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Numerical Term Logic

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Abstract This paper is an attempt to show that my work to establish numerically flexible quantifiers for the syllogism can be aptly combined with the term logic advanced by Sommers, Englebretsen, and others.

1 Introduction Sommers, followed by Englebretsen and others, has developed a comprehensive notational and deductive system in categorical logic which has the syllogism as its base (see Sommers [11]) but which extends far beyond the traditional logic. (See Kelley [4], chapter 14, for a textbook presentation.) Indeed, its power rivals that of the first-order predicate calculus and its defenders allege it to be superior to the calculus on various important counts. (For example, see Sommers [10] and Englebretsen [2].) Furthermore, I have developed a numerically expanded scheme of quantification in categorical logic of which the traditional syllogism turns out to be but one of infinitely many numerical instances (see Murphree [5] and [6]).

In this paper I propose to show that the two approaches can be aptly combined into a program more comprehensive than either. Specifically, I propose that with only minor adaptations, the symbolic and deductive mechanism developed by Sommers and Englebretsen (called "term logic" or TL) works for the propositions and inferences of my "numerical logic" (NL); and I propose that these latter propositions and inferences, in turn, reveal a vast field of applicability hitherto unavailable to the term logic. The preliminary tasks are those of summarizing the basic features of each system. The numerical logic (NL) is introduced first.

2 Numerical logic (NL) The numerically expanded logic is concerned with quantities between the extremes of "all" and "some". Specifically, it accommodates numerical differences in particular quantifiers and different numerical deviations from universal quantifiers.

2.1 Propositions of NL The point of departure allowing the move from traditional quantification to the numerically expanded system is the observation that "all" as in

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"All Ss are P," contains an implicit zero in it, viz., "All Ss—with zero exception—are P" or "All but zero Ss are P." And when this is made explicit it becomes obvious that the traditional A proposition is the terminal instance of an infinite series of possible claims, viz.,

All but 0 Ss are P,
All but 1 Ss are P,
All but 2 Ss are P,
All but 3 Ss are P,

and for any number *x*,

All but *x* Ss are P.

These forms, moreover, are to be understood as carrying an "at least" qualification, that is, as "At least all but *x* Ss are P." In addition, "No Ss are P" can easily be rendered "[At most] zero Ss are P"; and given this rendition, it is clear that the traditional E proposition is also the terminal instance of an infinite series of possible claims, viz.,

At most 0 Ss are P,
At most 1 S is P,
At most 2 Ss are P,

and for any number *x*,

At most *x* Ss are P.

Although these quantifiers, viz., "At least all but *x* . . . " and "At most *x* . . . ," may seem a bit incongruous initially, they do function perfectly together, as can be seen by noting their respective obverses. That is,

At least all but *x* Ss are P = At most *x* Ss are nonP

and,

At most *x* Ss are P = At least all but *x* Ss are nonP.

Furthermore, the modern rendition of "some" is already explicitly numerical, viz., "At least one S is [not] P"; accordingly, the traditional particulars are terminal instances of infinite series of possible particular claims, viz.,

At least 1 S is P,	At least 1 S is not P,
At least 2 Ss are P,	At least 2 Ss are not P,
At least 3 Ss are P,	At least 3 Ss are not P,

and for any number *x*,

At least *x* Ss are P, At least *x* Ss are not P.

It is convenient to symbolize these by prefixing a numeral (or variable) to the conventional symbols of SAP, SEP, SIP, and SOP, as follows:

0SAP = SAP	0SEP = SEP	1SIP =SIP	1SOP = SOP
1SAP	1SEP	2SIP	2SOP
2SAP	2SEP	3SIP	3SOP
3SAP	3SEP	4SIP	4SOP

and for any number x,

xSAP xSEP xSIP xSOP

(See [5], pp. 5–16, for a more systematic treatment of numerical propositions.)

Other than for the numerical quantifiers, the features of the forms remain essentially the same as before. For example, each form is equivalent to its obverse, the E and I are equivalent to their converses, and the A and O are equivalent to their contrapositives. And the subject terms of the universals (if they may still be so-called) and the predicates of the negatives are distributed, whereas the other terms are undistributed. (See Murphree [7], pp. 33–40, for further clarification of distribution in numerical propositions.)

2.2 Syllogisms in NL Also, except for the numerical quantifiers, the syllogisms operate the same as before. But given the many numerical distinctions now possible, a new stipulation is required to insure that the numerical value of the conclusion is not in error. The general rule that allows the strongest numerical conclusion warranted by the premises is:

> The numerical value of a universal conclusion must equal the sum of the numerical values of the premises, whereas the numerical value of a particular conclusion must equal the difference between the values of the particular and the universal premises, as the latter is subtracted from the former.

So, according to this rule the general form for *Barbara* is:

$$\begin{array}{l} x\text{MAP} \\ \underline{y\text{SAM}} \\ x+y\text{SAP} \end{array}$$

When x and y are both instantiated with 0 the result is *Barbara* of the traditional logic; but x and y may be instantiated with other quantities just as well. Some samples are as follows.

$$\begin{array}{lll} x=0 & x=3 & x=177 \\ y=0 & y=1 & y=438 \\ & & \\ 0\text{MAP}=\text{MAP} & 3\text{MAP} & 177\text{MAP} \\ \underline{0\text{SAM}=\text{SAM}} & \underline{1\text{SAM}} & \underline{438\text{SAM}} \\ 0\text{SAP}\ =\text{SAP} & 4\text{SAP} & 615\text{SAP} \end{array}$$

Furthermore, according to the rule above the general form for *Darii* is:

$$\begin{array}{r} x\text{MAP} \\ \underline{x+y\text{SIM}} \\ y\text{SIP} \end{array}$$

When x is instantiated with 0 and y with 1 the result is the *Darii* of the traditional logic; but x and y may be instantiated with other quantities just as well. Some samples are as follows.

$$\begin{array}{lll} x = 0 & x = 1 & x = 123 \\ y = 1 & y = 3 & y = 456 \end{array}$$

$$\begin{array}{lll} \text{0MAP} = \text{MAP} & \text{1MAP} & \text{123MAP} \\ \underline{\text{1SIM} = \text{SIM}} & \underline{\text{4SIM}} & \underline{\text{579SIM}} \\ \text{1SIP} = \text{SIP} & \text{3SIP} & \text{456SIP} \end{array}$$

And both *Barbara* and *Darii* can be instantiated with infinitely many more valid numerical forms; and each of the other syllogisms valid in the traditional logic can be expanded infinitely in this way as well.

Viewed from this perspective, the traditional syllogistic logic is seen to be arbitrarily restrictive in only allowing the instantiation values of 0 and 1; and conversely, the full potential of syllogistic applicability is only disclosed when this arbitrary limitation is rejected. Again, it is this expanded quantificational potential that, I propose, NL has to offer TL.

3 Term logic (TL) As was noted earlier, TL is comprehensive and it rivals the first order predicate calculus. Perhaps the major difference is that whereas the predicate calculus treats categorical logic as a special type of propositional logic (e.g., universals are special kinds of conditionals), TL interprets propositional logic as being a special type of categorical logic (e.g., conjunctions are special kinds of particulars). (For example, see [2].)

However, since the topic under consideration does not involve propositional logic, the introduction of TL below is limited to the consideration of the basic categorical propositions, categorical syllogisms, and some extensions of the system.

3.1 Basic propositions of TL The traditional propositions are symbolized by having the terms of the proposition, S and P, introduced by a plus or minus sign. This allows four combinations, since each term may be introduced by either sign. These combinations then symbolize the four basic propositions when the minus sign introduces the distributed terms, and the plus sign introduces the undistributed terms, as follows.

$$\begin{array}{lcl} -\text{S}+\text{P} & = & \text{SAP,} \\ -\text{S}-\text{P} & = & \text{SEP,} \\ +\text{S}+\text{P} & = & \text{SIP, and} \\ +\text{S}-\text{P} & = & \text{SOP.} \end{array}$$

(See [10], pp. 36–38, for a justification of these assignments.)

But these binary uses of the plus and minus signs (e.g., $+\cdots+\cdots=$ Some ... are ...) are supplemented by the unary plus and minus signs to indicate a term, S (+S), and its complement, nonS (−S). However, following the convention in mathematics, the unary plus signs are usually omitted; so the more detailed rendition of the basic propositions would be:

$$\begin{array}{lcl} -(+\text{S})+(+\text{P}) & = & \text{SAP,} \\ -(+\text{S})-(+\text{P}) & = & \text{SEP,} \\ +(+\text{S})+(+\text{P}) & = & \text{SIP, and} \\ +(+\text{S})-(+\text{P}) & = & \text{SOP.} \end{array}$$

Furthermore, another application of the unary signs is applicable on the propositional level, since each of the above proposition forms might be affirmed or denied. So, still following the convention in mathematics, the notation above is to be understood as the affirmation of the propositions with the leading plus signs omitted. But when these are made explicit the notation is:

$$
\begin{aligned}
+[-(+S)+(+P)] &= \text{It is the case that SAP,}\\
+[-(+S)-(+P)] &= \text{It is the case that SEP,}\\
+[+(+S)+(+P)] &= \text{It is the case that SIP, and}\\
+[+(+S)-(+P)] &= \text{It is the case that SOP.}
\end{aligned}
$$

And again, each form might be denied as well, in which case the notation would be:

$$
\begin{aligned}
-[-(+S)+(+P)] &= \text{It is not the case that SAP,}\\
-[-(+S)-(+P)] &= \text{It is not the case that SEP,}\\
-[+(+S)+(+P)] &= \text{It is not the case that SIP, and}\\
-[+(+S)-(+P)] &= \text{It is not the case that SOP.}
\end{aligned}
$$

Here it is instructive to note that denial of a proposition is algebraically equal to its contradiction—as is shown when its leading negation sign is driven inside.

$$
\begin{aligned}
-[-(+S)+(+P)] &= \text{It is not the case that SAP} = +(+S)-(+P) = +S-P = \text{SOP,}\\
-[-(+S)-(+P)] &= \text{It is not the case that SEP} = +(+S)+(+P) = +S+P = \text{SIP,}\\
-[+(+S)+(+P)] &= \text{It is not the case that SIP} = -(+S)-(+P) = -S-P = \text{SEP,}
\end{aligned}
$$

and

$$
-[+(+S)-(+P)] = \text{It is not the case that SOP} = -(+S)+(+P) = -S+P = \text{SAP.}
$$

Furthermore, any two categorical forms are equivalent if and only if they (i) are algebraically equal and (ii) have the same quantity (or *valence*). Accordingly, −S+P and −P+S (SAP and PAS) are not equivalent because they are not algebraically equal, whereas −S+P and +P−S (SAP and POS) are not equivalent because they are of different quantities.

The standard equivalences of the A and O forms are:

	SAP	**SOP**	
1.	−S+P	+S−P	Original
2.	−S−(−P)	+S+(−P)	Obverse of 1
3.	−(−P)−S	+(−P)+S	Converse of 2
4.	−(−P)+(−S)	+(−P)−(−S)	Obverse of 3 (Contrapositive of 1)

while those of the E and I forms are:

	SEP	**SIP**	
1.	−S−P	+S+P	Original
2.	−P−S	+P+S	Converse of 1
3.	−S+(−P)	+S−(−P)	Obverse of 1
4.	−P+(−S)	+P−(−S)	Obverse of 2 (Contrapositive of 3)

Again, each original form both has the same quantity and is algebraically equal to each of its equivalents.

3.2 Syllogisms in TL In [9], Sommers gives four conditions for syllogistic validity. First, a set of premises yields a conclusion if two of these conditions are met, viz., (i) that no more than one premise be particular and (ii) that the middle terms have opposite distribution values. Accordingly,

$$\begin{array}{lcl} -\mathrm{M}+\mathrm{P} & & -\mathrm{M}+\mathrm{P} \\ \underline{-\mathrm{S}+\mathrm{M}} & \text{and} & \underline{+\mathrm{S}+\mathrm{M}} \end{array}$$

both yield conclusions, whereas

$$\begin{array}{llcl} +\mathrm{M}+\mathrm{P} & +\mathrm{M}+\mathrm{P} & & +\mathrm{P}-\mathrm{M} \\ \underline{+\mathrm{S}-\mathrm{M},} & \underline{-\mathrm{S}+\mathrm{M}} & \text{and} & \underline{+\mathrm{S}-\mathrm{M}} \end{array}$$

do not. The first of these three fails to meet the first condition since both premises are particular; the second fails to meet the second condition since both middle terms have the same distribution value; and the third fails to meet either condition.

Further, a valid argument must not only have conclusion-yielding premises, but the conclusion drawn must also be the one that is entailed. And this is the case if the final two conditions are met, viz., (iii) that the conclusion be particular if and only if a premise is particular, and (iv) that a term be distributed in the conclusion if and only if it is distributed in the premises. Accordingly, what follows from the first set of conclusion-yielding premises above is −S+P (which is *Barbara*) and what follows from the second set is +S+P (which is *Darii*):

$$\begin{array}{lcl} -\mathrm{M}+\mathrm{P} & & -\mathrm{M}+\mathrm{P} \\ \underline{-\mathrm{S}+\mathrm{M}} & \text{and} & \underline{+\mathrm{S}+\mathrm{M}} \\ -\mathrm{S}+\mathrm{P} & & +\mathrm{S}+\mathrm{P} \end{array}$$

It may be said in these cases that the middle terms of opposite distribution values cancel each other out or that the conclusion is the algebraic sum of the premises. And the same can be said for each of the other valid syllogisms.

***3.2.1 The* dictum de omni** But middle terms of opposite distribution values do not automatically cancel each other out. Rather, −M and +M only cancel when one of the premises is universal since, for example, nothing follows from SOM and MIP:

$$\begin{array}{l} +\mathrm{M}+\mathrm{P} \\ \underline{+\mathrm{S}-\mathrm{M}} \end{array}$$

So the explanation as to why the middle terms cancel when they do must be found in something beyond the mere algebraic cancellation of opposites: the answer, according to term logicians, is found in the *dictum de omni* (DDO) which is the principle that:

> *Whatever is said of all M is said of all of whatever is an M;*

or worded slightly differently,

> *Whatever is said of all M is said of all of whatever M is said of.*

Accordingly, in *Barbara* and *Darii* the major claims that P is true of all M and the respective minors claim that all or some Ss are things that are M. So, it follows by

DDO that all or some Ss are P and this is what justifies the cancellation of the middle terms. And given this explanation, it may be more appropriate to think of inference as a case of instantiation rather than as a case of the cancellation of terms. That is, on the basis of DDO the major term instantiates the middle term of the minor, as "M" (of the minor) is replaced by "what is said of all M," or P.

It is clear that DDO requires that the distributed middle term be able to appear as the subject of a universal affirmative premise, since "P is true of all M" is "−M+P." But this merely reaffirms that the valid syllogisms are reducible to *Barbara* and *Darii*, where this occurrence is made explicit.

3.3 Some extensions of basic TL Such are some of the basic features of the system. However, as was mentioned earlier, TL is comprehensive and some additional features of the system need to be introduced before the attempt is made to combine NL with TL.

3.3.1 Sorites and existential assumptions in TL One such additional feature is the system's ability to handle sorites. On the one hand it can do this in the same fashion that it handles simple syllogisms, as in the case on the left below where the distributed and undistributed occurrences of B, C, and D simply cancel themselves out and leave +A+E as the conclusion. Or, the conclusion can be derived by successive applications of DDO, as is shown on the right.

$$\begin{array}{l}
+A+B \\
-B+C \\
-C+D \\
\underline{-D+E} \\
+A+E
\end{array}
\qquad
\begin{array}{lll}
1. & +A+B & \\
2. & -B+C & \\
3. & -C+D & \\
\underline{4.} & \underline{-D+E} & \\
5. & +A+C & \text{From 1 \& 2} \\
6. & +A+D & \text{From 3 \& 5} \\
7. & +A+E & \text{From 4 \& 6}
\end{array}$$

Furthermore, it is by way of sorites that term logicians handle arguments whose validity is based on existential assumptions. That is, one of the conditions of validity above prevents drawing a particular conclusion directly from universal premises, and this rules out such inferences as AAI-1 and AAI-3, even though these are valid unless the discourse involves the possibility of empty sets. So, in order to show the validity of these inferences, the suppressed assumption that a class C has membership is made explicit by the addition of the premise, +C+C, or "There exists a C that is a C". Then the universal premises together with this existential premise constitute a sorites. (See Englebretsen [1], pp. 118–20.) Here again, the middle terms can be seen to cancel out so that the conclusion follows directly, as in

$$\begin{array}{l}
-M+P \\
-S+M \\
\underline{+S+S} \\
+S+P
\end{array}
\qquad \text{and} \qquad
\begin{array}{l}
-M+P \\
-M+S \\
\underline{+M+M} \\
+S+P;
\end{array}$$

or these can be derived by successive applications of DDO as in:

1. $-M+P$	1. $-M+P$
2. $-S+M$	2. $-M+S$
3. $+S+S$	3. $+M+M$
4. $+S+M$ (From 2 & 3)	4. $+M+S$ (From 2 & 3)
5. $+S+P$ (From 1 & 4)	5. $+S+P$ (From 1 & 4)

3.3.2 Relationals in TL Finally, one of the great strengths of term logic is its ability to handle propositions containing relational terms. The additional notation required is merely the introduction of the relational term and the appending of subscripts to keep track of the relata. The examples below illustrate how this works for two-place relationals where the overall statement is of the I form.

$+J_1+(R_{12}+S_2)$	Some junior resents some senior.
$+J_1+(R_{12}-S_2)$	Some junior resents every senior.

And the same patterns hold, *mutatis mutandis,* for the permutations of the A, E, and O forms.

The examples below are some three-place relationals where the overall statement is of the A form.

$-T_1+((G_{123}+B_2)+S_3)$	Every teacher gave a book to some student.
$-T_1+((G_{123}+B_2)-S_3)$	Every teacher gave a book to every student.
$-T_1+((G_{123}-B_2)+S_3)$	Every teacher gave every book to some student.
$-T_1+((G_{123}-B_2)-S_3)$	Every teacher gave every book to every student.

And the same patterns hold, *mutatis mutandis*, for the permutations of the E, I, and O forms.

Furthermore, inferences from relational premises proceed by DDO in the same way they do from the less complex premises, as is illustrated in the proof below.

1. $-T_1+((G_{123}+B_2)-S_3)$	Every teacher gave a book to every student.
2. $-((G_{123}+E_2)+I_3)+F_1$	Every giver of something expensive to an ingrate is a fool.
3. $+L+T$	Some leaders are teachers.
4. $-B+E$	All books are expensive.
5. $+S+I$	Some students are ingrates.
6. $+L_1+((G_{123}+B_2)-S_3)$	Some leader gave a book to every student. (From 1 & 3)
7. $+L_1+((G_{123}+E_2)-S_3)$	Some leader gave something expensive to every student. (From 4 & 6)
8. $+L_1+((G_{123}+E_2)+I_3)$	Some leader gave something expensive to some ingrate. (From 5 & 7)
9. $+L+F$	Some leader is a fool. (From 2 & 8)

Each step in the inference is an instance of DDO. That is, line 6 results from 1 and 3 as the two occurrences of T cancel; line 7 results from 4 and 6 as the two occurrences of B cancel; line 8 results from 5 and 7 as the two occurrences of S cancel; and line 9 results from 2 and 8 as the two occurrences of the complex term, $((G_{123}+E_2)+I_3)$, cancel. In each case, "what is said of the distributed term" in the one line instantiates the undistributed occurrence of that term in the other line.

4 Numerical term logic (NTL) With this introduction, the way is prepared for the attempt to show that NL and TL fit aptly together to form a “numerical term logic (NTL).”

4.1 Basic propositions of NTL The symbolization that suggests itself for the combined program adds the numeral (or variable) of NL to the plus/minus notation of TL, as is shown for the basic forms below.

Proposition	**NL**	+	**TL**	$\Longrightarrow$	**NTL**
All Ss are P	(0SAP)	+	$(-\mathrm{S}+\mathrm{P})$	$\Longrightarrow$	$(-0\mathrm{S}+\mathrm{P})$
No Ss are P	(0SEP)	+	$(-\mathrm{S}-\mathrm{P})$	$\Longrightarrow$	$(-0\mathrm{S}-\mathrm{P})$
Some Ss are P	(1SIP)	+	$(+\mathrm{S}+\mathrm{P})$	$\Longrightarrow$	$(+1\mathrm{S}+\mathrm{P})$
Some Ss are not P	(1SOP)	+	$(+\mathrm{S}-\mathrm{P})$	$\Longrightarrow$	$(+1\mathrm{S}-\mathrm{P})$

and for any numerical value x, these can be indicated as:

At least all but x Ss are P	$-x\mathrm{S}+\mathrm{P}$
At most x Ss are P	$-x\mathrm{S}-\mathrm{P}$
At least x Ss are P	$+x\mathrm{S}+\mathrm{P}$
At least x Ss are not P	$+x\mathrm{S}-\mathrm{P}$

Furthermore, except for the addition of the numeral (variable), the notation for equivalences remains the same as before. That is, equivalences of the A and O forms are:

	***x*SAP**	***x*SOP**	
1.	$-x\mathrm{S}+\mathrm{P}$	$+x\mathrm{S}-\mathrm{P}$	Original
2.	$-x\mathrm{S}-(-\mathrm{P})$	$+x\mathrm{S}+(-\mathrm{P})$	Obverse of 1
3.	$-x(-\mathrm{P})-\mathrm{S}$	$+x(-\mathrm{P})+\mathrm{S}$	Converse of 2
4.	$-x(-\mathrm{P})+(-\mathrm{S})$	$+x(-\mathrm{P})-(-\mathrm{S})$	Obverse of 3 (Contrapositive of 1)

whereas the equivalences of the E and I forms are:

	***x*SEP**	***x*SIP**	
1.	$-x\mathrm{S}-\mathrm{P}$	$+x\mathrm{S}+\mathrm{P}$	Original
2.	$-x\mathrm{P}-\mathrm{S}$	$+x\mathrm{P}+\mathrm{S}$	Converse of 1
3.	$-x\mathrm{S}+(-\mathrm{P})$	$+x\mathrm{S}-(-\mathrm{P})$	Obverse of 1
4.	$-x\mathrm{P}+(-\mathrm{S})$	$+x\mathrm{P}-(-\mathrm{S})$	Obverse of 2 (Contrapositive of 3)

However, it should be noted that the denial of a proposition requires a change in its numerical value as, for example, the denial of 0SAP is 1SOP. And accordingly, the denial of a universal whose numerical value is x results in a particular whose numerical value is $x+1$, and the denial of a particular whose numerical value is $x+1$ results in a universal whose numerical value is x. This is shown below.

Basic Forms	**Other Sample Forms**	**General Form-Types**
$-[-0\mathrm{S}+\mathrm{P}] = +1\mathrm{S}-\mathrm{P}$	$-[-1\mathrm{S}+\mathrm{P}] = +2\mathrm{S}-\mathrm{P}$	$-[-x\mathrm{S}+\mathrm{P}] = +(x+1)\mathrm{S}-\mathrm{P}$
$-[-0\mathrm{S}-\mathrm{P}] = +1\mathrm{S}+\mathrm{P}$	$-[-5\mathrm{S}-\mathrm{P}] = +6\mathrm{S}+\mathrm{P}$	$-[-x\mathrm{S}-\mathrm{P}] = +(x+1)\mathrm{S}+\mathrm{P}$
$-[+1\mathrm{S}+\mathrm{P}] = -0\mathrm{S}-\mathrm{P}$	$-[+4\mathrm{S}+\mathrm{P}] = -3\mathrm{S}-\mathrm{P}$	$-[+(x+1)\mathrm{S}+\mathrm{P}] = -x\mathrm{S}-\mathrm{P}$

$$-[+1\mathrm{S}-\mathrm{P}] = -0\mathrm{S}+\mathrm{P} \qquad -[+3\mathrm{S}-\mathrm{P}] = -2\mathrm{S}+\mathrm{P} \qquad -[+(x+1)\mathrm{S}-\mathrm{P}] = -x\mathrm{S}+\mathrm{P}$$

4.2 Syllogisms in NTL Some sample syllogisms in NTL are presented below alongside their corresponding versions in NL. The traditional instantiations of *Barbara* and *Darii* now appear as:

$$\begin{array}{lcl} 0\mathrm{MAP} & = & -0\mathrm{M}+\mathrm{P} \\ \underline{0\mathrm{SAM}} & = & \underline{-0\mathrm{S}+\mathrm{M}} \\ 0\mathrm{SAP} & = & -0\mathrm{S}+\mathrm{P} \end{array} \qquad \begin{array}{lcl} 0\mathrm{MAP} & = & -0\mathrm{M}+\mathrm{P} \\ \underline{1\mathrm{SIM}} & = & \underline{+1\mathrm{S}+\mathrm{M}} \\ 1\mathrm{SIP} & = & +1\mathrm{S}+\mathrm{P} \end{array}$$

Furthermore, higher numerical values can also be introduced into the NTL notational format. First, the minor premises can have higher numerical values, as in the two arguments below.

$$\begin{array}{lcl} 0\mathrm{MAP} & = & -0\mathrm{M}+\mathrm{P} \\ \underline{15\mathrm{SAM}} & = & \underline{-15\mathrm{S}+\mathrm{M}} \\ 15\mathrm{SAP} & = & -15\mathrm{S}+\mathrm{P} \end{array} \qquad \begin{array}{lcl} 0\mathrm{MAP} & = & -0\mathrm{M}+\mathrm{P} \\ \underline{10\mathrm{SIM}} & = & \underline{+10\mathrm{S}+\mathrm{M}} \\ 10\mathrm{SIP} & = & +10\mathrm{S}+\mathrm{P} \end{array}$$

but in addition, both premises can have higher numerical values as well.

$$\begin{array}{lcl} 11\mathrm{MAP} & = & -11\mathrm{M}+\mathrm{P} \\ \underline{15\mathrm{SAM}} & = & \underline{-15\mathrm{S}+\mathrm{M}} \\ 26\mathrm{SAP} & = & -26\mathrm{S}+\mathrm{P} \end{array} \qquad \begin{array}{lcl} 11\mathrm{MAP} & = & -11\mathrm{M}+\mathrm{P} \\ \underline{30\mathrm{SIM}} & = & \underline{+30\mathrm{S}+\mathrm{M}} \\ 19\mathrm{SIP} & = & +19\mathrm{S}+\mathrm{P} \end{array}$$

Now the conclusions of the NL examples above are reached by applying the rule given earlier, viz., that value of each universal conclusion is reached by adding the values of the universal premises, whereas the value of each particular conclusion is reached by subtracting the value of the universal from that of the particular premise. However, it is instructive to note that *the value of each conclusion in the NTL version above is equal to the sum of the numerical values of the premises, regardless of the quantity.* And this holds generally. That is, once conclusion-yielding premises are cast in the plus/minus format, the value of the conclusion is already given by the signed values of the premises. Accordingly, the general rule that allows the strongest numerical conclusion warranted by the premises in NTL is simply:

> The numerical value of the conclusion must equal the sum of the numerical values of the premises.

$$\begin{array}{l} -x\mathrm{M}+\mathrm{P} \\ \underline{-y\mathrm{S}+\mathrm{M}\quad} \\ -(x+y)\mathrm{S}+\mathrm{P} \end{array} \qquad \begin{array}{l} -x\mathrm{M}+\mathrm{P} \\ \underline{+(x+y)\mathrm{S}+\mathrm{M}} \\ +y\mathrm{S}+\mathrm{P} \end{array}$$

(Of course, the result is the same in both methods, but the latter—in keeping with TL's general technique of deriving the conclusion by summing the premises—is more systematic and more elegant.)

***4.2.1 The expanded* dictum de omni** It is clear that the first two sample sets of syllogisms above work by DDO. That is, since P is said of all M in each case ($-0\mathrm{M}+\mathrm{P}$), then it is also said of whatever quantity of Ss the minor asserts to be M. However, in the third set, P is not said of all M (-11M$+$P) and so DDO—at least as stated above—does not apply. That is, here P is said of all but eleven Ms rather than of all M and therefore, the Ms do not "cancel each other all the way." Specifically, eleven Ms do not cancel, and so these eleven must be deducted from the number of Ss that P would have been predicated of in the conclusion otherwise, as the examples of this third set illustrate. And such is the case generally, as the above "summation of numerical values of the premises" rule reflects. Accordingly, it seems that either another principle is required to justify these inferences or else that DDO must be expanded to cover them.

I propose an expansion of DDO that follows the same lines as the expansion of the A proposition given earlier. That is, there "All Ss are P" was interpreted as "All but zero Ss are P," and generalized to "All but *x* Ss are P"; and here I propose that

> *Whatever is said of all M is said of all of whatever is an M,*
> *(Whatever is said of all M is said of all of whatever M is said of,)*

can be interpreted as

> *Whatever is said of all but zero Ms is said of all but zero of whatever is an M,*
> *(Whatever is said of all but zero Ms is said of all but zero of whatever M is said of,)*

and generalized to

> *Whatever is said of all but x Ms is said of all but x of whatever is an M.*
> *(Whatever is said of all but x Ms is said of all but x of whatever M is said of.)*

Then the numerical inferences considered above are all justified by appeal to this expanded dictum. So when P is said of all but 11 Ms, as in the two problem cases considered above, then P is said of all but 11 of whatever Ss that M is said of. Accordingly, the conclusion for *Barbara* considered there is "All but 11 of all but 15 Ss are P," or "All but 26 Ss are P"; and the conclusion for *Darii* is "All but 11 of at least 30 Ss are P," or "At least 19 Ss are P." And according to the expanded DDO in general, when P is said of all but *x* Ms, it is said of all but *x* of whatever quantity of Ss are said to be M.

4.3 Some extensions of NLT

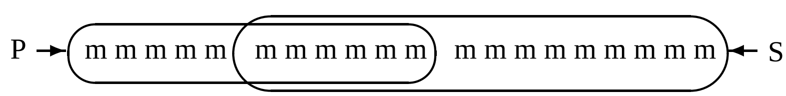
4.3.1 Sorites and existential assumptions in NTL Numerically expanded sorites can now be shown to be valid by the expanded DDO as well. For example, line 7 below follows from premises 1 – 4 by successive applications of it.

1.	$+8$A$+$B	Premise
2.	-1B$+$C	Premise
3.	-2C$+$D	Premise
4.	$\underline{-3\text{D}+\text{E}}$	Premise
5.	$+7$A$+$C	From 1 & 2
6.	-5C$+$E	From 3 & 4
7.	$+2$A$+$E	From 5 & 6

In the traditional logic—where the implicit numerical values of the propositions are limited to that of zero and one—the question of existential assumption is only concerned with whether a class might be assumed to have "at least one member," for the assumption of one member is all that is required to allow particular conclusions from universal premises. So, even though more members might be assumed, or known, to exist, the additional membership does not strengthen the logic at all. However, with the numerically expanded quantifiers, the assumption of alternative memberships becomes quite relevant to the logic. (See [5], chapter 7 and Murphree [8].) This can be seen in the two arguments below where different conclusions (line 5) are entailed by the same set of premises (lines 1 and 2) on the basis of different existential assumptions (line 3).

1.	$-2M+P$	$=$	$-2M+P$	Premise
2.	$-3S+M$	$=$	$-3S+M$	Premise
3.	$\underline{+11S+S}$	$\neq$	$\underline{+93S+S}$	Existential Assumption
4.	$+8S+M$	$\neq$	$+90S+M$	From 2 & 3
5.	$+6S+P$	$\neq$	$+88S+P$	From 1 & 4

4.3.1 (1) Maximum existential assumptions The assumptions above are "minimum assumptions" in that *at least* a certain membership is presupposed. However, as is shown in [8], with the introduction of numerically flexible quantifiers, "maximum" existential assumptions also become interesting. That is, it is possible to assume that a class has *at most* a certain membership and then to make inferences on the basis of that assumption. For example, if it is assumed that a class has at most 20 members (represented by the 20 `m`s below) then—when it is premised that at least 11 of them are philosophy majors ($+11M+P$) and that at least 15 of them are sophomores ($+15M+S$)—it follows that at least 6 sophomores are philosophy majors: $+6S+P$.



That is, under these conditions the overlap of the S-class and the P-class *must* include at least six members as shown above. (And, if there are fewer than 20 `m`s in fact, then the overlap is greater, so that more than 6 Ss are P.)

Following TL's clue for handling minimum assumptions as special I-form propositions, I suggest that maximum assumptions can be appropriately handled as special E-form propositions. Specifically, I suggest the assumption that there exist at most x `m`s can be appropriately rendered as "At most x Ms are M": $-xM-M$. Then a proof for the argument above might proceed as follows:

1.	$+11M+P$	At least 11 members are philosophy majors. (Premise)
2.	$+15M+S$	At least 15 members are sophomores. (Premise)
3.	$\underline{-20M-M}$	There are at most 20 members. (Maximum Existential Assumption)
4.	$-9M+P$	All but 9 members are philosophy majors. (From 1 & 3)
5.	$+6S+P$	At least 6 sophomores are philosophy majors. (From 2 & 4)

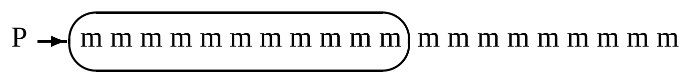
Now this is a good proof in the sense that lines 4 and 5 are indeed justified by the lines above them. That is, line 5 follows from lines 2 and 4 by DDO, and line 4 follows from lines 1 and 3; but the problem at this point is that *line 4 does not follow by DDO!* What follows by DDO instead is 4′.

4′. $+(-9)+\mathrm{P}-\mathrm{M}$ At least *negative* nine philosophy majors are not members.

(See [5], pp. 27–29, for a treatment of negative quantificational values.) Perhaps this can be more easily seen by casting this step in the form of *Darii*.

3.	$-20\mathrm{M}-\mathrm{M}$	$\Longleftarrow$	(obversion)	$\Longrightarrow$	$-20\mathrm{M}+(-\mathrm{M})$
2.	$\underline{+11\mathrm{M}+\mathrm{P}}$	$\Longleftarrow$	(conversion)	$\Longrightarrow$	$\underline{+11\mathrm{P}+\mathrm{M}}$
4′.	$+(-9)\mathrm{P}-\mathrm{M}$	$\Longleftarrow$	(obversion)	$\Longrightarrow$	$+(-9)\mathrm{P}+(-\mathrm{M})$

Yet line 4 above clearly does follow, since lines 2 and 3 together assert "At least 11 of at most 20 Ms are P"; and "at least 11 of at most 20" resolves into "at least all but 9":

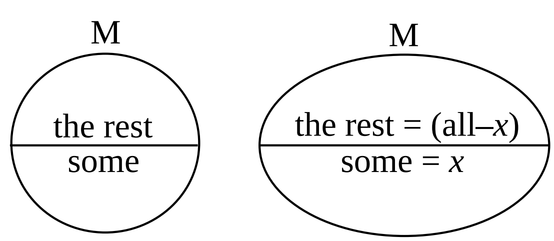


4. At least all but 9 Ms are P.

And since it is DDO that justifies the inference of line 4′ (but not line 4) above, it seems there must be some other principle that justifies the inference of line 4.

***4.3.1 (2) A new saying: The* dictum de aliquo** I suggest what that other principle is below and proudly dub it the *dictum de aliquo* (DDA), or the *saying concerning some*.

But first, in preparation, I suggest that whenever a claim is made about some Ms, the class of Ms can be thought of as divided into "some" and "the rest," in the sense that the "some" + "the rest" = "all." Hence, for any x, if the "some" = x, then "the rest" = all$-x$, for [x+(all$-x$) = all]. (See [5], pp. 7–8, for a further treatment of divided terms.)



Now, given this division in M, the following is an obvious truth:

> *Whatever is said of some Ms is said of all but the rest of Ms,*
> *(Whatever is said of some Ms is said of all of whatever all but the rest of the Ms are said of.)*

In fact, it is so obvious that it may seem silly ("What is said about the bottom half of the circle is said about all but the top half of the circle") or simply verbose ("Whatever is said about x Ms is said about *all but all but x* Ms").

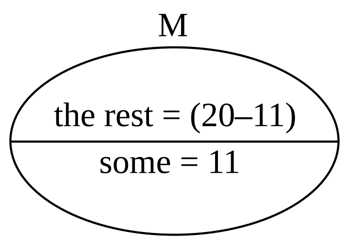
However, I maintain that it is significant when a maximum membership is assumed for a term; and again, I propose it to be the principle that justifies line 4 above: $-9M+P$; that is, P is said of at least 11 Ms,

1. $+11M+P$,

and since there are at most 20 Ms,

3. $-20M-M$,

"the rest" of the Ms are at most (20–11),



or, at most 9 Ms. So, since P is said of "all but the rest of the Ms" (by DDA), it is said of "all but at most 9 Ms" or equivalently, of "at least all but 9 Ms":

4. $-9M+P$.

But this is not to say that the inference by DDO that yields 4′ is invalid; rather, it is simply not applicable to the proof in this case. And conversely, although inferences based on DDA are always valid, they may not be applicable to a proof. In fact, it seems they might only be applicable when a maximum membership is assumed, for only then can the value of "the rest" be given a definite numerical interpretation.

4.3.2 Relationals in NTL Finally, the subjects and objects of relational propositions in TL also allow expanded numerical quantification. The following are samples of two-place relationships in NTL.

$+3J_1+(R_{12}+4S_2)$	At least three juniors resent four seniors.
$+3J_1+(R_{12}-4S_2)$	At least juniors resent all but four seniors.
$-3J_1+(R_{12}+4S_2)$	All but three juniors resent four seniors.
$-3J_1+(R_{12}-4S_2)$	All but three juniors resent all but four seniors.

In addition, such quantification in three-place relationships works equally well within the framework NTL, although there may be more ambiguity in the natural language at this level. For example, the utterance, "Three teachers gave two students four books," might be taken a number of different ways. On the one extreme, it might be taken as asserting that 24 books had been given, which would be the case if each of the three teachers gave four separate books to each of two students: $3 \times 4 \times 2 = 24$. This is symbolized as follows:

$$+3T_1 + ((G_{123} + 4B_2) + 2S_3)$$

And on the other extreme, it might be taken as asserting that only four books were given in all, which would be the case if the three teachers "went in together" and bought a set of four books which they gave as common property to two students: $1 \times 4 \times 1 = 4$. In this case, the set of teachers and the pair of students would be considered collective terms, so that the one group of three teachers gave four books to one pair of students. With the collective terms indicated by curly braces, this can be symbolized as follows.

$$1\{3T\}_1 + ((G_{123} + 4B_2) + 1\{2S\}_3)$$

And there are intermediate possibilities. If the group of teachers went in together and purchased four books for each of the two students, then eight books would have been given $1 \times 4 \times 2 = 8$,

$$+1\{3T\}_1 + ((G_{123} + 4B_2) + 2S_3);$$

or if each of the three teachers bought four books and gave them to the one couple, then twelve books would have been given $3 \times 4 \times 1 = 12$,

$$+3T_1 + ((G_{123} + 4B_2) + 1\{2S\}_3).$$

So again it seems that the problem likely to be encountered in the attempt to handle such statements would be that of identifying the specific proposition a speaker intends by such a locution as "Three teachers gave two students four books." But any of the propositions should be readily accommodated in the symbolism of NTL.

The sample proof below, which contains two existential assumptions as well as a three-place relationship, illustrates the logic at this level. Each step is justified by DDO.

1.	$-3T_1+((G_{123}+4B_2)-5S_3)$	All but 3 teachers gave 4 books to all but 5 students.
2.	$-2T+U$	All but 2 teachers are underpaid persons.
3.	$-0B+E$	Every book is expensive.
4.	$-7S+I$	All but 7 students are ingrates.
5.	$+50S+S$	There are at least 50 students.
6.	$+10T+T$	There are at least 10 teachers.
7.	$+7T_1+((G_{123}+4B_2)-5S_3)$	At least 7 teachers gave 4 books to all but 5 students. (From 1 & 6)
8.	$+7T_1+((G_{123}+4B_2)+45S_3)$	At least 7 teachers gave 4 books to 45 students. (From 5 & 7)
9.	$+5U_1+((G_{123}+4B_2)+45S_3)$	At least 5 underpaid persons gave 4 books to 45 students. (From 2 & 8)
10.	$+5U_1+((G_{123}+4E_2)+45S_3)$	At least 5 underpaid persons gave 4 expensive things to 45 students. (From 3 & 9)
11.	$+5U_1+((G_{123}+4E_2)+38I_3)$	At least 5 underpaid persons gave 4 expensive things to 38 ingrates. (From 4 & 10)

5 Conclusion Only minimum sketches of NL and TL have been presented above, although TL is very comprehensive and has far-reaching philosophical implications. (See Englebretsen [3] for a more thorough consideration of these.) But, although the introductions above are limited, I propose they are sufficient to show that the two systems are not only compatible but that they are complementary; furthermore, I propose that their combination—the resulting NTL—is much more powerful than either system alone or than both systems in isolation from each other.

It is clear, I suppose, that TL makes the greater contribution to NTL and conversely, that NL gains more in the union. Specifically, TL contributes the powerful notational and deductive mechanism which rescues NL from the narrow confines of syllogistic logic.

But on the other hand, the union with NL extends the applicability of TL immensely. Of course, what NL specifically adds is finer quantificational discrimination. And first, this finer discrimination often allows the conclusions of NTL to be more informative than those of TL. For example, whereas TL does not discriminate between the conclusions that "At least 1 S is P" and "At least 1 million Ss are P," NTL does so systematically.

But furthermore, this finer discrimination also allows the proof of many conclusions in NTL that are not possible in TL. For example, each set of premises below (along with the existential assumptions for the last two) entails "At least one S is P":

	All but 25 Ms are P	At least 200 Ms are P
All but 2 Ms are P	All but 30Ss are M	At least 300 Ms are S
At least 3 Ss are M	There exist at least 56 Ss	There exist at most 499 Ms
At least 1 S is P	At least 1 S is P	At least 1 S is P

But this entailed conclusion [+S+P] is not forthcoming for any of the three sets of premises in TL. And likewise, from the argument in the section above, "Some underpaid persons gave some expensive things to some ingrates" $[+U_1+((G_{123}+E_2)+I_3]$ cannot be derived by TL from the premises (and existential assumptions) given, although it is clearly entailed by them.

So, not only does NTL yield conclusions that are numerically finer than those of TL, but it also yields conclusions in cases where TL yields none at all.

Finally, it seems that neither TL nor NL sacrifices anything by the union into NTL. Certainly NL makes no sacrifices, and it seems that the union does not affect those wider aspects of TL not considered here, such as its treatment of propositional logic, at all. However, the union does require that DDO be expanded (to cover numerical deviations from full universal claims) and that DDO be supplemented by DDA (to cover implications based on maximum existential assumptions). But perhaps these are inherent refinements of TL; or, if they are nuisances instead, it still seems that TL would not give up anything of logical importance by allowing them.

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REFERENCES

[1] Englebretsen, G., "Notes on the new syllogistic," *Logique et Analyse*, vol. 85–86 (1979), pp. 111–20.

[2] Englebretsen, G., "The myth of modern logic," *Cogito*, Autumn (1990), pp. 150–58. 1

[3] Englebretsen, G., *Something to Reckon With: The Logic of Terms*, University of Ottawa Press, Ottawa, 1996.

[4] Kelley, D., *The Art of Reasoning*, expanded ed., W.W. Norton, New York, 1990. 1

[5] Murphree, W., *Numerically Exceptive Logic: A Reduction of the Classical Syllogism*, Peter Lang, New York, 1991. 1

[6] Murphree, W., "Expanding the traditional syllogism," *Logique et analyse*, vol. 141–42 (1993), pp. 105–20. MR 97b:03015 1

[7] Murphree, W., "The irrelevance of distribution for the syllogism," *Notre Dame Journal of Formal Logic*, vol. 35 (1994), pp. 433–49. Zbl 0920.03008 MR 96e:03004

[8] Murphree, W., "The numerical syllogism and existential presupposition," *Notre Dame Journal of Formal Logic*, vol. 38 (1997), pp. 49–64. Zbl 0900.03018 MR 98m:03012

[9] Sommers, F., "Distribution matters," *Mind*, vol. 84 (1975), pp. 27–46.

[10] Sommers, F., "Predication and term logic," *Notre Dame Journal of Formal Logic*, vol. 31 (1990), pp. 106–26. 1

[11] Sommers, F., "The world, the facts, and primary logic," *Notre Dame Journal of Formal Logic*, vol. 34 (1993), pp. 169–82. Zbl 0789.03006 MR 94g:03015 1

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