Supplement

In Section A we prove our remark on the validity of the Bernstein condition for higher-order derivatives in the case of kernels with faster spectral decay. The result extends the example of Gaussian kernels detailed in the main part of the paper.

A BERNSTEIN CONDITION FOR HIGHER-ORDER DERIVATIVES

We prove that in the case of kernels with spectral density decaying as $f_\Lambda(\omega) \propto e^{-\omega^{2\ell}}$ ($\ell \in \mathbb{N}^+$), the Bernstein condition (15) holds for $r \le 2\ell$-order derivatives. This example extends the case of Gaussian kernels where $\ell = 1$ and $r \le 2$. Let $\ell \in \mathbb{N}^+$ and the spectral measure associated with kernel k be absolutely continuous w.r.t. the Lebesgue measure with density

$$f_\Lambda(\omega) = c_\ell e^{-\omega^{2\ell}}$$

for some $c_\ell > 0$. f_Λ is positive and we determine c_ℓ as:

$$\begin{aligned} 1 &= \int_{\mathbb{R}} f_\Lambda(\omega)\mathrm{d}\omega = \int_{\mathbb{R}} c_\ell e^{-\omega^{2\ell}}\mathrm{d}\omega = 2c_\ell \int_0^\infty e^{-\omega^{2\ell}}\mathrm{d}\omega \\ &= \frac{c_\ell}{\ell}\int_0^\infty e^{-y} y^{\frac{1}{2\ell}-1}\mathrm{d}y = \frac{c_\ell}{\ell}\Gamma\left(\frac{1}{2\ell}\right) \quad \Rightarrow \\ c_\ell &= \frac{\ell}{\Gamma\left(\frac{1}{2\ell}\right)} \end{aligned}$$

where we used $y = \omega^{2\ell}$, $\omega = y^{\frac{1}{2\ell}}$, $\mathrm{d}\omega = \frac{1}{2\ell} y^{\frac{1}{2\ell}-1}\mathrm{d}y$ and the pdf of the Gamma distribution ($b = 1$, $a = \frac{1}{2\ell}$) $g(y;a,b) = \frac{b^a}{\Gamma(a)} y^{a-1}e^{-by}$, ($y > 0$, $a > 0$, $b > 0$) from which it follows that

$$\int_0^\infty y^{a-1}e^{-by}\mathrm{d}y = \frac{\Gamma(a)}{b^a}. \tag{24}$$

Consequently, one obtains

$$A_{r,n} = A_{r,n}(\Lambda) = \frac{\int_{\mathbb{R}} |\omega|^{rn}\mathrm{d}\Lambda(\omega)}{\left[\sqrt{\int_{\mathbb{R}} |\omega|^{2r}\mathrm{d}\Lambda(\omega)}\right]^n} = \frac{\frac{\Gamma\left(\frac{rn+1}{2\ell}\right)}{\Gamma\left(\frac{1}{2\ell}\right)}}{\left[\frac{\Gamma\left(\frac{2r+1}{2\ell}\right)}{\Gamma\left(\frac{1}{2\ell}\right)}\right]^{\frac{n}{2}}}$$

by using (24) with $b = 1$, $a = \frac{r+1}{2\ell}$ and the value of c_ℓ:

$$\begin{aligned} \int_{\mathbb{R}} |\omega|^r \mathrm{d}\Lambda(\omega) &= \int_{\mathbb{R}} |\omega|^r c_\ell e^{-\omega^{2\ell}}\mathrm{d}\omega \\ &= 2c_\ell \int_0^\infty \omega^r e^{-\omega^{2\ell}}\mathrm{d}\omega = \frac{c_\ell}{\ell}\int_0^\infty e^{-y} y^{\frac{r}{2\ell}} y^{\frac{1}{2\ell}-1}\mathrm{d}y \\ &= \frac{c_\ell}{\ell}\Gamma\left(\frac{r+1}{2\ell}\right) = \frac{\Gamma\left(\frac{r+1}{2\ell}\right)}{\Gamma\left(\frac{1}{2\ell}\right)}. \end{aligned}$$

Next we assume that $r \le 2\ell$ is fixed and apply induction to prove (15).

- For $n = 2$, by definition $A_{r,2} = 1$ ($\forall r \in \mathbb{N}^+$).
- The induction argument is as follows. By the inductive assumption it is sufficient to show the existence of $K_r \ge 1$ such that

$$B_{r,n} := \frac{A_{r,n+1}}{A_{r,n}} \le (n+1)K_r \tag{25}$$

since $A_{r,n} \le \frac{n!}{2}K_r^{n-2}$ and $\frac{A_{r,n+1}}{A_{r,n}} \le (n+1)K_r$ imply $A_{r,n+1} \le \frac{(n+1)!}{2}K_r^{n+1-2}$. By defining $c_r := \frac{\Gamma\left(\frac{2r+1}{2\ell}\right)}{\Gamma\left(\frac{1}{2\ell}\right)}$, we obtain

$$\begin{aligned} B_{r,n} &= \frac{\Gamma\left(\frac{r(n+1)+1}{2\ell}\right)}{\Gamma\left(\frac{1}{2\ell}\right)(c_r)^{\frac{n+1}{2}}}\frac{\Gamma\left(\frac{1}{2\ell}\right)(c_r)^{\frac{n}{2}}}{\Gamma\left(\frac{rn+1}{2\ell}\right)} = \frac{\Gamma\left(\frac{r(n+1)+1}{2\ell}\right)}{\sqrt{c_r}\,\Gamma\left(\frac{rn+1}{2\ell}\right)} \\ &= \frac{\Gamma\left(\frac{rn+1}{2\ell}+\frac{r}{2\ell}\right)}{\sqrt{c_r}\,\Gamma\left(\frac{rn+1}{2\ell}\right)} \overset{(a)}{\le} D_{r,n}\frac{\Gamma\left(\frac{rn+1}{2\ell}+\frac{2\ell}{2\ell}\right)}{\sqrt{c_r}\,\Gamma\left(\frac{rn+1}{2\ell}\right)} \\ &\overset{(b)}{=} D_{r,n}\frac{1}{\sqrt{c_r}}\frac{rn+1}{2\ell} \overset{(c)}{\le} D_{r,n}\frac{1}{\sqrt{c_r}}\underbrace{\frac{2\ell n+1}{2\ell}}_{n+\frac{1}{2\ell}} \\ &< D_{r,n}\frac{n+1}{\sqrt{c_r}}. \end{aligned}$$

Indeed,

– (a): The Gamma function has a global minima on the positive real line at $z_{min} \approx 1.46163$, it is strictly monotonically decreasing on $(0, z_{min})$ and strictly monotonically increasing on (z_{min}, ∞). The latter implies

$$\Gamma(z_1) \le \Gamma(z_2) \text{ for } z_{min} \le z_1 \le z_2. \tag{26}$$

Let us choose $z_1 = \frac{rn+1}{2\ell} + \frac{r}{2\ell}$ and $z_2 = \frac{rn+1}{2\ell} + \frac{2\ell}{2\ell}$. $z_1 \le z_2$ since $r \le 2\ell$. With this choice (26) guarantees (a) with $D_{r,n} = 1$ if

$$\begin{aligned} z_{min} &\overset{(d)}{\le} \frac{n+2}{2\ell} = \frac{rn+1}{2\ell} + \frac{r}{2\ell}\bigg|_{r=1} \le \\ &\le \frac{rn+1}{2\ell} + \frac{r}{2\ell} = z_1. \end{aligned}$$

If $n_s := \lceil 2\ell z_{min} - 2\rceil \le n$, then (d) holds. This means that (a) holds with

$$D_{r,n} = 1 \quad \text{if } n_s \le n.$$

For the remaining $n = 2, \ldots, n_s - 1$ values, (a) is fulfilled with equality using $D_{r,n} := \frac{\Gamma\left(\frac{rn+1}{2\ell}+\frac{2\ell}{2\ell}\right)}{\Gamma\left(\frac{rn+1}{2\ell}+\frac{r}{2\ell}\right)}$.

– (b): We applied the $\Gamma(z+1) = z\Gamma(z)$ property.

– (c): It follows from $r \le 2\ell$.

To sum up, we got that

$$B_{r,n} \le \frac{D_{r,n}}{\sqrt{c_r}}(n+1), \text{ with}$$

$$D_{r,n} = \begin{cases} 1 & \text{if } n_s \le n \\ \frac{\Gamma\left(\frac{rn+1}{2\ell}+\frac{2\ell}{2\ell}\right)}{\Gamma\left(\frac{rn+1}{2\ell}+\frac{r}{2\ell}\right)} & n = 2, \ldots, n_s - 1 \end{cases}.$$

Thus, one can choose

$$K_r = \max\left(\frac{D_{r,2}}{\sqrt{c_r}}, \ldots, \frac{D_{r,n_s-1}}{\sqrt{c_r}}, \frac{1}{\sqrt{c_r}}, 1\right)$$

in (25).