# Supplementary Material for "Efficient Linear Bandits through Matrix Sketching"

## A Proofs

We start with the proof of a simple lemma that is used in the definition of OFUL (see Algorithm 2).

**Lemma 5.** *For any positive definite $d \times d$ matrix $\boldsymbol{A}$, for any $\boldsymbol{w}_0, \boldsymbol{x} \in \mathbb{R}^d$ and $c > 0$, the solution of*

$$\max_{\boldsymbol{w}\in\mathbb{R}^d} \quad \boldsymbol{w}^\top \boldsymbol{x}$$
$$s.t. \quad \|\boldsymbol{w} - \boldsymbol{w}_0\|_{\boldsymbol{A}} \le c$$

*has value $\boldsymbol{w}_0^\top \boldsymbol{x} + c\,\|\boldsymbol{x}\|_{\boldsymbol{A}^{-1}}$.*

*Proof.* Let $\boldsymbol{u} = \boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{w} - \boldsymbol{w}_0)$ so that $\boldsymbol{w} = \boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{u} + \boldsymbol{w}_0$. Then the optimization problem can be equivalently rewritten as

$$\max_{\boldsymbol{w}\in\mathbb{R}^d} \quad \boldsymbol{u}^\top \boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{x} + \boldsymbol{w}_0^\top \boldsymbol{x}$$
$$\text{s.t.} \quad \|\boldsymbol{u}\| \le c$$

Then the solution is clearly $\boldsymbol{u} = c\,\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{x}/\,\|\boldsymbol{x}\|_{\boldsymbol{A}^{-1}}$, which achieves the claimed value. □

Our regret analyses follow (Abbasi-Yadkori et al., 2011; Abeille and Lazaric, 2017) and related works. However, due to the sketching of the correlation matrix, some key components of the proofs now depend on the spectral error (8). In Section A.2, we present tools specific to the analysis of linear bandits with FD-sketching. These tools are used to bound the instantaneous regret $\left(\boldsymbol{x}^\star - \boldsymbol{x}_t\right)^\top \boldsymbol{w}^\star$ in terms of the norm $\|\boldsymbol{w}^\star - \widetilde{\boldsymbol{w}}_t\|_{\widetilde{\boldsymbol{V}}_{t-1}}$ and the ridge leverage scores $\sum_{t=1}^T \|\boldsymbol{x}_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}$. Armed with these results, we then prove our regret bounds in Sections A.3 and A.4.

Next, we recall some standard tools from the analysis of linear bandits. All results in Section A.1 are from Abbasi-Yadkori et al. (2011).

### A.1 Tools from the analysis of linear contextual bandits

Recall that $\boldsymbol{V}_t = \sum_{s=1}^t \boldsymbol{x}_s \boldsymbol{x}_s^\top + \lambda \boldsymbol{I}$ with $\lambda > 0$.

**Lemma 6** (Determinant-trace inequality)**.**

$$\ln\det\left(\boldsymbol{V}_t\right) \le d \ln\left(\lambda + \frac{tL^2}{d}\right) \;.$$

**Lemma 7** (Ridge leverage scores)**.**

$$\sum_{t=1}^T \min\left\{1, \|\boldsymbol{x}_t\|^2_{\boldsymbol{V}_{t-1}^{-1}}\right\} \le 2\ln\left(\frac{\det\left(\boldsymbol{V}_T\right)}{\lambda \boldsymbol{I}}\right) \;. \tag{16}$$

*For $\lambda \ge \max\left\{1, L^2\right\}$, we also have that*

$$\sum_{t=1}^T \|\boldsymbol{x}_t\|^2_{\boldsymbol{V}_{t-1}^{-1}} \le 2d\ln\left(1 + \frac{TL^2}{\lambda d}\right) \;. \tag{17}$$

**Theorem 6** (Self-normalized bound for vector-valued martingales)**.** *Let*

$$S_t = \sum_{s=1}^t \eta_s \boldsymbol{x}_s \qquad t \ge 1$$

*where $\eta_1, \eta_2, \dots$ is a conditionally $R$-subgaussian real-valued stochastic process and $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots$ is any $\mathbb{R}^d$-valued stochastic process such that $\boldsymbol{x}_t$ is measurable with respect to the $\sigma$-algebra generated by $\eta_1, \dots, \eta_{t-1}$. Then, for any $\delta > 0$, with probability at least $1-\delta$, $\|S_t\|^2_{\boldsymbol{V}_t^{-1}} \le B_t(\delta)$ for all $t \ge 0$, where*

$$B_t(\delta) = 2R^2 \ln\left(\frac{1}{\delta} \det(\boldsymbol{V}_t)^{\frac{1}{2}} \det(\lambda \boldsymbol{I})^{-\frac{1}{2}}\right) . \tag{18}$$

Theorem 6 is key to showing that $\boldsymbol{w}^\star$ lies within the confidence ellipsoid centered at the estimate $\widetilde{\boldsymbol{w}}_t$ at time step $t$, this irrespective of the process that selected the $\boldsymbol{x}_s$ used to build $\widetilde{\boldsymbol{w}}_t$.

### A.2 Linear algebra and sketching tools

We start by introducting a basic relationship between the correlation matrix of actions $\boldsymbol{X}_s^\top \boldsymbol{X}_s$ and its FD-sketched estimate $\boldsymbol{S}_t^\top \boldsymbol{S}_t$ with sketch size $m \le d$. Recall that $\rho_t$ is the smallest eigenvalue of $\boldsymbol{S}_t^\top \boldsymbol{S}_t$ for $t = 1, \dots, T$ and $\bar{\rho}_t = \rho_1 + \dots + \rho_t$. Recall also that $\widetilde{\boldsymbol{V}} = \boldsymbol{S}_t^\top \boldsymbol{S}_t + \lambda \boldsymbol{I}$.

**Proposition 3.** *Let $\boldsymbol{S}_s$ be the matrix computed by FD-sketching at time step $s = 1, \dots, t$ (where $\boldsymbol{S}_0 = \boldsymbol{0}$). Then $\boldsymbol{X}_s^\top \boldsymbol{X}_s = \boldsymbol{S}_s^\top \boldsymbol{S}_s + \bar{\rho}_s \boldsymbol{I}$ .*

*Proof.* By construction, $\boldsymbol{S}_{s-1}^\top \boldsymbol{S}_{s-1} + \boldsymbol{x}_s \boldsymbol{x}_s^\top = \boldsymbol{U}_s \boldsymbol{\Sigma}_s \boldsymbol{U}_s^\top$ where $\boldsymbol{S}_s = (\boldsymbol{\Sigma}_s - \rho_s \boldsymbol{I}_{m \times m})^{\frac{1}{2}} \boldsymbol{U}_s$. Thus,

$$\boldsymbol{S}_s^\top \boldsymbol{S}_s = \boldsymbol{U}_s \boldsymbol{\Sigma}_s \boldsymbol{U}_s^\top - \rho_s \boldsymbol{I} = \boldsymbol{S}_{s-1}^\top \boldsymbol{S}_{s-1} + \boldsymbol{x}_s \boldsymbol{x}_s^\top - \rho_s \boldsymbol{I}$$

Summing both sides of the above over $s = 1, \dots, t$ we get

$$\boldsymbol{S}_t^\top \boldsymbol{S}_t = \sum_{s=1}^t \boldsymbol{x}_s \boldsymbol{x}_s^\top - \sum_{s=1}^t \rho_s \boldsymbol{I}$$

which implies the desired result. □

In the following lemma, we show a sketch-specific version of the determinant-trace inequality (Lemma 6). When the spectral error is small, the right-hand side of the inequality depends on the sketch size $m$ rather than the ambient dimension $d$.

**Lemma 8.**

$$\ln\left(\frac{\det(\boldsymbol{V}_t)}{\det(\lambda \boldsymbol{I})}\right) \le d \ln\left(1 + \frac{\bar{\rho}}{\lambda}\right) + m \ln\left(1 + \frac{tL^2}{m\lambda}\right) .$$

*Proof.* Let $\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_d \ge 0$ be the eigenvalues of $\boldsymbol{S}_t^\top \boldsymbol{S}_t$. We start by looking at the ratio of determinants. Using Proposition 3 we can write

$$\begin{aligned} \frac{\det(\boldsymbol{V}_t)}{\det(\lambda \boldsymbol{I})} &= \frac{\det\left(\boldsymbol{S}_s^\top \boldsymbol{S}_s + \bar{\rho}_s \boldsymbol{I} + \lambda \boldsymbol{I}\right)}{\det(\lambda \boldsymbol{I})} = \prod_{i=1}^d \left(\frac{\widetilde{\lambda}_i}{\lambda} + 1 + \frac{\bar{\rho}}{\lambda}\right) \\ &= \left(1 + \frac{\bar{\rho}}{\lambda}\right)^{d-m} \prod_{i=1}^m \left(\frac{\widetilde{\lambda}_i}{\lambda} + 1 + \frac{\bar{\rho}}{\lambda}\right) \end{aligned} \tag{19}$$

since $\widetilde{\lambda}_{m+1} = \dots = \widetilde{\lambda}_d = 0$ because $\boldsymbol{S}_t^\top \boldsymbol{S}_t$ has rank at most $m$. We now use the AM-GM inequality, stating that

$$\left(\prod_{i=1}^m \alpha_i\right)^{\frac{1}{m}} \le \frac{1}{m} \sum_{i=1}^m \alpha_i \qquad \forall\, \alpha_1, \dots, \alpha_m \ge 0 .$$

Using the AM-GM inequality, the product in (19) can be bounded as

$$\begin{aligned}
\prod_{i=1}^{m}\left(\frac{\widetilde{\lambda}_i}{\lambda}+1+\frac{\bar{\rho}}{\lambda}\right) &\le \left(1+\frac{\bar{\rho}}{\lambda}+\frac{1}{m\lambda}\sum_{i=1}^{m}\widetilde{\lambda}_i\right)^m \\
&= \left(1+\frac{\bar{\rho}}{\lambda}+\frac{\operatorname{tr}(\boldsymbol{S}_t^\top \boldsymbol{S}_t)}{m\lambda}\right)^m \\
&\le \left(1+\frac{\bar{\rho}}{\lambda}+\frac{tL^2}{m\lambda}\right)^m \qquad (20)
\end{aligned}$$

where the last inequality holds because

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{S}_t^\top \boldsymbol{S}_t) &= \operatorname{tr}\left(\widetilde{\boldsymbol{V}}_t-\lambda\boldsymbol{I}\right) \\
&\le \operatorname{tr}\left(\boldsymbol{V}_t-\lambda\boldsymbol{I}\right) && \text{(by Proposition 3)} \\
&= \sum_{s=1}^{t}\operatorname{tr}(\boldsymbol{x}_s\boldsymbol{x}_s^\top) && \text{(by definition of } \boldsymbol{V}_t\text{)} \\
&\le tL^2\ .
\end{aligned}$$

Finally, substituting (20) into (19) and taking logs on both sides gives

$$\begin{aligned}
\ln\left(\frac{\det(\boldsymbol{V}_t)}{\det(\lambda\boldsymbol{I})}\right) &\le (d-m)\ln\left(1+\frac{\bar{\rho}}{\lambda}\right)+m\ln\left(1+\frac{\bar{\rho}}{\lambda}+\frac{tL^2}{m\lambda}\right) \\
&= d\ln\left(1+\frac{\bar{\rho}}{\lambda}\right)+m\ln\left(1+\frac{\frac{tL^2}{m\lambda}}{1+\frac{\bar{\rho}}{\lambda}}\right) \\
&\le d\ln\left(1+\frac{\bar{\rho}}{\lambda}\right)+m\ln\left(1+\frac{tL^2}{m\lambda}\right)
\end{aligned}$$

concluding the proof. □

The next lemma is similar to (Abbasi-Yadkori et al., 2011, Lemma 11). However, now the statement depends on the sketched matrix $\widetilde{\boldsymbol{V}}_{t-1}$ instead of $\boldsymbol{V}_{t-1}$. Although we pay in terms of the spectral error $\varepsilon_m$, we also improve the dependence on the dimension from $d$ to $m$ whenever $\varepsilon_m$ is sufficiently small.

**Lemma 9** (Sketched leverage scores)**.**

$$\sum_{t=1}^{T}\min\left\{1,\|\boldsymbol{x}_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\} \le 2\left(1+\varepsilon_m\right)\left(d\ln\left(1+\varepsilon_m\right)+m\ln\left(1+\frac{TL^2}{m\lambda}\right)\right)\ . \qquad (21)$$

*Proof.* Throughout the proof, unless stated explicitly, we drop the subscripts containing $t$. Therefore, $\boldsymbol{V}=\boldsymbol{V}_{t-1}$, $\widetilde{\boldsymbol{V}}=\widetilde{\boldsymbol{V}}_{t-1}$, $\boldsymbol{x}=\boldsymbol{x}_t$, and $\bar{\rho}=\bar{\rho}_{t-1}$. Now suppose that $(\widetilde{\lambda}_i+\lambda,\widetilde{\boldsymbol{u}}_i)$ is an $i$-th eigenpair of $\widetilde{\boldsymbol{V}}$. Then, Proposition 3 implies that a corresponding eigenpair of $\boldsymbol{V}$ is $(\widetilde{\lambda}_i+\lambda+\bar{\rho},\widetilde{\boldsymbol{u}}_i)$. Using this fact we have that

$$\begin{aligned}
\|\boldsymbol{x}\|^2_{\boldsymbol{V}^{-1}} &= \boldsymbol{x}^\top\widetilde{\boldsymbol{V}}\widetilde{\boldsymbol{V}}^{-1}\boldsymbol{V}^{-1}\boldsymbol{x} \\
&= \boldsymbol{x}^\top\left(\sum_{i=1}^{d}\widetilde{\boldsymbol{u}}_i\widetilde{\boldsymbol{u}}_i^\top\frac{1}{\widetilde{\lambda}_i+\lambda}\,\frac{\widetilde{\lambda}_i+\lambda}{\widetilde{\lambda}_i+\lambda+\bar{\rho}}\right)\boldsymbol{x} \\
&\ge \frac{\lambda}{\lambda+\bar{\rho}}\boldsymbol{x}^\top\left(\sum_{i=1}^{d}\widetilde{\boldsymbol{u}}_i\widetilde{\boldsymbol{u}}_i^\top\frac{1}{\widetilde{\lambda}_i+\lambda}\right)\boldsymbol{x} \\
&= \frac{\lambda}{\lambda+\bar{\rho}}\|\boldsymbol{x}\|^2_{\widetilde{\boldsymbol{V}}^{-1}}\ .
\end{aligned}$$

Furthermore, this implies that

$$\begin{aligned}
&\min\left\{1, \frac{\lambda}{\lambda+\bar{\rho}}\|\boldsymbol{x}\|^2_{\widetilde{\boldsymbol{V}}^{-1}}\right\} \le \min\left\{1, \|\boldsymbol{x}\|^2_{\boldsymbol{V}^{-1}}\right\} \\
\implies\quad &\min\left\{1+\frac{\bar{\rho}}{\lambda}, \|\boldsymbol{x}\|^2_{\widetilde{\boldsymbol{V}}^{-1}}\right\} \le \left(1+\frac{\bar{\rho}}{\lambda}\right)\min\left\{1, \|\boldsymbol{x}\|^2_{\boldsymbol{V}^{-1}}\right\} && \text{(multiply both sides by } 1+\tfrac{\bar{\rho}}{\lambda}\text{)} \\
\implies\quad &\min\left\{1, \|\boldsymbol{x}\|^2_{\widetilde{\boldsymbol{V}}^{-1}}\right\} \le \left(1+\frac{\bar{\rho}}{\lambda}\right)\min\left\{1, \|\boldsymbol{x}\|^2_{\boldsymbol{V}^{-1}}\right\} \ .
\end{aligned}$$

Finally, combining the above with Lemma 7, equation (17), and using the fact that $\bar{\rho}_{t-1} \le \bar{\rho}_T$, we obtain

$$\begin{aligned}
\sum_{t=1}^T \min\left\{1, \|\boldsymbol{x}_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\} &\le 2\left(1+\frac{\bar{\rho}_T}{\lambda}\right)\ln\left(\frac{\det(\boldsymbol{V}_T)}{\det(\lambda I)}\right) \\
&\le 2\left(1+\frac{\bar{\rho}_T}{\lambda}\right)\left(d\ln\left(1+\frac{\bar{\rho}_T}{\lambda}\right) + m\ln\left(1+\frac{TL^2}{m\lambda}\right)\right) && \text{(by Lemma 8)} \\
&\le 2\left(1+\varepsilon_m\right)\left(d\ln\left(1+\varepsilon_m\right) + m\ln\left(1+\frac{TL^2}{m\lambda}\right)\right)
\end{aligned}$$

where the last inequality follows from Proposition 1. □

Now we prove Theorem 2, characterizing the confidence ellipsoid generated by the sketched estimate.

**Theorem 2** (Sketched confidence ellipsoid – restated). *For any $\delta \in (0,1)$, the optimal parameter $\boldsymbol{w}^\star$ belongs to the set*

$$\widetilde{C}_t \equiv \left\{\boldsymbol{w} \in \mathbb{R}^d \ : \ \|\boldsymbol{w} - \widetilde{\boldsymbol{w}}_t\|_{\widetilde{\boldsymbol{V}}_t} \le \widetilde{\beta}_t(\delta)\right\}$$

*with probability at least $1-\delta$, where*

$$\begin{aligned}
\widetilde{\beta}_t(\delta) &= R\sqrt{m\ln\left(1+\frac{tL^2}{m\lambda}\right) + 2\ln\left(\frac{1}{\delta}\right) + d\ln\left(1+\frac{\bar{\rho}_t}{\lambda}\right)}\sqrt{1+\frac{\bar{\rho}_t}{\lambda}} + S\sqrt{\lambda}\left(1+\frac{\bar{\rho}_t}{\lambda}\right) \\
&\overset{\widetilde{\mathcal{O}}}{=} R\sqrt{\left(m + d\ln(1+\varepsilon_m)\right)\left(1+\varepsilon_m\right)} + S\sqrt{\lambda}\left(1+\varepsilon_m\right) \ .
\end{aligned}$$

*Proof.* Throughout the proof we frequently use Proposition 3, implying $\boldsymbol{X}_t^\top \boldsymbol{X}_t = \boldsymbol{S}_t^\top \boldsymbol{S}_t + \bar{\rho}_t \boldsymbol{I}$. For brevity, in the following we drop subscripts containing $t$ in matrices. Let $\boldsymbol{\eta}_t = (\eta_1, \eta_2 \dots, \eta_t)$, and by definition of the sketched estimate we have that

$$\begin{aligned}
\widetilde{\boldsymbol{w}}_t &= \left(\boldsymbol{S}_t^\top \boldsymbol{S}_t + \lambda \boldsymbol{I}\right)^{-1} \boldsymbol{X}_t^\top \left(\boldsymbol{X}_t \boldsymbol{w}^\star + \boldsymbol{\eta}_t\right) \\
&= \left(\boldsymbol{S}_t^\top \boldsymbol{S}_t + \lambda \boldsymbol{I}\right)^{-1} \boldsymbol{X}_t^\top \boldsymbol{\eta}_t + \left(\boldsymbol{S}_t^\top \boldsymbol{S}_t + \lambda \boldsymbol{I}\right)^{-1} \boldsymbol{X}_t^\top \boldsymbol{X}_t \boldsymbol{w}^\star \\
&= \left(\boldsymbol{S}_t^\top \boldsymbol{S}_t + \lambda \boldsymbol{I}\right)^{-1} \boldsymbol{X}_t^\top \boldsymbol{\eta}_t \\
&\quad + \left(\boldsymbol{S}_t^\top \boldsymbol{S}_t + \lambda \boldsymbol{I}\right)^{-1} \left(\boldsymbol{X}_t^\top \boldsymbol{X}_t + (\lambda - \bar{\rho}_t)\boldsymbol{I}\right)\boldsymbol{w}^\star - (\lambda - \bar{\rho}_t)\left(\boldsymbol{S}_t^\top \boldsymbol{S}_t + \lambda \boldsymbol{I}\right)^{-1} \boldsymbol{w}^\star \\
&= \left(\boldsymbol{S}_t^\top \boldsymbol{S}_t + \lambda \boldsymbol{I}\right)^{-1} \boldsymbol{X}_t^\top \boldsymbol{\eta}_t + \boldsymbol{w}^\star - (\lambda - \bar{\rho}_t)\left(\boldsymbol{S}_t^\top \boldsymbol{S}_t + \lambda \boldsymbol{I}\right)^{-1} \boldsymbol{w}^\star \\
&= \widetilde{\boldsymbol{V}}_t^{-1} \boldsymbol{X}_t^\top \boldsymbol{\eta}_t + \boldsymbol{w}^\star - (\lambda - \bar{\rho}_t)\widetilde{\boldsymbol{V}}_t^{-1} \boldsymbol{w}^\star \ . \qquad (22)
\end{aligned}$$

Then, by (22), for any $\boldsymbol{x} \in \mathbb{R}^d$ we have that

$$\begin{aligned}
\boldsymbol{x}^\top\left(\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\right) &= \boldsymbol{x}^\top \widetilde{\boldsymbol{V}}_t^{-1} \boldsymbol{X}_t^\top \boldsymbol{\eta}_t - (\lambda - \bar{\rho}_t)\boldsymbol{x}^\top \widetilde{\boldsymbol{V}}_t^{-1} \boldsymbol{w}^\star && (23) \\
&\le \left\|\boldsymbol{x}^\top \widetilde{\boldsymbol{V}}_t^{-1}\right\|_{\boldsymbol{V}_t} \|\boldsymbol{X}_t^\top \boldsymbol{\eta}_t\|_{\boldsymbol{V}_t^{-1}} - (\lambda - \bar{\rho}_t)\left\langle \boldsymbol{x}, \boldsymbol{w}^\star \right\rangle_{\widetilde{\boldsymbol{V}}_t^{-1}} && \text{(by Cauchy-Schwartz)} \\
&\le \left\|\boldsymbol{x}^\top \widetilde{\boldsymbol{V}}_t^{-1}\right\|_{\boldsymbol{V}_t} \|\boldsymbol{X}_t^\top \boldsymbol{\eta}_t\|_{\boldsymbol{V}_t^{-1}} + |\lambda + \bar{\rho}_t|\left|\left\langle \boldsymbol{x}, \boldsymbol{w}^\star \right\rangle_{\widetilde{\boldsymbol{V}}_t^{-1}}\right| && \text{(by the triangle inequality.)}
\end{aligned}$$

We now choose $\boldsymbol{x} = \widetilde{\boldsymbol{V}}_t(\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star)$ and proceed by bounding terms in the above. By the choice of $\boldsymbol{x}$, we have that $\boldsymbol{x}^\top \left(\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\right) = \|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|^2_{\widetilde{\boldsymbol{V}}_t}$, $\left\|\boldsymbol{x}^\top \widetilde{\boldsymbol{V}}_t^{-1}\right\|_{\boldsymbol{V}_t} = \|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\boldsymbol{V}_t}$ and

$$\begin{aligned}
\langle \boldsymbol{x}, \boldsymbol{w}^\star \rangle_{\widetilde{\boldsymbol{V}}_t^{-1}} &= \left(\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\right)^\top \boldsymbol{w}^\star \\
&\le \|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_2 \, \|\boldsymbol{w}^\star\|_2 && \text{(by Cauchy-Schwartz)} \\
&\le \|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_2 \, S \ .
\end{aligned}$$

Finally, by Theorem 6, for any $\delta > 0$, with probability at least $1 - \delta$,

$$\|\boldsymbol{X}^\top \boldsymbol{\eta}_t\|_{\boldsymbol{V}_t^{-1}} \le \sqrt{B_t(\delta)} \qquad \forall t \ge 0 \ .$$

The left-hand side of (23) can now upper bounded as

$$\begin{aligned}
&\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|^2_{\widetilde{\boldsymbol{V}}_t} \le \sqrt{B_t(\delta)} \|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\boldsymbol{V}_t} + S(\lambda + \bar{\rho}_t)\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_2 \\
\Longrightarrow \quad &\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\widetilde{\boldsymbol{V}}_t} \le \sqrt{B_t(\delta)} \frac{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\boldsymbol{V}_t}}{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\widetilde{\boldsymbol{V}}_t}} + S(\lambda + \bar{\rho}_t) \frac{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_2}{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\widetilde{\boldsymbol{V}}_t}} \ .
\end{aligned} \tag{24}$$

Now we handle the ratios of norms in the right-hand side of (24). First,

$$\begin{aligned}
\frac{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\boldsymbol{V}_t}}{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\widetilde{\boldsymbol{V}}_t}} &= \sqrt{\frac{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|^2_{\widetilde{\boldsymbol{V}}_t} + \bar{\rho}_t \|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_2^2}{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|^2_{\widetilde{\boldsymbol{V}}_t}}} \\
&= \sqrt{1 + \bar{\rho}_t \frac{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_2^2}{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|^2_{\widetilde{\boldsymbol{V}}_t}}} \\
&\le \sqrt{1 + \frac{\bar{\rho}_t}{\lambda}}
\end{aligned}$$

since $\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|^2_{\widetilde{\boldsymbol{V}}_t} \ge \lambda \|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_2^2$ and, using the same reasoning,

$$\frac{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_2}{\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\widetilde{\boldsymbol{V}}_t}} \le \frac{1}{\sqrt{\lambda}} \ .$$

Substituting these into (24) gives

$$\|\widetilde{\boldsymbol{w}}_t - \boldsymbol{w}^\star\|_{\widetilde{\boldsymbol{V}}_t} \le \sqrt{B_t(\delta)\left(1 + \frac{\bar{\rho}_t}{\lambda}\right)} + S\sqrt{\lambda}\left(1 + \frac{\bar{\rho}_t}{\lambda}\right) \ .$$

Now we provide a deterministic bound on $B_t(\delta)$. Using Lemma 8 we have

$$\begin{aligned}
\sqrt{B_t(\delta)} &= R\sqrt{2 \ln\left(\frac{1}{\delta} \det\left(\boldsymbol{V}_t\right)^{\frac{1}{2}} \det\left(\lambda \boldsymbol{I}\right)^{-\frac{1}{2}}\right)} \\
&\le R\sqrt{d \ln\left(1 + \frac{\bar{\rho}_t}{\lambda}\right) + m \ln\left(1 + \frac{tL^2}{m\lambda}\right) + 2\ln\left(\frac{1}{\delta}\right)} \ .
\end{aligned}$$

This proves the first statement (10). Finally, (11) follows by Proposition 1, that is $1 + \bar{\rho}_t/\lambda \le 1 + \varepsilon_m$. $\square$

We close this section by computing a closed form for $\widetilde{\boldsymbol{V}}_t^{-\frac{1}{2}}$, the square root of the inverse of the sketched correlation matrix. This is used by sketched linear TS for selecting actions. We make use of the following result —see. e.g., (Higham, 2008, Theorem 1.35).

**Theorem 7** (Generalized Woodbury matrix identity)**.** *Let $\boldsymbol{A} \in \mathbb{C}^{d\times m}$ and $\boldsymbol{B} \in \mathbb{C}^{m\times d}$, with $d \ge m$, and assume that $\boldsymbol{BA}$ is nonsingular. Let $f$ be defined on the spectrum of $\alpha \boldsymbol{I}_{d\times d} + \boldsymbol{AB}$, and if $d = m$ let $f$ be defined at $\alpha$. Then $f(\alpha \boldsymbol{I}_{d\times d} + \boldsymbol{AB}) = f(\alpha \boldsymbol{I}_{d\times d}) + \boldsymbol{A}(\boldsymbol{BA})^{-1}\left(f(\alpha \boldsymbol{I}_{m\times m} + \boldsymbol{BA}) - f(\alpha \boldsymbol{I}_{m\times m})\right)\boldsymbol{B}$.*

This is used to prove the following.

**Corollary 1.** *For $\lambda > 0$, let*

$$\boldsymbol{S}'_t = \left(\boldsymbol{\Sigma}_t + \left(\frac{\lambda}{2} - \rho_t\right)\boldsymbol{I}_{m\times m}\right)^{\frac{1}{2}}\boldsymbol{U}_t \ .$$

*Then*

$$\widetilde{\boldsymbol{V}}_t^{-\frac{1}{2}} = \boldsymbol{S}_t^{'\top}\left(\boldsymbol{S}'_t\boldsymbol{S}_t^{'\top}\right)^{-1}\left(\frac{\lambda}{2}\boldsymbol{I} + \boldsymbol{S}'_t\boldsymbol{S}_t^{'\top}\right)^{-\frac{1}{2}}\boldsymbol{S}'_t \ .$$

*Proof.* We apply Theorem 7 with $f(\widetilde{\boldsymbol{V}}) = \widetilde{\boldsymbol{V}}_t^{-\frac{1}{2}}$. However, since $\boldsymbol{S}_t\boldsymbol{S}_t^\top$ is singular by design, we apply the theorem with $\boldsymbol{B}$ set the non-singular proxy matrix $\boldsymbol{S}'_t$, $\boldsymbol{A}$ set to $\boldsymbol{S}_t^{'\top}$, and $\alpha$ set to $\lambda/2$. Thus $\widetilde{\boldsymbol{V}}_t = \boldsymbol{S}_t^{'\top}\boldsymbol{S}'_t + \frac{\lambda}{2}\boldsymbol{I}_{d\times d}$ and

$$\begin{aligned}
\left(\boldsymbol{S}_t^{'\top}\boldsymbol{S}'_t + \frac{\lambda}{2}\boldsymbol{I}_{d\times d}\right)^{-\frac{1}{2}} &= \sqrt{\frac{2}{\lambda}}\boldsymbol{I}_{m\times m} + \boldsymbol{S}_t^{'\top}\left(\boldsymbol{S}'_t\boldsymbol{S}_t^{'\top}\right)^{-1}\left(\left(\frac{\lambda}{2}\boldsymbol{I}_{m\times m} + \boldsymbol{S}'_t\boldsymbol{S}_t^{'\top}\right)^{-\frac{1}{2}} - \sqrt{\frac{2}{\lambda}}\boldsymbol{I}_{m\times m}\right)\boldsymbol{S}'_t \\
&= \boldsymbol{S}_t^{'\top}\left(\boldsymbol{S}'_t\boldsymbol{S}_t^{'\top}\right)^{-1}\left(\frac{\lambda}{2}\boldsymbol{I}_{m\times m} + \boldsymbol{S}'_t\boldsymbol{S}_t^{'\top}\right)^{-\frac{1}{2}}\boldsymbol{S}'_t \qquad (25)
\end{aligned}$$

where (25) follows since $\boldsymbol{S}_t^{'\top}\left(\boldsymbol{S}'_t\boldsymbol{S}_t^{'\top}\right)^{-1}\boldsymbol{S}'_t = \boldsymbol{I}_{m\times m}$. □

### A.3 Proof of the regret bound for SOFUL (Theorem 3)

We start with a preliminary lemma.

**Lemma 10.** *For any $\delta > 0$, the instantaneous regret of SOFUL satisfies*

$$(\boldsymbol{x}_t^\star - \boldsymbol{x}_t)^\top\boldsymbol{w}^\star \le 2\widetilde{\beta}_{t-1}(\delta)\|\boldsymbol{x}_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \qquad t = 1,\dots,T \ .$$

*Proof.* Let $\widetilde{\boldsymbol{w}}_{t-1}^{\mathrm{SO}}$ be the FD-sketched RLS estimate of OFUL (Algorithm 5). Recall that the optimal action at time $t$ is $\boldsymbol{x}_t^\star = \arg\max_{\boldsymbol{x}\in D_t}\boldsymbol{x}^\top\boldsymbol{w}^\star$, whereas

$$\left(\boldsymbol{x}_t, \widetilde{\boldsymbol{w}}_{t-1}^{\mathrm{SO}}\right) = \mathop{\arg\max}_{(\boldsymbol{x},\boldsymbol{w})\in D_t\times\widetilde{C}_{t-1}} \boldsymbol{x}^\top\boldsymbol{w} \ .$$

We use these facts to bound the instantaneous regret,

$$\begin{aligned}
\left(\boldsymbol{x}_t^\star - \boldsymbol{x}_t\right)^\top\boldsymbol{w}^\star &\le \boldsymbol{x}_t^\top\widetilde{\boldsymbol{w}}_{t-1}^{\mathrm{SO}} - \boldsymbol{x}_t^\top\boldsymbol{w}^\star \\
&= \boldsymbol{x}_t^\top\left(\widetilde{\boldsymbol{w}}_{t-1}^{\mathrm{SO}} - \boldsymbol{w}^\star\right) \\
&= \boldsymbol{x}_t^\top\left(\widetilde{\boldsymbol{w}}_{t-1}^{\mathrm{SO}} - \widetilde{\boldsymbol{w}}_{t-1}\right) + \boldsymbol{x}_t^\top\left(\widetilde{\boldsymbol{w}}_{t-1} - \boldsymbol{w}^\star\right) \\
&\le \|\boldsymbol{x}_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\left(\|\widetilde{\boldsymbol{w}}_{t-1}^{\mathrm{SO}} - \widetilde{\boldsymbol{w}}_{t-1}\|_{\widetilde{\boldsymbol{V}}_{t-1}} + \|\widetilde{\boldsymbol{w}}_{t-1} - \boldsymbol{w}^\star\|_{\widetilde{\boldsymbol{V}}_{t-1}}\right) && \text{(by Cauchy-Schwartz)} \\
&\le 2\widetilde{\beta}_{t-1}(\delta)\|\boldsymbol{x}_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} && \text{(by Theorem 2)}
\end{aligned}$$

concluding the proof. □

Now we are ready to prove the regret bound.

*Proof of Theorem 3.* Bounding the regret using Lemma 10 gives

$$
\begin{aligned}
R_T &= \sum_{t=1}^{T} \left(\boldsymbol{x}_t^\star - \boldsymbol{x}_t\right)^\top \boldsymbol{w}^\star \\
&\leq 2 \sum_{t=1}^{T} \min\left\{LS, \widetilde{\beta}_{t-1}(\delta) \|\boldsymbol{x}_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\} && \left(\text{since } \max_{t=1,\dots,T} \max_{\boldsymbol{x}\in D_t} |\boldsymbol{x}^\top \boldsymbol{w}^\star| \leq LS \text{ by Cauchy-Schwartz}\right) \\
&\leq 2 \sum_{t=1}^{T} \widetilde{\beta}_{t-1}(\delta) \min\left\{\frac{L}{\sqrt{\lambda}}, \|\boldsymbol{x}_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\} && \left(\text{since } \min_{t=0,\dots,T-1} \min_{\delta\in[0,1]} \widetilde{\beta}_t(\delta) \geq S\sqrt{\lambda}\right) \\
&\leq 2 \left(\max_{t=0,\dots,T-1} \widetilde{\beta}_t(\delta)\right) \sum_{t=1}^{T} \min\left\{\frac{L}{\sqrt{\lambda}}, \|\boldsymbol{x}_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\} \\
&\leq 2 \max\left\{1, \frac{L}{\sqrt{\lambda}}\right\} \left(\max_{t=0,\dots,T-1} \widetilde{\beta}_t(\delta)\right) \sum_{t=1}^{T} \min\left\{1, \|\boldsymbol{x}_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\} \\
&\leq 2 \max\left\{1, \frac{L}{\sqrt{\lambda}}\right\} \left(\max_{t=0,\dots,T-1} \widetilde{\beta}_t(\delta)\right) \sqrt{T \sum_{t=1}^{T} \min\left\{1, \|\boldsymbol{x}_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\}}\ . && (\text{by Cauchy-Schwartz})
\end{aligned}
$$

Now we finish by further bounding the terms in the above. In particular, we bound $\widetilde{\beta}_t(\delta)$ by (11)

$$
\max_{t=0,\dots,T-1} \widetilde{\beta}_t(\delta) \overset{\widetilde{\mathcal{O}}}{=} R\sqrt{\big(m + d\ln(1+\varepsilon_m)\big)\left(1+\varepsilon_m\right)} + S\sqrt{\lambda}\left(1+\varepsilon_m\right)
$$

while the bound on the summation term uses Lemma 9,

$$
\sqrt{\sum_{t=1}^{T} \min\left\{1, \|X_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\}} \overset{\widetilde{\mathcal{O}}}{=} \sqrt{\left(1+\varepsilon_m\right)\big(d\ln\left(1+\varepsilon_m\right) + m\big)}\ .
$$

Then, using $C_\lambda = \max\left\{1, \frac{L}{\sqrt{\lambda}}\right\}$ and $\widetilde{m} = m + d\ln(1+\varepsilon_m)$,

$$
\begin{aligned}
R_T &\overset{\widetilde{\mathcal{O}}}{=} C_\lambda \sqrt{T}\left(R\sqrt{\widetilde{m}\left(1+\varepsilon_m\right)} + S\sqrt{\lambda}\left(1+\varepsilon_m\right)\right)\sqrt{\widetilde{m}\left(1+\varepsilon_m\right)} \\
&\overset{\widetilde{\mathcal{O}}}{=} C_\lambda \sqrt{T}\Big(R\,\widetilde{m}\left(1+\varepsilon_m\right) + S\sqrt{\lambda}\left(1+\varepsilon_m\right)^{\frac{3}{2}}\sqrt{\widetilde{m}}\Big) \\
&\overset{\widetilde{\mathcal{O}}}{=} C_\lambda \left(1+\varepsilon_m\right)^{\frac{3}{2}} \widetilde{m}\left(R + S\sqrt{\lambda}\right)\sqrt{T}
\end{aligned}
$$

which completes the proof. □

*Proof of Theorem 4.* Recall that

$$
\Delta \leq \min_{t=1,\dots,T} \left(\boldsymbol{x}_t^\star - \boldsymbol{x}_t\right)^\top \boldsymbol{w}^\star\ .
$$

Similarly to the proof of Theorem 3, we use Lemma 10 to bound the instantaneous regret. However, we first use the gap assumption to bound the regret in terms of the sum of squared instantaneous regrets,

$$
\begin{aligned}
R_T &= \sum_{t=1}^{T} \left(\boldsymbol{x}_t^\star - \boldsymbol{x}_t\right)^\top \boldsymbol{w}^\star \\
&\le \frac{1}{\Delta} \sum_{t=1}^{T} \left( \left(\boldsymbol{x}_t^\star - \boldsymbol{x}_t\right)^\top \boldsymbol{w}^\star \right)^2 \\
&\le \frac{2}{\Delta} \sum_{t=1}^{T} \min\left\{ 2L^2 S^2, \widetilde{\beta}_{t-1}(\delta)^2 \|\boldsymbol{x}_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \right\} && (26) \\
&\le \frac{2}{\Delta} \left( \max_{t=0,\dots,T-1} \widetilde{\beta}_t(\delta)^2 \right) \sum_{t=1}^{T} \min\left\{ \frac{2L^2}{\lambda}, \|\boldsymbol{x}_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \right\} && (27) \\
&\le \frac{2}{\Delta} \max\left\{1, \frac{2L^2}{\lambda}\right\} \left( \max_{t=0,\dots,T-1} \widetilde{\beta}_t(\delta)^2 \right) \sum_{t=1}^{T} \min\left\{ 1, \|\boldsymbol{x}_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \right\} && (28)
\end{aligned}
$$

where (27) holds because $\min_t \min_\delta \widetilde{\beta}_t(\delta)^2 \ge S^2\lambda$. Inequality (26) holds because

$$
\begin{aligned}
\left( \left(\boldsymbol{x}_t^\star - \boldsymbol{x}_t\right)^\top \boldsymbol{w}^\star \right)^2 &\le 2(\boldsymbol{x}_t^{\star\top} \boldsymbol{w}^\star)^2 + 2(\boldsymbol{x}_t^\top \boldsymbol{w}^\star)^2 \\
&\le 4L^2 S^2 && \text{(by Cauchy-Schwartz)}
\end{aligned}
$$

and because of Lemma 10.

We now finish bounding the regret by further bounding the individual terms in (28). In particular, we use (11) to bound $\widetilde{\beta}_t(\delta)$ as follows

$$
\begin{aligned}
\max_{t=0,\dots,T-1} \widetilde{\beta}_t(\delta)^2 &\stackrel{\widetilde{\mathcal{O}}}{=} R^2 \left( \sqrt{\big(m + d\ln(1+\varepsilon_m)\big)\left(1+\varepsilon_m\right)} + S\sqrt{\lambda}\left(1+\varepsilon_m\right) \right)^2 \\
&\stackrel{\widetilde{\mathcal{O}}}{=} R^2 \big(m + d\ln(1+\varepsilon_m)\big)\left(1+\varepsilon_m\right) + S^2\lambda\left(1+\varepsilon_m\right)^2 \ .
\end{aligned}
$$

Lemma 9 gives

$$
\sum_{t=1}^{T} \min\left\{1, \|X_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\} \stackrel{\widetilde{\mathcal{O}}}{=} \left(1+\varepsilon_m\right) \Big( m\ln(T) + d\ln\left(1+\varepsilon_m\right) \Big) \ .
$$

Then, using again $C_\lambda = \max\left\{1, \frac{L}{\sqrt{\lambda}}\right\}$ and $\widetilde{m} = m + d\ln(1+\varepsilon_m)$,

$$
\begin{aligned}
R_T &\stackrel{\widetilde{\mathcal{O}}}{=} \frac{C_\lambda^2}{\Delta} \left( R^2 \widetilde{m}\left(1+\varepsilon_m\right) + S^2\lambda\left(1+\varepsilon_m\right)^2 \right)\left(1+\varepsilon_m\right)\widetilde{m} \\
&\stackrel{\widetilde{\mathcal{O}}}{=} \frac{C_\lambda^2}{\Delta} \left(\widetilde{m} R^2 + S^2\lambda\right)\left(1+\varepsilon_m\right)^3 \widetilde{m} \\
&\stackrel{\widetilde{\mathcal{O}}}{=} \frac{C_\lambda^2}{\Delta} \left(R^2 + S^2\lambda\right)\left(1+\varepsilon_m\right)^3 \widetilde{m}^2
\end{aligned}
$$

concluding the proof. $\square$

### A.4 Proof of the regret bound for Sketched Linear TS (Theorem 5)

Here $\widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}}$ is used to denote the FD-sketched RLS estimate of linear TS (Algorithm 6). As in (Abeille and Lazaric, 2017), we split the regret as follows

$$\begin{aligned} R_T &= \sum_{t=1}^{T} \left(\boldsymbol{x}_t^\star - \boldsymbol{x}_t\right)^\top \boldsymbol{w}^\star \\ &= \sum_{t=1}^{T} \left(\boldsymbol{x}_t^{\star\top} \boldsymbol{w}^\star - \boldsymbol{x}_t^\top \widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}}\right) + \sum_{t=1}^{T} \left(\boldsymbol{x}_t^\top \widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}} - \boldsymbol{x}_t^\top \boldsymbol{w}^\star\right) \\ &= \sum_{t=1}^{T} \left(J_t(\boldsymbol{w}^\star) - J_t(\widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}})\right) + \sum_{t=1}^{T} \left(\boldsymbol{x}_t^\top \widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}} - \boldsymbol{x}_t^\top \boldsymbol{w}^\star\right) \end{aligned} \tag{29}$$

where

$$J_t(\boldsymbol{w}) = \max_{\boldsymbol{x} \in D_t} \boldsymbol{x}^\top \boldsymbol{w}$$

is an "optimistic" reward function. Most of the proof is concerned with bounding the first term in (29). The second term is instead obtained in way similar to the analysis of OFUL. Fix any $\delta \in (0,1)$, let $\delta' = \frac{\delta}{4T}$, and introduce events

$$\begin{aligned} \widetilde{E}_t &\equiv \left\{ \|\widetilde{\boldsymbol{w}}_s - \boldsymbol{w}^\star\| \le \widetilde{\beta}_s(\delta'),\ s = 1, \dots, t \right\} \\ \widetilde{E}_t^{\text{TS}} &\equiv \left\{ \|\widetilde{\boldsymbol{w}}_s^{\text{TS}} - \widetilde{\boldsymbol{w}}_s\| \le \widetilde{\gamma}_s(\delta'),\ s = 1, \dots, t \right\} \end{aligned}$$

and $E_t \equiv \widetilde{E}_t \cap \widetilde{E}_t^{\text{TS}}$. Observe that, by definition,

$$\widetilde{E}_T \subset \cdots \subset \widetilde{E}_1 \qquad \text{and} \qquad \widetilde{E}_T^{\text{TS}} \subset \cdots \subset \widetilde{E}_1^{\text{TS}} \tag{30}$$

We also use the following lower bound on the probability of $E_T$.

**Lemma 11.** $\mathbb{P}\left(E_T\right) \ge 1 - \dfrac{\delta}{2}$.

*Proof.* The proof is identical to the proof of (Abeille and Lazaric, 2017, Lemma 1), the only difference being that we use the confidence ellipsoid defined in Theorem 2. □

We study the regret when $E_T$ occurs,

$$\begin{aligned} \mathbb{I}\{E_T\}\, R_T &= \sum_{t=1}^{T} \mathbb{I}\{E_T\} \left(J_t(\boldsymbol{w}^\star) - J_t(\widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}})\right) + \sum_{t=1}^{T} \mathbb{I}\{E_T\} \left(\boldsymbol{x}_t^\top \widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}} - \boldsymbol{x}_t^\top \boldsymbol{w}^\star\right) \\ &\le \sum_{t=1}^{T} \mathbb{I}\{E_{t-1}\} \left(J_t(\boldsymbol{w}^\star) - J_t(\widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}})\right) + \sum_{t=1}^{T} \mathbb{I}\{E_{t-1}\} \left(\boldsymbol{x}_t^\top \widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}} - \boldsymbol{x}_t^\top \boldsymbol{w}^\star\right) && \text{(using (30))} \\ &= \sum_{t=1}^{T} r_t^{\text{TS}} + \sum_{t=1}^{T} r_t^{\text{RLS}} && (31) \end{aligned}$$

where we introduced the notation

$$r_t^{\text{TS}} = \mathbb{I}\{E_{t-1}\} \left(J_t(\boldsymbol{w}^\star) - J_t(\widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}})\right) \qquad \text{and} \qquad r_t^{\text{RLS}} = \mathbb{I}\{E_{t-1}\} \left(\boldsymbol{x}_t^\top \widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}} - \boldsymbol{x}_t^\top \boldsymbol{w}^\star\right)\ .$$

First we focus on $r_t^{\text{TS}}$, and get that

$$\begin{aligned} r_t^{\text{TS}} &= \left(J_t(\boldsymbol{w}^\star) - J_t(\widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}})\right) \mathbb{I}\{E_{t-1}\} \\ &\le \left(J_t(\boldsymbol{w}^\star) - \inf_{\boldsymbol{w} \in \widetilde{C}_{t-1}^{\text{TS}}} J_t(\boldsymbol{w})\right) \mathbb{I}\{E_{t-1}\} && \left(\text{because } E_{t-1} \text{ implies } \widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}} \in \widetilde{C}_{t-1}^{\text{TS}}\right) \\ &\le \left(J_t(\boldsymbol{w}^\star) - \inf_{\boldsymbol{w} \in \widetilde{C}_{t-1}^{\text{TS}}} J_t(\boldsymbol{w})\right) \mathbb{I}\left\{\widetilde{E}_{t-1}\right\}\ . && \text{(using (30))} \end{aligned}$$

Consider the following set of "optimistic" coefficients $\boldsymbol{w}$ such that $J_t(\boldsymbol{w}^\star) \le J_t(\boldsymbol{w})$ and, moreover, $\boldsymbol{w}$ belongs to the sketched TS confidence ellipsoid,

$$W_t^{\text{OPT-TS}} \equiv \left\{\boldsymbol{w} \in \mathbb{R}^d \ : \ J_t(\boldsymbol{w}^\star) \le J_t(\boldsymbol{w})\right\} \cap \widetilde{C}_t^{\text{TS}} \ .$$

Then, for $\widetilde{\boldsymbol{w}}^{\text{TS}} \in W_{t-1}^{\text{OPT-TS}}$

$$r_t^{\text{TS}} \le \left(J_t(\widetilde{\boldsymbol{w}}^{\text{TS}}) - \inf_{\boldsymbol{w} \in \widetilde{C}_{t-1}^{\text{TS}}} J_t(\boldsymbol{w})\right) \mathbb{I}\left\{\widetilde{E}_{t-1}\right\} \ . \tag{32}$$

We now use (Abeille and Lazaric, 2017, Proposition 3 and Lemma 2) (restated below here for convenience) to argue about the convexity of $J$ and relate its gradient to the chosen action.

**Proposition 4.** *For any finite set $D$ of actions $\boldsymbol{x}$ such that $\|\boldsymbol{x}\| \le 1$, $\max_{\boldsymbol{x} \in D} \boldsymbol{x}^\top \boldsymbol{w}$ is convex on $\mathbb{R}^d$. Moreover, it is continuous with continuous first derivatives (except for a zero-measure set w.r.t. the Lebesgue measure).*

**Lemma 12.** *For any $\boldsymbol{w} \in \mathbb{R}^d$, we have*

$$\nabla\Big(\max_{\boldsymbol{x} \in D} \boldsymbol{x}^\top \boldsymbol{w}\Big) = \arg\max_{\boldsymbol{x} \in D} \boldsymbol{x}^\top \boldsymbol{w}$$

*(except for a zero-measure w.r.t. the Lebesgue measure).*

Relying on the two results above, we can proceed as follows. Introduce $J_t^{/L}(\boldsymbol{w}) = J_t(\boldsymbol{w})/L = \max_{\boldsymbol{x} \in D_t} (\boldsymbol{x}/L)^\top \boldsymbol{w}$. Then by Proposition 4, $J_t^{/L}(\boldsymbol{w})$ is convex for $\boldsymbol{w} \in \mathbb{R}^d$ since $\|\boldsymbol{x}/L\| \le 1$. Then, by letting $\boldsymbol{x}^\star(\widetilde{\boldsymbol{w}}^{\text{TS}}) = \nabla J_t(\widetilde{\boldsymbol{w}}^{\text{TS}})$, for any $\widetilde{\boldsymbol{w}}^{\text{TS}} \in W_{t-1}^{\text{OPT-TS}}$ we have

$$\begin{aligned}
J_t(\widetilde{\boldsymbol{w}}^{\text{TS}}) - \inf_{\boldsymbol{w} \in \widetilde{C}_{t-1}^{\text{TS}}} J_t(\boldsymbol{w}) &= L\left(J_t^{/L}(\widetilde{\boldsymbol{w}}^{\text{TS}}) - \inf_{\boldsymbol{w} \in \widetilde{C}_{t-1}^{\text{TS}}} J_t^{/L}(\boldsymbol{w})\right) \\
&\le L \sup_{\boldsymbol{w} \in \widetilde{C}_{t-1}^{\text{TS}}} \left\{\nabla J_t^{/L}(\widetilde{\boldsymbol{w}}^{\text{TS}})^\top \left(\widetilde{\boldsymbol{w}}^{\text{TS}} - \boldsymbol{w}\right)\right\} \\
&= L \sup_{\boldsymbol{w} \in \widetilde{C}_{t-1}^{\text{TS}}} \left\{\left(\frac{\boldsymbol{x}^\star(\widetilde{\boldsymbol{w}}^{\text{TS}})}{L}\right)^\top \left(\widetilde{\boldsymbol{w}}^{\text{TS}} - \boldsymbol{w}\right)\right\} \\
&\le \|\boldsymbol{x}^\star(\widetilde{\boldsymbol{w}}^{\text{TS}})\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \sup_{\boldsymbol{w} \in \widetilde{C}_{t-1}^{\text{TS}}} \|\widetilde{\boldsymbol{w}}^{\text{TS}} - \boldsymbol{w}\|_{\widetilde{\boldsymbol{V}}_{t-1}} && \text{(by Cauchy-Schwartz)} \\
&\le 2\widetilde{\gamma}_{t-1}(\delta') \|\boldsymbol{x}^\star(\widetilde{\boldsymbol{w}}^{\text{TS}})\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}
\end{aligned}$$

where the last inequality holds for all $\widetilde{\boldsymbol{w}}^{\text{TS}} \in \widetilde{C}_{t-1}^{\text{TS}}$ and by the triangle inequality. Substituting this into (32), and taking expectation with respect to $\widetilde{\boldsymbol{w}}^{\text{TS}}$ yields

$$r_t^{\text{TS}} \le 2\widetilde{\gamma}_{t-1}(\delta') \, \mathbb{E}\left[\|\boldsymbol{x}^\star(\widetilde{\boldsymbol{w}}^{\text{TS}})\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \mathbb{I}\left\{\widetilde{E}_{t-1}\right\} \,\middle|\, \widetilde{\boldsymbol{w}}^{\text{TS}} \in W_{t-1}^{\text{OPT-TS}}, \mathcal{F}_{t-1}\right] \ . \tag{33}$$

where we use $\mathcal{F}_t$ to denote the $\sigma$-algebra generated by the random variables $\eta_1, \boldsymbol{Z}_1, \ldots, \eta_{t-1}, \boldsymbol{Z}_{t-1}$. Now we further upper bound $r_t^{\text{TS}}$ while bounding the probability of event $\widetilde{\boldsymbol{w}}^{\text{TS}} \in W_{t-1}^{\text{OPT-TS}}$ occurring in (33). This is done in the following lemma, whose proof (omitted here) is identical to the proof of (Abeille and Lazaric, 2017, Lemma 3), where ellipsoids are replaced by their sketched counterparts.

**Lemma 13.** *Assume that $\mathcal{D}^{\text{TS}}$ is a TS-sampling distribution with anti-concentration parameter $p$. Then, for $\boldsymbol{Z} \sim \mathcal{D}^{\text{TS}}$ we have that*

$$\mathbb{P}\left(\widetilde{\boldsymbol{w}}^{\text{TS}} \in W_{t-1}^{\text{OPT-TS}} \,\middle|\, \widetilde{E}_{t-1}, \mathcal{F}_{t-1}\right) \ge \frac{p}{2} \qquad t = 1, \ldots, T \ .$$

We now proceed with the main argument of the proof. Using $g(\widetilde{\boldsymbol{w}}^{\text{TS}}) = \|\boldsymbol{x}^\star(\widetilde{\boldsymbol{w}}^{\text{TS}})\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}$,

$$\begin{aligned}
\mathbb{E}\left[g(\widetilde{\boldsymbol{w}}^{\text{TS}}) \,\middle|\, \widetilde{E}_{t-1}, \mathcal{F}_{t-1}\right] &\ge \mathbb{E}\left[g(\widetilde{\boldsymbol{w}}^{\text{TS}})\mathbb{I}\left\{\widetilde{\boldsymbol{w}}^{\text{TS}} \in W_{t-1}^{\text{OPT-TS}}\right\} \,\middle|\, \widetilde{E}_{t-1}, \mathcal{F}_{t-1}\right] \\
&= \mathbb{E}\left[g(\widetilde{\boldsymbol{w}}^{\text{TS}}) \,\middle|\, \widetilde{\boldsymbol{w}}^{\text{TS}} \in W_{t-1}^{\text{OPT-TS}}, \widetilde{E}_{t-1}, \mathcal{F}_{t-1}\right] \mathbb{P}\left(\widetilde{\boldsymbol{w}}^{\text{TS}} \in W_{t-1}^{\text{OPT-TS}} \,\middle|\, \widetilde{E}_{t-1}, \mathcal{F}_{t-1}\right) \\
&\ge \mathbb{E}\left[g(\widetilde{\boldsymbol{w}}^{\text{TS}}) \,\middle|\, \widetilde{\boldsymbol{w}}^{\text{TS}} \in W_{t-1}^{\text{OPT-TS}}, \widetilde{E}_{t-1}, \mathcal{F}_{t-1}\right] \frac{p}{2} && \text{(by Lemma 13.)}
\end{aligned}$$

The above combined with (33) implies that

$$\begin{aligned}
r_t^{\mathrm{TS}} &\leq 2\widetilde{\gamma}_{t-1}(\delta')\,\mathbb{E}\left[g(\widetilde{\boldsymbol{w}}^{\mathrm{TS}})\mathbb{I}\left\{\widetilde{E}_{t-1}\right\} \,\middle|\, \widetilde{\boldsymbol{w}}^{\mathrm{TS}} \in W_{t-1}^{\mathrm{OPT\text{-}TS}}, \mathcal{F}_{t-1}\right] \\
&= 2\widetilde{\gamma}_{t-1}(\delta')\,\mathbb{E}\left[g(\widetilde{\boldsymbol{w}}^{\mathrm{TS}}) \,\middle|\, \widetilde{\boldsymbol{w}}^{\mathrm{TS}} \in W_{t-1}^{\mathrm{OPT\text{-}TS}}, \widetilde{E}_{t-1}, \mathcal{F}_{t-1}\right] \mathbb{P}\big(\widetilde{E}_{t-1}\big) \\
&\leq \frac{4}{p}\widetilde{\gamma}_{t-1}(\delta')\,\mathbb{E}\left[g(\widetilde{\boldsymbol{w}}^{\mathrm{TS}}) \,\middle|\, \widetilde{E}_{t-1}, \mathcal{F}_{t-1}\right] \ . \qquad (34)
\end{aligned}$$

Finally, summing (34) over time we get

$$\sum_{t=1}^{T} r_t^{\mathrm{TS}} \leq \frac{4}{p}\left(\max_{t=0,\dots,T}\{\widetilde{\gamma}_t(\delta')\}\right)\sum_{t=1}^{T}\mathbb{E}\left[\|\boldsymbol{x}^\star(\widetilde{\boldsymbol{w}}^{\mathrm{TS}})\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \,\middle|\, \mathcal{F}_{t-1}\right] \ .$$

Note that we can already bound $\widetilde{\gamma}_t$ using (12). However, we cannot bound the expectation right away, so we rewrite the above as follows

$$\sum_{t=1}^{T} r_t^{\mathrm{TS}} \leq \frac{4}{p}\left(\max_{t=0,\dots,T}\{\widetilde{\gamma}_t(\delta')\}\right)\left(\sum_{t=1}^{T}\|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} + M_T\right) \qquad (35)$$

where we introduce the martingale

$$M_T = \sum_{t=1}^{T}\left(\mathbb{E}\left[\|\boldsymbol{x}^\star(\widetilde{\boldsymbol{w}}^{\mathrm{TS}})\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \,\middle|\, \mathcal{F}_{t-1}\right] - \|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right) \ .$$

Next, we use the Azuma-Hoeffding inequality to upper-bound $M_T$.

**Theorem 8** (Azuma-Hoeffding inequality)**.** *If a supermartingale $Y_t$ corresponding to a filtration $\mathcal{F}_t$ satisfies $|Y_t - Y_{t-1}| \leq c_t$ for some constant $c_t$ for $t = 1, 2, \dots,$ then for any $\alpha$,*

$$\mathbb{P}\left(Y_T - Y_0 \geq \alpha\right) \leq \exp\left(-\frac{\alpha^2}{2\sum_{t=1}^{T} c_t^2}\right) \ .$$

Now verify that for any $t = 1, \dots, T$,

$$M_t - M_{t-1} = \mathbb{E}\left[\|\boldsymbol{x}^\star(\widetilde{\boldsymbol{w}}^{\mathrm{TS}})\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \,\middle|\, \mathcal{F}_{t-1}\right] - \|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \leq \frac{2L}{\sqrt{\lambda}} \ .$$

Thus, by the Azuma-Hoeffding inequality, with probability at least $1 - \delta/2$ we have

$$M_T \leq \sqrt{\frac{4LT}{\lambda}\ln\left(\frac{4}{\delta}\right)} \ . \qquad (36)$$

Now we focus our attention on the remaining term:

$$\begin{aligned}
\sum_{t=1}^{T}\|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} &\leq \sum_{t=1}^{T}\min\left\{\frac{L}{\sqrt{\lambda}}, \|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\} \\
&\leq \max\left\{1, \frac{L}{\sqrt{\lambda}}\right\}\sum_{t=1}^{T}\min\left\{1, \|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\} \\
&\leq \max\left\{1, \frac{L}{\sqrt{\lambda}}\right\}\sqrt{T\sum_{t=1}^{T}\min\left\{1, \|X_t\|^2_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}}\right\}} \qquad \text{(by Cauchy-Schwartz)} \\
&\overset{\widetilde{\mathcal{O}}}{=} \max\left\{1, \frac{1}{\sqrt{\lambda}}\right\}\sqrt{(1+\varepsilon_m)\left(d\ln(1+\varepsilon_m) + m\right)T} \qquad (37)
\end{aligned}$$

where the last step is due to Lemma 9.

For brevity denote $\widetilde{m} = m + d\ln(1+\varepsilon_m)$. Now, we substitute into (35) the bound (36) on $M_T$, the bound (37), and the bound (12) on $\widetilde{\gamma}_t$. This gives

$$\sum_{t=1}^{T} r_t^{\text{TS}} \overset{\widetilde{\mathcal{O}}}{=} \sqrt{d}\left(R\sqrt{\widetilde{m}\,(1+\varepsilon_m)} + S\sqrt{\lambda}\cdot(1+\varepsilon_m)\right)\left(\max\left\{1, \frac{1}{\sqrt{\lambda}}\right\}\sqrt{(1+\varepsilon_m)\widetilde{m}T} + \sqrt{\frac{T}{\lambda}}\right)$$
$$\overset{\widetilde{\mathcal{O}}}{=} \max\left\{1, \frac{1}{\sqrt{\lambda}}\right\}\widetilde{m}\,(1+\varepsilon_m)^{\frac{3}{2}}\left(R + S\sqrt{\lambda}\right)\sqrt{dT} \tag{38}$$

which holds with high probability (due to Azuma-Hoeffding inequality).

Now we bound the remaining RLS term of the regret. In particular,

$$\begin{aligned}
\sum_{t=1}^{T} r_t^{\text{RLS}} &= \sum_{t=1}^{T} \mathbb{I}\{E_{t-1}\}\left(X_t^\top \widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}} - X_t^\top \boldsymbol{w}^\star\right) \\
&= \sum_{t=1}^{T} \mathbb{I}\{E_{t-1}\}\left(X_t^\top \widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}} - X_t^\top \widetilde{\boldsymbol{w}}_{t-1}\right) + \sum_{t=1}^{T} \mathbb{I}\{E_{t-1}\}\left(X_t^\top \widetilde{\boldsymbol{w}}_{t-1} - X_t^\top \boldsymbol{w}^\star\right) \\
&\le \sum_{t=1}^{T} \mathbb{I}\{E_{t-1}\}\,\|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}}\|\widetilde{\boldsymbol{w}}_{t-1}^{\text{TS}} - \widetilde{\boldsymbol{w}}_{t-1}\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} \\
&+ \sum_{t=1}^{T} \mathbb{I}\{E_{t-1}\}\,\|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}}\|\widetilde{\boldsymbol{w}}_{t-1} - \boldsymbol{w}^\star\|_{\widetilde{\boldsymbol{V}}_{t-1}^{-1}} && \text{(by Cauchy-Schwartz)} \\
&\le \sum_{t=1}^{T} \|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}}\widetilde{\gamma}_{t-1}(\delta') && \text{(by definition of event } \widetilde{E}_{t-1}^{\text{TS}}) \\
&+ \sum_{t=1}^{T} \|X_t\|_{\widetilde{\boldsymbol{V}}_{t-1}}\widetilde{\beta}_{t-1}(\delta') && \text{(by definition of event } \widetilde{E}_{t-1}) \\
&\overset{\widetilde{\mathcal{O}}}{=} \max\left\{1, \frac{1}{\sqrt{\lambda}}\right\}\sqrt{\widetilde{m}(1+\varepsilon_m)T} && \text{(using (37))} \\
&\cdot\, d\left(R\sqrt{\widetilde{m}\,(1+\varepsilon_m)} + S\sqrt{\lambda}\cdot(1+\varepsilon_m)\right) && \text{(using Theorem 2 to bound } \widetilde{\beta} \text{ and (12) to bound } \widetilde{\gamma}) \\
&\overset{\widetilde{\mathcal{O}}}{=} \max\left\{1, \frac{1}{\sqrt{\lambda}}\right\}\left(R\widetilde{m}\,(1+\varepsilon_m) + S\sqrt{\lambda}\sqrt{\widetilde{m}}\,(1+\varepsilon_m)^{\frac{3}{2}}\right)\sqrt{dT} \\
&\overset{\widetilde{\mathcal{O}}}{=} \max\left\{1, \frac{1}{\sqrt{\lambda}}\right\}\widetilde{m}\,(1+\varepsilon_m)^{\frac{3}{2}}\left(R + S\sqrt{\lambda}\right)\sqrt{dT}\,. && (39)
\end{aligned}$$

Hence, combining (31), (38), and (39) gives, with high probability,

$$\mathbb{I}\{E_T\}\,R_T = \sum_{t=1}^{T} r_t^{\text{TS}} + \sum_{t=1}^{T} r_t^{\text{RLS}} \overset{\widetilde{\mathcal{O}}}{=} \max\left\{1, \frac{1}{\sqrt{\lambda}}\right\}\widetilde{m}\,(1+\varepsilon_m)^{\frac{3}{2}}\left(R + S\sqrt{\lambda}\right)\sqrt{dT}$$

The proof is concluded by observing that Lemma 11 proves that $E_T$ also holds with high probability.

## B Experiments

In this section we present experiments on six publicly available classification datasets.

**Setup.** The idea of our experimental setup is similar to the one described by Cesa-Bianchi et al. (2013). Namely, we convert a $K$-class classification problem into a contextual bandit problem as follows: given a dataset of labeled instances $(\boldsymbol{x}, y) \in \mathbb{R}^d \times \{1, \dots, K\}$, we partition it into $K$ subsets according to the class labels. Then we create $K$ sequences by drawing a random permutation of each subset. At each step $t$ the decision set $D_t$ is obtained by picking the $t$-th instance from each one of these $K$ sequences. Finally, rewards are determined by choosing a class $y \in \{1, \dots, K\}$ and then consistently assigning reward 1 to all instances labeled with $y$ and reward 0 to all remaining instances.

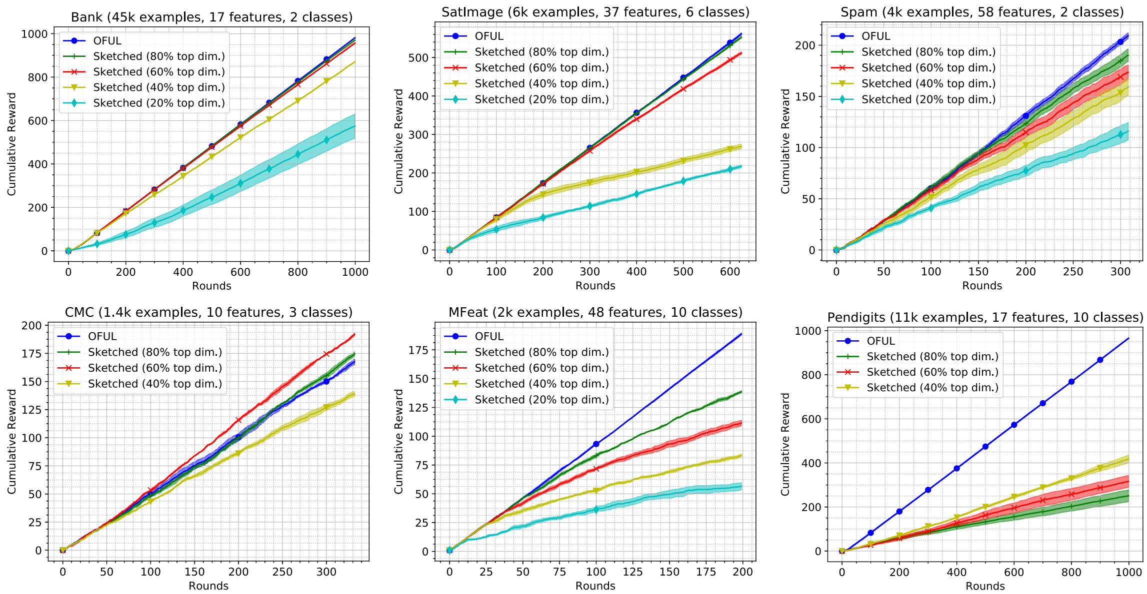


Figure 3: Comparison of SOFUL to OFUL on six real-world datasets and for different sketch sizes. Note that, in some cases, a sketch size equal to 80% and even 60% of the context space dimension does not significantly affect the perfomance.

**Datasets.** We perform experiments on six publicly available datasets for multiclass classification from the `openml` repository (Vanschoren et al., 2013) —dataset IDs 1461, 23, 32, 182, 22, and 44, see the table below here for details.

| Dataset | Examples | Features | Classes |
|---|---|---|---|
| Bank | 45k | 17 | 2 |
| SatImage | 6k | 37 | 6 |
| Spam | 4k | 58 | 2 |
| Pendigits | 11k | 17 | 10 |
| MFeat | 2k | 48 | 10 |
| CMC | 1.4k | 10 | 3 |

**Baselines.** The hyperparameters $\beta$ (confidence ellipsoid radius) and $\lambda$ (RLS regularization parameter) are selected on a validation set of size 100 via grid search on $(\beta, \lambda) \in \{1, 10^2, 10^3, 10^4\} \times \{10^{-2}, 10^{-1}, 1\}$ for OFUL, and $\{1, 10^2, 10^3\} \times \{10^{-2}, 10^{-1}, 1, 10^2\}$ for linear TS.

**Results** We observe that on three datasets, Figure 3, sketched algorithms indeed do not suffer a substantial drop in performance when compared to the non-sketched ones, even when the sketch size amounts to 60% of the context space dimension. This demonstrates that sketching successfully captures relevant subspace information relatively to the goal of maximizing reward.

Because the FD-sketching procedure considered in this paper is essentially performing online PCA, it is natural to ask how our sketched algorithms would compare to their non-sketched version run on the best $m$-dimensional subspace (computed by running PCA on the entire dataset). In Figure 4, we compare SOFUL and sketched linear TS to their non-sketched versions. In particular, we keep 60%, 40%, and 20% of the top principal components, and notice that, like in Figure 3, there are cases with little or no loss in performance.

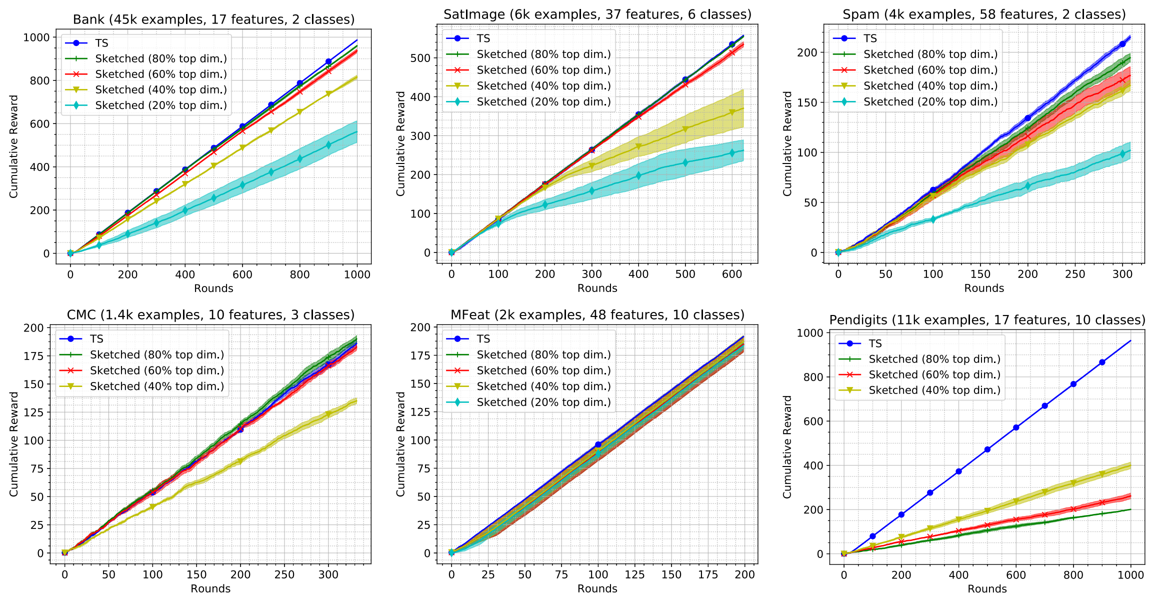


Figure 3: Comparison of sketched linear TS to linear TS on six real-world datasets and for different sketch sizes. Note that, in some cases, a sketch size equal to 80% and even 60% of the context space dimension does not significantly affect the perfomance.

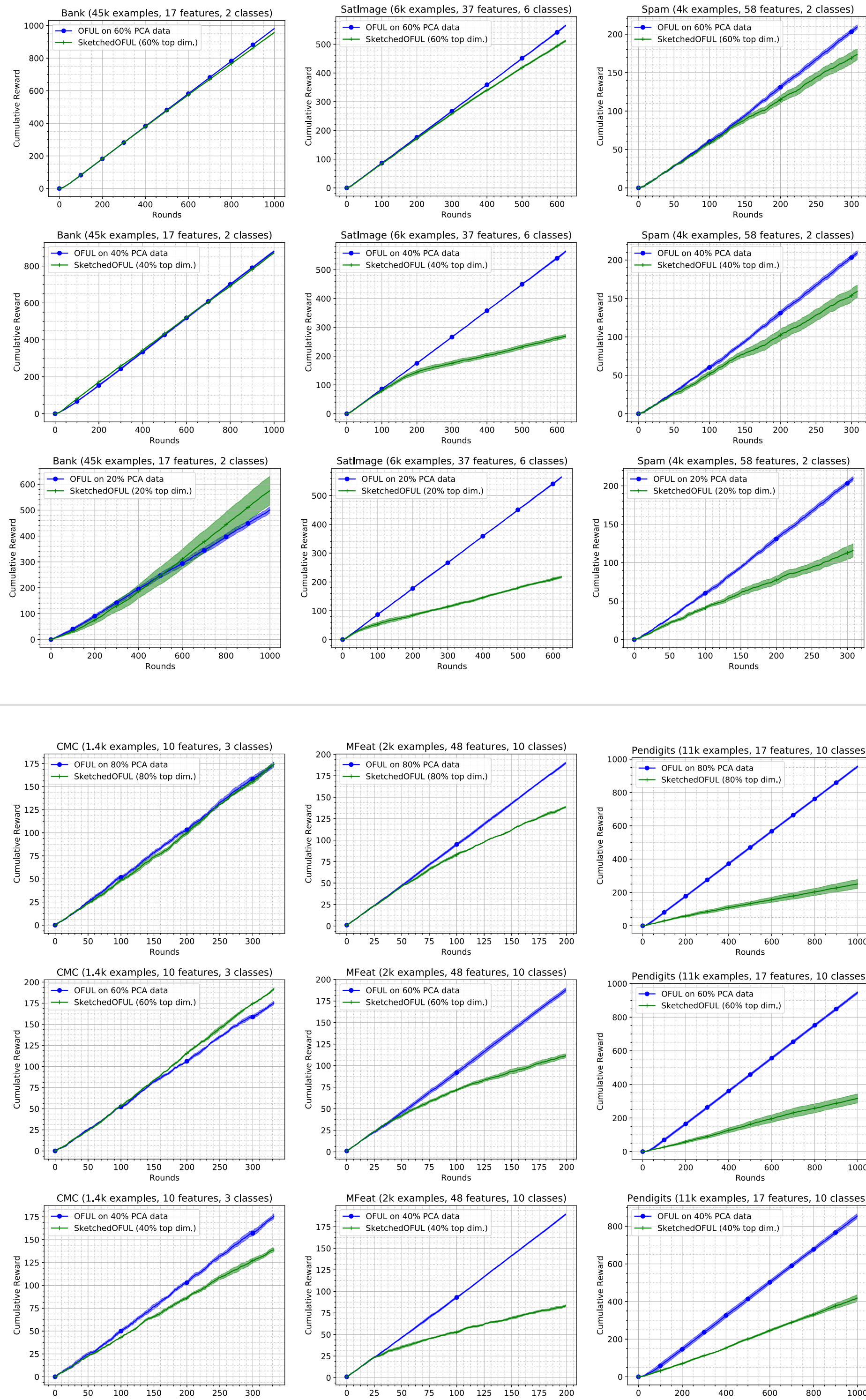


Figure 4: Comparison of OFUL run on the best $m$-dimensional subspace against SOFUL run with sketch size $m$. Rows show $m$ as a fraction of the context space dimension: $60\%, 40\%, 20\%$ (for the first three datasets), while columns correspond to different datasets. Note that, in some cases (with sketch size $m$ of size at least $60\%$), SOFUL performs as well as if the best $m$-dimensional subspace had been known in hindsight.

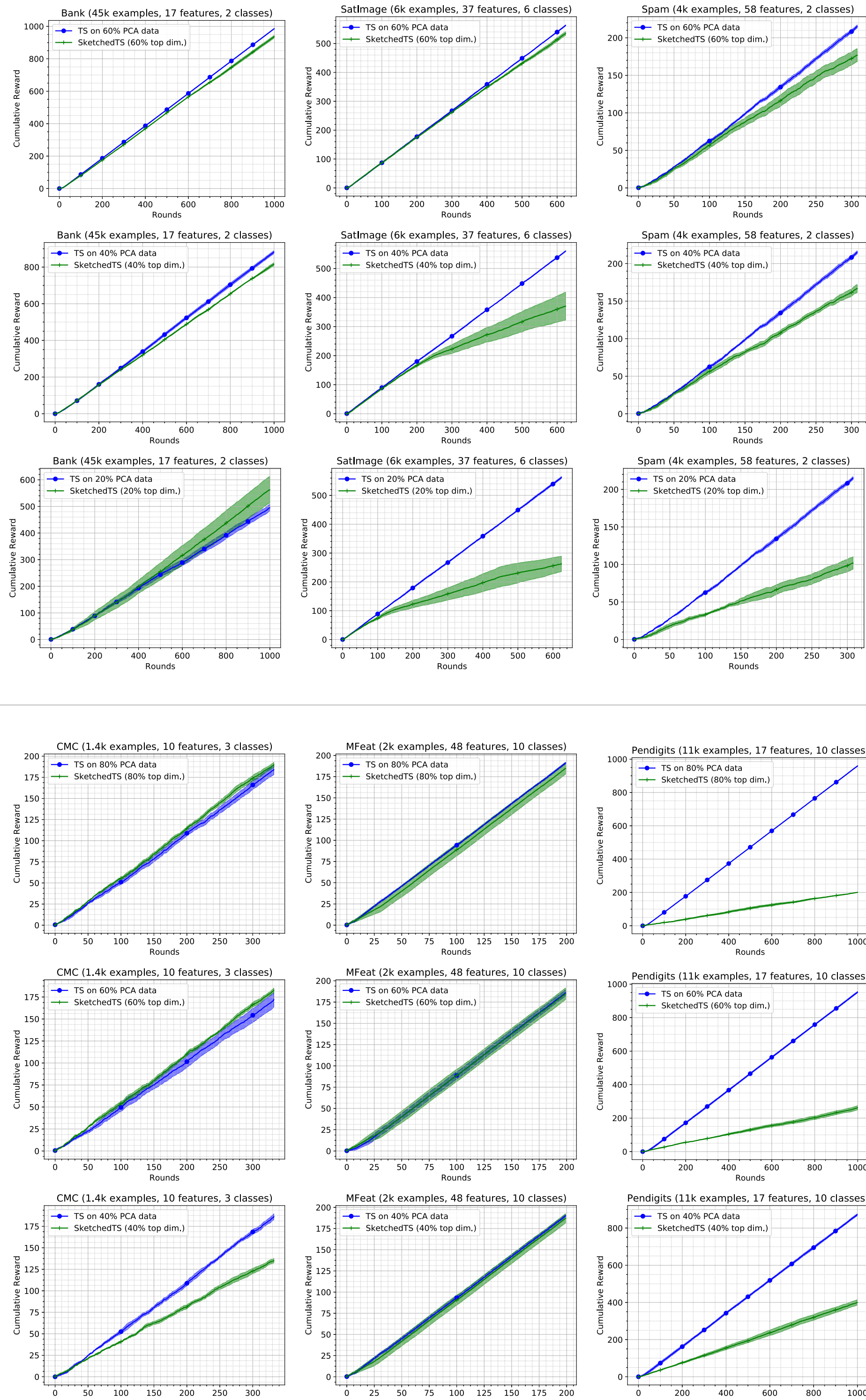


Figure 5: Comparison of linear TS run on the best $m$-dimensional subspace against sketched linear TS run with sketch size $m$. Rows show $m$ as a fraction of the context space dimension: $60\%, 40\%, 20\%$ (for the first three datasets), while rows correspond to different datasets. Note that, in some cases (with sketch size $m$ of size at least $60\%$), sketched linear TS performs as well as if the best $m$-dimensional subspace had been known in hindsight.