A Supplementary material for Sections 3-4

A.1 Proof of Remark 1

Since f is monotone, $f(\mathbf{x}+k\mathbf{e}_i) \geq f(\mathbf{x})$ and $f(k\mathbf{e}_i) \geq f(\mathbf{0})$ for any $\mathbf{x} \in \boldsymbol{\mathcal{Z}}$, $i \in [n]$, and $k \in \mathbb{R}_+$. Hence, $\alpha(\boldsymbol{\mathcal{Z}}) \leq 1$. Moreover, $\inf_{\substack{\mathbf{x}\in\boldsymbol{\mathcal{Z}}\\ i\in[n]}} \lim_{k\to 0^+} \frac{f(\mathbf{x}+k\mathbf{e}_i)-f(\mathbf{x})}{f(k\mathbf{e}_i)-f(\mathbf{0})} \leq 1$, since the considered ratio equals 1 when $\mathbf{x} = \mathbf{0}$. Hence, $\alpha(\boldsymbol{\mathcal{Z}}) \geq 0$. □

A.2 Proof of Remark 2

The proof is obtained simply noting that the curvature $\alpha(\tilde{\boldsymbol{\mathcal{S}}})$ of γ is always upper bounded by 1. □

A.3 Proof of Proposition 1

We first show that the budget allocation game of Example 1 is a valid utility game with continuous strategies. In fact, for any $l \in [Nd]$

$$[\nabla\gamma(\mathbf{s})]_l = \sum_{t\in\mathcal{T}:m\in\Gamma(t)} -\ln(1-p_j(m,t)) \prod_{i=1}^{N}(1-P_i(\mathbf{s}_i,t)) \,,$$

where $j \in [N]$ and $m \in [d]$ are the indexes of advertiser and channel corresponding to coordinate $l \in [Nd]$, respectively. Hence, γ is monotone since $[\nabla\gamma(\mathbf{s})]_l \geq 0$ for any $l \in [Nd]$ and $\mathbf{s} \in \mathbb{R}_+^{Nd}$. Moreover, γ is DR-submodular since $\gamma(\mathbf{s}) = \sum_{t\in\mathcal{T}} \gamma_t(\mathbf{s})$ where $\gamma_t(\mathbf{s}) = 1 - \prod_{i=1}^N (1 - P_i(\mathbf{s}_i, t))$ is such that for any $j, l \in [N], m, n \in [d]$, $\frac{\partial^2\gamma_t(\mathbf{s})}{\partial[\mathbf{s}_j]_m \partial[\mathbf{s}_l]_n} = -\ln(1 - p_j(m,t) \ln(1-p_l(n,t)) \prod_{i=1}^N (1-P_i(\mathbf{s}_i,t)) \leq 0$ for any $\mathbf{s} \in \mathbb{R}_+^{Nd}$. Finally, condition ii) can be verified equivalently as in [23, Proof of Proposition 5] and condition iii) holds with equality.

The set $\tilde{\boldsymbol{\mathcal{S}}} := \{\mathbf{x} \in \mathbb{R}_+^{Nd} \mid \mathbf{0} \leq \mathbf{x} \leq \mathbf{s}_{max}\}$ with $\mathbf{s}_{max} = 2(\bar{s}_1, \ldots, \bar{s}_N)$ is such that $\mathbf{s} + \mathbf{s}' \leq \mathbf{s}_{max}$ for any pair $\mathbf{s}, \mathbf{s}' \in \boldsymbol{\mathcal{S}}$. Moreover, using the expression of $\nabla\gamma(\mathbf{s})$, the curvature of γ with respect to $\tilde{\boldsymbol{\mathcal{S}}}$ is

$$1 - \alpha(\tilde{\boldsymbol{\mathcal{S}}}) = \inf_{\substack{\mathbf{s}\in\tilde{\boldsymbol{\mathcal{S}}}\\ l\in[Nd]}} \frac{[\nabla\gamma(\mathbf{s})]_l}{[\nabla\gamma(0)]_l} = \min_{i\in[N],r\in[d]} \frac{\sum\limits_{t\in\mathcal{T}:r\in\Gamma(t)} \ln(1-p_i(r,t)) \prod\limits_{j\in[N]} (1-P_j(2\bar{\mathbf{s}}_j,t))}{\sum\limits_{t\in\mathcal{T}:r\in\Gamma(t)} \ln(1-p_i(r,t))}$$

$$=: 1-\alpha > 0.$$

Hence, using Theorem 1 we conclude that $PoA_{CCE} \leq 1 + \alpha$. □

A.4 Proof of Fact 1

Condition i) holds since γ is monotone DR-submodular by definition. Also, condition ii) holds with equality. Moreover, defining (with abuse of notation) $[\mathbf{s}]_1^i = (\mathbf{s}_1, \ldots, \mathbf{s}_i, \mathbf{0}, \ldots, \mathbf{0})$ for $i \in [N]$ with $[\mathbf{s}]_1^0 = \mathbf{0}$, condition iii) holds since by DR-submodularity one can verify that $\sum_{i=1}^N \hat{\pi}_i(\mathbf{s}) = \sum_{i=1}^N \gamma(\mathbf{s}) - \gamma(\mathbf{0}, \mathbf{s}_{-i}) \leq \gamma([\mathbf{s}]_1^i) - \gamma([\mathbf{s}]_1^{i-1}) = \gamma(\mathbf{x}) - \gamma(\mathbf{0}) = \gamma(\mathbf{x})$. □

A.5 Proof of Corollary 1

By definition of α, and according to Theorem 1, $\hat{\mathcal{G}}$ is such that $PoA_{CCE} \leq (1 + \alpha)$. In other words, letting $\mathbf{s}^\star = \arg\max_{\mathbf{s}\in\boldsymbol{\mathcal{S}}} \gamma(\mathbf{s})$, any CCE σ of $\hat{\mathcal{G}}$ satisfies $\mathbb{E}_{\mathbf{s}\sim\sigma}[\gamma(\mathbf{s})] \geq 1/(1+\alpha)\gamma(\mathbf{s}^\star)$. Moreover, since players simultaneously use no-regret algorithms D-noRegret converges to one of such CCE [15, 28]. Hence, the statement of the remark follows. □

A.6 Proof of Proposition 2

Consider the sensor coverage problem with continuous assignments defined in Example 2. We first show that γ is a monotone DR-submodular function. In fact, for any $i \in [Nd]$, $[\nabla\gamma(\mathbf{x})]_i = -\ln(1-p_l^m) \prod_{i\in[N]} (1 - p_i^m)^{[\mathbf{x}_i]_m} \geq 0$, where l and m and the indexes of sensor and location corresponding to coordinate i, respectively. Moreover, for any pair of sensors $j, l \in [N]$, $\frac{\partial^2\gamma(\mathbf{x})}{\partial[\mathbf{x}_j]_m \partial[\mathbf{x}_l]_n} = -\ln(1-p_j^m)\ln(1-p_l^n) \prod_{i\in[N]}(1 - p_i^m)^{[\mathbf{x}_i]_m} \leq 0$ if $m = n$, and 0 otherwise. The problem of maximizing γ subject to $\boldsymbol{\mathcal{X}} = \prod_{i=1}^N \mathcal{X}_i$, hence, is one of maximizing a monotone DR-submodular function subject to decoupled constraints discussed in Section 3.2. Thus, as outlined in Section 3.2, we can setup a valid utility game $\hat{\mathcal{G}}$.

The vector $\mathbf{x}_{max} = 2\bar{\mathbf{x}} = 2(\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_N)$ is such that $\forall \mathbf{x}, \mathbf{x}' \in \boldsymbol{\mathcal{X}}$, $\mathbf{x} + \mathbf{x}' \leq \mathbf{x}_{max}$. Moreover, defining $\tilde{\boldsymbol{\mathcal{X}}} := \{\mathbf{x} \in \mathbb{R}_+^{Nd} \mid \mathbf{0} \leq \mathbf{x} \leq \mathbf{x}_{max}\}$, the curvature of γ with respect to $\tilde{\boldsymbol{\mathcal{X}}}$, satisfies $\alpha(\tilde{\boldsymbol{\mathcal{X}}}) = 1 - \inf_{\substack{\mathbf{x}\in\tilde{\boldsymbol{\mathcal{X}}}\\ l\in[Nd]}} \frac{[\nabla\gamma(\mathbf{x})]_l}{[\nabla\gamma(0)]_l} = 1 - \min_{r\in[d]} \prod_{i\in[N]}(1-p_i^r)^{2\bar{\mathbf{x}}_i} = \max_{r\in[d]} P(r, 2\bar{\mathbf{x}}) = \alpha$. Hence, by Corollary 1, any no-regret distributed algorithm has expected approximation ratio of $1/(1+\alpha)$. In addition, γ is also concave in each $\mathcal{X}_i$, since the $(d \times d)$ blocks on the diagonal of its Hessian are diagonal and negative, hence online gradient ascent ensures no-regret for each player [12] and can be run in a distributed manner. □

A.7 Equivalent characterizations of DR properties

To prove the main results of the paper, the following two propositions provide equivalent characterizations of `weak DR` and `DR` properties, respectively[4].

Proposition 4. A function $f : \boldsymbol{\mathcal{X}} \subseteq \mathbb{R}^n \to \mathbb{R}$ is *weakly DR-submodular* (Definition 4) if and only if for all $\mathbf{x} \leq \mathbf{y} \in \boldsymbol{\mathcal{X}}$, $\forall \mathbf{z} \in \mathbb{R}^n_+$ s.t. $(\mathbf{x}+\mathbf{z})$ and $(\mathbf{y}+\mathbf{z})$ are in $\boldsymbol{\mathcal{X}}$, with $z_i = 0 \ \forall i \in [n] : y_i > x_i$,

$$f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}) \geq f(\mathbf{y}+\mathbf{z}) - f(\mathbf{y})\,.$$

Proof. (property of Proposition 4 $\to$ `weak DR`)
We want to prove that for all $\mathbf{x} \leq \mathbf{y} \in \boldsymbol{\mathcal{X}}$, $\forall i$ s.t. $x_i = y_i$, $\forall k \in \mathbb{R}_+$ s.t. $(\mathbf{x}+k\mathbf{e}_i)$ and $(\mathbf{y}+k\mathbf{e}_i)$ are in $\boldsymbol{\mathcal{X}}$,

$$f(\mathbf{x}+k\mathbf{e}_i) - f(\mathbf{x}) \geq f(\mathbf{y}+k\mathbf{e}_i) - f(\mathbf{y})\,.$$

This is trivially done choosing $\mathbf{z} = k\mathbf{e}_i$. Note that $\mathbf{z}$ is such that $z_i = 0, \forall i \in \{i | y_i > x_i\}$, so the property of Proposition 4 can indeed be applied.

(`weak DR` $\to$ property of Proposition 4)
For all $\mathbf{x} \leq \mathbf{y} \in \boldsymbol{\mathcal{X}}$, $\forall \mathbf{z} \in \mathbb{R}^n_+$ s.t. $(\mathbf{x}+\mathbf{z})$ and $(\mathbf{y}+\mathbf{z})$ are in $\boldsymbol{\mathcal{X}}$, with $z_i = 0 \ \forall i \in [n] : y_i > x_i$, we have

$$\begin{aligned}
f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}) &= \sum_{i=1}^n f(\mathbf{x}+[\mathbf{z}]_1^i) - f(\mathbf{x}+[\mathbf{z}]_1^{i-1}) \\
&= \sum_{i:x_i=y_i} f(\mathbf{x}+[\mathbf{z}]_1^{i-1}+z_i\mathbf{e}_i) - f(\mathbf{x}+[\mathbf{z}]_1^{i-1}) \\
&\geq \sum_{i:x_i=y_i} f(\mathbf{y}+[\mathbf{z}]_1^{i-1}+z_i\mathbf{e}_i) - f(\mathbf{y}+[\mathbf{z}]_1^{i-1}) \\
&= \sum_{i=1}^n f(\mathbf{y}+[\mathbf{z}]_1^i) - f(\mathbf{y}+[\mathbf{z}]_1^{i-1}) \\
&= f(\mathbf{y}+\mathbf{z}) - f(\mathbf{y})\,.
\end{aligned}$$

The first equality is obtained from a telescoping sum, the second equality follows since when $y_i > x_i$, $z_i = 0$. The inequality follows from `weak DR` property of f and the last two equalities are similar to the first two. ☐

Proposition 5. A function $f : \boldsymbol{\mathcal{X}} \subseteq \mathbb{R}^n \to \mathbb{R}$ is *DR-submodular* (Definition 1) if and only if for all $\mathbf{x} \leq \mathbf{y} \in \boldsymbol{\mathcal{X}}$, $\forall \mathbf{z} \in \mathbb{R}^n_+$ s.t. $(\mathbf{x}+\mathbf{z})$ and $(\mathbf{y}+\mathbf{z})$ are in $\boldsymbol{\mathcal{X}}$,

$$f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}) \geq f(\mathbf{y}+\mathbf{z}) - f(\mathbf{y})\,.$$

Proof. (property of Proposition 5 $\to$ `DR`)
We want to prove that for all $\mathbf{x} \leq \mathbf{y} \in \boldsymbol{\mathcal{X}}$, $\forall i \in [n]$, $\forall k \in \mathbb{R}_+$ s.t. $(\mathbf{x}+k\mathbf{e}_i)$ and $(\mathbf{y}+k\mathbf{e}_i)$ are in $\boldsymbol{\mathcal{X}}$,

$$f(\mathbf{x}+k\mathbf{e}_i) - f(\mathbf{x}) \geq f(\mathbf{y}+k\mathbf{e}_i) - f(\mathbf{y})\,.$$

This is trivially done choosing $\mathbf{z} = k\mathbf{e}_i$ and applying the property of Proposition 5.

(`DR` $\to$ property of Proposition 5)
For all $\mathbf{x} \leq \mathbf{y} \in \boldsymbol{\mathcal{X}}$, $\forall \mathbf{z} \in \mathbb{R}^n_+$ s.t. $(\mathbf{x}+\mathbf{z})$ and $(\mathbf{y}+\mathbf{z})$ are in $\boldsymbol{\mathcal{X}}$, we have

$$\begin{aligned}
f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}) &= \sum_{i=1}^n f(\mathbf{x}+[\mathbf{z}]_1^i) - f(\mathbf{x}+[\mathbf{z}]_1^{i-1}) \\
&= \sum_{i=1}^n f(\mathbf{x}+[\mathbf{z}]_1^{i-1}+z_i\mathbf{e}_i) - f(\mathbf{x}+[\mathbf{z}]_1^{i-1}) \\
&\geq \sum_{i=1}^n f(\mathbf{y}+[\mathbf{z}]_1^{i-1}+z_i\mathbf{e}_i) - f(\mathbf{y}+[\mathbf{z}]_1^{i-1}) \\
&= \sum_{i=1}^n f(\mathbf{y}+[\mathbf{z}]_1^i) - f(\mathbf{y}+[\mathbf{z}]_1^{i-1}) \\
&= f(\mathbf{y}+\mathbf{z}) - f(\mathbf{y})\,.
\end{aligned}$$

The first and last equalities are telescoping sums and the inequality follows from the `DR` property of f. ☐

A.8 Properties of (twice) differentiable submodular functions

As mentioned in Section 4, submodular continuous functions are defined on subsets of $\mathbb{R}^n$ of the form $\boldsymbol{\mathcal{X}} = \prod_{i=1}^n \mathcal{X}_i$, where each $\mathcal{X}_i$ is a compact subset of $\mathbb{R}$. From the `weak DR` property (Definition 4) it follows that, when f is differentiable, it is submodular iff

$$\forall \mathbf{x}, \mathbf{y} \in \boldsymbol{\mathcal{X}} : \mathbf{x} \leq \mathbf{y}, \forall i \text{ s.t. } x_i = y_i,\ \nabla_i f(\mathbf{x}) \geq \nabla_i f(\mathbf{y})\,.$$

That is, the gradient of f is a weak antitone mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Moreover, we saw that a function $f : \boldsymbol{\mathcal{X}} \to \mathbb{R}$ is *submodular* iff for all $\mathbf{x} \in \boldsymbol{\mathcal{X}}$, $\forall i \neq j$ and $a_i, a_j > 0$ s.t. $x_i + a_i \in \mathcal{X}_i$, $x_j + a_j \in \mathcal{X}_j$, we have [1]

$$f(\mathbf{x}+a_i\mathbf{e}_i) - f(\mathbf{x}) \geq f(\mathbf{x}+a_i\mathbf{e}_i+a_j\mathbf{e}_j) - f(\mathbf{x}+a_j\mathbf{e}_j)\,.$$

As visible from the latter condition, when f is twice-differentiable, it is submodular iff all the off-diagonal entries of its Hessian are non-positive [1]:

$$\forall \mathbf{x} \in \boldsymbol{\mathcal{X}}, \quad \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \leq 0, \quad \forall i \neq j\,.$$

Hence, the class of submodular continuous functions contains a subset of both convex and concave functions.

Similarly, from the `DR` property (Definition 1) it follows that for a differentiable continuous function DR-submodularity is equivalent to

$$\forall \mathbf{x} \leq \mathbf{y}, \nabla f(\mathbf{x}) \geq \nabla f(\mathbf{y})\,.$$

That is, the gradient of f is an antitone mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. More precisely, [4, Proposition 2] showed

[4]The introduced properties are the continuous versions of the 'group DR property'[5] of submodular set functions.

that a function f is DR-submodular iff it is submodular (weakly DR-submodular) and *coordinate-wise concave*. A function $f : \boldsymbol{\mathcal{X}} \to \mathbb{R}$ is coordinate-wise concave if, for all $\mathbf{x} \in \boldsymbol{\mathcal{X}}$, $\forall i \in [n], \forall k, l \in \mathbb{R}_+$ s.t. $(\mathbf{x} + k\mathbf{e}_i)$, $(\mathbf{x} + l\mathbf{e}_i)$, and $(\mathbf{x} + (k+l)\mathbf{e}_i)$ are in $\boldsymbol{\mathcal{X}}$, we have

$$f(\mathbf{x} + k\mathbf{e}_i) - f(\mathbf{x}) \geq f(\mathbf{x} + (k+l)\mathbf{e}_i) - f(\mathbf{x} + l\mathbf{e}_i)\,,$$

or equivalently, if twice differentiable, $\frac{\partial^2 f(\mathbf{x})}{\partial x_i^2} \leq 0$ $\forall i \in [n]$. Hence, as stated in Section 3, a twice-differentiable function is DR-submodular iff all the entries of its Hessian are non-positive:

$$\forall \mathbf{x} \in \boldsymbol{\mathcal{X}}, \quad \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \leq 0, \quad \forall i, j\,.$$

A.9 Proof of Proposition 3

By Definition 2, the curvature $\alpha(\boldsymbol{\mathcal{Z}})$ of f w.r.t. $\boldsymbol{\mathcal{Z}}$ satisfies

$$f(\mathbf{x} + k\mathbf{e}_i) - f(\mathbf{x}) \geq (1 - \alpha(\boldsymbol{\mathcal{Z}}))[f(k\mathbf{e}_i) - f(\mathbf{0})]\,, \quad (1)$$

for any $\mathbf{x} \in \boldsymbol{\mathcal{Z}}, i \in [n]$ s.t. $\mathbf{x} + k\mathbf{e}_i \in \boldsymbol{\mathcal{Z}}$ with $k \to 0_+$. We firstly show that condition (1) indeed holds for any $\mathbf{x} \in \boldsymbol{\mathcal{Z}}, i \in [n]$, and $k \in \mathbb{R}_+$ s.t. $\mathbf{x} + k\mathbf{e}_i \in \boldsymbol{\mathcal{Z}}$, by using monotonicity and coordinate-wise concavity of f. As seen in Appendix A.8, DR-submodularity implies coordinate-wise concavity. To this end, we define

$$\alpha_i^k(\boldsymbol{\mathcal{Z}}) = 1 - \inf_{\substack{\mathbf{x} \in \boldsymbol{\mathcal{Z}}:\\ \mathbf{x} + k\mathbf{e}_i \in \boldsymbol{\mathcal{Z}}}} \frac{f(\mathbf{x} + k\mathbf{e}_i) - f(\mathbf{x})}{f(k\mathbf{e}_i) - f(\mathbf{0})}\,.$$

Hence, it sufficies to prove that, for any $i \in [n]$, $\alpha_i^k(\boldsymbol{\mathcal{Z}})$ is non-increasing in k. Note that by DR-submodularity,

$$\alpha_i^k(\boldsymbol{\mathcal{Z}}) = 1 - \frac{f(\mathbf{z}_{max}) - f(\mathbf{z}_{max} - k\mathbf{e}_i)}{f(k\mathbf{e}_i) - f(\mathbf{0})}\,.$$

Hence, for any pair $l, m \in \mathbb{R}_+$ with $l < m$, $\alpha_i^m(\boldsymbol{\mathcal{Z}}) \geq \alpha_i^l(\boldsymbol{\mathcal{Z}})$ is true whenever

$$\frac{f(\mathbf{z}_{max}) - f(\mathbf{z}_{max} - m\mathbf{e}_i)}{f(m\mathbf{e}_i) - f(\mathbf{0})} \geq \frac{f(\mathbf{z}_{max}) - f(\mathbf{z}_{max} - l\mathbf{e}_i)}{f(l\mathbf{e}_i) - f(\mathbf{0})}\,.$$

The last inequality is satisfied since, by coordinate-wise concavity, $[f(\mathbf{z}_{max}) - f(\mathbf{z}_{max} - m\mathbf{e}_i)]/m \geq [f(\mathbf{z}_{max}) - f(\mathbf{z}_{max} - l\mathbf{e}_i)]/l$ and $[f(m\mathbf{e}_i) - f(\mathbf{0})]/m \leq [f(l\mathbf{e}_i) - f(\mathbf{0})]/l$. This is because, given a concave function $g : \mathbb{R} \to \mathbb{R}$, the quantity

$$R(x_1, x_2) := \frac{g(x_2) - g(x_1)}{x_2 - x_1}$$

is non-increasing in x_1 for fixed x_2, and vice versa. Moreover, monotonicity ensures that all of the above ratios are non-negative.

To conclude the proof of Proposition 3 we show that if condition (1) holds for any $\mathbf{x} \in \boldsymbol{\mathcal{Z}}, i \in [n]$, and $k \in \mathbb{R}_+$ s.t. $\mathbf{x} + k\mathbf{e}_i \in \boldsymbol{\mathcal{Z}}$, then the result of the proposition follows. Indeed, for any $\mathbf{x}, \mathbf{y}$ s.t. $\mathbf{x} + \mathbf{y} \in \boldsymbol{\mathcal{Z}}$ we have

$$\begin{aligned}
f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x}) &= \sum_{i=1}^{n} f(\mathbf{x} + [\mathbf{y}]_1^i) - f(\mathbf{x} + [\mathbf{y}]_1^{i-1}) \\
&= \sum_{i=1}^{n} f(\mathbf{x} + [\mathbf{y}]_1^{i-1} + y_i\mathbf{e}_i) - f(\mathbf{x} + [\mathbf{y}]_1^{i-1}) \\
&\geq (1 - \alpha(\boldsymbol{\mathcal{Z}})) \sum_{i=1}^{n} f(y_i\mathbf{e}_i) - f(\mathbf{0}) \\
&\geq (1 - \alpha(\boldsymbol{\mathcal{Z}})) \sum_{i=1}^{n} f([\mathbf{y}]_1^i) - f([\mathbf{y}]_1^{i-1}) \\
&= (1 - \alpha(\boldsymbol{\mathcal{Z}}))(f(\mathbf{y}) - f(\mathbf{0}))\,,
\end{aligned}$$

where the first inequality follows by condition (1) and the second one from f being weakly DR-submodular (and using Proposition 4). □

B Supplementary material for Section 5

In the first part of this appendix we generalize the submodularity ratio defined in [11] for set functions to continuous domains and discuss its main properties. We compare it to the ratio by [16] and we relate it to the generalized submodularity ratio defined in Definition 5. Then, we provide a class of social functions with generalized submodularity ratio $0 < \eta < 1$ and we report the proof of Theorem 2. Finally, we analyze the sensor coverage problem with the non-submodular objective defined in Section 5.

B.1 Submodularity ratio of a monotone function on continuous domains

We generalize the class of submodular continuous functions, defining the *submodularity ratio* $\eta \in [0,1]$ of a monotone function defined on a continuous domain.

Definition 7 (submodularity ratio)**.** The submodularity ratio of a monotone function $f : \boldsymbol{\mathcal{X}} \subseteq \mathbb{R}^n_+ \to \mathbb{R}$ is the largest scalar η such that for all $\mathbf{x}, \mathbf{y} \in \boldsymbol{\mathcal{X}}$ such that $\mathbf{x} + \mathbf{y} \in \boldsymbol{\mathcal{X}}$,

$$\sum_{i=1}^{n} \big[f(\mathbf{x} + y_i\mathbf{e}_i) - f(\mathbf{x})\big] \geq \eta\big[f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})\big] .$$

It is straightforward to show that $\eta \in [0,1]$ and, when restricted to binary sets $\boldsymbol{\mathcal{X}} = \{0,1\}^n$, Definition 7 coincides with the submodularity ratio defined in [11] for set functions. A set function is submodular iff it has submodularity ratio $\eta = 1$ [11]. However, functions with submodularity ratio $0 < \eta < 1$ still preserve 'nice' properties in term of maximization guarantees. Similarly to [11], we can affirm the following.

Proposition 6. A function $f : \boldsymbol{\mathcal{X}} \subseteq \mathbb{R}^n_+ \to \mathbb{R}$ is weakly DR-submodular (Definition 4) iff it has submodularity ratio $\eta = 1$.

Proof. If f is weakly DR-submodular (Definition 4), then for any $\mathbf{x}, \mathbf{y} \in \boldsymbol{\mathcal{X}}$,

$$\begin{aligned}&\sum_{i=1}^{d} f(\mathbf{x} + y_i\mathbf{e}_i) - f(\mathbf{x})\\ &\geq \sum_{i=1}^{d} f(\mathbf{x} + [\mathbf{y}]_1^i) - f(\mathbf{x} + [\mathbf{y}]_1^{i-1}) = f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}).\end{aligned}$$

Assume now f has submodularity ratio $\eta = 1$. We prove that f is weakly DR-submodular by proving that it is submodular. Hence, we want to prove that for all $\mathbf{x} \in \boldsymbol{\mathcal{X}}$, $\forall i \neq j$ and $a_i, a_j > 0$ s.t. $x_i + a_i \in \mathcal{X}_i$, $x_j + a_j \in \mathcal{X}_j$,

$$\begin{aligned}&f(\mathbf{x} + a_i\mathbf{e}_i) - f(\mathbf{x}) \geq \\ &\qquad f(\mathbf{x} + a_i\mathbf{e}_i + a_j\mathbf{e}_j) - f(\mathbf{x} + a_j\mathbf{e}_j) .\end{aligned} \tag{2}$$

Consider $\mathbf{y} = a_i\mathbf{e}_i + a_j\mathbf{e}_j \in \boldsymbol{\mathcal{X}}$. Since f has submodularity ratio $\eta = 1$, we have

$$\begin{aligned}&f(\mathbf{x} + a_i\mathbf{e}_i) - f(\mathbf{x}) + f(\mathbf{x} + a_j\mathbf{e}_j) - f(\mathbf{x})\\ &\qquad \geq f(\mathbf{x} + a_i\mathbf{e}_i + a_j\mathbf{e}_j) - f(\mathbf{x}) ,\end{aligned}$$

which is equivalent to the submodularity condition (2). ☐

An example of functions with submodularity ratio $\eta > 0$ is the product between an affine and a weakly DR-submodular function, as stated in the following proposition.

Proposition 7. Let $f, \rho : \boldsymbol{\mathcal{X}} \subseteq \mathbb{R}^n_+ \to \mathbb{R}_+$ be two monotone functions, with f weakly DR-submodular, and g affine such that $\rho(\mathbf{x}) = \mathbf{a}^\top\mathbf{x} + b$ with $\mathbf{a} \geq \mathbf{0}$ and $b > 0$. Then, provided that $\boldsymbol{\mathcal{X}}$ is bounded, the product $g(\mathbf{x}) := f(\mathbf{x})\rho(\mathbf{x})$ has submodularity ratio $\eta = \inf_{i\in[n],\mathbf{x}\in\boldsymbol{\mathcal{X}}} \frac{b}{b+\sum_{j\neq i} a_j x_j} > 0$.

Proof. Note that since ρ is affine, for any $\mathbf{x}, \mathbf{y} \in \boldsymbol{\mathcal{X}}$ we have that $g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}) = f(\mathbf{x}+\mathbf{y})\rho(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})\rho(\mathbf{x}) = \rho(\mathbf{x}+\mathbf{y})[f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})] + f(\mathbf{x})\,(\mathbf{a}^\top\mathbf{y})$. For any pair $\mathbf{x}, \mathbf{y} \in \boldsymbol{\mathcal{X}}$ we have:

$$\begin{aligned}&\sum_{i=1}^{n} \big[g(\mathbf{x} + y_i\mathbf{e}_i) - g(\mathbf{x})\big]\\ &= \sum_{i=1}^{n} \rho(\mathbf{x} + y_i\mathbf{e}_i)[f(\mathbf{x} + y_i\mathbf{e}_i) - f(\mathbf{x})] + f(\mathbf{x})\,(y_i\mathbf{a}^\top\mathbf{e}_i)\\ &\geq \min_{i\in[n]} \rho(\mathbf{x} + y_i\mathbf{e}_i) \sum_{i=1}^{n} f(\mathbf{x} + y_i\mathbf{e}_i) - f(\mathbf{x}) + f(\mathbf{x})\,(\mathbf{a}^\top\mathbf{y})\\ &\geq \underbrace{\frac{\min_{i\in[n]} \rho(\mathbf{x} + y_i\mathbf{e}_i)}{\rho(\mathbf{x}+\mathbf{y})}}_{:=\eta(\mathbf{x},\mathbf{y})} \Big(\rho(\mathbf{x}+\mathbf{y})[f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})]\\ &\qquad\qquad + f(\mathbf{x})\,(\mathbf{a}^\top\mathbf{y})\Big)\\ &= \eta(\mathbf{x},\mathbf{y})\,[g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x})] .\end{aligned}$$

The first inequality follows since ρ is affine non-negative and f is non-negative. The second inequality is due to f being weakly DR-submodular (f has submodularity ratio $\eta = 1$) and $0 < \eta(\mathbf{x},\mathbf{y}) \leq 1$, which holds because $b > 0$ and $\mathbf{a} \geq \mathbf{0}$. Hence, it follows that γ has submodularity ratio

$$\eta := \inf_{\substack{\mathbf{x},\mathbf{y}\in\boldsymbol{\mathcal{X}}:\\ \mathbf{x}+\mathbf{y}\in\boldsymbol{\mathcal{X}}}} \eta(\mathbf{x},\mathbf{y}) = \inf_{i\in[n],\mathbf{y}\in\boldsymbol{\mathcal{X}}} \frac{b}{b + \sum_{j\neq i} a_j y_j} > 0 .$$

☐

B.1.1 Related notion by [16]

A generalization of submodular continuous functions was also provided in [16] together with provable maximization guarantees. However, it has different implications than the submodularity ratio defined above. In fact, [16] considered the class of differentiable functions $f : \boldsymbol{\mathcal{X}} \subseteq \mathbb{R}^n_+ \rightarrow \mathbb{R}$ with parameter η defined as

$$\eta = \inf_{\mathbf{x},\mathbf{y} \in \boldsymbol{\mathcal{X}}, \mathbf{x} \leq \mathbf{y}} \inf_{i \in [n]} \frac{[\nabla f(\mathbf{x})]_i}{[\nabla f(\mathbf{y})]_i} .$$

For monotone functions $\eta \in [0, 1]$, and a differentiable function is DR-submodular iff $\eta = 1$ [16]. Note that the parameter η of [16] generalizes the `DR` property of f, while our submodularity ratio η generalizes the `weak DR` property.

B.2 Relations with the generalized submodularity ratio of Definition 5

In Proposition 6 we saw that submodularity ratio $\eta = 1$ is a necessary and sufficient condition for weak DR-submodularity. In contrast, a generalized submdoularity ratio (Definition 5) $\eta = 1$ is only necessary for the social function γ to be weakly DR-submodular. This is stated in the following proposition. For non submodular γ, no relation can be established between submodularity ratio of Definition 7 and generalized submodularity ratio of Definition 5.

Proposition 8. Given a game $\mathcal{G} = (N, \{\mathcal{S}_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma)$. If γ is weakly DR-submodular, then γ has generalized submodularity ratio $\eta = 1$.

Proof. Consider any pair of outcomes $\mathbf{s}, \mathbf{s}' \in \boldsymbol{\mathcal{S}}$. For $i \in \{0, \ldots, N\}$, with abuse of notation we define $[\mathbf{s}']_1^i := (\mathbf{s}'_1, \ldots, \mathbf{s}'_i, \mathbf{0}, \ldots, \mathbf{0})$ with $[\mathbf{s}']_1^0 = \mathbf{0}$. We have,

$$\sum_{i=1}^N \gamma(\mathbf{s}_i + \mathbf{s}'_i, \mathbf{s}_{-i}) - \gamma(\mathbf{s})$$
$$\geq \sum_{i=1}^N \gamma(\mathbf{s} + [\mathbf{s}']_1^i) - \gamma(\mathbf{s} + [\mathbf{s}']_1^{i-1})$$
$$= \gamma(\mathbf{s} + \mathbf{s}') - \gamma(\mathbf{s}) ,$$

where the inequality follows since γ is weakly DR-submodular and the equality is a telescoping sum. □

Similarly to Proposition 7 in the previous section, in the following proposition we show that social functions γ defined as product of weakly DR-submodular functions and affine functions have generalized submodularity ratio $\eta > 0$.

Proposition 9. Given a game $\mathcal{G} = (N, \{\mathcal{S}_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma)$. Let γ be defined as $\gamma(\mathbf{s}) := f(\mathbf{x})\rho(\mathbf{x})$ with $f, \rho : \mathbb{R}_+^{Nd} \rightarrow \mathbb{R}_+$ be two monotone functions, with f weakly DR-submodular, and g affine such that $\rho(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} + b$ with $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_N) \geq \mathbf{0}$ and $b > 0$. Then, γ has generalized submodularity ratio $\eta = \inf_{i \in [N], \mathbf{s} \in \boldsymbol{\mathcal{S}}} \frac{b}{b + \sum_{j \neq i} \mathbf{a}_j^\top \mathbf{s}_j} > 0$.

Proof. The proof is equivalent to the proof of Proposition 7, with the only difference that $\mathbf{s}'_i$ belong to $\mathbb{R}^d_+$ instead of $\mathbb{R}_+$. □

Note that for the game considered in the previous proposition, using Proposition 7 one could also affirm that γ has submodularity ratio $\eta = \inf_{i \in [Nd], \mathbf{s} \in \boldsymbol{\mathcal{S}}} \frac{b}{b + \sum_{j \neq i} [\mathbf{a}]_j [\mathbf{s}]_j} > 0$ which, unless $d = 1$, is strictly smaller than its generalized submodularity ratio.

B.3 Proof of Theorem 2

The proof is equivalent to the proof of Theorem 1, with the only difference that here we prove that $\mathcal{G}$ is a *(η, η)-smooth* game in the framework of [28]. Then, it follows that $PoA_{CCE} \leq (1 + \eta)/\eta$.

For the smoothness proof, consider any pair of outcomes $\mathbf{s}, \mathbf{s}^\star \in \boldsymbol{\mathcal{S}}$. We have:

$$\sum_{i=1}^N \pi_i(\mathbf{s}_i^\star, \mathbf{s}_{-i}) \geq \sum_{i=1}^N \gamma(\mathbf{s}_i^\star, \mathbf{s}_{-i}) - \gamma(0, \mathbf{s}_{-i})$$
$$\geq \sum_{i=1}^N \gamma(\mathbf{s}_i^\star + \mathbf{s}_i, \mathbf{s}_{-i}) - \gamma(\mathbf{s})$$
$$= \eta\, \gamma(\mathbf{s} + \mathbf{s}^\star) - \eta\, \gamma(\mathbf{s}) .$$

The first inequality is due to condition ii) of Definition 3. The second inequality follows since γ is player-wise DR-submodular (applying Proposition 5 for each player i) and the second inequality from γ having generalized submodularity ratio η. □

B.4 Analysis of the sensor coverage problem with non-submodular objective

We analyze the sensor coverage problem with non-submodular objective defined in Section 5, where $\gamma(\mathbf{x}) = \sum_{r \in [d]} w_r(\mathbf{x})\ P(r, \mathbf{x})$ with $w_r(\mathbf{x}) = \mathbf{a}_r \frac{\sum_{i=1}^N [\mathbf{x}_i]_r}{N} + b_r$. Note that by Proposition 9, the function $\gamma_r(\mathbf{x}) := w_r(\mathbf{x})\ P(r, \mathbf{x})$ has generalized submodularity ratio $\eta > 0$, hence it is not hard to show that $\gamma(\mathbf{x}) = \sum_{r \in [d]} \gamma_r(\mathbf{x})$ shares the same property. Moreover, there exist parameters $\mathbf{a}_r, b_r$ for which γ is not submodular. Interestingly, γ is convave in each $\mathcal{X}_i$.

In fact, γ_r's are concave in each $\mathcal{X}_i$ since $P(r, \mathbf{x})$'s are concave in each $\mathcal{X}_i$ and w_r's are positive affine functions. Moreover, γ is playerwise DR-submodular since the $(d \times d)$ blocks on the diagonal of its Hessian are diagonal (and their entries are non-positive, by concavity of γ in each $\mathcal{X}_i$).

To maximize γ, as outlined in Section 3.2, we can set up a game $\mathcal{G} = (N, \{\mathcal{S}_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N, \gamma)$ where for each player i, $\mathcal{S}_i = \mathcal{X}_i$, and $\pi_i(\mathbf{s}) = \gamma(\mathbf{s}) - \gamma(\mathbf{0}, \mathbf{s}_{-i})$ for every outcome $\mathbf{s} \in \boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{X}}$. Hence, condition ii) of Definition 3 is satisfied with equality. Following the proof of Theorem 2, we have that:

$$\sum_{i=1}^N \pi_i(\mathbf{s}_i^\star, \mathbf{s}_{-i}) \geq \eta\, \gamma(\mathbf{s} + \mathbf{s}^\star) - \eta\, \gamma(\mathbf{s})$$

In order to bound PoA_{CCE}, the last proof steps of Section 4.2 still ought to be used. Such steps rely on condition iii), which in Section 3.2 was proved using submodularity of γ. Although γ is not submodular, we prove a weaker version of condition iii) as follows. By definition of γ_r and for every outcome $\mathbf{x}$ we have $\sum_{i=1}^N \gamma_r(\mathbf{s}) - \gamma_r(\mathbf{0}, \mathbf{s}_{-i}) = \sum_{i=1}^N w_r(\mathbf{x})[P(r, \mathbf{s}) - P(r, (\mathbf{0}, \mathbf{s}_{-i}))] + [w_r(\mathbf{s}_i, \mathbf{0}) - w_r(\mathbf{0})]P(r, (\mathbf{0}, \mathbf{s}_{-i})) \leq w_r(\mathbf{x})P(r, \mathbf{s}) + P(r, \mathbf{s}) \sum_{i=1}^N [w_r(\mathbf{s}_i, \mathbf{0}) - w_r(\mathbf{0})] = (1 + \frac{w_r(\mathbf{x}) - w_r(\mathbf{0})}{w_r(\mathbf{x})})\gamma_r(\mathbf{x}) \leq 2\gamma_r(\mathbf{x})$. The equalities are due to w_r being affine, the first inequality is due to $P(r, \cdot)$ being submodular and monotone, and the last inequality holds since w_r is positive and monotone. Hence, from the inequalities above we have $\sum_{i=1}^N \pi_i(\mathbf{x}) = \sum_{i=1}^N \gamma(\mathbf{x}) - \gamma(\mathbf{0}, \mathbf{x}_{-i}) \leq 2\gamma(\mathbf{x})$. Note that a tighter condition can also be derived depending on the functions w_r's, using $(1 + \max_{\mathbf{x} \in \boldsymbol{\mathcal{X}}, r \in [d]} \frac{w_r(\mathbf{x}) - w_r(\mathbf{0})}{w_r(\mathbf{x})})$ in place of 2. We will now use such condition in the same manner condition iii) was used in Section 4.2. Let $\mathbf{s}^\star = \arg\max_{\mathbf{s} \in \boldsymbol{\mathcal{S}}} \gamma(\mathbf{s})$. Then, for any CCE σ of $\mathcal{G}$ we have

$$\begin{aligned}\mathbb{E}_{\mathbf{s}\sim\sigma}[\gamma(\mathbf{s})] &\geq \frac{1}{2}\sum_{i=1}^N \mathbb{E}_{\mathbf{s}\sim\sigma}[\pi_i(\mathbf{s})] \geq \frac{1}{2}\sum_{i=1}^N \mathbb{E}_{\mathbf{s}\sim\sigma}[\pi_i(\mathbf{s}_i^\star, \mathbf{s}_{-i})] \\ &\geq \frac{\eta}{2}\gamma(\mathbf{s}^\star) - \frac{\eta}{2}\,\mathbb{E}_{\mathbf{s}\sim\sigma}[\gamma(\mathbf{s})]\,.\end{aligned}$$

Hence, $PoA_{CCE} \leq (1 + 0.5\eta)/0.5\eta$.