbb{I}\{\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)), \mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))\} \middle| \eta_{q,k} \le \tau_j < \eta_{q+1,k}\right] \\
&\le \sum_{j_1=0}^{\infty} \Pr[\eta_{q,k} = \tau_{j_1}] \sum_{j_2=0}^{\infty} p^*(1-p^*)^{j_2} \sum_{j=j_1}^{j_1+j_2} \left(\prod_{i\in Z} B_q - 1\right) \qquad (26) \\
&= \sum_{j_1=0}^{\infty} \Pr[\eta_{q,k} = \tau_{j_1}] \sum_{j_2=0}^{\infty} p^*(j_2+1)(1-p^*)^{j_2} \left(\prod_{i\in Z} B_q - 1\right) \\
&= \sum_{j_1=0}^{\infty} \Pr[\eta_{q,k} = \tau_{j_1}] \frac{1}{p^*} \left(\prod_{i\in Z} B_q - 1\right) \\
&= \frac{1}{p^*} \left(\prod_{i\in Z} B_q - 1\right)
\end{aligned}
$$

where (26) holds due to our observations and (16) in Fact 3.

Finally, we have

$$
\begin{aligned}
\sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)), \mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))\}] &\le \sum_{q=0}^{T} \sum_{k=0}^{|Z|-1} \mathbb{E}\left[\sum_{t=\eta_{q,k}+1}^{\eta_{q,k+1}} \mathbb{I}\{\mathcal{E}_{Z,1}(\boldsymbol{\theta}(t)), \mathcal{E}_{Z,2}(\boldsymbol{\theta}(t))\}\right] \\
&\le \sum_{q=0}^{T} \sum_{k=0}^{|Z|-1} \frac{1}{p^*} \left(\prod_{i\in Z} B_q - 1\right) \\
&= \frac{|Z|}{p^*} \sum_{q=0}^{T} \left(\prod_{i\in Z} B_q - 1\right) \\
&\le 13\alpha_2' \cdot \frac{|Z|}{p^*} \cdot \left(\frac{2^{2|Z|+3} \log \frac{|Z|}{\varepsilon^2}}{\varepsilon^{2|Z|+2}}\right) \qquad (27)
\end{aligned}
$$

where (27) holds due to (17) in Fact 3.