Supplement Material: Domain-Size Aware Markov Logic Networks

Happy Mittal
IIT Delhi

Ayush Bhardwaj
IIT Delhi

Vibhav Gogate
UT Dallas

Parag Singla
IIT Delhi

This document is a supplement to the paper "Domain-size Aware Markov Logic Networks". This contains the detailed proofs of all the propositions and corollaries stated in the main paper.

We first provide the proof for the counterexample for the proposition 0 presented in the main paper. Again, as stated in the main paper also, we assume all the weights of the formulas are strictly positive.

Counterexample for Proposition 0: Consider an MLN M consisting of one formula $w : Q(x) \vee R(y) \vee P$, where $|\Delta_x| = 1$ and $|\Delta_y| = n$. Then,

$$P_M(Q(1)) = \frac{2^{n+1}e^{wn}}{(2^n e^{wn} + (1+e^w)^n) + 2^{n+1}e^{wn}}$$

Proof. We have

$$P_M(Q(1)) = \frac{Z_{Q(1)=1}}{Z_{Q(1)=0} + Z_{Q(1)=1}} = \frac{1}{1 + \frac{Z_{Q(1)=0}}{Z_{Q(1)=1}}} \quad (1)$$

Now

$$Z_{Q(1)=0} = (1+e^w)^n + 2^n e^{wn} \quad (2)$$

and

$$Z_{Q(1)=1} = 2^{n+1}e^{wn} \quad (3)$$

Substituting eq (2) and eq (3) in eq (1), we get

$$P_M(Q(1)) = \frac{2^{n+1}e^{wn}}{(2^n e^{wn} + (1+e^w)^n) + 2^{n+1}e^{wn}}$$

Also, $\lim_{n\to\infty} P_M(Q(1)) = 2/3$ □

Now we prove proposition 1 in the main paper. Before proving proposition 1, we present a lemma, which will be useful for the proof.

Lemma 1. *Consider an MLN M with a single formula of the form $w : R_1(y_1) \vee R_2(y_2) \vee \cdots \vee R_k(y_k)$, and let $|\Delta_{y_1}| = |\Delta_{y_2}| = \cdots = |\Delta_{y_k}| = n$, then partition function*

$$Z_M \leqslant e^{wn^k} 2^{n(k-1)}(k-1+(1+e^{-w})^n)$$

Proof. Proof follows from repeatedly applying binomial rule [1] on all the predicates. First applying binomial rule on R_1, we get

$$Z_M = \sum_{j_1} \binom{n}{j_1} e^{wn^{k-1}(n-j_1)} Z_{M_1}$$

where M_1 has a single formula $j_1 w : R_2(y_2) \vee \cdots \vee R_k(y_k)$

$$Z_M = e^{wn^k} \sum_{j_1} \binom{n}{j_1} e^{-wj_1 n^{k-1}} Z_{M_1}$$

Now applying binomial rule on R_2, we get

$$Z_M = e^{wn^k} \sum_{j_1} \tbinom{n}{j_1} e^{-wj_1 n^{k-1}} \sum_{j_2} \tbinom{n}{j_2} e^{wj_1 n^{k-2}(n-j_2)} Z_{M_2}$$

where M_2 has a single formula $j_1 j_2 w : R_3(y_3) \vee \cdots \vee R_k(y_k)$

$$Z_M = e^{wn^k} \sum_{j_1} \binom{n}{j_1} \sum_{j_2} \binom{n}{k_2} e^{-wj_1 j_2 n^{k-2})} Z_{M_2}$$

Similarly, after applying binomial till R_{k-1}, we get

$$Z_M = e^{wn^k} \sum_{j_1} \binom{n}{j_1} \sum_{j_2} \binom{n}{j_2} \cdots \sum_{j_{k-1}} \binom{n}{j_{k-1}} e^{-wnj_1 j_2 \cdots j_{k-1}} Z_{M_{k-1}}$$

where M_{k-1} has a single formula $wj_1 j_2 \ldots j_{k-1} : R_k(y_k)$ Now $Z_{M_{k-1}} = (1 + e^{wj_1 j_2 \cdots j_{k-1}})^n$ (By Decomposer [1]), so finally

$$\begin{aligned} Z_M &= e^{wn^k} \sum_{j_1} \binom{n}{j_1} \sum_{j_2} \binom{n}{j_2} \cdots \sum_{j_{k-1}} \binom{n}{j_{k-1}} e^{-wnj_1 j_2 \cdots j_{k-1}} (1 + e^{wj_1 j_2 \cdots j_{k-1}})^n \\ &= e^{wn^k} \sum_{j_1} \binom{n}{j_1} \sum_{j_2} \binom{n}{j_2} \cdots \sum_{j_{k-1}} \binom{n}{j_{k-1}} (1 + e^{-wj_1 j_2 \cdots j_{k-1}})^n \end{aligned}$$

Now when any one of the $j_1, j_2, \ldots j_{k-1}$ becomes 0, then we get

$$Z'_M = e^{wn^k}(k-1) * 2^{n(k-1)}$$

On the other hand, if all of the $j_1, j_2, \dots j_{k-1}$ are non-zero, then we get

$$\begin{aligned} Z''_M &= e^{wn^k} \sum_{j_1=1}^{n} \binom{n}{j_1} \sum_{j_2=1}^{n} \binom{n}{j_2} \\ &\dots \sum_{j_{k-1}=1}^{n} \binom{n}{j_{k-1}} (1+e^{-wj_1 j_2 \dots j_{k-1}})^n \\ &\leqslant (1+e^{-w})^n e^{wn^k} \sum_{j_1=1}^{n} \binom{n}{j_1} \sum_{j_2=1}^{n} \binom{n}{j_2} \\ &\dots \sum_{j_{k-1}=1}^{n} \binom{n}{j_{k-1}} \\ &= (1+^{-w})^n e^{wn^k} (2^n-1)^{k-1} \\ &\leqslant (1+e^{-w})^n e^{wn^k} (2^n)^{k-1} \end{aligned}$$

Adding Z'_M and Z''_M, we get

$$\begin{aligned} Z_M &\leqslant (k-1) * 2^{n(k-1)} + (1+e^{-w})^n e^{wn^k} (2^n)^{k-1} \\ &= e^{wn^k} 2^{n(k-1)} (k-1+(1+e^{-w})^n) \end{aligned}$$

□

Proposition 1. *Consider an MLN M with a single formula of the form $w : Q(x) \vee R_1(y_1) \vee \dots \vee R_k(y_k)$. Here $k \geqslant 1$. Also $|\Delta y_1| = \dots = |\Delta y_k| = n$, and $|\Delta x| = r$, where $r \geqslant 1$ is some constant. $\lim_{n\to\infty} P_M(Q(1))$ is 1.*

Proof. We have

$$P_M(Q(1)) = \frac{Z_{Q(1)=1}}{Z_{Q(1)=0} + Z_{Q(1)=1}} = \frac{1}{1+\frac{Z_{Q(1)=0}}{Z_{Q(1)=1}}} \quad (4)$$

Now $Z_{Q(1)=0}$ is the partition function of the MLN $w : R_1(y_1) \vee R_2(y_2) \vee \dots \vee R_k(y_k); w : Q(x) \vee R_1(y_1) \vee R_2(y_2) \vee \dots \vee R_k(y_k)$, and $|\Delta(x)| = r-1$.
Applying binomial rule on $Q(x)$, we get,

$$Z_{Q(1)=0} = \sum_{j=0}^{r-1} \binom{r-1}{j} e^{n^k(r-1-j)} Z'_{Q(1)=0}$$

where $Z'_{Q(1)=0}$ is the partition function of the MLN $(k+1)w : R_1(y_1) \vee R_2(y_2) \vee \dots \vee R_k(y_k)$
Now using Lemma 1, we get

$$\begin{aligned} Z_{Q(1)=0} &\leqslant \sum_{j=0}^{r-1} \binom{r-1}{j} e^{wn^k(r-1-j)} \\ &\quad e^{w(j+1)n^k} 2^{n(k-1)} (k-1+(1+e^{-w(j+1)})^n) \end{aligned}$$

Also, $Z_{Q(1)=1} = e^{wn^k} Z'_{Q(1)=1}$, where $Z'_{Q(1)=1}$ is the partition function of the MLN $w : Q(x) \vee R_1(y_1) \vee R_2(y_2) \vee \dots \vee R_k(y_k)$, and $|\Delta_x| = r-1$.
Applying binomial rule on $Q(x)$, we get,

$$Z'_{Q(1)=1} = \sum_{j=0}^{r-1} \binom{r-1}{j} e^{n^k(r-1-j)} Z''_{Q(1)=1}$$

where $Z''_{Q(1)=1}$ is the partition function of the MLN $jw : R_1(y_1) \vee R_2(y_2) \vee \dots \vee R_k(y_k)$

$$\begin{aligned} Z_{Q(1)=1} = \sum_{j=0}^{r-1} \binom{r-1}{j} e^{wn^k(r-1-j)} e^{wjn^k} \sum_{j_1} \binom{n}{j_1} \sum_{j_2} \\ \binom{n}{j_2} \dots \sum_{j_{k-1}} \binom{n}{j_{k-1}} (1+e^{-wjj_1 j_2 \dots j_{k-1}})^n \end{aligned}$$

Let's evaluate $Z_{Q(1)=1}$ when $j=0$. Let us denote that by Z_{j_0}

$$Z_{j_0} = e^{wn^k(r-1)} 2^{kn}$$

so

$$Z_{Q(1)=1} \geqslant Z_{j_0} = e^{wn^k(r-1)} 2^{kn}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{Z_{Q(1)=0}}{Z_{Q(1)=1}} &\leqslant \lim_{n\to\infty} \frac{\sum_{j=0}^{r-1} \binom{r-1}{j} (k-1+(1+e^{-wj-w})^n))}{2^n} \\ &\leqslant \lim_{n\to\infty} \frac{(k-1+(1+e^{-w})^n) 2^{r-1}}{2^n} \\ &= \lim_{n\to\infty} \frac{k-1}{2^n} + \left(\frac{1+e^{-w}}{2}\right)^n 2^{r-1} \\ &= 0 \end{aligned}$$

Hence from eq (4), $\lim_{n\to\infty} P_M(Q(1)) = 1$ □

Proposition 2. *Consider an MLN M having single formula of the form $w : Q(x) \vee R(y) \vee P_1 \vee P_2 \dots \vee P_m$, where $|\Delta_x| = 1$ and $|\Delta_y| = n$. Then $\lim_{n\to\infty} P_M(Q(1)) = \frac{2^m}{2^{m+1}-1}$*

Proof. We evaluate $Z_{Q(1)=0}$ and $Z_{Q(1)=1}$ with the help of Binomial and Decomposer rules.

$$Z_{Q(1)=0} = (2^m - 1)\, 2^n e^{wn} + (1+e^w)^n$$

Every grounding is satisfied when $Q(1) = 1$. Hence,

$$Z_{Q(1)=1} = 2^m 2^n e^{wn}$$

Thus,

$$\begin{aligned} \frac{Z_{Q(1)=0}}{Z_{Q(1)=1}} &= \frac{(2^m-1)\, 2^n e^{wn} + (1+e^w)^n}{2^m 2^n e^{wn}} \\ &= \left(1-\frac{1}{2^m}\right) + \frac{1}{2^m-1} \left(\frac{1+e^w}{2e^w}\right)^n \\ &= \left(1-\frac{1}{2^m}\right) + \frac{1}{2^m-1} \left(\frac{1+e^{-w}}{2}\right)^n \end{aligned}$$

As $\lim_{n\to\infty}\left(\frac{1+e^{-w}}{2}\right)^n = 0$, we get

$$\lim_{n\to\infty}\frac{Z_{Q(1)=0}}{Z_{Q(1)=1}} = \left(1-\frac{1}{2^m}\right)$$

Hence

$$\lim_{n\to\infty} P_M(Q(1)) = \frac{2^m}{2^{m+1}-1}$$

□

Proposition 3. *Consider an MLN M with single formula of the form $w : Q(x) \vee R(y)$, where $|\Delta_x| = |\Delta_y| = n$. Then $\lim_{n\to\infty} P_M(Q(1)) = \frac{3}{4}$*

Proof. Again we evaluate $Z_{Q(1)=0}$ and $Z_{Q(1)=1}$ with the help of Binomial and Decomposer rules. First $Z_{Q(1)=0}$

$$\begin{aligned}
&= e^{n^2w}\sum_{k=0}^{n-1}\binom{n-1}{k}\sum_{j=0}^{n}\binom{n}{j}e^{-(k+1)jw} \\
&= e^{n^2w}\left(2^{n-1}+\sum_{k=0}^{n-1}\binom{n-1}{k}\sum_{j=1}^{n}\binom{n}{j}e^{-(k+1)jw}\right) \\
&= e^{n^2w}\left(2^{n-1}+\sum_{k=0}^{n-1}\binom{n-1}{k}\sum_{j=0}^{n-1}\binom{n}{j+1}\right. \\
&\qquad \left. e^{-(k+1)(j+1)w}\right) \qquad (5)
\end{aligned}$$

Thus,

$$Z_{Q(1)=0} \geqslant e^{n^2w} * 2^{n-1} \qquad (6)$$

Also, note

$$(k+1)(j+1) \geqslant k+j+1$$

From (5), we have

$$\begin{aligned}
Z_{Q(1)=0} &\leqslant e^{n^2w}\left(2^{n-1}+\left(\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{k}\right.\right. \\
&\qquad \left.\left.\binom{n}{j+1}e^{-(k+j+1)w}\right)\right) \\
&\leqslant e^{n^2w}\left(2^{n-1}+e^{-w}\left(1+e^{-w}\right)^n\right. \\
&\qquad \left.\left(\left(1+e^{-w}\right)^{n+1}-1\right)\right) \qquad (7)
\end{aligned}$$

As $1 << (1+e^{-w})^{n+1}$, From eq (6) and (7)

$$1 \leqslant \frac{Z_{Q(1)=0}}{2^{n-1}e^{n^2w}} \leqslant \left(1+e^{-w}(1+e^{-w})\left(\frac{(1+e^{-w})^2}{2}\right)^n\right)$$

Assume w $\geqslant$ 1 and as $(1+e^{-w})^2 \leqslant 2$,

$$\lim_{n\to\infty} Z_{Q(1)=0} \approx 2^{n-1}e^{n^2w} \qquad (8)$$

Similarly with help of binomial, we get $Z_{Q(1)=1}$

$$\begin{aligned}
&= e^{n^2w}\sum_{k=0}^{n-1}\binom{n-1}{k}\sum_{j=0}^{n}\binom{n}{j}e^{-kjw} \\
&= e^{n^2w}\left(2^{n-1}+2^n-1+\sum_{k=1}^{n-1}\binom{n-1}{k}\right. \\
&\qquad \left.\sum_{j=1}^{n}\binom{n}{j}e^{-kjw}\right) \\
&\approx e^{n^2w}\left(3*2^{n-1}+\sum_{k=0}^{n-2}\binom{n-1}{k+1}\right. \\
&\qquad \left.\sum_{j=0}^{n-1}\binom{n}{j+1}e^{-(k+1)(j+1)w}\right) \\
&= e^{n^2w}\left(3*2^{n-1}+\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{k+1}\binom{n}{j+1}\right. \qquad (9) \\
&\qquad \left. e^{-(k+1)(j+1)w}\right)
\end{aligned}$$

Thus,

$$Z_{Q(1)=1} \geqslant 3 * e^{n^2w} * 2^{n-1} \qquad (10)$$

Also, note

$$(k+1)(j+1) \geqslant k+j+1$$

From (9), analogous to previous part, we have $Z_{Q(1)=1}$

$$\begin{aligned}
&\leqslant e^{n^2w}\left(3*2^{n-1}+\left(\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{k+1}\binom{n}{j+1}e^{-(k+j+1)w}\right)\right) \\
&\leqslant e^{n^2w}\left(3*2^{n-1}+e^{w}\left(1+e^{-w}\right)^n\left(\left(1+e^{-w}\right)^{n+1}\right)\right) \qquad (11)
\end{aligned}$$

Therefore from eq (10) and (11)

$$1 \leqslant \frac{Z_{Q(1)=1}}{3*2^{n-1}e^{n^2w}} \leqslant \left(1+\frac{e^{-w}(1+e^{-w})}{3}\left(\frac{(1+e^{-w})^2}{2}\right)^n\right)$$

Assume w $\geqslant$ 1 and as $(1+e^{-w})^2 \leqslant 2$,

$$\lim_{n\to\infty} Z_{Q(1)=1} \approx 3*2^{n-1}e^{n^2w} \qquad (12)$$

So from eq (8) and (12)

$$\begin{aligned}
\lim_{n\to\infty}\frac{Z_{Q(1)=0}}{Z_{Q(1)=1}} &\leqslant \lim_{n\to\infty}\frac{2^{n-1}e^{n^2w}}{3*2^{n-1}e^{n^2w}} \\
&= \frac{1}{3} \qquad (13)
\end{aligned}$$

For $0 < w \leqslant 1$, we found the limit analytically, and it also comes down to $\frac{1}{3}$. Hence

$$\lim_{n\to\infty} P_M(Q(1) = 1) = \frac{3}{4}$$

□

Proposition 4. *Consider an MLN M with single formula of the form $w : Q(x) \vee P(x, y) \vee R(y)$. Here $|\Delta x| = r$, where $r \geqslant 1$ is some constant, and $|\Delta y| = n$. Then $\lim_{n\to\infty} P_M(Q(1))$ is 1.*

Proof. Again we evaluate $Z_{Q(1)=0}$ and $Z_{Q(1)=1}$ with the help of Binomial and Decomposer rules.

$$Z_{Q(1)=0} = e^{nw} \sum_{k=0}^{r-1} \binom{r-1}{k} 2^{-nk} \left((1 + e^{-w})^{k+1} + 2^{k+1}\right)^n \tag{14}$$

Also,

$$Z_{Q(1)=1} = 2^n e^{nw} \sum_{k=0}^{r-1} \binom{r-1}{k} 2^{-nk} \left((1 + e^{-w})^k + 2^k\right)^n \tag{15}$$

From eq (14) and (15), we have $\lim_{n\to\infty} \frac{Z_{Q(1)=0}}{Z_{Q(1)=1}}$

$$= \lim_{n\to\infty} \frac{\sum_{k=0}^{r-1} \binom{r-1}{k} 2^{-nk} \left((1 + e^{-w})^{k+1} + 2^{k+1}\right)^n}{2^n \sum_{k=0}^{r-1} \binom{r-1}{k} 2^{-nk} \left((1 + e^{-w})^k + 2^k\right)^n}$$

We could not evaluating above limit analytically, so we computed it numerically : for different w, we calculated the value in the limit of $n \to \infty$, and it comes out to be zero for every w. Hence we get $\lim_{n\to\infty} P_M(Q(1)) = 1$. □

Proposition 5. *Consider a DA-MLN D with a single formula of the form $w : Q(x) \vee R(y)$. Let $|\Delta_x| = 1$. Further, let $|\Delta_y| = n$. Then, $\lim_{n\to\infty} P_D(Q(1)) = \frac{1}{1+e^{\frac{-w}{2}}}$.*

Proof. In order to compute $P_D(Q(1))$, we need to compute $\frac{Z_{Q(1)=0}}{Z_{Q(1)=1}}$: We have

$$\lim_{n\to\infty} \frac{Z_{Q(1)=0}}{Z_{Q(1)=1}} = \lim_{n\to\infty} \frac{(1 + e^{\frac{w}{n}})^n}{e^w * 2^n} = e^{\frac{-w}{2}}$$

So $\lim_{n\to\infty} P_D(Q(1)) = \frac{1}{1+e^{\frac{-w}{2}}}$. □

Proposition 6. *Consider a DA-MLN D having single formula of the form $w : Q(x) \vee R(y) \vee P_1 \vee P_2 ... \vee P_m$, where $|\Delta_x| = 1$ and $|\Delta y| = n$. Then $\lim_{n\to\infty} P_D(Q(1))$ is a function of w.*

Proof. We evaluate $Z_{Q(1)=0}$ and $Z_{Q(1)=1}$ with the help of Binomial and Decomposer rules.

$$Z_{Q(1)=0} = (2^m - 1)\, 2^n e^{\frac{w}{n} n} + \left(1 + e^{\frac{w}{n}}\right)^n$$

Every grounding of theory is satisfied when $Q(1) = 1$. Hence,

$$Z_{Q(1)=1} = 2^m 2^n e^{\frac{w}{n} n}$$

Thus,

$$\begin{aligned}\frac{Z_{Q(1)=0}}{Z_{Q(1)=1}} &= \frac{(2^m - 1)\, 2^n e^{\frac{w}{n} n} + \left(1 + e^{\frac{w}{n}}\right)^n}{2^m 2^n e^{\frac{w}{n} n}} \\ &= \left(1 - \frac{1}{2^m}\right) + \frac{1}{2^m - 1} \left(\frac{1 + e^{\frac{w}{n}}}{2e^{\frac{w}{n}}}\right)^n \\ &= \left(1 - \frac{1}{2^m}\right) + \frac{1}{2^m - 1} \left(\frac{1 + e^{-\frac{w}{n}}}{2}\right)^n\end{aligned}$$

As $\lim_{n\to\infty} \left(\frac{1+e^{-\frac{w}{n}}}{2}\right)^n = e^{-\frac{w}{2}}$,

$$\lim_{n\to\infty} \frac{Z_{Q(1)=0}}{Z_{Q(1)=1}} = \left(1 - \frac{1}{2^m}\right) + \left(\frac{e^{-\frac{w}{2}}}{2^m - 1}\right)$$

Hence,

$$\lim_{n\to\infty} P_D(Q(1) = 1) = \frac{1}{1 + \left(1 - \frac{1}{2^m}\right) + \frac{1}{2^m - 1} e^{-\frac{w}{2}}}$$

Clearly the marginal probability of Q(1) is dependent on w. □

Proposition 7. *Consider a DA-MLN D with single formula of the form $w : Q(x) \vee R(y)$, where $|\Delta x| = |\Delta y| = n$. Then $\lim_{n\to\infty} P_D(Q(1)) = f(w)$, where $f(w)$ is a (non constant) function of w.*

Proof. We evaluate $Z_{Q(1)=0}$ and $Z_{Q(1)=1}$ with the help of Binomial and Decomposer rules. First $Z_{Q(1)=0}$

$$\begin{aligned} &= e^{n^2 w} \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{j=0}^{n} \binom{n}{j} e^{\frac{-(k+1)jw}{n}} \\ &= e^{n^2 f(w)} \left(2^{n-1} + \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{j=1}^{n} \binom{n}{j} e^{\frac{-(k+1)jw}{n}}\right) \\ &= e^{n^2 f(w)} \Bigg(2^{n-1} + \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{j=0}^{n-1} \binom{n}{j+1} \\ &\qquad e^{\frac{-(k+1)(j+1)w}{n}}\Bigg) \\ &= e^{n^2 f(w)} \Bigg(2^{n-1} + \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \binom{n-1}{k} \binom{n}{j+1} \\ &\qquad e^{\frac{-(k+1)(j+1)w}{n}}\Bigg) \end{aligned} \tag{16}$$

Similarly, we get $Z_{Q(1)=1}$

$$\begin{aligned}
&= e^{n^2 f(w)} \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{j=0}^{n} \binom{n}{j} e^{\frac{-kjw}{n}} \\
&= e^{n^2 f(w)} \Bigg(2^{n-1} + 2^n - 1 + \sum_{k=1}^{n-1} \binom{n-1}{k} \\
&\qquad \sum_{j=1}^{n} \binom{n}{j} e^{\frac{-kjw}{n}} \Bigg) \\
&\approx e^{n^2 f(w)} \Bigg(3 * 2^{n-1} + \sum_{k=0}^{n-2} \binom{n-1}{k+1} \\
&\qquad \sum_{j=0}^{n-1} \binom{n}{j+1} e^{\frac{-(k+1)(j+1)w}{n}} \Bigg) \\
&= e^{n^2 f(w)} \Bigg(3 * 2^{n-1} + \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \binom{n-1}{k+1} \binom{n}{j+1} \\
&\qquad e^{\frac{-(k+1)(j+1)w}{n}} \Bigg)
\end{aligned} \tag{17}$$

Evaluating the limit for Eq (16) and (17) analytically in this case is difficult, so again, we evaluated that numerically. For different values of w, we numerically computed the value of the probability in the limit (the limit does exist). These values are plotted in Figure 1. Unlike Proposition 3, there is a clear dependence of the limiting probability on w in this case. □

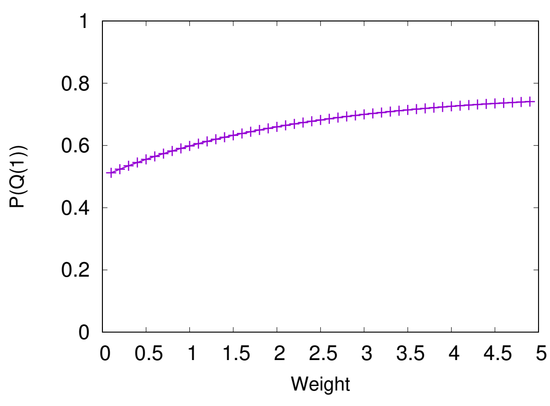

Figure 1: Query Marginal vs Weight (Proposition 7)

Proposition 8. *Consider a DA-MLN D with single formula of the form $w : Q(x) \vee P(x, y) \vee R(y)$. Here $|\Delta x| = r$, where $r \geqslant 1$ is some constant, and $|\Delta y| = n$. Then $\lim_{n\to\infty} P_D(Q(1)) = f(w)$, where $f(w)$ is a (non constant) function of w.*

Proof. We proceed same as proposition 4, and thus $\lim_{n\to\infty} \frac{Z_{Q(1)=0}}{Z_{Q(1)=1}}$ becomes

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{r-1} \binom{r-1}{k} 2^{-nk} \left((1 + e^{-w/n})^{k+1} + 2^{k+1} \right)^n}{2^n \sum_{k=0}^{r-1} \binom{r-1}{k} 2^{-nk} \left((1 + e^{-w/n})^{k} + 2^{k} \right)^n}$$

Again we evaluated the limit numerically, and the probability values are plotted in the figure 2, which clearly shows the dependence of the marginal on the weight w. □

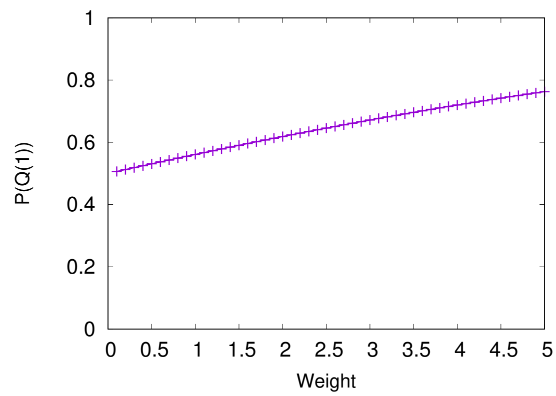

Figure 2: Query Marginal vs Weight (Proposition 8)

Proposition 9. *Consider an MLN M with single formula of the form $w : Q(x) \vee P(y)$. Here $|\Delta y| = n$. Let $|\Delta x| = 1$. Suppose P is evidence predicate, i.e., all its groundings are given to be true or false. If the ratio of true and false groundings of P remains constant with respect to n, then $\lim_{n\to\infty} P_M(Q(1)) = 1$.*

Proof. Let the ratio of true and false groundings of P is some constant r. Then We have

$$\lim_{n\to\infty} \frac{Z_{Q(1)=0}}{Z_{Q(1)=1}} = \lim_{n\to\infty} \frac{e^{rnw}}{e^{nw}} = e^{(r-1)nw} \tag{18}$$

Eq (18) evaluates to 0, and hence $\lim_{n\to\infty} P_M(Q(1)) = 1$. □

Proposition 10. *Consider a DA-MLN D with single formula of the form $w : Q(x) \vee P(y)$. Here $|\Delta y| = n$. Let $|\Delta x| = 1$. Suppose P is evidence predicate. If the ratio of true and false groundings (denoted by r) of P remains constant with respect to n, then $\lim_{n\to\infty} P_D(Q(1)) = \frac{1}{1+e^{(r-1)w}}$.*

Proof. Let the ratio of true and false groundings of P is some constant r. Then we have

$$\lim_{n\to\infty} \frac{Z_{Q(1)=0}}{Z_{Q(1)=1}} = \lim_{n\to\infty} \frac{e^{rw}}{e^{w}} = e^{(r-1)w}$$

Hence $\lim_{n\to\infty} P_D(Q(1)) = \frac{1}{1+e^{(r-1)w}}$. □

References

[1] V. Gogate and P. Domingos. Probabilisitic theorem proving. In *Proc. of UAI-11*, pages 256–265, 2011.