Strategic Residential Segregation

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Abstract

Residential segregation is a widespread phenomenon that can be observed in almost every major city. In these urban areas, residents with different ethnical or socioeconomic backgrounds tend to form homogeneous clusters. In Schelling's classical segregation model two types of agents are placed on a grid. An agent is content with its location if the fraction of its neighbors, which have the same type as the agent, is at least τ, for some $0 < \tau \leq 1$. Discontent agents simply swap their location with a randomly chosen other discontent agent or jump to a random empty location. The model gives a coherent explanation of how clusters can form even if all agents are tolerant, i.e., if they agree to live in mixed neighborhoods. For segregation to occur, all it needs is a slight bias towards agents preferring similar neighbors.

Although the model is well studied, previous research focused on a random process point of view. However, it is more realistic to assume instead that the agents strategically choose where to live. We close this gap by introducing and analyzing game-theoretic models of Schelling segregation, where rational agents strategically choose their locations.

As the first step, we introduce and analyze a generalized game-theoretic model that allows more than two agent types and more general underlying graphs modeling the residential area. We introduce different versions of Swap and Jump Schelling Games. Swap Schelling Games assume that every vertex of the underlying graph serving as a residential area is occupied by an agent and pairs of discontent agents can swap their locations, i.e., their occupied vertices, to increase their utility. In contrast, for the Jump Schelling Game, we assume that there exist empty vertices in the graph and agents can jump to these vacant vertices if this increases their utility. We show that the number of agent types as well as the structure of underlying graph heavily influence the dynamic properties and the tractability of finding an optimal strategy profile.

As a second step, we significantly deepen these investigations for the swap version with $\tau = 1$ by studying the influence of the underlying topology modeling the residential area on the existence of equilibria, the Price of Anarchy, and the

dynamic properties. Moreover, we restrict the movement of agents locally. As a main takeaway, we find that both aspects influence the existence and the quality of stable states.

Furthermore, also for the swap model, we follow sociological surveys and study, asking the same core game-theoretic questions, non-monotone single-peaked utility functions instead of monotone ones, i.e., utility functions that are not monotone in the fraction of same-type neighbors. Our results clearly show that moving from monotone to non-monotone utilities yields novel structural properties and different results in terms of existence and quality of stable states.

In the last part, we introduce an agent-based saturated open-city variant, the Flip Schelling Process, in which agents, based on the predominant type in their neighborhood, decide whether to change their types. We provide a general framework for analyzing the influence of the underlying topology on residential segregation and investigate the probability that an edge is monochrome, i.e., that both incident vertices have the same type, on random geometric and Erdős–Rényi graphs. For random geometric graphs, we prove the existence of a constant $c > 0$ such that the expected fraction of monochrome edges after the Flip Schelling Process is at least $1/2+c$. For Erdős–Rényi graphs, we show the expected fraction of monochrome edges after the Flip Schelling Process is at most $1/2 + o(1)$.

Zusammenfassung

Die Segregation von Wohngebieten ist ein weit verbreitetes Phänomen, das in fast jeder größeren Stadt zu beobachten ist. In diesen städtischen Gebieten neigen Bewohner mit unterschiedlichem ethnischen oder sozioökonomischen Hintergrund dazu, homogene Nachbarschaften zu bilden. In Schellings klassischem Segregationsmodell werden zwei Arten von Agenten auf einem Gitter platziert. Ein Agent ist mit seinem Standort zufrieden, wenn der Anteil seiner Nachbarn, die denselben Typ wie er haben, mindestens τ beträgt, für $0 < \tau \leq 1$. Unzufriedene Agenten tauschen einfach ihren Standort mit einem zufällig ausgewählten anderen unzufriedenen Agenten oder springen auf einen zufälligen leeren Platz. Das Modell liefert eine kohärente Erklärung dafür, wie sich Cluster bilden können, selbst wenn alle Agenten tolerant sind, d.h. wenn sie damit einverstanden sind, in gemischten Nachbarschaften zu leben. Damit es zu Segregation kommt, genügt eine leichte Tendenz, dass die Agenten ähnliche Nachbarn bevorzugen.

Obwohl das Modell gut untersucht ist, lag der Schwerpunkt der bisherigen Forschung eher auf dem Zufallsprozess. Es ist jedoch realistischer, davon auszugehen, dass Agenten strategisch ihren Wohnort aussuchen. Wir schließen diese Lücke, indem wir spieltheoretische Modelle der Schelling-Segregation einführen und analysieren, in welchen rationale Akteure ihre Standorte strategisch wählen.

In einem ersten Schritt führen wir ein verallgemeinertes spieltheoretisches Modell ein, das mehr als zwei Agententypen und allgemeinere zugrundeliegende Graphen zur Modellierung des Wohngebiets zulässt und analysieren es. Zu diesem Zweck untersuchen wir verschiedene Versionen von Tausch- und Sprung-Schelling-Spielen. Bei den Tausch-Schelling-Spielen gehen wir davon aus, dass jeder Knoten des zugrunde liegenden Graphen, der als Wohngebiet dient, von einem Agenten besetzt ist und dass Paare von unzufriedenen Agenten ihre Standorte, d.h. ihre besetzten Knoten, tauschen können, um ihren Nutzen zu erhöhen. Im Gegensatz dazu gehen wir beim Sprung-Schelling-Spiel davon aus, dass es leere Knoten im Graphen gibt und die Agenten zu diesen unbesetzten Knoten springen können, wenn dies ihren Nutzen erhöht. Wir zeigen, dass die Anzahl der Agententypen sowie die zugrundeliegende Struktur des Graphen,

die dynamischen Eigenschaften und die Komplexität der Berechenbarkeit eines optimalen Strategieprofils stark beeinflussen.

In einem zweiten Schritt vertiefen wir diese Untersuchungen für die Tauschvariante mit $\tau = 1$ erheblich, indem wir den Einfluss der zugrunde liegenden Topologie, die das Wohngebiet modelliert, auf die Existenz von Gleichgewichten, den Preis der Anarchie und die dynamischen Eigenschaften hin untersuchen. Darüber hinaus schränken wir die Bewegung der Agenten lokal ein. Die wichtigste Erkenntnis ist, dass beide Aspekte die Existenz als auch die Qualität stabiler Zustände beeinflussen.

Desweiteren folgen wir, auch für das Tauschmodell, soziologischen Untersuchungen und untersuchen für dieselben zentralen spieltheoretischen Fragen nicht-monotone einspitzige Nutzenfunktionen anstelle von monotonen, d.h. Nutzenfunktionen, die nicht monoton bezüglich des Anteils der gleichartigen Nachbarn sind. Unsere Ergebnisse zeigen deutlich, dass der Übergang von monotonen zu nicht-monotonen Nutzenfunktionen zu neuen strukturellen Eigenschaften und anderen Ergebnissen in Bezug auf die Existenz und Qualität stabiler Zustände führt.

Im letzten Teil führen wir eine agentenbasierte gesättigte Offene-Stadt-Variante ein, den Flip-Schelling-Prozess, bei dem Agenten auf der Grundlage des vorherrschenden Typs in ihrer Nachbarschaft entscheiden, ob sie ihren Typ wechseln. Wir stellen einen allgemeinen Rahmen für die Analyse des Einflusses der zugrundeliegenden Topologie auf die Wohnsegregation zur Verfügung und untersuchen die Wahrscheinlichkeit, dass eine Kante einfarbig auf zufälligen geometrischen und Erdős–Rényi-Graphen ist, d.h. dass beide inzidenten Knoten denselben Typ haben. Für zufällige geometrische Graphen beweisen wir die Existenz einer Konstante $c > 0$, so dass der erwartete Anteil einfarbiger Kanten nach dem Flip-Schelling-Prozess mindestens $1/2 + c$ beträgt. Für Erdős–Rényi-Graphen zeigen wir, dass der erwartete Anteil einfarbiger Kanten nach dem Flip-Schelling-Prozess höchstens $1/2 + o(1)$ ist.

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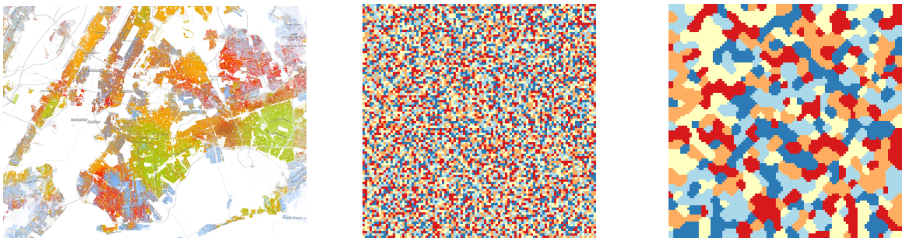
Contents

1 Introduction

Today's metropolitan areas are populated by a diverse set of residential groups, which differ along racial, socioeconomic, and other traits. A common finding is that social groups within cities are not well-mixed, i.e., the different groups of inhabitants tend to separate themselves into largely homogeneous neighborhoods. This remarkable phenomenon is well-known as *residential segregation* [MD88; Whi86]. The causes of residential segregation are complex and range from discriminatory laws to individual action. Neighborhood segregation and ghetto formation are serious social and political issues. In particular, segregation has many negative consequences for the inhabitants of a city, for example, it negatively impacts their health [AL03], their mortality [Jac+00], and, in general, their socioeconomic conditions [MD93].

The phenomenon of residential segregation has been widely studied by social scientists, mathematicians, and, recently, also by computer scientists. In these studies local and myopic location choices by many individuals with preferences over their direct residential neighborhood yield cityscapes that are severely segregated along racial and ethnic lines, see Figure 1.1 (a). Hence, local strategic choices on the micro level lead to an emergent phenomenon on the macro level. This paradigm of "micro motives" versus "macrobehavior" [Sch78] was first investigated and modeled by Thomas Schelling in the 1970s, who proposed a very simple stylized agent-based model for analyzing residential segregation [Sch69; Sch71]. His work specifies a spatial setting where individual agents with a bias toward favoring similar agents care only about the composition of their respective local neighborhoods. This model gives a coherent explanation for the widespread phenomenon of residential segregation since it shows that local choices by the agents yield globally segregated states [Cla86].

With the use of two types of coins as two types of individual agents placed on a line or a checkerboard that models some residential area, Schelling demonstrated the emergence of segregated neighborhoods under the simple assumption of the following threshold behavior: agents are content with their current location if the fraction of agents of their own type in their neighborhood is at least τ, where



(a) Residential segregation in New York City, color-coded by ethnicity.

(b) Initial random strategy profile on a grid in Schelling's model.

(c) Equilibrium found for the instance in (b) with $\tau = \frac{1}{2}$ via improving response dynamics.

Figure 1.1: Residential segregation in New York City and Schelling's segregation model. Residential segregation along racial lines in many urban areas in the US is the most famous example of this phenomenon. Shown in (a) is a snippet from the Racial Dot Map [Cab13] based on data from the 2010 US Census where every dot corresponds to a citizen.

$0 < \tau \leq 1$ is a global parameter which applies to all agents. Otherwise, they are discontent. Content agents do not move, but discontent agents will swap their location with some other random discontent agent or perform a random jump to an unoccupied place.[1] While the jumps model moves from one house to another empty house in residential areas, i.e., only one agent is affected, the swaps model housing swaps without empty houses, i.e., two agents are involved.

Using this basic model, Schelling demonstrated experimentally that starting from a uniformly random distribution of the agents, see Figure 1.1 (b), the induced random process yields a residential pattern that shows strong segregation, see Figure 1.1 (c). While this is to be expected for intolerant agents, i.e., $\tau > \frac{1}{2}$, the astonishing finding of Schelling was that this also happens for tolerant agents, i.e., $\tau \leq \frac{1}{2}$. This counter-intuitive observation explains why even in a very tolerant population residential segregation along racial/ethnical, religious, or socio-economical lines can emerge, and is one of the main reasons why Schelling's elegant model became one of the landmarks and acclaimed models in sociology and economics. The emergence of residential segregation seems to be intriguing. Only a slight bias towards favoring similar neighbors at the local level leads to the emergence of residential segregation at the global level,

1 A playful interactive demonstration can be found in [HC16].

although no individual agent strictly prefers this. Moreover, the model is very simple but still manages to carve out the unintended consequences resulting from the interaction between individuals.

Against this background, Schelling's model spurred a significant number of research articles that studied and motivated variants of the model, the works by Clark [Cla86], Alba & Logan [AL93], Benard & Willer [BW07], Henry et al. [HPZ11] and Bruch [Bru14], to name only a few. Interestingly, also a physical analog of Schelling's model was found by Vinković & Kirman [VK06], but it was argued by Clark & Fosset [CF08] that such models do not enhance the understanding of the underlying social dynamics. In contrast, they promote simulation studies via agent-based models where the agents' utility function is inspired by real-world behavior. Since Schelling's model as an agent-based system can be easily simulated on a computer, many such empirical simulation studies were conducted to investigate the influence of various parameters on the obtained segregation, e.g., see the works by Fossett [Fos06], Epstein & Axtell [EA96], Gaylord & d'Andria [Gd98], Singh et al. [SVW09], Benenson et al. [BHO09], Roger & McKane [RM11], Carver & Turrini [CT18], and Chapter 4 in Easley & Kleinberg [EK10]. All these empirical studies consider essentially an induced random process, i.e., discontent agents are activated at random and active agents then swap or jump to other randomly selected positions. Note, that Schelling proposed his model as a random process as well. However, it is more realistic to assume instead that the agents are strategic. In some frameworks, like SimSeg [Fos98] or the model by Pancs & Vriend [PV07], agents only change their location if this yields an improvement according to some utility function. This assumption of having rational agents who act strategically matches the behavior of real-world agents, who would only move if this improves their situation and maximizes their utility.

To address this selfish behavior, we depart in this thesis from the assumption of a random process by introducing and analyzing game-theoretic versions of Schelling's model. The residential area is thereby modeled as a multi-agent system consisting of selfish agents who occupy vertices of an underlying graph and try to maximize their utility, which depends on the agents' types in their immediate neighborhood, by strategically selecting locations.

We initiate the study of different variants of an agent-based model. To this end, we distinguish between *saturated* and *non-saturated*, and *open* and *closed* city models, respectively. Except for Chapter 6, we focus on closed city models, which require a fixed population, while open cities allow residents to move away and new residents to arrive. In the saturated city model, vertices are not allowed to be unoccupied, hence, a new agent enters as soon as one agent vacates a vertex. In non-saturated city models, vertices are allowed to be unoccupied. In closed city models, the saturated version corresponds to swap games, i.e., discontent agents swap their locations, while for the non-saturated version we consider jump games, i.e., a discontent agent performs a jump to an unoccupied place to improve.

1.1 Outline

Chapter 2 is concerned with the background, i.e., notation and basic concepts, for the following chapters. We briefly introduce the necessary game-theoretic notation and provide a detailed definition of the models we study in the rest of this thesis. Moreover, we give an overview of the latest results for Schelling's model.

In Chapter 3, we introduce and analyze a generalized game-theoretic model of Schelling segregation that allows more than two agent types and more general underlying graphs modeling the residential area. We show that both aspects heavily influence the dynamic properties and the tractability of finding an optimal strategy profile. In particular, we introduce different versions of *Swap* and *Jump Schelling Games*. We map the boundary when improving response dynamics, i.e., the natural approach for finding equilibrium states, are guaranteed to converge. To this end, we prove several sharp threshold results where guaranteed improving response dynamics convergence suddenly turns into the most robust possible non-convergence result: a violation of weak acyclicity. In particular, we show such threshold results also for Schelling's original model, which is in contrast to the standard assumption in many empirical papers where it is generally assumed that convergence is guaranteed. Furthermore, we show that in the case of convergence, improving response dynamics find an equilibrium in $O(m)$ steps, where m is the number of edges in the underlying graph. Moreover, we provide empirical results that indicate that geometry is essential for strong segregation. Finally, as a new interesting direction, we generalize Schelling's model such that agents have preferences over the different locations. We provide first preliminary results for this extended version and show that convergence is guaranteed on regular graphs if τ is high, i.e., $\tau \geq \frac{1}{2}$ or if all agents have common favorite vertex preferences.

In Chapter 4, we significantly deepen the investigations of Chapter 3 for the Swap Schelling Game with high τ, i.e., $\tau = 1$, of the resulting strategic multi-agent system, by studying the influence of the underlying topology modeling the residential area on the existence of equilibria, the dynamic properties and on the *Price of Anarchy*, a concept that measures how the efficiency of a system degrades due to selfish behavior of its agents. Moreover, as a new conceptual contribution, we consider the influence of locality, i.e., the location swaps are restricted to swaps of neighboring agents. We give improved, almost tight bounds on the Price of Anarchy for arbitrary underlying graphs, and we present (almost) tight bounds for regular graphs, paths, and cycles. Moreover, we give almost tight bounds for grids, which are commonly used in empirical studies. For grids, we also show that locality has a severe impact on the game dynamics. Finally, we provide some results on the *Price of Stability*, a concept that measures the ratio between the best objective function value of one of its stable outcomes and that of an optimal outcome.

In Chapter 5, we refrain from our assumption in Chapter 3 and Chapter 4 that agents are equipped with a monotone utility function that ensures higher utility if an agent has more same-type neighbors. Sociological polls [Smi+19] suggest that real-world agents are actually favoring mixed-type neighborhoods, and hence should be modeled via non-monotone utility functions. To address this, we study Swap Schelling Games with two types of agents with single-peaked utility functions. Our main finding is that tolerance, i.e., agents favoring fifty-fifty neighborhoods or being in the minority, is necessary for the existence of equilibria on almost regular or bipartite graphs. Regarding the quality of equilibria, we derive (almost) tight bounds on the Price of Anarchy and the Price of Stability. In particular, we show that the latter is constant on bipartite and almost regular graphs.

In Chapter 6, we stick with the assumption that an agent, placed on a graph, has one out of two types, and we introduce an agent-based saturated open-city variant, the *Flip Schelling Process*, in which agents, based on the predominant type in their neighborhood, decide whether to change their types; similar to a new agent arriving as soon as another agent leaves the vertex. We investigate the probability that an edge $\{u, v\}$ is monochrome, i.e., that both vertices u and v have the same type in the Flip Schelling Process, and we provide a general framework for analyzing the influence of the underlying graph topology on residential segregation. In particular, for two adjacent vertices, we show that a highly decisive common neighborhood, i.e., a common neighborhood where the absolute value of the difference between the number of adjacent vertices with different types is high, supports segregation and that large common neighborhoods are more decisive.

As an application, we study the expected behavior of the Flip Schelling Process on two standard random-graph models with and without geometry: (1) For random geometric graphs, we show that the existence of an edge $\{u, v\}$ makes a highly decisive common neighborhood for u and v more likely. Based on this, we prove the existence of a constant $c > 0$ such that the expected fraction of monochrome edges after the Flip Schelling Process is at least $1/2 + c$. (2) For Erdős–Rényi graphs, we show that large common neighborhoods are unlikely and that the expected fraction of monochrome edges after the Flip Schelling Process is at most $1/2 + o(1)$. Our results indicate that the underlying graph's cluster structure significantly impacts the obtained segregation strength.

We conclude this thesis with Chapter 7, where we emphasize the most important findings. Moreover, we outline the most promising ideas for future research directions and point to the most pressing open questions.

2 Preliminaries

Game theory studies mathematical models of strategic interaction among rational agents. It has various applications in many fields, whereas *algorithmic game theory* specifies the area at the intersection of game theory and computer science.

We assume that the reader is familiar with basic concepts of game and graph theory, respectively, and we only provide details for topics we consider more advanced. However, we recap certain key notations. For a more detailed introduction, we refer for game theory to the books by Myerson [Mye91], Osborne & Rubinstein [OR94], Nisan et al. [Nis+07], and Shoham & Leyton-Brown [SL08]. The work by von Neumann and Morgenstern [NM44] introduces *Utility Theory* which maps a state of the world to a real number and allows modeling the behavior of selfish agents. For omitted definitions concerning graph theory, we refer to the book by West [Wes17].

In this chapter, we first introduce the notation that we use throughout this thesis, cf. Section 2.1, and provide a very brief introduction about the key concepts in non-cooperative game theory that we need and use in this thesis, cf. Section 2.2. Next, we define the Schelling Game and its variants together with all terms and notations necessary to understand the thesis, cf. Section 2.3. We give a brief overview of how we measure the quality of strategy profiles in Section 2.4, and how we investigate dynamic properties in Section 2.5. We close this chapter by providing an overview of the related work on Schelling Games, cf. Section 2.6.

2.1 Notation

We use $\mathbf{N}$ to denote the set of all natural numbers including 0, $\mathbf{N}^+$ to denote the set of all natural numbers without 0, and $\mathbf{R}$ to denote the set of all real numbers. For $x, y \in \mathbf{N}$, we define $[x..y] = [x, y] \cap \mathbf{N}$ and for $x \in \mathbf{N}^+$, we define $[x] = [1..x]$. We write $X \sim \mathrm{Bin}(n, p)$ to denote that X follows the binomial distribution with $n \in \mathbf{N}^+$ independent Bernoulli trials and success probability $p \in [0, 1]$ for each of these n trials.

We use standard graph-theoretic notation. Let $G = (V, E)$ be an unweighted, undirected and connected graph, with vertex set V and edge set E. We denote the cardinalities of V and E with n and m, respectively. The distance $\operatorname{dist}_G(u, v)$ between two vertices $u, v \in V$ is the number of edges on a shortest path between u and v. The diameter of G is the length of a longest shortest path between any pair of vertices and is denoted by $D(G)$. For any vertex $v \in V$, we denote the *(open) neighborhood* of v in G as

$$N(v) = \{u \in V : \{v, u\} \in E\},$$

and $\deg_v = |N(v)|$ denotes the degree of v in G, i.e., the number of its neighbors. We call a vertex of degree one a *leaf.* Let $\Delta = \max_{v \in V} \deg_v$ and $\delta = \min_{v \in V} \deg_v$ be the maximum and minimum degree of vertices in G, respectively. We denote with α the *independence number* of G, i.e., the cardinality of a maximum independent set in G. For any vertex $v \in V$, let the *closed neighborhood* of v in G be

$$N[v] = \{v\} \cup N(v).$$

Note that the neighborhood as well as the independence number depend on the graph G. However, since the graph will be clear from the context, we remove it from the notation for the sake of simplicity.

A *clique* is a graph such that every two distinct vertices are adjacent. A *path* is represented as a non-empty, finite sequence of edges which joins a sequence of distinct vertices. A *cycle* or *ring* is a non-empty, finite sequence of distinct edges in which, in contrast to a path, only the first and last vertices are equal. A *tree* is a connected acyclic graph. A *star* is a tree that contains one central vertex connected to $n - 1$ leaf vertices. We call a graph G *β-almost regular* if $\Delta - \delta = \beta$ and we call β-almost regular graphs *regular* if $\beta = 0$ and *almost regular* when $\beta = 1$. If the vertex degree is of importance, we call a regular graph *Δ-regular graph*, i.e., every vertex has the same degree Δ. Grid graphs will play a prominent role. We consider *grid graphs with* 4*-neighbors (4-grids)* which are formed by a two-dimensional lattice with l rows and h columns and every vertex is connected to the vertex on its left, top, right, and bottom, respectively, if they exist. In *grid graphs with* 8*-neighbors (8-grids)*, vertices are additionally also connected to their top-left, top-right, bottom-left, and bottom-right vertices, respectively, if they exist.

In Chapter 6, we consider *random geometric graphs* and *Erdős–Rényi graphs* with a total of $n \in \mathbf{N}^+$ vertices and an *expected average degree* $\overline{\deg} > 0$. For a given $r \in \mathbf{R}^+$, a random geometric graph $G \sim \mathcal{G}(n, r)$ is obtained by distributing n vertices uniformly at random in some geometric ground space and connecting vertices u and v if and only if $\operatorname{dist}(u, v) \leq r$. Note that we here, in contrast to $\operatorname{dist}_G(u, v)$ which is defined on a graph, consider the Euclidean distance in the Euclidean space. For a given $p \in [0, 1]$, let $\mathcal{G}(n, p)$ denote an Erdős–Rényi graph. Each edge $\{u, v\}$ is included with probability p, independently from every other edge. It holds that $\overline{\deg} = (n-1)p$.

2.2 Non-Cooperative Game-Theory

Non-cooperative game theory studies rational and selfish agents acting under different settings and choosing strategies to maximize their payoff. A game consists of a set of $n' \in \mathbf{N}^+$ numbered agents. In *finite strategic games* for each $i \in [n']$, an agent i has a finite set of possible actions A_i. We call a vector $a = (a_i)_{i \in [n']} \in A_1 \times A_2 \times \cdots \times A_{n'}$ an *action profile*. Thus, the set of all action profiles is $\mathcal{A} = A_1 \times A_2 \times \cdots \times A_{n'}$. An agent's *strategy* is any of the options which they choose in a setting where the outcome depends not only on their own actions but on the actions of others. In this thesis, we focus on *pure strategies*, where agents choose a single action from their action profile and are not allowed to randomize between different actions. Let S_i be the set of all strategies of agent i. The *strategy space* $\mathcal{S} = S_1 \times S_2 \times \ldots S_{n'}$ is the Cartesian product of the strategies of all agents. A *strategy profile* $s \in \mathcal{S}$ is an n'-dimensional vector of strategies where the i-th entry s_i specifies the strategy chosen by agent i. Each player assigns a *utility* value to each action profile. We denote the function $\mathsf{U}_i = \mathcal{A} \to \mathbf{R}$ as the *utility function* of agent i. Thus, an agent evaluates its utility based on its own action and the actions of all other agents.

Agents can change their strategies if it is profitable. Let $s \in \mathcal{S}$ be a strategy profile and let $s' \in \mathcal{S}$ be the strategy profile obtained from s after agent i changed its strategy from $s_i \in S_i$ to some other strategy $s'_i \in S_i$. We say that this is an *improving response* if $\mathsf{U}_i(s') > \mathsf{U}_i(s)$, i.e., if agent i strictly increases its utility. If $\forall s^* \in S_i \colon \mathsf{U}_i(s') \geq \mathsf{U}_i(s^*)$, we denote the strategy s'_i as the *best response* of agent i. A strategy profile s is *stable* or in a *pure Nash Equilibrium* [Nas50] if no agent can strictly increase its utility by unilaterally changing its strategy, i.e., for any agent $i \in [n']$ any strategy profile s' $\mathsf{U}_i(s) \geq \mathsf{U}_i(s')$. Moreover, we also investigate 2-*coalitional pure Nash Equilibria*. Let $s \in \mathcal{S}$ be a strategy profile and let $s'' \in \mathcal{S}$ be the strategy profile obtained from s after agent i changed its strategy from $s_i \in S_i$ to some other strategy $s'_i \in S_i$ and agent j changed its strategy from $s_j \in S_j$ to some other strategy $s'_j \in S_j$. This is beneficial for both agents if for all agents $i, j \in [n']$ $\mathsf{U}_i(s'') > \mathsf{U}_i(s)$ and $\mathsf{U}_j(s'') > \mathsf{U}_j(s)$. A 2-coalitional pure Nash Equilibrium is a Nash Equilibrium where no coalition of 2 agents can cooperatively deviate in a way that benefits all members of the coalition.

2.3 The Schelling Game

Given a graph $G = (V, E)$, let $\mathcal{T}_k(G)$ denote the set of k-tuples of positive integers summing up to $n' \leq |V|$. An instance $(G, \boldsymbol{t})$ of a *Schelling Game with k types (k-SG)* is defined by a graph $G = (V, E)$ and a k-tuple $\boldsymbol{t} = (t_1, \ldots, t_k) \in \mathcal{T}_k(G)$. The graph G serves as the underlying topology modeling the residential area in which the agents select a location. There are n' strategic agents that need to choose vertices in V. Every agent belongs to exactly one of the k *types*, which model racial/ethnic, religious, or socio-economic groups, and there are t_i agents of type i, for every $i \in [k]$. When $|t_i| = |t_j|$ for each $i, j \in [k]$, we say that the game is *balanced*. For convenience and in all of our illustrations, we associate each agent type $i \in [k]$ with a color. For $k = 2$ this corresponds to Schelling's original model [Sch69; Sch71] with two different types of agents. In this case, we use the colors blue and orange and denote by b and $o = n' - b$ the number of blue and orange agents, respectively. Additionally, in the case of a game with $k = 2$, we assume that $o \leq b$, i.e., orange is the color of the minority type. For any graph G and any k-dimensional type vector $\boldsymbol{t} \in \mathcal{T}_k(G)$, let $c : [n'] \to [k]$ denote the function that maps any agent $i \in [n']$ to its color $c(i) \in [k]$.

The strategy of an agent is its location on the graph, i.e., a vertex of G. A *feasible strategy profile $\boldsymbol{\sigma}$* is an n'-dimensional vector whose i-th entry corresponds to the strategy of the i-th agent and where all strategies are pairwise disjoint. Let $\boldsymbol{\sigma}^{-1}$ be its inverse function, mapping a vertex $v \in V$ to the agent i choosing v as its strategy. We denote by $\ominus$ if a vertex is empty. Hence, $\boldsymbol{\sigma}^{-1}$ is equal $\ominus$ if v is empty, i.e., no agents choose v as its strategy. Thus, any feasible strategy profile $\boldsymbol{\sigma}$ corresponds to a coloring of G such that for each $i \in [k]$ exactly t_i vertices of G are colored with the i-th color and all vertices which are not selected as the strategy of an agent remain uncolored. We say that agent *i occupies vertex v in $\boldsymbol{\sigma}$* if the i-th entry of $\boldsymbol{\sigma}$, denoted as $\boldsymbol{\sigma}(i)$, is v and, equivalently, if $\boldsymbol{\sigma}^{-1}(v) = i$. It will become important to distinguish if two agents i, j occupy neighboring vertices under $\boldsymbol{\sigma}$. For this, we use the notation $1_{ij}(\boldsymbol{\sigma})$ with $1_{ij}(\boldsymbol{\sigma}) = 1$ if the agents i and j occupy neighboring vertices, i.e., if agent i and j are *adjacent*, under $\boldsymbol{\sigma}$, and $1_{ij}(\boldsymbol{\sigma}) = 0$ otherwise.

For an agent i and any feasible strategy profile $\boldsymbol{\sigma}$, we denote by

$$C_i(\boldsymbol{\sigma}) = \{v \in V \mid c(\boldsymbol{\sigma}^{-1}(v)) = c(i)\}$$

the set of vertices of G which are occupied by agents having the same color as agent i. We call agent i and j a *colored pair* if $1_{ij}(\boldsymbol{\sigma}) = 1$ and $c(i) = c(j)$. Agents care about the fraction of colored pairs in their surrounding area. In this thesis, we consider different approaches concerning the surrounding area, in particular the (closed) neighborhood, but also only parts of the neighborhood. Let $f_i(\sigma)$ denote this fraction. We define the different variants we are considering later. The utility of an agent i in $\boldsymbol{\sigma}$ is defined as

$$\mathsf{U}_i(\boldsymbol{\sigma}) = p(f_i(\boldsymbol{\sigma})),$$

where p is a function with domain $[0, 1]$. Each agent aims at maximizing its utility. Again, we define the different variants defining p later.

An agent can change its strategy either via swapping with another agent who agrees or via jumping to another unoccupied vertex in the graph. This yields the *Swap Schelling Game* and the *Jump Schelling Game*.

Swap Schelling Games In an instance $(G, \boldsymbol{t})$ of a *Swap Schelling Game with k types (k-SSG)*, there are $n' = n$ strategic agents that need to choose vertices in V in such a way that every vertex is occupied by exactly one agent, i.e., $\boldsymbol{\sigma}$ is a permutation of V, and we treat $\boldsymbol{\sigma}$ as a bijective function mapping agents to vertices. Agents can change their strategies only by swapping vertex occupation with another agent. Consider two strategic agents i and j which occupy vertices $\boldsymbol{\sigma}(i)$ and $\boldsymbol{\sigma}(j)$, respectively. After performing a *swap* both agents exchange their occupied vertex which yields a new feasible strategy profile $\boldsymbol{\sigma}_{ij}$, which is identical to $\boldsymbol{\sigma}$ except that the i-th and the j-th entries are exchanged. Thus, in the induced coloring of G, the coloring corresponding to $\boldsymbol{\sigma}_{ij}$ is identical to the coloring corresponding to $\boldsymbol{\sigma}$ except that the colors of vertices $\boldsymbol{\sigma}(i)$ and $\boldsymbol{\sigma}(j)$ are exchanged. We say that a swap is *local* if the swapping agents occupy neighboring vertices, i.e., if $1_{ij}(\boldsymbol{\sigma}) = 1$.

As agents are strategic and want to maximize their utility, we only consider *profitable swaps* by agents, i.e., swaps that strictly increase the utility of both agents involved in the swap. It follows that profitable swaps can only occur between agents of different colors. We call a feasible strategy profile $\boldsymbol{\sigma}$ a *swap equilibrium*, or simply *equilibrium*, if $\boldsymbol{\sigma}$ does not admit profitable swaps, that is, if for each pair of agents i, j, we have

$$\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_{ij}) \text{ or } \mathsf{U}_j(\boldsymbol{\sigma}) \geq \mathsf{U}_j(\boldsymbol{\sigma}_{ij}).$$

Hence, stable placements correspond to 2-coalitional pure Nash equilibria.

If agents are restricted to performing only local swaps, then we call the corresponding strategic game *Local Swap Schelling Game with k types (local k-SSG)*. We call $\boldsymbol{\sigma}$ a *local swap equilibrium* if no profitable local swap exists under $\boldsymbol{\sigma}$. Clearly, any swap equilibrium $\boldsymbol{\sigma}$ is also a local swap equilibrium but the converse is not true. Thus, the set of local swap equilibria is a superset of the set of swap equilibria. See Example 4.1 in Chapter 4 for an illustration of the (local) k-SSG.

Jump Schelling Games In an instance $(G, \boldsymbol{t})$ of a *Jump Schelling Game with k types (k-JSG)*, there are $n' < n$ strategic agents that need to choose vertices in V in such a way that every vertex is occupied by at most one agent. Hence, $\boldsymbol{\sigma}$ is an injective function mapping agents to distinct vertices. In the JSG, we assume the existence of *empty* vertices in the underlying graph, i.e., vertices which are not occupied by an agent, and an agent can change its strategy to any currently empty vertex, which we refer to as a *jump* to that vertex. Consider the strategic agent i, which occupies vertex $\boldsymbol{\sigma}(i)$. After performing a jump to an empty vertex $v \in V$, this yields a new feasible strategy profile $\boldsymbol{\sigma}_i$, which is identical to $\boldsymbol{\sigma}$ except that the i-th entry changed. Thus, in the induced coloring of G, the coloring corresponding to $\boldsymbol{\sigma}_i$ is identical to the coloring corresponding to $\boldsymbol{\sigma}$ except that the colors of the vertex $\boldsymbol{\sigma}(i)$ and v are exchanged, i.e., $\boldsymbol{\sigma}(i)$ is now uncolored while v is occupied by an agent.

An agent only jumps to another empty vertex if this strictly increases its utility. We call a feasible strategy profile $\boldsymbol{\sigma}$ *jump equilibrium* if $\boldsymbol{\sigma}$ does not admit profitable jumps, that is, if for each agent i, we have

$$\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_i).$$

Here, a jump equilibrium corresponds to a pure Nash Equilibrium.

If the game is clear from the context, we will simply say that a feasible strategy profile $\boldsymbol{\sigma}$ is *stable* or an *equilibrium*.

Different Variants There are several ways how we can define the utility of an agent i in $\boldsymbol{\sigma}$. Throughout this thesis, we will investigate different variants.

In Chapter 3, we consider two different calculation methods of the fraction $f_i(\sigma)$, called the *one-versus-all* and *one-versus-one* versions. In the one-versus-all version, an agent wants a certain fraction of agents of its own type in its neighborhood, regardless of the specific types of adjacent agents with other types. To this end, let $\tau \in (0, 1)$ be the *intolerance parameter*. Similar to Schelling's model we say that an agent i is *content* with a feasible strategy profile $\boldsymbol{\sigma}$ if at least a τ-fraction of same-type agents is in agent i's neighborhood. Otherwise i is *discontent*. An agent aims to find a vertex in the given graph where it is content or, if this is not possible, where it has the highest possible utility.

▶ **Definition 2.1.** For an instance $(G, \boldsymbol{t}, \tau)$ of the *one-versus-all Swap Schelling Game (1-k-SSG)*, we define

$$f_i(\sigma) = \frac{|N(\boldsymbol{\sigma}(i)) \cap C_i(\boldsymbol{\sigma})|}{\deg_{\boldsymbol{\sigma}(i)}}.$$

The utility of an agent i in $\boldsymbol{\sigma}$ is defined as $\mathsf{U}_i(\boldsymbol{\sigma}) = \min\left\{1, \frac{f_i(\sigma)}{\tau}\right\}$. ◀

Turning our focus to the JSG, we only consider vertices that are occupied by agents. To this end, we denote the set of empty vertices in a feasible strategy profile as $E(\sigma) = \{v \in V \mid \boldsymbol{\sigma}^{-1}(v) = \ominus\}$. If an agent i has no neighboring agents, i.e., $N(\boldsymbol{\sigma}(i)) \setminus E(\boldsymbol{\sigma}) = \emptyset$, we say that i is *isolated*, otherwise i is *un-isolated*. We assume that isolated agents are always discontent.

▶ **Definition 2.2.** For an instance $(G, \boldsymbol{t}, \tau)$ of the *one-versus-all Jump Schelling Game (1-k-JSG)*, we define

$$f_i(\sigma) = \frac{|N(\boldsymbol{\sigma}(i)) \cap C_i(\boldsymbol{\sigma})|}{|N(\boldsymbol{\sigma}(i)) \setminus E(\boldsymbol{\sigma})|}$$

if agent i is un-isolated and $f_i(\sigma) = 0$ if agent i is isolated. The utility of an agent i in $\boldsymbol{\sigma}$ is defined as $\mathsf{U}_i(\boldsymbol{\sigma}) = \min\left\{1, \frac{f_i(\sigma)}{\tau}\right\}$. ◀

In contrast to this, in the one-versus-one version, an agent only compares the number of own-type agents to the number of agents in the largest group of agents with different type in its neighborhood. Thus, we only consider a part of the neighborhood of agent i. Let

$$C^d(\boldsymbol{\sigma}) = \{v \in V \mid c(\boldsymbol{\sigma}^{-1}(v)) = d\}$$

be the set of vertices for G which are occupied by agents having color $d \in [k]$. Let $|C_i^{\max}(\boldsymbol{\sigma})|$ be the number of neighboring agents of the type $d' \neq c(i)$ that make up the largest proportion among all neighbors, i.e.,

$$|C_i^{\max}(\boldsymbol{\sigma})| = \max_{d' \neq c(i)} |N(\boldsymbol{\sigma}(i)) \cap C^{d'}(\boldsymbol{\sigma})|.$$

Note that we only consider colored vertices, i.e., vertices which are occupied by an agent. Hence, in the case that agent i is isolated, we have $|C_i^{\max}(\boldsymbol{\sigma})| = 0$.

▶ **Definition 2.3.** For an instance $(G, \boldsymbol{t}, \tau)$ of the *one-versus-one Swap Schelling Game (1-1-SSG)* and the *one-versus-one Jump Schelling Game (1-1-JSG)*, we define

$$f_i(\sigma) = \frac{|N(\boldsymbol{\sigma}(i)) \cap C_i(\boldsymbol{\sigma})|}{|N(\boldsymbol{\sigma}(i)) \cap C_i(\boldsymbol{\sigma})| + |C_i^{\max}(\boldsymbol{\sigma})|}$$

if agent i is un-isolated and $f_i(\sigma) = 0$ if agent i is isolated. The utility of an agent i in $\boldsymbol{\sigma}$ is defined as $\mathsf{U}_i(\boldsymbol{\sigma}) = \min\left\{1, \frac{f_i(\sigma)}{\tau}\right\}$. ◀

Notice, that the one-versus-all and one-versus-one versions coincide for $k = 2$, thus both versions generalize the two-type case. Hence, in the two-type case, we only talk about Swap and Jump Schelling Games.

In Chapter 4, we assume that every vertex of the underlying graph serving as a residential area is occupied by an agent, and pairs of discontent agents can swap their locations, i.e., their occupied vertices, to maximize the fraction of own-type neighbors. Hence, we set $\tau = 1$ in the one-versus-all Swap Schelling version. Thus, the utility of agent i in $\boldsymbol{\sigma}$ is defined as

$$\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{|N(\boldsymbol{\sigma}(i)) \cap C_i(\boldsymbol{\sigma})|}{\deg_{\boldsymbol{\sigma}(i)}},$$

i.e., as the ratio of the number of agents with the same type which occupies neighboring vertices and the total number of neighboring vertices, and each agent aims at maximizing its utility.

In Chapter 5, we turn our focus to the closed neighborhood, i.e., we consider for agent i the fraction of agents of its own color in i's neighborhood including itself. Thus, agents are aware of their own contribution to the diversity of their neighborhood. We investigate a variant of the 2-SSG with single-peaked utility functions, called the *Single-Peaked Swap Schelling Game (SP-2-SSG)*. In an instance of the SP-2-SSG (G, o, Λ) there are o orange agents and a peak position Λ. Remember that orange is the number of the minority type, i.e., $o \leq n/2$.

▶ **Definition 2.4.** For an instance $(G, \boldsymbol{t}, \Lambda)$ of the *Single-Peaked Swap Schelling Game (SP-2-SSG)*, we define

$$f_i(\boldsymbol{\sigma}) = \frac{|N[\boldsymbol{\sigma}(i)] \cap C_i(\boldsymbol{\sigma})|}{|N[\boldsymbol{\sigma}(i)]|}.$$

The function p to compute the utility $\mathsf{U}_i(\boldsymbol{\sigma})$ of an agent i is a single-peaked function with domain $[0, 1]$ and peak at $\Lambda \in (0, 1)$ that satisfies the following two properties:

(i) p is a strictly monotonically increasing function in the interval $[0, \Lambda]$ with $p(0) = 0$;

(ii) for each $x \in [\Lambda, 1]$, $p(x) = p\left(\frac{\Lambda(1-x)}{1-\Lambda}\right)$ and $p(\Lambda) = 1$.

The utility of an agent i in $\boldsymbol{\sigma}$ is defined as $\mathsf{U}_i(\boldsymbol{\sigma}) = p\left(\frac{|N[\boldsymbol{\sigma}(i)] \cap C_i(\boldsymbol{\sigma})|}{|N[\boldsymbol{\sigma}(i)]|}\right)$. ◀

In Chapter 6, we investigate a model, called the *Flip Schelling Process (FSP)*, which differs from the models introduced so far. In the FSP, agents have binary types, i.e., $k = 2$. An agent is content if the fraction of agents in its neighborhood with the same type is larger than $\frac{1}{2}$. Otherwise, if the fraction is smaller than $\frac{1}{2}$, an agent is discontent and is willing to flip its type to become content. If the fraction of the same type of agents in its neighborhood is exactly $\frac{1}{2}$, an agent flips its type with probability $\frac{1}{2}$. Hence, the FSP is defined as follows: an agent i whose type is aligned with the type of more than $\deg_{\boldsymbol{\sigma}(i)}/2$ of its neighbors keeps its type. If more than $\deg_{\boldsymbol{\sigma}(i)}/2$ neighbors have a different type, then agent i changes its type. In case of a tie, i.e., if exactly $\deg_{\boldsymbol{\sigma}(i)}/2$ neighbors have a different type, then i changes its type with probability $\frac{1}{2}$. FSP is a simultaneous-move, one-shot process, i.e., all agents make their decision at the same time and, moreover, only once.

2.4 Game Efficiency

In Chapter 4, we measure the quality of a feasible strategy profile $\boldsymbol{\sigma}$ by its *social welfare* $\mathsf{U}(\boldsymbol{\sigma})$, which is the sum over the utilities of all agents, i.e.,

$$\mathsf{U}(\boldsymbol{\sigma}) = \sum_{i=1}^{n'} \mathsf{U}_i(\boldsymbol{\sigma}).$$

For any game $(G, \boldsymbol{t})$, let $\boldsymbol{\sigma}^*(G, \boldsymbol{t})$ denote a feasible strategy profile that maximizes the social welfare, and let $\mathsf{SE}(G, \boldsymbol{t})$ and $\mathsf{LSE}(G, \boldsymbol{t})$ denote the set of swap equilibria and local swap equilibria for $(G, \boldsymbol{t})$, respectively. Note that the set of equilibria depends also on the choice of τ. Since we fix in Chapter 4 $\tau = 1$, we remove it from the notation for the sake of simplicity. We will study the impact of the agents' selfishness on the obtained social welfare for games played on a given class of underlying graphs $\mathcal{G}$ with k agent types by analyzing the *Price of Anarchy* (PoA) [KP09], which is defined as

$$\mathrm{PoA}(\mathcal{G}, k) = \max_{G \in \mathcal{G}} \max_{\boldsymbol{t} \in \mathcal{T}_k(G)} \frac{\mathsf{U}(\boldsymbol{\sigma}^*(G, \boldsymbol{t}))}{\min_{\boldsymbol{\sigma} \in \mathsf{SE}(G, \boldsymbol{t})} \mathsf{U}(\boldsymbol{\sigma})}.$$

Analogously, we define the *Local Price of Anarchy* (LPoA) as the same ratio but concerning local swap equilibria. It follows that, for any $k \geq 2$ and class of graphs $\mathcal{G}$, we have $\mathrm{PoA}(\mathcal{G}, k) \leq \mathrm{LPoA}(\mathcal{G}, k)$.

The *Price of Stability* (PoS) [Ans+08] for games with k types played on a family of graphs $\mathcal{G}$ is defined as

$$\mathrm{PoS}(\mathcal{G}, k) = \max_{G \in \mathcal{G}} \max_{\boldsymbol{t} \in \mathcal{T}_k(G)} \frac{\mathsf{U}(\boldsymbol{\sigma}^*(G, \boldsymbol{t}))}{\max_{\boldsymbol{\sigma} \in SE(G, \boldsymbol{t})} \mathsf{U}(\boldsymbol{\sigma})}$$

and is thus the best-case equivalent of the Price of Anarchy. We define the *Local Price of Stability* (LPoS) similar to the LPoA by replacing the set of swap equilibria with that of local swap equilibria. In this case, as the set of local swap equilibria of a game is a superset of that of its swap equilibria, it follows that $\mathrm{LPoS}(\mathcal{G}, k) \leq \mathrm{PoS}(\mathcal{G}, k)$ for any class of graphs $\mathcal{G}$ and integer $k \geq 2$.

Analogously, we define the Price of Anarchy and the Price of Stability considering jump equilibria.

Another variant to measure the quality of a strategy profile $\boldsymbol{\sigma}$ is via the *Degree of Integration* (DoI) [Aga+21], defined by the number of non-segregated agents. We use it for analyzing the Single-Peaked Swap Schelling Game in Chapter 5 and prefer it to the standard utilitarian welfare since it measures segregation independently of the value of Λ. In the context of Swap Schelling Games, we define the DoI as follows[2],

$$\mathsf{DoI} = |\{i \in [n'] \mid f_i(\boldsymbol{\sigma}) < 1\}|.$$

The DoI is a simple segregation measure that captures how many agents have contact with other-type agents. For any fixed game (G, b, Λ), let $\boldsymbol{\sigma}^*$ denote a feasible strategy profile maximizing the DoI and let $\mathsf{SE}(G, b, \Lambda)$ denote the set of swap equilibria for (G, b, Λ). We study the impact of the agents' selfishness by analyzing the Price of Anarchy concerning the DoI ($\mathrm{PoA}_{\mathsf{DoI}}$), which is defined as

$$\mathrm{PoA}_{\mathsf{DoI}}(G, b, \Lambda) = \frac{\mathsf{DoI}(\boldsymbol{\sigma}^*)}{\min_{\boldsymbol{\sigma} \in \mathsf{SE}(G,b,\Lambda)} \mathsf{DoI}(\boldsymbol{\sigma})}$$

and the Price of Stability concerning the DoI ($\mathrm{PoS}_{\mathsf{DoI}}$), which is defined as

$$\mathrm{PoS}_{\mathsf{DoI}}(G, b, \Lambda) = \frac{\mathsf{DoI}(\boldsymbol{\sigma}^*)}{\max_{\boldsymbol{\sigma} \in \mathsf{SE}(G,b,\Lambda)} \mathsf{DoI}(\boldsymbol{\sigma})}.$$

2.5 Game Dynamics

We also investigate the dynamic properties of our game variants, i.e., we analyze if a game has the *finite improvement property (FIP)* [MS96]. So far, we focused on stable states themselves. However, we study the process as well and how the process evolves if we start with a non-stable feasible strategy profile and let the agents perform profitable moves. In particular, our main question is whether any sequence of improving swaps or jumps is finite. Such games often have many attractive properties, like the guaranteed existence of pure equilibria, and often a fast convergence to such a stable state.

2 Note that in the context of Jump Schelling Games, we have to factor out isolated agents from the number.

To this end, we analyze whether *improving response dynamics (IRD)*, i.e., the natural approach for finding equilibrium states where agents sequentially try to change towards better strategies until no agent can further improve, converge. For showing this, we employ *generalized ordinal potential functions*. Such a function Φ maps placements to real numbers such that if an agent (or a pair of agents) in a feasible strategy profile $\boldsymbol{\sigma}$ can improve by a jump (or a swap) which results in a feasible strategy profile $\boldsymbol{\sigma}'$, then $\Phi(\boldsymbol{\sigma}') > \Phi(\boldsymbol{\sigma})$ holds. That is, any improving strategy change also increases the potential function value. The existence of a generalized ordinal potential function shows that a game is a *potential game* [MS96], which guarantees the existence of pure equilibria and that IRD must terminate in an equilibrium. In contrast, the FIP can be disproved by showing the existence of an *improving response cycle (IRC)*, which is a sequence of improving strategy changes that visits the same state of the game twice. Hence, there exists a sequence of feasible strategy profiles $\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^\ell$, with $\boldsymbol{\sigma}^\ell = \boldsymbol{\sigma}^0$, where $\boldsymbol{\sigma}^{q+1}$ is obtained by a profitable jump (or a swap) by one (or by two) agents in $\boldsymbol{\sigma}^q$, for $q \in [0..\ell - 1]$.

The existence of an IRC directly implies that a potential function cannot exist and, thus, that IRD may not terminate. However, even with existing IRCs, it is still possible that from any state of the game, there exists a finite sequence of improving strategy changes that leads to an equilibrium. In this case, the game is *weakly acyclic* [You93]. Thus, the strongest possible non-convergence result is a proof that a game is not weakly acyclic.

For investigating the FIP, the following function Φ mapping feasible strategy profiles to natural numbers is important:

$$\Phi(\boldsymbol{\sigma}) = \left|\left\{\{u,v\} \in E \mid c(\boldsymbol{\sigma}^{-1}(u)) = c(\boldsymbol{\sigma}^{-1}(v))\right\}\right|.$$

Hence, $\Phi(\boldsymbol{\sigma})$ is the number of edges of G whose endpoints are occupied by agents of the same color under the feasible strategy profile $\boldsymbol{\sigma}$. We denote such edges as *monochrome edges* and $\Phi(\boldsymbol{\sigma})$ as the *potential of* $\boldsymbol{\sigma}$.

2.6 Related Work

There is a huge body of work on Schelling's model and variations thereof, see e.g. [BW07; BHO09; Cla86; Sch69; Sch78; VK06; Whi86]. Most related work is purely empirical and provides simulation results. We focus here on the amount of related work, which rigorously proves properties of (variants of) the Schelling's model.

On the theoretical side, the underlying stochastic process leading to segregation was rigorously studied [Ger+08; OF18a; OF18b; ZR22]. Young [You98] was the first to analyze a variant of the one-dimensional segregation model. He considered the specific dynamics where a pair of agents are chosen at random and they swap places with a suitably chosen probability. He analyzed the induced Markov chain and proved that under certain conditions total segregation is with a high probability a stochastically stable state. The first rigorous analysis of the original Schelling model was achieved by Brandt et al. [Bra+12] for the case where agents with tolerance parameter $\tau = \frac{1}{2}$ are located on a ring and agents can only swap positions. They proved that the process converges with high probability to a state where the average size of monochromatic neighborhoods is polynomial in w, where w is the window size for determining the neighborhood. Interestingly, Barmpalias et al. [BEL14] have proven a drastically different behavior for $0.3531 < \tau < \frac{1}{2}$ where the size of monochromatic neighborhoods is exponential in w. Later, Barmpalias et al. [BEL16] analyzed a 2-dimensional variant, where both agent types have different tolerance parameters and agents may change their type if they are discontent. Finally, Immorlica et al. [Imm+17] considered the random Schelling dynamics on a 2-dimensional toroidal grid with $\tau = \frac{1}{2} - \varepsilon$, for some small $\epsilon > 0$. Their main result is a proof that the average size of monochromatic neighborhoods is exponential in w.

The focus of the above-mentioned works is on investigating the expected size of the obtained homogeneous regions, but it is also shown that the stochastic processes starting from a uniform random agent placement converge with high probability to a stable placement. The convergence time was considered by Mobius & Rosenblat [MR00], who observed that the Markov chain analyzed in [You98] has a very high mixing time. Bhakta et al. [BMR14] showed in the two-dimensional grid case a dichotomy in mixing times for high τ and very low τ values.

All these models of segregation are essentially random processes where discontent agents choose their new location at random. However, in reality, agents would not move randomly, instead, they move to a location that maximizes their utility. Pancs & Vriend [PV07] used different types of utility functions for their agents in extensive simulation experiments. Furthermore, Schelling's model recently gained traction within the algorithmic game theory, artificial intelligence, and multi-agent systems.

Zhang [Zha04; Zha11] proved similar results to [You98] in 2-dimensional models. Moreover, he was the first who introduced an (evolutionary) game-theoretic model related to Schelling's original model. In particular, Zhang analyzed a model where the agents are endowed with a noisy single-peaked utility function that depends on the ratio of the numbers of the two agent types in any local neighborhood and is a departure from the threshold behavior proposed by Schelling. The highest utility is attained in perfectly balanced neighborhoods and agents slightly prefer being in the majority over being in the minority. In contrast to our models, Zhang's model [Zha04] assumes transferable utilities and it can happen that after a randomly chosen swap one or both agents are worse off. Grauwin et al. [GGJ12] generalized the results.

Agarwal et al. [Aga+21] studied a variant of our model in Chapter 3 and Chapter 4 with k types, where the agents are partitioned into stubborn and strategic agents. The former agents do not move and the latter agents try to maximize the fraction of same-type agents in their neighborhood by jumping to a suitable empty location. This corresponds to a variant of the Schelling Game with $\tau = 1$. They showed that equilibria are not guaranteed to exist, in particular, that equilibria are not guaranteed to exist on underlying trees, and that deciding equilibrium existence or the existence of an agent strategy profile with certain social welfare is NP-hard. Moreover, the authors studied the Price of Anarchy in terms of utilitarian social welfare and in terms of the newly introduced Degree of Integration, which counts the number of non-segregated agents and is inspired by the work of [LC82]. For the former, i.e., utilitarian social welfare, they showed that the Price of Anarchy and the Price of Stability can be unbounded for $k \geq 2$ for jump games and $k \geq 3$ for the swap version, even in balanced games. They also established that the Price of Anarchy is in $\Theta(n)$ on underlying star graphs if there are at least two agents of each type. For two types in the swap version, they showed that the Price of Anarchy is between 2.058 and 4 for balanced games on any graph. For the Price of Stability, they give a constant lower bound under certain conditions and showed that it equals 1 on regular graphs. For the latter, i.e., the Degree of Integration, they give a tight bound of $\frac{n}{2}$ on the Price of Anarchy and the Price of Stability that is achieved on a tree. In contrast, they derived a constant Price of Stability on paths. The complexity results were extended by Kreisel et al. [Kre+22]. Kanellopulos et al. [KKV22] considered a generalized variant, where an ordering of the agent types exists and agents are more tolerant towards agents of types that are closer according to the ordering.

Chan et al. [CIT20] studied a variant of the Jump Schelling Game with $\tau = 1$ where the agents' utility is a function of the composition of their neighborhood and of the social influence by agents that select the same location. In their work, the social influence is defined by an auxiliary directed graph that models the social network. This idea of additional social influence was earlier proposed by Agarwal et al. [Aga+21] using an undirected social network. Another novel variant of the Jump Schelling Game was investigated by Kanellopoulos et al. [KKV21]. There the main new aspect is that an agent is included when counting its neighborhood size. This subtle change leads to agents preferring locations with more own-type neighbors. Also very recently, Bullinger et al. [BSV21] studied welfare guarantees in Schelling Games. They showed results on computing assignments with high social welfare as well as on other optimality notions, such as Pareto optimality and two newly introduced measures. In particular, they considered the number of agents with non-zero utility as social welfare function. They proved hardness results for computing the social optimal state and they discussed other stability notations. Deligkas et al. [DEG22] dealt with the same questions albeit under the prism of parameterized complexity. Strategic segregation was also considered in social network formation [ACT19], where agents are grouped into types and can choose to create or sever links whilst maximizing their own private interests.

We note that hedonic games [BJ02; DG80], where selfish agents form coalitions, are also related to Schelling's model, but the utility of an agent only depends on its chosen coalition. In Schelling's model, the neighborhood of an agent could be considered as its coalition, but then not all agents in a coalition derive the same utility from it. Nevertheless, Schelling games are similar to fractional hedonic games [Azi+19; Bil+18; CMM19; MMV19; MMV20], hedonic diversity games [BE20; BEI19], and FEN-hedonic games [FKR19; Iga+19; Ker+20; KR19]. In these games, the utility of an agent only depends on the coalition containing that agent. In FEN-hedonic games, every agent partitions the set of agents into friends, enemies, and neutral agents, and the value of a coalition for an agent then depends on the distribution of these types within the coalition. This is similar to Schelling games, where the neighborhood of an agent can be considered as its coalition and the utility of an agent depends on the type distribution within its neighborhood. Even closer to Schelling games are fractional hedonic games and hedonic diversity games. Fractional hedonic games are additively separable hedonic games in which the total value of a coalition is divided by the cardinality of the coalition. Thus, if the value that agent i ascribes to another agent j is 1 if i and j are of the same type and 0 otherwise, then fractional hedonic games and Schelling games share the same utility function. However, they heavily differ on the feasibility of coalition structures: in fractional hedonic games, coalitions are unrestricted and pairwise disjoint, whereas in Schelling games they overlap and are superimposed by the topology of the underlying graph. Hedonic diversity games account for a mixture of both homophilic and heterophilic agents. More precisely, there are two different types of agents and the utility of an agent for being in a coalition depends on the distribution of same-type agents in a coalition and its cardinality.

Close to hedonic games are the very recently introduced topological distance games [BS22], where agents are assigned to vertices of an underlying graph and the utility of an agent depends on both the agent's inherent utilities for other agents as well as its distance from these agents on the graph.

Cooperative games with overlapping coalitions, called OCF-games, from the cooperative game theory literature are related. There, agents can be contained in many coalitions and coalitions may overlap, as in Schelling games. OCF-games are introduced in Chalkiadakis et al. [Cha+10] and different variants of the core are defined and analyzed. In [Zic+19; ZME14], other stability concepts are considered and the tractability of the involved computational problems is studied.

3 Convergence and Hardness of Strategic Schelling Segregation

This chapter is based on joint work with Ankit Chauhan, Hagen Echzell, Tobias Friedrich, Pascal Lenzner, Marcus Pappik, Friedrich Schöne, Fabian Sommer, and David Stangl [CLM18; Ech+19].

Real-world agents move if this is beneficial. Hence, we analyze a game-theoretic version of Schelling's model of real-world agents who would only move if this improves their situation. To this end, we introduce a model which is close to Schelling's formulation, but, we extend the model to more than two agent types. That is, in our model, there are k types of agents and the utility of an agent depends on the type ratio in its neighborhood. An agent is content if the fraction of own-type neighbors is above $\tau \in (0, 1)$. To improve their utility, agents can either swap with another agent who is willing to swap, that is the Swap Schelling Game, or jump to an unoccupied vertex, which is the Jump Schelling Game. Empirically, our model yields outcomes that are very similar to Schelling's original model - see Figure 3.1 for an example.

This chapter sets out to explore the properties of the strategic dynamic processes and the tractability of the induced optimization problems. Our main

Figure 3.1: An instance of a Swap Schelling Game with two different types, with $o = b = 5\,000$ and $\tau = \frac{1}{2}$. Left: An initial feasible strategy profile of the agents. The agents are distributed uniformly at random. Middle: A sample swap equilibrium of the 2-SSG showing significant segregation. Right: For comparison, a simulation of the Schelling Process with $\tau = \frac{1}{2}$ [Hay13] where on each turn, every discontent agent moves to a randomly chosen vacant site.

Table 3.1: Results regarding IRD. "reg." stands for regular graphs, "arb" for arbitrary graphs, which model the residential area. "ow" is an abbreviation for otherwise. "✓" denotes that IRD converge to an equilibrium, "o" denotes the existence of an IRC. "×" denotes that the version is not weakly acyclic. If τ is omitted, the result holds for any $0 < \tau < 1$. Let Δ be the degree of the vertices in a regular graph.

	1-k-SSG	1-1-SSG	1-k-JSG	1-1-JSG
reg.	✓(Thm. 3.4)	✓(Thm. 3.5) $\tau \le \frac{1}{\Delta}$ o (Thm. 3.6) $\tau \ge \frac{5}{\Delta-1}$	✓(Thm. 3.8) $\tau \le \frac{2}{\Delta}$ o (Thm. 3.9) $\tau > \frac{2}{\Delta}$	✓(Thm. 3.11) $\tau \le \frac{1}{\Delta}$ o (Thm. 3.12) $\tau > \frac{2}{\Delta}$
arb.	✓(Thm. 3.1) $k = 2$, $\tau \le \frac{1}{2}$ × (Thm. 3.2, 3.3) ow	× (Thm. 3.7)	× (Thm. 3.10)	× (Thm. 3.13)

contribution is thereby an investigation of the convergence behavior of many variants of Schelling's model. This corresponds to analyzing improving response dynamics. Previous work, including Schelling's original papers and all the mentioned empirical simulation studies, assume that IRD always converge to an equilibrium or converge with high probability [BEL16; BEL14; Bra+12; Imm+17]. We challenge the assumption of guaranteed convergence by precisely mapping the boundary of when IRD are assured to find an equilibrium. We show that IRD behave radically differently in the swap version, cf. Section 3.2, compared to the jump version, cf. Section 3.3. Moreover, we show that contrasting behavior can even be found within these two variants. We demonstrate the extreme cases of guaranteed IRD convergence, i.e., the existence of a generalized ordinal potential function, and the strongest possible non-convergence result, i.e., that even weak acyclicity is violated. For this, we provide sharp threshold results where for some τ^* IRD are guaranteed to converge for $\tau \le \tau^*$ and we have non-weak-acyclicity for $\tau > \tau^*$, depending on the underlying graph. See Table 3.1 for an overview of our results. One of our main results, cf. Theorem 3.1, is a proof that IRD in the Swap Schelling Game converge if there are two types of agents and agents are tolerant, that is, $k = 2$ and $\tau \le \frac{1}{2}$, for any underlying connected graph as a residential area. This is in sharp contrast to the convergence results for large τ, i.e., $\tau > \frac{1}{2}$ or more agent types, i.e., $k \ge 3$, cf. Theorem 3.2 and Theorem 3.3. If the underlying graph is regular then IRD convergence is guaranteed for arbitrary k and τ in $O(|E|)$ moves. For the Jump Schelling Game, we exactly characterize when IRD convergence is ensured. In the case of IRD convergence, we show that this happens after $O(|E|)$ many jumps on an underlying graph $G = (V, E)$.

As a further conceptual contribution, we start a discussion about segregation models with more than two agent types. Besides the simple generalization of differentiating only between own type and other types, i.e., the 1-k-SSG, cf. Section 3.2.1, and 1-k-JSG, cf. Section 3.3.1, we propose a more natural alternative, called the 1-1-SSG, cf. Section 3.2.2, and the 1-1-JSG, cf. Section 3.3.2, where agents compare the type ratios only with the largest subgroup in their neighborhood. The idea here is that a minority group mainly cares about if there is a dominant other group within the neighborhood.

Moreover, we investigate the influence of the underlying graph on the hardness of computing an optimal feasible strategy profile, cf. Section 3.4. We show that computing this is NP-hard for arbitrary underlying graphs if $\tau = \frac{1}{2}$, cf. Theorem 3.14 or if τ is close to the maximum degree in the graph, cf. Theorem 3.15. In contrast to this, we provide an efficient algorithm for computing the optimum feasible strategy profile on a 2-regular graph with two agent types, cf. Theorem 3.16. The number of agent types also has an influence: we establish NP-hardness even on 2-regular graphs if there are sufficiently many agent types, cf. Theorem 3.18.

Regarding the influence on the obtained segregation, we present experimental results which measure the obtained segregation on grids with 8-neighborhood, on random unit-disc graphs with an expected vertex degree of 8, and on random 8-regular graphs, cf. Section 3.5. Our experiments reveal that geometry seems to have a significant influence on the segregation strength since the process on random 8-regular graphs yields significantly less segregation than the process on grids or unit-disc graphs. In Chapter 6 we investigate the influence of the underlying topology in more detail for a related process.

Last, we introduce a variant of Schelling's model which takes into account the critical aspect of individual location differentiation which has a significant influence on residential decisions in real life, cf. Section 3.6. Agents want to stay close to their working place or close to their current residence. Hence, we assume that agents have preferences regarding their exact position in the graph. As before, the most important goal of an agent is to live in a neighborhood such that its neighborhood type preference is satisfied. However, the vertices that satisfy this property are no longer all equally good. For example, it may find agents that prefer a central vertex in the graph, while others prefer to live on the edge. We provide first results for IRD convergence for the Swap Schelling Game with two types of agents and show that if the underlying graph is regular and if all agents have a common favorite vertex, cf. Theorem 3.20, or if $\tau \geq \frac{1}{2}$, cf. Theorem 3.21, then convergence is guaranteed.

3.1 Model

We consider Schelling Games with k types and intolerance parameter $\tau \in (0, 1)$. In particular, we investigate the Swap Schelling Game and the Jump Schelling Game in the one-versus-all and one-versus-one versions. In the swap setting agents change their strategy via swapping with another agent while in the jump setting an agent changes its strategy via jumping to an empty vertex. Remember that in the one-versus-all version an agent wants a certain fraction of agents of its own type in its neighborhood, regardless of the specific other types, while in the one-versus-one version an agent is only aware of the largest group of agents with a different type. Note that both versions coincide for $k = 2$. The utility of an agent i in $\boldsymbol{\sigma}$ is defined as $\mathsf{U}_i(\boldsymbol{\sigma}) = \min\left\{1, \frac{f_i(\sigma)}{\tau}\right\}$. For the definition of $f_i(\sigma)$, we refer to Definition 2.1, Definition 2.2 and Definition 2.3. An agent i is content if $\mathsf{U}_i(\boldsymbol{\sigma}) = 1$. For investigating the dynamic properties, we mainly use the potential function $\Phi(\boldsymbol{\sigma}) = \left|\left\{\{u, v\} \in E \mid c(\boldsymbol{\sigma}^{-1}(u)) = c(\boldsymbol{\sigma}^{-1}(v))\right\}\right|$.

3.2 Schelling Dynamics for the Swap Schelling Game

We analyze the convergence behavior of the Swap Schelling Game. Our main goal is to investigate under which conditions a generalized ordinal potential function exists. To this end, we prove for various special cases of the SSG that they are actually generalized ordinal potential games. For this, we analyze the change in the potential function value for a suitably chosen potential function for an arbitrary location swap of two agents i and j. Such a swap changes the current feasible strategy profile $\boldsymbol{\sigma}$ only in the locations of agents i and j and yields a new feasible strategy profile $\boldsymbol{\sigma}_{ij}$. We start with investigating two types of agents which is similar to the original formulation of Schelling's model. We prove initial results, in particular that the 2-SSG converges for $\tau \leq \frac{1}{2}$ on arbitrary graphs. Moreover, we present a matching non-convergence bound for $\tau > \frac{1}{2}$.

▶ **Theorem 3.1.** For $\tau \leq \frac{1}{2}$, any 2-SSG played on an arbitrary graph possesses the FIP. ◀

Proof. We prove the statement by showing that $\Phi(\boldsymbol{\sigma})$ is a generalized ordinal potential function. Remember that $\Phi(\boldsymbol{\sigma})$ is the number of edges of G whose endpoints are occupied by agents of the same color under $\boldsymbol{\sigma}$. First, note that two agents i and j only swap if both agents are discontent and of different types since a swap between agents of the same type cannot be an improvement for at least one of the involved agents. Assume, without loss of generality, that i is orange and j is blue. Let o_i and b_i be the number of orange and blue neighbors of $\boldsymbol{\sigma}(i)$ and o_j and b_j be the number of orange and blue neighbors of $\boldsymbol{\sigma}(j)$, respectively. It holds that

$$\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{o_i}{\deg_{\boldsymbol{\sigma}(i)}} < \tau, \ \mathsf{U}_j(\boldsymbol{\sigma}) = \frac{b_j}{\deg_{\boldsymbol{\sigma}(j)}} < \tau.$$

Since $\tau \leq \frac{1}{2}$ and $\deg_{\boldsymbol{\sigma}(i)} = o_i + b_i$ it follows that $o_i < b_i$. Analogously, we get for agent j that $b_j < o_j$. Thus, $o_i + b_j < b_i + o_j$. If the swap is not local, i.e., agents i and j are not adjacent, this implies for the change in the potential function value that $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) = b_i + o_j - \left(o_i + b_j\right) > 0$. Hence, the number of monochrome edges increases. If the swap is local, it holds that

$$\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = \frac{o_j - 1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(j)}}, \ \mathsf{U}_j(\boldsymbol{\sigma}_{ij}) = \frac{b_i - 1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(i)}} < \tau,$$

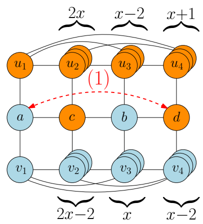
with $1_{ij}(\boldsymbol{\sigma}) = 1$. Thus, this implies for the change in the potential function value that $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) = b_i + o_j - \left(o_i + b_j\right) - 2$. If $b_i - o_i > 1$ or $o_j - b_j > 1$ $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) > 0$. Hence, we are left with the case that $b_i - o_i = 1$ and $o_j - b_j = 1$. Since we consider a profitable swap for agent i and j and we assume $1_{ij}(\boldsymbol{\sigma}) = 1$, it holds that

$$\mathsf{U}_i(\boldsymbol{\sigma}) < \mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = \frac{o_j - 1}{\deg_{\boldsymbol{\sigma}(j)}} = \frac{b_j}{\deg_{\boldsymbol{\sigma}(j)}} = \mathsf{U}_j(\boldsymbol{\sigma}).$$

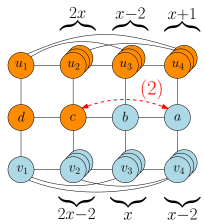
The same is true for agent j

$$\mathsf{U}_j(\boldsymbol{\sigma}) < \mathsf{U}_j(\boldsymbol{\sigma}_{ij}) = \frac{b_i - 1}{\deg_{\boldsymbol{\sigma}(i)}} = \frac{o_i}{\deg_{\boldsymbol{\sigma}(i)}} = \mathsf{U}_i(\boldsymbol{\sigma}).$$

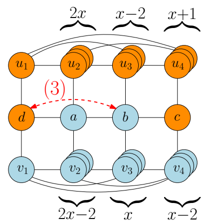
$\mathsf{U}_i(\boldsymbol{\sigma}) < \mathsf{U}_j(\boldsymbol{\sigma}) < \mathsf{U}_i(\boldsymbol{\sigma})$ is clearly a contradiction. Hence, the number of monochrome edges increases with every profitable swap. ■



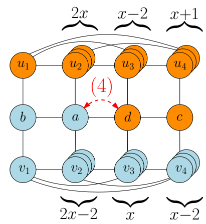
(a) Initial feasible strategy profile



(b) Feasible strategy profile after the first swap



(c) Feasible strategy profile after the second swap



(d) Feasible strategy profile after the third swap

Figure 3.2: An IRC for the 2-SSG with $x = \max\left(\lceil\frac{1}{\tau-0.5}\rceil, \left\lceil\frac{1}{2-2\tau}\right\rceil\right)$ for $\tau \in \left(\frac{1}{2}, 1\right)$. See the proof of Theorem 3.2 for more details. Multiple vertices in series represent a clique of vertices of the stated size. Edges between cliques or between a clique and single vertices represent that all involved vertices are completely interconnected.

We now show that this bound is tight, i.e., that for $\tau > \frac{1}{2}$ IRD may not converge.

▶ **Theorem 3.2.** For $\tau \in \left(\frac{1}{2}, 1\right)$, IRD are not guaranteed to converge in the 2-SSG on arbitrary graphs. Moreover, weak acyclicity is violated. ◀

Proof. We prove the statement by providing an improving response cycle where in every step exactly one improving swap is possible. The construction is shown in Figure 3.2 and we assume that x is sufficiently large, that is,

$$x = \max\left(\left\lceil\frac{1}{\tau - 0.5}\right\rceil, \left\lceil\frac{1}{2 - 2\tau}\right\rceil\right).$$

With $z \in [4]$, the orange agents occupying the set of vertices of u_z, which consist of 1, $2x$, $x - 2$, and $x + 1$ vertices, respectively, and the blue agents occupying the set of vertices of v_z, which consist of 1, $2x - 2$, x and $x - 2$ vertices, respectively, are interconnected and form a clique. During the entire cycle the agents occupying u_z and v_z, respectively, are content. To this end, note that $\lceil\frac{1}{\tau-0.5}\rceil \geq \lceil\frac{1}{2-2\tau}\rceil$ for $0.5 < \tau \leq 0.8\overline{3}$. An orange agent $i \in u_z$ has $4x$ neighbors and at most one neighbor is blue. Hence, for any feasible strategy profile $\boldsymbol{\sigma}$ depicted in Figure 3.2 $f_i(\sigma) = \frac{4x-1}{4x}$ which yields $\mathrm{U}_i(\boldsymbol{\sigma}) = 1$ by our choice of x since $\frac{4x-1}{4x}$ is larger or equal than τ with the corresponding selected x. The same applies to a blue agent $j \in v_z$ who has $4x - 3$ neighbors in total and at most one neighbor is orange. It holds that $\frac{4x-4}{4x-3}$ is larger or equal than τ with the corresponding selected x. Therefore, an agent $i \in u_z$ and an agent $j \in v_z$, respectively, never have the incentive to swap their position with any other agent, since they are content.

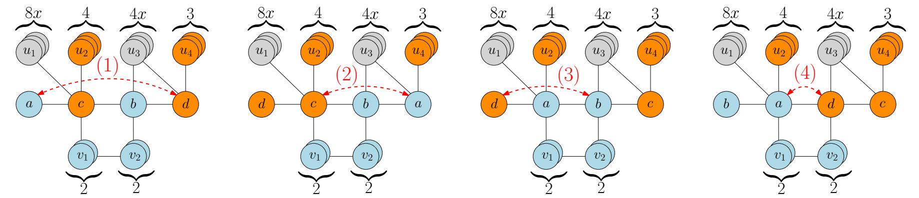
In the initial strategy profile, cf. Figure 3.2 (a), both agents a and d are discontent. By swapping their vertices, agent a can increase its utility from $\frac{1}{3\tau}$ to $\frac{x-1}{2x\tau}$ and agent d increases its utility from $\frac{x+1}{2x\tau}$ to $\min\left(1, \frac{2}{3\tau}\right)$. This is the only possible swap since neither b nor c have the opportunity to improve their utility via swapping with c, d, and a, b, respectively. After the first swap, cf. Figure 3.2 (b), agent a is still not content. A swap with agent c increases agent a's utility to $\frac{2x-1}{4x\tau}$, and agent c can increase its utility from $\frac{2x+1}{4x\tau}$ to $\frac{x+1}{2x\tau}$. Again, no other swap is possible since agent b decreases its utility by swapping with agent c or d. After this, cf. Figure 3.2 (c), agent b and d have the opportunity to swap and increase their utility from $\frac{x+1}{2x\tau}$ to $\min\left(1, \frac{2}{3\tau}\right)$ and $\frac{1}{3\tau}$ to $\frac{x-1}{2x\tau}$, respectively. Once more there is no other profitable swap. Agent a does not want to swap with agent d and agent b not with agent c. Finally, cf. Figure 3.2 (d), agent a and d swap and both agents increase their utility to $\frac{1}{2\tau}$. Neither does agent b want to swap with agent c nor can agent c improve its utility by swapping with agent a. After the fourth step, the obtained strategy profile is equivalent to the initial feasible strategy profile, cf. Figure 3.2 (a), only the blue agents a and b, and the orange agents c and d, respectively, have exchanged positions. Since all the executed swaps were the only possible strategy changes, this proves that the 2-SSG is not weakly acyclic, since, starting with the given initial feasible strategy profile, there is no possibility to reach a swap equilibrium via profitable swaps. ■

3.2.1 IRD Convergence for the One-versus-All Version

The 1-k-variant seems to be a straightforward generalization of the two types case. An agent simply compares the number of neighbors of its own type with the total number of neighbors. Interestingly, our IRD convergence results for the 1-k-SSG with $k > 2$ for arbitrary graphs for $\tau \leq \frac{1}{2}$ are in sharp contrast to the results for $k = 2$: On arbitrary graphs with tolerant agents, i.e., with $\tau \leq \frac{1}{2}$, and $k > 2$ types IRD convergence is no longer guaranteed. This emphasizes the influence of the number of agent types on the convergence behavior of the IRD.

▶ **Theorem 3.3.** IRD are not guaranteed to converge in the 1-k-SSG with $k > 2$ for $\tau \in (0, 1)$ on arbitrary graphs. Moreover, weak acyclicity is violated. ◀

Proof. We give an example of an improving response cycle, where in every step



(a) Initial feasible strategy profile

(b) Feasible strategy profile after the first swap

(c) Feasible strategy profile after the second swap

(d) Feasible strategy profile after the third swap

Figure 3.3: An IRC for the 1-k-SSG with $x > \frac{3}{4\tau} - 1$ for any $\tau \in (0, 0.5]$. Please refer to Theorem 3.3 for more details. Multiple vertices in series represent a clique of vertices of the stated size. Edges between cliques or between a clique and single vertices represent that all involved vertices are completely interconnected.

exactly one improving swap exists, for any $\tau \le 0.5$. Together with the improving response cycle given in Theorem 3.2 for $\tau > 0.5$, this yields the statement.

Consider the instance in Figure 3.3 with a sufficiently high x, that is, $x > \frac{3}{4\tau} - 1$ and let $\tau \le 0.5$. We have $k = 3$ types (orange, blue, gray). With $y \in [4]$ and $z \in [2]$, the agents occupying the set of vertices of u_y, which consist of $8x$, 4, $4x$ and 3 vertices, and the set of vertices of v_z, consisting of two times 2 vertices, respectively, are interconnected and each form a clique. During the entire cycle the agents occupying u_y and v_z, respectively, are content. An agent occupying a vertex of $u_y \cup v_z$ has at most two neighboring agents of different types and at least two agents of its own type. Since $\tau \le 0.5$, these agents are content. Therefore, they have no incentive to swap.

In the initial strategy profile, cf. Figure 3.3 (a), agents a and d are discontent and want to swap. Agent a increases its utility from 0 to $\frac{1}{4(x+1)\tau}$ while agent d is content after the swap, i.e, $\mathrm{U}_j(\boldsymbol{\sigma}_{ij}) = 1$. This is the only possible swap. Agent c does not want to swap with agent a or b since such a swap decreases its utility, as well as agent b cannot improve its utility by swapping with c or d. After the first swap, cf. Figure 3.3 (b), agent a can further increase its utility via a profitable swap with agent c. Such a swap increases the utility of agent a to $\frac{3}{8(x+1)\tau}$, while agent c can improve from $\frac{5}{8(x+1)\tau}$ to $\frac{3}{4(x+1)\tau}$. Again, this is the only possible swap, since agent d is content and c still cannot perform a profitable swap with agent b. After this, cf. Figure 3.3 (c), agent d has no neighbor of its own type, so it swaps with agent b who becomes content. Agent d increases its utility from 0 to $\frac{1}{4(x+1)\tau}$. Agent a cannot perform a profitable swap with agent d and agent b not with agent c since both, b and c, would have no agent of their own type in their neighborhood. Finally, cf. Figure 3.3 (d), agents a and d can swap. Agent d increases its utility to $\frac{1}{2(x+1)\tau}$ and agent a increases its utility from $\frac{3}{8(x+1)\tau}$ to $\frac{1}{2(x+1)\tau}$. No other two agents have any incentive to swap their position, since neither agent c nor d can perform a profitable swap with agent b since they would not have a neighboring agent of their own type. For the same reason agent a is not interested in swapping with agent c. The resulting feasible strategy profile is equivalent to the initial one, only the blue agents a and b and the orange agents c and d exchanged positions.

Since all swaps are the only ones possible, this shows that the 1-k-SSG is not weakly acyclic as there is no possibility to reach a stable feasible strategy profile. ■

On the positive side, we can show that convergence is guaranteed for the 1-k-SSG for any $k \geq 2$ on regular graphs.

▶ **Theorem 3.4.** IRD are guaranteed to converge in $\mathcal{O}(|E|)$ moves for the 1-k-SSG with $\tau \in (0, 1)$ on any regular graph. ◀

Proof. We prove the statement by showing that $\Phi(\boldsymbol{\sigma})$ is a generalized ordinal potential function. Note, that a content agent never has the incentive to swap. Moreover, profitable swaps can only occur between agents of different colors. Since we consider a regular graph $G = (V, E)$, it holds for all $v, w \in V$ $\deg_v = \deg_w = \Delta$.

Consider a swap performed by agents i and j. Assume, without loss of generality, that i is orange and j is blue. Let o_i and b_i be the number of orange and blue neighbors of $\boldsymbol{\sigma}(i)$ and o_j and b_j be the number of orange and blue neighbors of $\boldsymbol{\sigma}(j)$, respectively. Since we consider a profitable swap for agent i and j, it holds that

$$\frac{o_i}{\Delta} < \frac{o_j - 1_{ij}(\boldsymbol{\sigma})}{\Delta} \text{ and } \frac{b_j}{\Delta} < \frac{b_i - 1_{ij}(\boldsymbol{\sigma})}{\Delta}.$$

This implies for the change in the potential function value

$$\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) = b_i + o_j - (o_i + b_j) - 2 \cdot 1_{ij}(\boldsymbol{\sigma}) > 0.$$

Hence, the number of monochrome edges increases. Since $\Phi(\boldsymbol{\sigma}) \leq |E|$ and $\Phi(\boldsymbol{\sigma})$

increases after every swap by at least 1 the IRD ends in a swap equilibrium in $\mathcal{O}(|E|)$ steps. ■

3.2.2 IRD Convergence for the One-versus-One Version

For the 1-1-variant an agent compares the number of neighboring agents of its type with the size of the largest group of agents with a different type in its neighborhood. This captures the realistic setting where agents simply try to avoid being in a neighborhood where another group of agents dominates. We show that even on regular graphs an improving response cycle exists for the 1-1-SSG for sufficiently high τ. We start with a simple positive result.

▶ **Theorem 3.5.** IRD are guaranteed to converge in $\mathcal{O}(n)$ moves for the 1-1-SSG with $\tau \leq \frac{1}{\Delta}$ on any Δ-regular graph. ◀

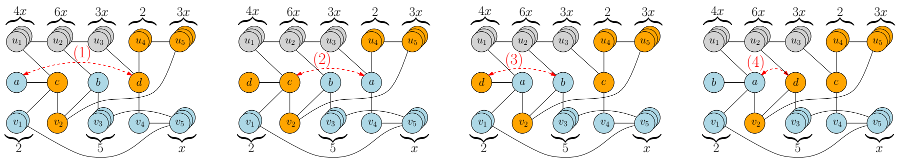
Proof. Any agent i of type t_i who has a neighbor j with the same type t_i is already content, since $\tau \leq \frac{1}{\Delta}$. Hence, neither agent i nor agent j will be involved in a profitable swap. Moreover, it follows that any discontent agent i cannot have a same-type neighbor, that is, $\mathrm{U}_i(\boldsymbol{\sigma}) = 0$. Agent i only swaps vertex occupation with another agent j if it is adjacent to at least one other same-type agent. It follows that $\mathrm{U}_i(\boldsymbol{\sigma}_{ij}) = 1$. Thus, each agent participates in at most one swap, and the game converges after at most n swaps. ■

If τ is high enough, then the 1-1-SSG is no longer a potential game on regular graphs.

▶ **Theorem 3.6.** IRD are not guaranteed to converge for the 1-1-SSG with $\tau \geq \frac{5}{\Delta-1}$ on any Δ-regular graph. ◀

Proof. Consider the instance in Figure 3.4 with $x > \frac{5(1-\tau)}{6\tau}$. We omit the edges between the cliques u_1, u_2, and u_3, of gray agents. Now, the highest degree in the graph is $6(x+1)$, cf. in Figure 3.4 (a) the vertex which is occupied by the blue agent b. To make the graph regular, we insert new vertices filled with agents such that each new agent is the only agent of its type and connect these new vertices with existing vertices and each other as needed.

In the initial feasible strategy profile, cf. Figure 3.4 (a), agents a and d are discontent and want to swap. Agent a increases its utility from 0 to $\frac{1}{(3x+1)\tau}$ while agent d increases its utility from $\frac{2}{(3x+2)\tau}$ to either 1 if $\tau > \frac{1}{2}$ and agent d is thus



(a) Initial feasible strategy profile **(b)** Feasible strategy profile after the first swap **(c)** Feasible strategy profile after the second swap **(d)** Feasible strategy profile after the third swap

Figure 3.4: Multiple vertices in a series represent a clique of vertices of the stated size. Edges between cliques or between a clique and single vertices represent that all involved vertices are completely interconnected. For the proof of Theorem 3.6, we omit the edges between the cliques u_1, u_2 and u_3, of gray agents such that the highest degree in the graph is $6(x+1)$. To make the graph regular, we insert new vertices filled with agents such that each new agent is the only agent of its type and connect these new vertices with existing vertices and each other as needed. For the proof of Theorem 3.7, the figure shows an IRC with exactly one improving swap per step for the 1-1-SSG with $x > \max\left(\frac{5(1-\tau)}{6\tau}, \frac{\tau}{1-\tau}\right)$ for any $\tau \in (0,1)$.

content after the swap or to a utility of $\frac{1}{2\tau}$. After the first swap, cf. Figure 3.4 (b), agent a currently only has one same-type neighbor and $3x$ adjacent gray agents, and, thus, is still discontent since $\frac{1}{3x+1} < \frac{5}{6x+5} \leq \tau$ for $\tau \in (0,1)$. A swap with agent c increases agent a's utility to $\frac{2}{(4x+2)\tau}$ while agent c can improve from $\frac{2}{(4x+2)\tau}$ to $\frac{2}{(3x+2)\tau}$. In the next step, cf. Figure 3.4 (c), agent d has no adjacent agent of its own type. Therefore, it swaps with agent b who becomes content, if $\tau \leq \frac{1}{2}$, after the swap or has a utility equals $\frac{1}{2\tau}$. Agent d increases its utility from 0 to $\frac{1}{(6x+1)\tau}$. Finally, cf. Figure 3.4 (d), agent a and agent d can perform a profitable swap. Agent d has the possibility to increase its utility to $\frac{1}{(4x+1)\tau}$ and agent a can increase its own utility from $\frac{3}{(4x+3)\tau}$ to $\frac{5}{(6x+5)\tau}$.

From $x > \frac{5(1-\tau)}{6\tau}$ and $\Delta = 6(x+1)$, we obtain $\tau \geq \frac{5}{\Delta - 1}$, where equality is reached if x is chosen as low as possible. ■

The situation is even worse on arbitrary graphs as the following theorem shows.

▶ **Theorem 3.7.** IRD are not guaranteed to converge in the 1-1-SSG for $\tau \in (0,1)$ on arbitrary graphs. Moreover, weak acyclicity is violated. ◀

Proof. We show the statement by giving an example of an improving response cycle where in every step exactly one improving swap exists. Consider the instance in Figure 3.4 with $x > \max\left(\frac{5(1-\tau)}{6\tau}, \frac{\tau}{1-\tau}\right)$. We have three types of agents, orange, blue and gray. With $z \in [5]$, the agents occupying the vertices the set of vertices of u_z, which consists of $4x$, $6x$, $3x$, 2 and $3x$ vertices, and the agents occupying the set of vertices of v_z, which consist of 2, 1, 5, 1 and x vertices, respectively, are interconnected and each form a clique. During the entire cycle the agents occupying u_z and v_z, respectively, are content. The orange agent occupying vertex v_2 has at most 2 neighbors of any type other than orange and at least $3x$ neighbors of its own orange type. All the other agents occupying vertex $v \in u_z \cup v_z$ have at most one neighbor of a different type and at least x adjacent agents of their own type. Therefore, an agent i with $\boldsymbol{\sigma}(i) \in u_z \cup v_z$ has no incentive to swap since $\mathsf{U}_i(\boldsymbol{\sigma}) = 1$ by our choice of x.

In the initial feasible strategy profile, cf. Figure 3.4 (a), agent a and d are discontent and want to swap. Agent a increases its utility from 0 to $\frac{1}{(3x+1)\tau}$ while agent d is content after the swap. This is the only possible swap. Agent c cannot perform a profitable swap with agent a or b and agent b cannot improve by swapping with d. After the first swap, cf. Figure 3.4 (b), agent a is still discontent. A swap with agent c increases its utility to $\frac{2}{(4x+2)\tau}$, and agent c can improve its utility from $\frac{2}{(4x+2)\tau}$ to $\frac{2}{(3x+2)\tau}$. Again, this is the only possible swap, since agent d is content and agent c decreases its utility by swapping with agent b. After this, cf. Figure 3.4 (c), agent d has no adjacent agent of its own type. Therefore, d swaps with agent b who becomes content while agent d increases its utility from 0 to $\frac{1}{(6x+1)\tau}$. Agent a won't perform a swap with agent d since $\mathsf{U}_a(\boldsymbol{\sigma}_{ad}) = 0$ and agent b cannot perform a profitable swap with agent c since this wouldn't be an improvement for b. Finally, cf. Figure 3.4 (d), agent a and agent d swap. Agent d increases its utility to $\frac{1}{(4x+1)\tau}$ and agent a increases its utility from $\frac{3}{(4x+3)\tau}$ to $\frac{5}{(6x+5)\tau}$. No other two agents have the incentive to swap their vertices. We end up in a feasible strategy profile that is equivalent to the initial one, with only the blue agents a and b and the orange agents c and d exchanged positions. Since all swaps were the only ones possible, this proves that the 1-1-SSG is not weakly acyclic as there is no possibility to reach a swap equilibrium via profitable swaps. ■

3.3 Schelling Dynamics for the Jump Schelling Game

We now analyze the convergence behavior of IRD for the strategic segregation process via jumps. We show that convergence is not guaranteed on arbitrary graphs and prove that the threshold for convergence on Δ-regular graphs is at $\tau = \frac{2}{\Delta}$.

We first turn our focus to the 1-k-JSG, where an agent only distinguishes between its own and other types. Hence, an agent simply compares the number of neighbors of its own type with the total number of neighbors.

3.3.1 IRD Convergence for the One-versus-All Version

In the following, we prove a sharp threshold result, with the threshold being at $\tau = \frac{2}{\Delta}$, for the convergence of IRD for the 1-k-JSG on Δ-regular graphs, for any $\Delta \geq 2$. Moreover, we show that the game is not weakly acyclic on arbitrary graphs.

▶ **Theorem 3.8.** IRD are guaranteed to converge in $O(|E|)$ steps for the 1-k-JSG with $\tau \leq \frac{2}{\Delta}$ on any Δ-regular graph. ◀

Proof. For any Δ-regular graph $G = (V, E)$ with $\frac{1}{2} - \frac{1}{2\Delta} < c < \frac{1}{2}$, we define the weight $w_{\boldsymbol{\sigma}}(e)$ of any edge $e = \{u, v\} \in E$ as:

$$w_{\boldsymbol{\sigma}}(e) = \begin{cases} 1, & \text{if } c(\boldsymbol{\sigma}^{-1}(u)) \neq c(\boldsymbol{\sigma}^{-1}(v)), \\ c, & \text{if } c(\boldsymbol{\sigma}^{-1}(u)) = \ominus \text{ and } c(\boldsymbol{\sigma}^{-1}(v)) = i \text{ with } i \in [k] \text{ or} \\ & \quad c(\boldsymbol{\sigma}^{-1}(u)) = i \text{ with } i \in [k] \text{ and } c(\boldsymbol{\sigma}^{-1}(v)) = \ominus, \\ 0, & \text{otherwise.} \end{cases}$$

We prove that $\Psi(\boldsymbol{\sigma}) = \sum_{e \in E} w_{\boldsymbol{\sigma}}(e)$ is a generalized ordinal potential function. Note that τ is sufficiently small, such that an agent is content if it has at least two neighbors of its own type. Therefore, an agent who is willing to jump to another vertex has at most one neighbor of the same type. Without loss of generality, we assume the existence of a discontent agent i under a feasible strategy profile $\boldsymbol{\sigma}$. Let $a_{\boldsymbol{\sigma}}$ be the number of adjacent agents to i with the same type like agent i under $\boldsymbol{\sigma}$ and let $b_{\boldsymbol{\sigma}}$ be the number of adjacent agents to i with a different type from agent i under $\boldsymbol{\sigma}$. Moreover, let $\varepsilon_{\boldsymbol{\sigma}}$ be the number of empty vertices in the neighborhood of $\boldsymbol{\sigma}(i)$. Let $a_{\boldsymbol{\sigma}_i}$ and $b_{\boldsymbol{\sigma}_i}$ be the number of agents of the same type and different type, respectively, under $\boldsymbol{\sigma}_i$ and let ε_{σ_i} be the number of empty vertices in the neighborhood of $\boldsymbol{\sigma}_i(i)$. We show that if an agent jumps, Ψ changes such it holds that

$$\begin{aligned}\Psi(\boldsymbol{\sigma}) - \Psi(\boldsymbol{\sigma}_i) =& \big(0a_\sigma + 1b_\sigma + c\varepsilon_\sigma + ca_{\sigma_i} + cb_{\sigma_i} + 0\varepsilon_{\sigma_i}\big)\\ &- \big(c(a_\sigma + b_\sigma) + 0\varepsilon_\sigma + 0a_{\sigma_i} + 1b_{\sigma_i} + c\varepsilon_{\sigma_i}\big)\\ =& - ca_\sigma + (1-c)b_\sigma + c\varepsilon_\sigma + ca_{\sigma_i} + (c-1)b_{\sigma_i} - c\varepsilon_{\sigma_i} > 0,\end{aligned}$$

and therefore Ψ decreases with every improving jump of an agent as we prove below.

First, note that there is no incentive for agent i to decrease the number of same-type neighbors since decreasing it would mean that either $a_\sigma \geq 2$, i.e., agent i is content and has no incentive to jump, or $a_\sigma = 1$ and thus $a_{\sigma_i} = 0$ which cannot be an improvement since in this case $\mathrm{U}_i(\boldsymbol{\sigma}_i) = 0$. Hence, we have to distinguish between two cases:

If $a_\sigma < a_{\sigma_i}$, then agent i increases the number of neighbors of its own type. Since we consider a Δ-regular graph, we have $a_\sigma + b_\sigma + \varepsilon_\sigma = a_{\sigma_i} + b_{\sigma_i} + \varepsilon_{\sigma_i} = \Delta$ and therefore $b_\sigma = \Delta - a_\sigma - \varepsilon_\sigma$ and $b_{\sigma_i} = \Delta - a_{\sigma_i} - \varepsilon_{\sigma_i}$. Hence,

$$\begin{aligned} & - ca_\sigma + (1-c)b_\sigma + c\varepsilon_\sigma + ca_{\sigma_i} + (c-1)b_{\sigma_i} - c\varepsilon_{\sigma_i}\\ = & - ca_\sigma + (1-c)(\Delta - a_\sigma - \varepsilon_\sigma) + c\varepsilon_\sigma + ca_{\sigma_i} + (c-1)(\Delta - a_{\sigma_i} - \varepsilon_{\sigma_i}) - c\varepsilon_{\sigma_i}\\ = & - ca_\sigma + (1-c)(-a_\sigma - \varepsilon_\sigma) + c\varepsilon_\sigma + ca_{\sigma_i} + (c-1)(-a_{\sigma_i} - \varepsilon_{\sigma_i}) - c\varepsilon_{\sigma_i}\\ = & - ca_\sigma - a_\sigma - \varepsilon_\sigma + ca_\sigma + c\varepsilon_\sigma + c\varepsilon_\sigma + ca_{\sigma_i} - ca_{\sigma_i} - c\varepsilon_{\sigma_i} + a_{\sigma_i} + \varepsilon_{\sigma_i} - c\varepsilon_{\sigma_i}\\ = & (2c-1)\varepsilon_\sigma + (1-2c)\varepsilon_{\sigma_i} - a_\sigma + a_{\sigma_i}\\ \geq & (2c-1)\varepsilon_\sigma - a_\sigma + a_{\sigma_i},\end{aligned}$$

since $1 - 2c > 0$ and $\varepsilon_{\sigma_i} \geq 0$. If $\varepsilon_\sigma = 0$, we obtain $(2c-1)\varepsilon_\sigma - a_\sigma + a_{\sigma_i} = -a_\sigma + a_{\sigma_i} > 0$. If $\varepsilon_\sigma > 0$, we have

$$(2c-1)\varepsilon_\sigma - a_\sigma + a_{\sigma_i} > \left(2\left(\frac{1}{2} - \frac{1}{2\Delta}\right) - 1\right)\varepsilon_\sigma - a_\sigma + a_{\sigma_i} = \frac{-\varepsilon_\sigma}{\Delta} - a_\sigma + a_{\sigma_i} \geq 0,$$

since $\frac{\varepsilon_\sigma}{\Delta} \leq 1 \leq a_{\sigma_i} - a_\sigma$.

If $a_\sigma = a_{\sigma_i}$, then the number of same type neighbors of agent i stays the same. Since i improves its utility the number of neighbors of i with a different type has to decrease and therefore $b_{\sigma_i} < b_\sigma$. We denote the difference as β with

$b_\sigma = b_{\sigma_i} + \beta$. With $\beta > 0$ and since we consider a Δ-regular graph, it follows that $\varepsilon_{\sigma_i} = \varepsilon_\sigma + \beta$. Hence,

$$\begin{aligned}
& -ca_\sigma + (1-c)b_\sigma + c\varepsilon_\sigma + ca_{\sigma_i} + (c-1)b_{\sigma_i} - c\varepsilon_{\sigma_i} \\
= & -ca_\sigma + (1-c)(b_{\sigma_i} + \beta) + c\varepsilon_\sigma + ca_{\sigma_i} + (c-1)b_{\sigma_i} - c(\varepsilon_\sigma + \beta) \\
= & -ca_\sigma + (1-c)\beta + ca_{\sigma_i} - c\beta \\
= & \ (1-c)\beta - c\beta \\
= & \ (1-2c)\beta > 0,
\end{aligned}$$

where the second to last equality holds since $a_\sigma = a_{\sigma_i}$.

Since $\Psi(\boldsymbol{\sigma}) \leq |E|$ and $\Psi(\boldsymbol{\sigma})$ decreases after every jump by at least $(1-2c)$ the IRD find an equilibrium in $\mathcal{O}(|E|)$. ■

Actually, Theorem 3.8 is tight and convergence is not guaranteed if $\tau > \frac{2}{\Delta}$.

▶ **Theorem 3.9.** IRD are not guaranteed to converge in the 1-k-JSG for $\tau > \frac{2}{\Delta}$ on Δ-regular graphs. ◀

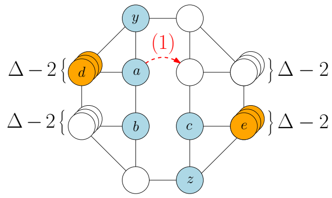
Proof. We prove the statement by providing an improving response cycle. See Figure 3.5. If we have more than two types of different agents, all agents of types dissimilar from orange and blue can be placed outside of the neighborhood of the agents a, b and c who are involved in the IRC.

Let $\tau > \frac{2}{\Delta}$. In the initial feasible strategy profile, cf. Figure 3.5 (a), agent a is discontent and has a utility of $\frac{2}{\Delta\tau}$. By jumping next to agent c it becomes content. Because of this jump, agent b is now isolated, cf. Figure 3.5 (b). Jumping next to the agents d and y increases its utility from 0 to $\frac{1}{(\Delta-1)\tau}$. After the second step, the obtained feasible strategy profile is equivalent to the initial feasible strategy profile, cf. Figure 3.5 (c). Hence, the next two jumps from agents c and a are similar to the first two: First, agent c jumps next to agent b such that c is content, then agent a jumps next to agents c and z to avoid an isolated position. We end up with a feasible strategy profile equivalent to the initial one. ■

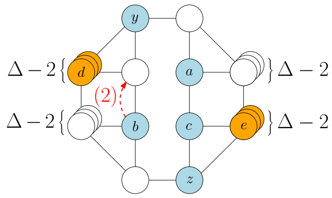
If the underlying graph is an arbitrary graph the situation is even worse.

▶ **Theorem 3.10.** IRD are not guaranteed to converge in the 1-k-JSG for $\tau \in (0, 1)$ on arbitrary graphs. Moreover, weak acyclicity is violated. ◀

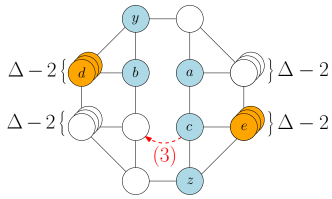
Proof. We show the statement by giving an example of an improving response cycle where in every step exactly one agent has exactly one improving jump.



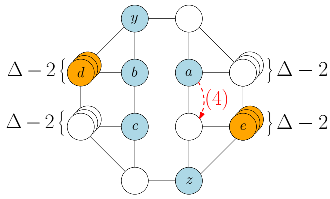
(a) Initial feasible strategy profile



(b) Feasible strategy profile after the first jump



(c) Feasible strategy profile after the second jump

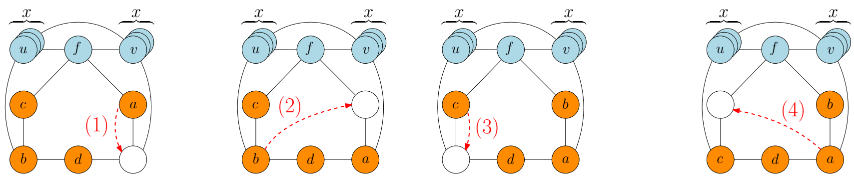


(d) Feasible strategy profile after the third jump

Figure 3.5: An IRC for the JSG for $\tau > \frac{2}{\Delta}$ on a Δ-regular graph. See the proof of Theorem 3.9 for more details. Empty vertices are white. Multiple vertices in series represent a clique of $\Delta - 2$ vertices. An edge between a clique and a single vertex denotes that each clique vertex is connected to that single vertex. An edge between two cliques represents that each clique vertex has exactly one neighbor in the other clique. With this, the graph is indeed Δ-regular: Each vertex is connected to all vertices of exactly one group of size $\Delta - 2$ and to two other vertices.

Consider the instance in Figure 3.6. We assume that x is sufficiently high, e.g., $x > \max\left(\frac{2}{\tau}, \frac{1}{1-\tau}\right)$. If we have more than two different types of agents, all agents of types dissimilar to orange and blue can be placed in cliques outside of the neighborhood of all the agents involved in the IRC. If these cliques are placed inside graph components that are neither connected to the IRC vertices, nor to each other, the agents of these types never become discontent. Hence, the jumps of the given IRC are the only ones possible. In our construction we have four orange agents, a, b, c, d, and $2x+1$ blue agents which occupy the vertices in the vertex sets u and v and vertex f. Moreover, we have one empty vertex. All vertices which are occupied by the blue agents are interconnected and each forms a clique.

During the entire cycle, all blue agents are content. A blue agent i has $2x+1$ and $2x+2$ neighbors, respectively, of whom at least $2x$ are of the same blue type.



(a) Initial feasible strategy profile **(b)** Feasible strategy profile after the first jump **(c)** Feasible strategy profile after the second jump **(d)** Feasibel strategy profile after the third jump

Figure 3.6: An IRC with exactly one improving jump per step for the JSG for $x > \max(\frac{2}{\tau}, \frac{1}{1-\tau})$ for any $\tau \in (0, 1)$. See the proof of Theorem 3.10 for more details. The empty vertex is white. Multiple vertices in a series represent a clique of vertices of the stated size. Edges between cliques or between a clique and single vertices represent that all involved vertices are completely interconnected.

Hence, for any feasible strategy profile $\boldsymbol{\sigma}$ depicted in Figure 3.6 $\mathrm{U}_i(\boldsymbol{\sigma}) = 1$ and thus i has no incentive to jump to another currently empty vertex. Also, the orange agent d remains content during the entire cycle since it is never isolated and has never an adjacent agent of a different type.

In the initial feasible strategy profile, cf. Figure 3.6 (a), the orange agent a is discontent, since its only adjacent agent f is blue. Therefore, a jumps to the empty vertex. Agent b and, depending on the value of τ, agent c is discontent. However, jumping to the empty vertex next to agent d is not an improvement for them. After the first jump, cf. Figure 3.6 (b), agent b is discontent, since x is chosen sufficiently high such that $\mathrm{U}_b(\boldsymbol{\sigma}_a) < 1$. Hence, jumping to the empty vertex next to agent a improves the utility of b from $\frac{2}{(x+2)\tau}$ to $\min(1, \frac{1}{2\tau})$. Again, this is the only valid jump, since agent c would still have exactly one blue agent and one orange agent in its neighborhood by jumping next to agent a. After two further jumps, cf. Figure 3.6 (c) and Figure 3.6 (d), by the agents c and a, which are equivalent to those shown in Figure 3.6 (a) and Figure 3.6 (b), we obtain a feasible strategy profile which is equivalent to the initial one. Since all executed jumps were the only ones possible, this shows that the JSG is not weakly acyclic as there is no possibility to reach an equilibrium via improving jumps. ■

3.3.2 IRD Convergence for the One-versus-One Version

Now we turn our focus to the 1-1-JSG. By using the same proof as in Theorem 3.5 with jumps instead of swaps, we get the following positive result.

▶ **Theorem 3.11.** IRD are guaranteed to converge in $\mathcal{O}(n')$ moves for the 1-1-JSG with $\tau \leq \frac{1}{\Delta}$ on any Δ-regular graph. ◀

The same IRC which proves Theorem 3.9 for the 1-k-JSG yields the next result.

▶ **Theorem 3.12.** IRD are not guaranteed to converge in the 1-1-JSG for $\tau > \frac{2}{\Delta}$ on Δ-regular graphs. ◀

Finally, the proof of Theorem 3.10 works for the following result as well.

▶ **Theorem 3.13.** IRD are not guaranteed to converge in the 1-1-JSG for $\tau \in (0, 1)$ on arbitrary graphs. Moreover, weak acyclicity is violated. ◀

3.4 Computational Complexity

We now investigate the computational hardness of computing an optimal feasible strategy profile with respect to the number of content agents, i.e., a feasible strategy profile where as many agents as possible are content.

3.4.1 Hardness Properties for Two Types of Agents

We start with two types of agents and show that finding an optimal feasible strategy profile for the 2-SSG in an arbitrary graph G is NP-hard by giving a reduction from the Balanced Satisfactory Problem (BSP), which was introduced in [GK98; GK00] and proven to be NP-hard in [BTV06]. This result directly implies that finding an optimal feasible strategy profile for the balanced 2-JSG with no empty vertices is NP-hard as well.

▶ **Theorem 3.14.** Finding a feasible strategy profile which maximizes the number of content agents for the balanced 2-SSG on an arbitrary graph G is NP-hard for $\tau = \frac{1}{2}$. ◀

Proof. We prove the statement by giving a reduction from the BSP. Given a graph $G = (V, E)$ with an even number of vertices. Let $v \in V$ and $V' \subseteq V$. We denote by $\deg_{V'}(v)$ the number of vertices in V' which are adjacent to v. A balanced satisfactory partition exists if there is a non-trivial partition V_1, V_2 of the vertices V with $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ and $|V_1| = |V_2|$ such that each vertex $v \in V_i$ with $i \in \{1, 2\}$ has at least $\deg_{V_i}(v) \geq \frac{\deg_v}{2}$, i.e., each vertex has at least as many neighbors in its own part as in the other. Let $\boldsymbol{\sigma}^*$ be a feasible strategy profile that maximizes the number of content agents. If such a partition exists, we can find it by computing $\boldsymbol{\sigma}^*$ in the graph G for two different types of agents of size $\frac{|V|}{2}$ and $\tau = \frac{1}{2}$.

Obviously, a feasible strategy profile $\boldsymbol{\sigma}$ without discontent agents is maximal concerning the number of content agents. For a content agent i we have

$$\frac{|N(\boldsymbol{\sigma}(i)) \cap C_i(\boldsymbol{\sigma})|}{\deg_{\boldsymbol{\sigma}(i)}} \geq \frac{1}{2} = \tau,$$

and thus, since there are not empty vertices,

$$|N(\boldsymbol{\sigma}(i)) \cap C_i(\boldsymbol{\sigma})| \geq \frac{\deg_{\boldsymbol{\sigma}(i)}}{2}.$$

If we have a feasible strategy profile where all agents are content we can gather all vertices which are occupied by orange agents to the subset V_1 and all agents which are occupied by blue agents to the subset V_2. It holds for every agent i that

$$\deg_{V_i}(\boldsymbol{\sigma}^*(i)) = |N(\boldsymbol{\sigma}(i)) \cap C_i(\boldsymbol{\sigma})| \geq \frac{\deg_{\boldsymbol{\sigma}(i)}}{2}.$$

Hence, calculating an optimal feasible strategy profile must be NP-hard. ■

The above proof relies on the fact that there are no empty vertices. The computational hardness of the JSG changes if many empty vertices exist. Obviously, it is easy to find an optimal feasible strategy profile if there are enough empty vertices to separate both types of agents and a suitable separator is known. Mapping the boundary for the transition from NP-hardness to efficient computation is a challenging question for future work.

Next, we show that finding an optimal feasible strategy profile is hard for high τ via a reduction from Minimum Cut Into Equal Size (MCIES), which was proven to be NP-hard in [GJS76].

▶ **Theorem 3.15.** Finding a feasible strategy profile which maximizes the number of content agents in the balanced 2-SSG on an arbitrary graph G is NP-hard for $\tau > \frac{3\Delta}{3\Delta+1}$. ◀

Proof. We prove the statement by giving a reduction from MCIES. Given a graph $G = (V, E)$ and an integer $W \in \mathbf{N}$. MCIES is the decision whether there is a non-trivial partition V_1, V_2 with $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ and $|V_1| = |V_2|$ such that

$|\{\{v_1, v_2\} \in V \mid v_1 \in V_1, v_2 \in V_2\}| \leq W$, i.e., there are at most W edges between the two parts.

We create a graph $G' = (V', E')$ in which every vertex $v \in V$ is replaced by a clique C_v in G' of size $3\Delta + 1$. We replace each edge $\{u, v\} \in E$ by an edge $\{u', v'\}$ between two vertices $u' \in C_u$ and $v' \in C_v$ such that each vertex in G' has at most one neighbor outside its clique. Therefore, the degree of vertices in G' is either 3Δ or $3\Delta + 1$, and thus, the maximum vertex degree $\Delta_{G'}$ in G' is $3\Delta + 1$. We have two different agent types, each consisting of $|V'|/2$ agents. Let

$$\tau > \frac{\Delta_{G'} - 1}{\Delta_{G'}} = \frac{3\Delta}{3\Delta + 1}.$$

Hence, an agent is content in G' if it has no neighbors of a different type. For a feasible strategy profile $\boldsymbol{\sigma}^*$ in G' to maximize the number of content agents, all cliques C have to be uniform, i.e., assign agents of the same type to each vertex in C. Otherwise, another non-uniform clique C' has to exist and we can re-assign the agents in both cliques in a feasible strategy profile $\boldsymbol{\sigma}$ to make C uniform. In $\boldsymbol{\sigma}^*$ all agents of both cliques are discontent, while under $\boldsymbol{\sigma}$ at least $2\Delta + 1$ agents in C that have no neighbors outside C are content. Since each clique is only connected to at most Δ other vertices, at most 2Δ agents are discontent under $\boldsymbol{\sigma}$ that were content in $\boldsymbol{\sigma}^*$. Therefore, $\boldsymbol{\sigma}^*$ cannot be optimal.

If we have a feasible strategy profile that maximizes the number of content agents with $2W'$ discontent agents, we can gather all $v \in V$ where C_v is occupied by orange agents into V_1, and similarly into V_2 for blue agents. We then have W' edges between the two sets V_1 and V_2. Hence, a feasible strategy profile with $2W'$ discontent agents correspond to an MCIES with $W = W'$ edges between the partitions and vice versa. ■

For the above theorems, we use a welfare function that counts the number of discontent agents. However, we remark that even if we use the social welfare, the above hardness results still hold. This relates to the hardness results from Agarwal et al. [Aga+21], which hold for the JSG with $\tau = 1$ in the presence of stubborn agents who are unwilling to move.

We contrast the above results by providing an efficient algorithm for computing an optimal feasible strategy profile for the 2-SSG and the 2-JSG on a 2-regular graph by employing a well-known dynamic programming algorithm for Subset Sum [Cor+09; GJ79].

▶ **Theorem 3.16.** Finding a feasible strategy profile which maximizes the number of content agents in the 2-SSG on a 2-regular graph G can be done in $O(n^2)$ for $\tau > \frac{1}{2}$. ◀

Proof. Let $G = (V, E)$ be a 2-regular graph, consisting of r rings. Ring i has r_i vertices. Let o be the number of orange agents and b be the number of blue agents.

For finding a feasible strategy profile that maximizes the number of content agents, we take the multiset $r_1, \ldots, r_m$ as elements and o as target sum as an instance of Subset Sum. We can solve this in $O(n^2)$ since $o \leq n$. In case of a Yes-instance, we can place the orange agents on the rings indicated by the selected elements. Thus no agents of different types are on the same ring. If the instance is a No-instance, then in the optimal feasible strategy profile there is exactly one ring with agents of a different type. This implies that at least 3 and at most 4 agents are discontent. To check if an optimal feasible strategy profile with 3 discontent agents is possible, we solve the Subset Sum instance with target sum $o + 1$. If this is possible, then we place the o orange agents on the respective rings such that exactly one vertex is empty. Then all empty vertices are filled with blue agents. If the instance with target sum $o + 1$ is a No-instance, we greedily fill the rings with consecutive orange agents such that we get one ring with empty vertices. Then, we fill all the empty vertices with blue agents to obtain exactly 4, discontent agents. ■

Optimal feasible strategy profiles for the JSG can be found with an analogous algorithm. Note that we can pack the empty vertices as a barrier between blue and orange agents to reduce the number of discontent agents.

▶ **Theorem 3.17.** Finding a feasible strategy profile which maximizes the number of content agents in the 2-JSG on a 2-regular graph G can be done in $O(n^2)$ for $\tau > \frac{1}{2}$. ◀

3.4.2 Hardness Properties for More Types of Agents

Compared to the previous subsection, we now show that also the number of different agent types influences the computational hardness of finding an optimal feasible strategy profile concerning the number of content agents. We establish NP-hardness even on 2-regular graphs if there are sufficiently many agent types, by giving a reduction from 3-Partition, which was proven to be NP-hard in [GJ79].

▶ **Theorem 3.18.** Finding a feasible strategy profile that maximizes the number of content agents in the balanced 1-1-SSG and balanced 1-k-SSG on a 2-regular graph with $\tau > \frac{1}{2}$ is NP-hard. ◀

Proof. We prove the statement by giving a polynomial time reduction from 3-Partition. Given a multiset S of $3k$ positive integers. 3-Partition concerns whether S can be partitioned into k disjoint sets S_i with $i \in \{1, \ldots, k\}$ of size three, such that the sum of the numbers in each subset is equal, i.e.,

$$\sum_{s_i \in S_1} s_i = \sum_{s_i \in S_2} s_i = \cdots = \sum_{s_i \in S_k} s_i .$$

As these sets are disjoint, we already know that each of them sums up to $\frac{\sum_{s_i \in S} s_i}{k}$. 3-Partition keeps its NP-hardness if the integers in S are encoded unary. Moreover, it remains NP-hard if we assume for all $s_i \in S$

$$\frac{\sum_{s_i \in S} s_i}{4k} < s_i < \frac{\sum_{s_i \in S} s_i}{2k}.$$

Based on a 3-Partition instance, we generate a 2-regular graph, containing a ring for each $s_i \in S$ with s_i vertices. Thus our graph has $n = \sum_{s_i \in S} s_i$ vertices in total. We can assume $s_i \geq 3$ for all $s_i \in S$, since adding a constant to all elements does not change the existence of a solution.

Note that each type consists of $\frac{n}{k}$ agents. Let $\boldsymbol{\sigma}^*$ be a feasible strategy profile without discontent agents for $\tau > \frac{1}{2}$. Hence, no ring contains agents of different types, since an agent is discontent if it has an adjacent agent of a different type. Thus, we have a disjoint partitioning of the rings, such that the number of vertices in each partition adds up to

$$\frac{n}{k} = \frac{\sum_{s_i \in S} s_i}{k}.$$

Furthermore, we assumed that $\frac{n}{4k} < s_i < \frac{n}{2k}$, thus all agents of a type t_i have to be placed on exactly three rings. This directly implies a solution for the 3-Partition instance. If the corresponding 3-Partition instance has a solution $S_1, \ldots, S_k$, this produces a partitioning of the rings, such that each partition contains

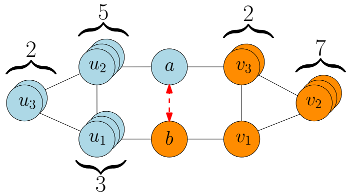
$$\frac{\sum_{s_i \in S} s_i}{k} = \frac{n}{k}$$

vertices. Placing the agent types according to this partitioning won't produce any ring with agents of different types on it. Such a feasible strategy profile does not contain any discontent agent and has to be optimal. Since our reduction can be done in polynomial time for unary encoded instances of 3-PARTITION, this proves the NP-hardness of finding an optimal feasible strategy profile. ■

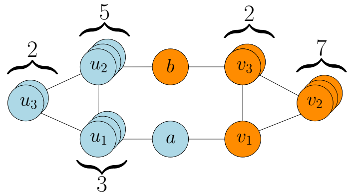
Kreisel et al. [Kre+22] showed that deciding the existence of a feasible strategy profile with certain minimum social welfare is NP-hard. This is in line with our hardness results for computing socially optimal states that maximize the number of content agents. To conclude the section on computational hardness, we want to emphasize that solving the question of whether finding a feasible strategy profile that maximizes the number of content agents is easy or hard does not allow us to make equivalent statements for computing stable feasible strategy profiles. The following example illustrates the rather counter-intuitive fact that a feasible strategy profile that maximizes the number of content agents is not necessarily stable. However, Kreisel et al. [Kre+22] prove that deciding the existence of a swap equilibrium and a jump equilibrium for two types of agents is NP-hard as well. In the case of equilibrium existence, it remains open how hard finding an integrated one is.

▶ **Theorem 3.19.** For the 2-SSG there is a graph G where no feasible strategy profile which maximizes the number of content agents is stable. ◀

Proof. We prove the statement by giving an example. Consider Figure 3.7. The pictured graph has two cliques u_i and v_i with $i \in [3]$ of size 10. Let $\tau > 0.9$. The feasible strategy profile $\boldsymbol{\sigma}$ depicted in Figure 3.7 (a) has 7 discontent agents, and the feasible strategy profile $\boldsymbol{\sigma}_{ab}$ in Figure 3.7 (b) has 8 discontent agents. The former is optimal since every feasible strategy profile $\boldsymbol{\sigma}'$ other than the given two has to place agents of different types in at least one of the cliques. This would cause all agents in the clique to become discontent and thus yield more than 10 discontent agents. However, the agents a and b want to swap under the feasible strategy profile $\boldsymbol{\sigma}$. Hence, the unique feasible strategy profile which maximizes the number of content agents $\boldsymbol{\sigma}$ is not stable. ■



(a) feasible strategy profile which maximizes the number of content agents σ



(b) feasible strategy profile σ_{ab} after the swap

Figure 3.7: A graph where the feasible strategy profile which maximizes the number of content agents σ is not stable for $\tau > 0.9$. See the proof of Theorem 3.19 for more details. Multiple vertices in a series represent a clique of vertices of the stated size. Edges between cliques or between a clique and single vertices represent that all involved vertices are completely interconnected.

3.5 Empirical Study of the Segregation Strength

Most of the literature focuses on the segregation strength of the stable states. Thus quantifying the segregation of stable feasible strategy profiles via a suitable segregation measure is important. Many such measures have been proposed, see e.g. the survey by Massey & Denton [MD88], but most of them are restricted to grids or are not compatible with more than two agent types.

To evaluate the magnitude of segregation in our experiments, we calculate the *Freeman Segregation Index (FSI)* [Fre78]. It is defined as $\frac{E[X]-X}{E[X]}$, with

$$X = \sum_{i\in[n']} |N(\sigma(i)) \setminus C_i(\sigma)|$$

being a random variable, representing twice the number of edges between agents of different types and $E[X]$ being the expected value of X for a random feasible strategy profile. By definition, the FSI is upper bounded by 1 for high segregation and is expected to be 0 for a random feasible strategy profile. We use the FSI since it can be applied to general graphs, it is easy to interpret, and can be extended to multiple types of agents straightforwardly. We adjust the calculation of $E[X]$

for an arbitrary set of k types as

$$\begin{aligned}
E[X] &= \sum_{i\in[n']} E[|N(\boldsymbol{\sigma}(i)) \setminus C_i(\boldsymbol{\sigma})|] \\
&= \sum_{i\in[n']} \sum_{t_j\in\mathbf{t}} E[|N(\boldsymbol{\sigma}(i)) \setminus C_i(\boldsymbol{\sigma})| \mid i \in t_j] \cdot \Pr\left[i \in t_j\right] \\
&= \sum_{i\in[n']} \sum_{t_j\in\mathbf{t}} |N(\boldsymbol{\sigma}(i))| \frac{n' - |t_j|}{n' - 1} \cdot \frac{|t_i|}{n'} \\
&= \sum_{i\in[n']} |N(\boldsymbol{\sigma}(i))| \sum_{t_j\in\mathbf{t}} \frac{|t_j|(n' - |t_j|)}{n'(n' - 1)}.
\end{aligned}$$

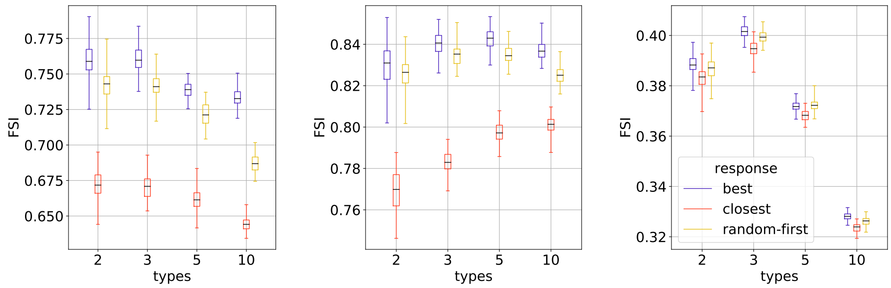
Note that the inner sum is a constant, independent of the agent i. Thus, even in the jumping game it only has to be computed once.

We find that geometry seems to have a significant influence on the segregation strength while the specific choice of the model seems to have little influence, although in the one-versus-one version agents tend to be happier. In the following, we explain our experimental setup and then go into detail about our observations.

Experimental Setup For our simulations, we consider three different graph topologies: toroidal grids with the Moore neighborhood, i.e., the vertices have diagonal edges and all inner vertices have degree 8, random 8-regular graphs and random unit-disc graphs with expected degree 8.

We generate grids with 100×100 up to 300×300 vertices where the grid sides increase in steps of 20. To have comparable random 8-regular graphs we generate them with the same number of vertices. For each configuration, we run the IRD starting from 100 random initial feasible strategy profiles to derive the results.

To get the initial feasible strategy profiles, the agents are placed uniformly at random on the vertices of the graph and we assume equal proportions of each agent type, i.e., that the game is balanced. For the jump game, we use 1% empty vertices. In each round the discontent agents are activated in random order and each activated agent iterates randomly over all possible locations for a swap or a jump and chooses the vertex which maximizes its utility ("best"), the first one

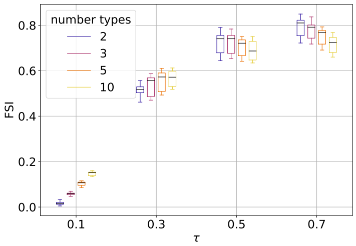


(a) grid **(b)** unit disc **(c)** 8-regular

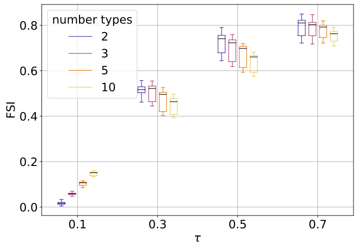
Figure 3.8: Segregation of 1-k-*SSG* for different response strategies on several graphs with several number of types for $\tau = 0.5$. Please refer to Section 3.5.1 for more details.

which yields an improvement ("random-first") or the improving vertex which is closest to the current vertex ("closest").

From our perspective all three variants of how agents find a new location have a good motivation. Clearly, finding the best vertex, so ideally a vertex where the agent becomes content, is the highest goal for an agent. However, also the two other variants are plausible. People are lazy and do not want to or maybe need more time or opportunity to check all possible vertices and then decide which one is the best. Therefore moving to the first vertex which improves the agent's situation is a realistic scenario. Schelling [Sch71] mentioned a limit on travel distance. He motivated this by the idea that agents may become unable to move to where their demands are satisfied. We think that this is a nice idea since people have restrictions in their everyday life like the place of their work, schools, family, and friends. Hence, they cannot or do not want to move far away from their current position. In contrast to Schelling, where an agent does not move at all if there is no improvement possible in a defined radius, an agent in our implementation moves even if the new vertex is far away, but only if there is no closer vertex which would yield an improvement. In Chapter 4 we investigate the influence of such a restriction, that is, agents are restricted to local movements.



(a) 1-*k*-*SSG*



(b) 1-1-*SSG*

Figure 3.9: Segregation for different τ and different numbers of types in 1-k-*SSG* and 1-1-*SSG*. Please refer to Section 3.5.1 for more details.

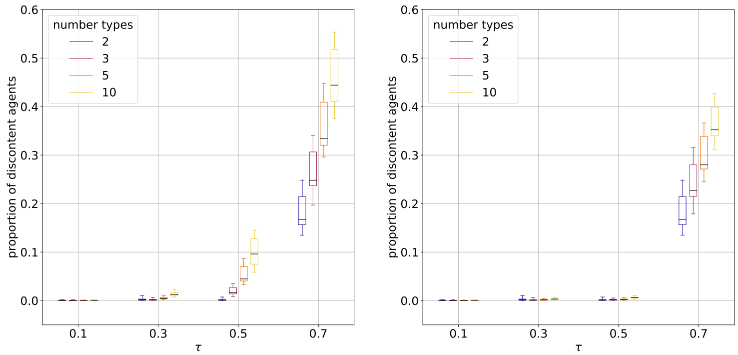
3.5.1 Segregation

We only depict two main results. Figure 3.8 shows that without geometry, i.e., on a random 8-regular graph, the stable states are much less segregated. This is consistent with our findings in Chapter 6 for a related process where our results indicate that the underlying graph's cluster structure significantly impacts the obtained segregation strength.

Consider Figure 3.9, which shows a comparison of multiple simulations concerning the FSI. Note that if we consider two agent types, the 1-k-SSG and the 1-1-SSG are identical and are the 2-SSG. The results confirm the visual impression. It is not surprising that the level of segregation increases with higher τ whereas the number of types just has a small impact. However, we cannot see any differences among the different variants besides the small observation that for $t = 0.3$ in the 1-1-SSG the FSI decreases with a greater number of types. This similarity of 1-k-SSG and 1-1-SSG is very interesting and counter-intuitive. In 1-1-SSG agents are satisfied in neighborhoods where they would be discontent in the 1-k-SSG since they don't necessarily compare themselves with their whole neighborhood. For 1-k-JSG and 1-1-JSG, the results look very similar.

3.5.2 Number of Discontent Agents

Furthermore, we investigate the number of discontent agents in the final feasible strategy profile. We measure the final proportion of discontent agents. Figure 3.10 shows a comparison of the proportion of discontent agents in 1-k-SSG and 1-1-SSG. As expected, this proportion increases in both cases with τ, especially



(a) 1-*k*-*SSG* **(b)** 1-1-*SSG*

Figure 3.10: Proportion of discontent agents for different τ and different numbers of types in 1-k-SSG and 1-1-SSG. Please refer to Section 3.5.2 for more details.

for a high number of types. However, in contrast to the FSI, the social welfare behave differently for 1-k-SSG and 1-1-SSG. As we increase τ and the number of types, the proportion of discontent agents grows faster for 1-k-SSG. This fits our intuition since, in the 1-1-SSG, agents only compare with the largest subgroup in their neighborhood. Therefore, generally, they are satisfied with a lower absolute number of neighbors of their own type and thus they are more likely to be content, even with a higher number of types and higher τ. For 1-k-JSG and 1-1-JSG, the results look very similar.

3.6 Agents with Location Preferences

An interesting direction is to generalize Schelling's model such that agents have preferences over the different locations in the residential area, i.e., agents strive for being close to their favorite vertex, additionally to their preferences over their neighborhood. Let π_i be the preferred vertex of agent i. Thus, the utility function of our agents is based on two main assumptions:

(1) An agent's high-priority goal is to find a location where it is content in terms of the neighborhood type ratio.
(2) An agent's low-priority goal is to find a location that is as close as possible to its favorite vertex.

Hence, a content agent i strives for locations where it is content, but as close as possible to π_i. This introduces the critical aspect of individual location differentiation which has a significant influence on residential decisions in real life. People have preferences about where they live, depending on, for example, their job, important amenities, and personal preferences. Agarwal et al. [Aga+21] incorporate as well the idea that agents may have preferences over locations. However, instead of assuming that optimizing the distance to the preferred location is the secondary goal, the authors introduce stubborn agents which stay at their chosen vertex irrespective of their surrounding agents.

We incorporate our assumptions as follows in our utility function for the 2-SSG and 2-JSG, respectively,

$$\mathrm{U}_i(\boldsymbol{\sigma}) = \left(\min\left\{1, \frac{f_i(\sigma)}{\tau}\right\}, D(G) - \mathrm{dist}_G(\boldsymbol{\sigma}(i), \pi_i)\right).$$

Note that the utility function is now a vector instead of a single value. We choose the lexicographic order $\leq_{lex}$ [3] for comparing utility vectors. Agents want to maximize their utility vector lexicographically, i.e., it is more important for an agent to be content than to be close to its favorite vertex.[4]

Note, that location preferences have a severe impact on the properties of the 2-SSG and 2-JSG, respectively. See Figure 3.11 which gives an example that our potential function $\Phi(\boldsymbol{\sigma})$, used, e.g., in the proof of Theorem 3.1 to show convergence for the 2-SSG for $\tau \leq \frac{1}{2}$ on arbitrary graphs, breaks. Clearly, any feasible strategy profile $\boldsymbol{\sigma}$ which is stable under the 2-SSG and 2-JSG with vertex preferences is also stable for the 2-SSG and 2-JSG itself, respectively. The converse is not true. In the following, we provide first insights on the properties of the 2-SSG with vertex preferences.

If all agents have the same favorite vertex, i.e., $\pi_i = \pi_j$ for $i, j \in [n]$, the 2-SSG behaves nicely on regular graphs.

► **Theorem 3.20.** The 2-SSG with common favorite vertex preferences, i.e., $\pi_i = \pi_j$ for $i, j \in [n]$, with $\tau \in (0, 1)$, is a potential game on any regular graph. ◄

3 $(\alpha, \beta) <_{lex} (\gamma, \delta)$, if $\alpha < \gamma$ or $\alpha = \gamma$ and $\beta < \delta$. $(\alpha, \beta) =_{lex} (\gamma, \delta)$, if $\alpha = \gamma$ and $\beta = \delta$. $(\alpha, \beta) >_{lex} (\gamma, \delta)$, if $\alpha > \gamma$ or $\alpha = \gamma$ and $\beta > \delta$.

4 We add the diameter $D(G)$ into the utility vector to enforce that an agent strives for maximizing both entries, instead of maximizing its utility while minimizing the distance.

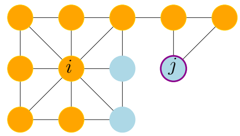


Figure 3.11: Consider the two agents i and j for $\tau \in (0, 1)$. Let the current vertex $\boldsymbol{\sigma}(j)$ of j be the favorite vertex $\pi_i = \pi_j = \boldsymbol{\sigma}(j)$ of agent i and agent j, respectively. A swap is an improving move for both agents, since agent i increases its utility from $\left(\min\left\{1, \frac{6}{8\tau}\right\}, 1\right)$ to $(1, 4)$ and agent j increases its utility from $(0, 4)$ to $\left(\min\left\{1, \frac{2}{8\tau}\right\}, 1\right)$. However, the change in potential function value $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) = 2 + 2 - 6 - 0 = -2 < 0$, i.e., the number of monochrome edges decreases. Please refer to Section 3.6 for more details.

Proof. Let Δ be the degree of the vertices in the regular graph G. We prove the statement by showing that $\Phi(\boldsymbol{\sigma})$ is a generalized ordinal potential function. Consider a swap performed by agents i and j. If i and j are both discontent we already showed in the proof of Theorem 3.4 that the value of the potential function increases. Note, moreover, that if i and j are both content no profitable swap is possible since it wouldn't be profitable for one of the involved agents concerning its distance to its favorite vertex. Hence, without loss of generality, we are left with the case that the orange agent i is content while the blue agent j is discontent. Let o_i and b_i be the number of orange and blue neighbors of $\boldsymbol{\sigma}(i)$ and o_j and b_j be the number of orange and blue neighbors of $\boldsymbol{\sigma}(j)$, respectively. Since agent i is content, i only swaps if a swap reduces its distance to its favorite vertex π_i. Thus, agent j increases the distance to $\pi_j = \pi_i$ and it holds that $\frac{b_j}{\Delta} < \frac{b_i - 1_{ij}(\boldsymbol{\sigma})}{\Delta}$. Since $\Delta = o_i + b_i = o_j + b_j$ it follows that $o_i < o_j - 1_{ij}(\boldsymbol{\sigma})$. This implies

$$\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) = b_i + o_j - (o_i + b_j) - 2 \cdot 1_{ij}(\boldsymbol{\sigma}) > 0.$$

Hence, the number of monochrome edges increases. ■

If we do not make any assumptions on the choice of the favorite vertices, we can show for large τ that the 2-SSG is guaranteed to converge on regular graphs.

▶ **Theorem 3.21.** The 2-SSG with vertex preferences and $\tau \geq \frac{1}{2}$ is a potential game on any regular graph. ◀

Proof. We prove the theorem by showing that

$$\Psi(\boldsymbol{\sigma}) = \left(\Phi(\boldsymbol{\sigma}), \sum_{i\in[n]} D(G) - \mathrm{dist}_G(\boldsymbol{\sigma}(i), \pi_i)\right)$$

is a generalized ordinal potential function. We show that if the agents i and j swap, the value of the potential function Ψ increases lexicographically. Let Δ be the degree of the vertices in the regular graph G. From the proof of Theorem 3.20 we already know that whenever a swapping agent increases the first entry of its utility vector $\Phi(\boldsymbol{\sigma})$, and therefore $\Psi(\boldsymbol{\sigma})$, increases as well. Hence, we are left with the case that the agents i and j solely swap to get closer to their favorite vertices π_i and π_j, i.e., $\mathrm{dist}_G(\boldsymbol{\sigma}(i), \pi_i) > \mathrm{dist}_G(\boldsymbol{\sigma}_{ij}(i), \pi_i)$ and $\mathrm{dist}_G(\boldsymbol{\sigma}(j), \pi_j) > \mathrm{dist}_G(\boldsymbol{\sigma}_{ij}(j), \pi_j)$, respectively. In this case, it holds for the change in the value of the potential function that $\Psi(\boldsymbol{\sigma}_{ij}) - \Psi(\boldsymbol{\sigma})$

$$\begin{aligned}
&= \Big(\bullet, 2D(G) - \mathrm{dist}_G(\boldsymbol{\sigma}_{ij}(i), \pi_i) - \mathrm{dist}_G(\boldsymbol{\sigma}_{ij}(j), \pi_j) - \\
&\quad \big(2D(G) - \mathrm{dist}_G(\boldsymbol{\sigma}(i), \pi_i) - \mathrm{dist}_G(\boldsymbol{\sigma}(j), \pi_j)\big)\Big) \\
&= \big(\bullet, \mathrm{dist}_G(\boldsymbol{\sigma}(i), \pi_i) - \mathrm{dist}_G(\boldsymbol{\sigma}_{ij}(i), \pi_i) + \mathrm{dist}_G(\boldsymbol{\sigma}(j), \pi_j) - \mathrm{dist}_G(\boldsymbol{\sigma}_{ij}(j), \pi_j)\big) \\
&> (\bullet, 0)
\end{aligned}$$

To show that Ψ increases lexicographically, we have to show that in this case $\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}_{ij})$. If both agents i and j are of the same type, a swap doesn't change the number of monochrome edges, thus, $\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}_{ij})$. Assume, without loss of generality, agent i is orange and agent j blue. If both agents are content and $\tau > \frac{1}{2}$, the swap between both agents cannot be profitable since the two agents involved would become discontent. In the case of $\tau = \frac{1}{2}$, if both agents are content and, moreover, can perform a profitable swap, a swap doesn't change the number of monochrome edges since both neighborhoods must look the same, thus, $\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}_{ij})$. Hence, assume, without loss of generality, that the orange agent i is discontent. Agent i solely swaps to get closer to π_i if the fraction of adjacent same-type agents in its neighborhood does not decrease. Since we consider regular graphs this is equivalent to that the number of adjacent same-type agents does not decrease for both involved agents which implies that the number of monochrome edges does not decrease, i.e., $\Phi(\boldsymbol{\sigma}) \leq \Phi(\boldsymbol{\sigma}_{ij})$. ■

For tolerant agents, we can show convergence for the 2-SSG on rings.

▶ **Theorem 3.22.** The 2-SSG with vertex preferences and $\tau < \frac{1}{2}$ is a potential game on rings. ◀

Proof. We use an argument similar to the one in the proof of Theorem 3.21 and use the same generalized ordinal potential function $\Psi(\boldsymbol{\sigma})$. The case where at least one discontent agent is involved in the swap is analogous to the proof of Theorem 3.21. Thus, we are left to consider the case where two content agents i and j swap. Two content agents swap if and only if they remain content after the swap and if they get closer to their favorite vertices, i.e., $\text{dist}_G(\boldsymbol{\sigma}(i), \pi_i) > \text{dist}_G(\boldsymbol{\sigma}_{ij}(i), \pi_i)$ and $\text{dist}_G(\boldsymbol{\sigma}(j), \pi_j) > \text{dist}_G(\boldsymbol{\sigma}_{ij}(j), \pi_j)$, respectively. Similar to the proof of Theorem 3.21 it holds in this case that $\Psi(\boldsymbol{\sigma}_{ij}) - \Psi(\boldsymbol{\sigma}) > (\,\bullet\,, 0)$. We show that after such a swap $\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}_{ij})$ which implies that Ψ increases lexicographically. If both agents i and j are of the same type, a swap doesn't change the number of monochrome edges, thus, $\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}_{ij})$. If two content agents i and j of different types perform a profitable swap, then both of them must have exactly one adjacent agent of their own type and one of the other type, respectively. Moreover, agent i and j cannot be neighbors since otherwise, both would be discontent after the swap. Thus, also in this case, the number of monochrome edges does not change. ■

3.7 Conclusion and Open Problems

We conducted a thorough analysis of the dynamic properties of the first truly game-theoretic version of Schelling's segregation model where agents choose strategically their location and provided tight threshold results for the IRD convergence for several versions of the game. Furthermore, we found that the number of agent types and the underlying graph has a severe impact on the computational hardness of computing optimal feasible strategy profiles.

It remains open whether IRD always converge for the 1-1-SSG with $\tau \in \left(\frac{1}{\Delta}, \frac{5}{\Delta-1}\right)$, and for the 1-1-JSG with $\tau \in \left(\frac{1}{\Delta}, \frac{2}{\Delta}\right)$ on regular graphs. Since most versions are not guaranteed to converge via IRD, the existence of stable feasible strategy profiles for all graph types is not given. Agarwal et al. [Aga+21] showed for the 1-k-JSG that stable feasible strategy profiles exist if the underlying graph is a star or a graph with maximum degree 2 and $\tau = 1$. Furthermore, they proved that if the underlying graph is a tree the existence of stable feasible strategy profiles may fail to exist for $\tau = 1$ in the 1-k-JSG. However, in general, it remains an open question in terms of different values of τ and for different underlying

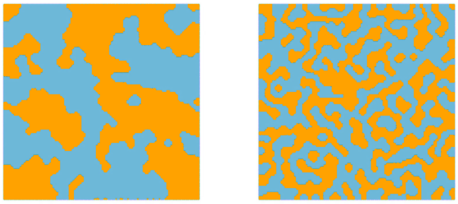


Figure 3.12: An instance of a Swap Schelling Game with two different types, with $o = b = 5\,000$ and $\tau = \frac{1}{2}$. Note, that we start with an initial feasible strategy profile of the agents, similar to the one depicted in Figure 3.1. Left: A sample swap equilibrium of the 2-SSG with common favorite vertex preferences, i.e., $\pi_i = \pi_j$ for $i, j \in [n']$. Right: A sample swap equilibrium of the 2-SSG with vertex preferences chosen uniformly at random. The favorite vertices of the agents are chosen uniformly at random.

graphs whether stable feasible strategy profiles exist and whether they can be computed efficiently. We conjecture the following:

▶ **Conjecture 3.23.** Equilibria are not guaranteed to exist in all cases for which we constructed IRCs. ◀

Also, the computational hardness of finding optimal feasible strategy profiles for some variants deserves further study and could be extended to study the existence of other interesting states, e.g., stable states with low segregation.

We emphasize that there are many possible ways to model Schelling segregation with at least three agent types. For example, types could have preferences over other types which then yields a rich unexplored setting. [KKV22] introduced a generalization of Schelling Games considering such an ordering of the types where agents are tolerant to other agents even if they are not of the same type if they are close enough.

Moreover, it remains open how vertex preferences of the agents change the properties of different variants of the Schelling game. We provide some preliminary results for the swap version. However, the jump version with location preferences remains completely open. We emphasize that there are again many ways to model vertex preferences. For example, an ordering over the vertices is also conceivable here.

Last, an ambitious endeavor is to prove bounds on the size of the monochrome regions similar to the works [BEL16; BEL14; Bra+12; Imm+17]. In particular, it would be interesting to explore the impact of location preferences on the induced stable feasible strategy profile. Preliminary experimental results show that different location preferences indeed may have an impact on the size of the monochrome regions and the segregation, cf. Figure 3.12.

4 Topological Influence and Locality in Swap Schelling Games

This chapter is based on joint work with Davide Bilò, Vittorio Bilò, and Pascal Lenzner [Bil+22b]. Moreover, I want to thank one anonymous reviewers who spotted an error in one of the statements and gave helpful comments for fixing it.

In the following chapter, we deepen the investigations of Chapter 3 for Swap Schelling Games. To this end, we follow the model of Agarwal et al. [Aga+21], that is, we consider Swap Schelling Games with $\tau = 1$, where the focus of our analysis is on the influence of the underlying graph and on the impact of restricting the agents to perform only local swaps.

In particular, we investigate the influence of the given topology that models the residential area on core game-theoretic questions like the existence of equilibria and game dynamics, the Price of Anarchy, and the Price of Stability. We thereby focus on popularly studied topologies like grids, (almost) regular graphs, paths, and cycles which were used in many empirical studies that simulated Schelling's process, see, e.g., [BEL16; BEL14; BMR14; Imm+17; OF18a; OF18b]. Concerning the existence of equilibria and game dynamics, cf. Section 4.2, and the Price of Anarchy, cf. Section 4.3, see Table 4.1 and Table 4.2 for a detailed result overview. Moreover, a more condensed overview of the achieved asymptotic bounds on the Price of Anarchy can be found in Table 4.3 in Section 4.5. Furthermore, we also show how our existential results can be used to derive some non-trivial upper bounds for the Price of Stability in both the general and the local version of the model. The characterization of the Price of Stability in Swap Schelling Games is quite a challenging task and very few results are currently known, see the discussion in Section 4.4.

While in [Aga+21] it was proven that equilibria may fail to exist for arbitrary underlying graphs and in Theorem 3.4 equilibrium existence was shown for regular graphs, we extend and refine these results by investigating almost regular graphs as well as paths, 4-grids and 8-grids. We establish equilibrium existence for all these graph classes and all our results yield polynomial time algorithms for computing an equilibrium. Moreover, we study the Price of Anarchy in-depth.

Table 4.1: Result overview. We investigate the existence of equilibria and the finite improvement property for arbitrary k, except for 8-grids where we focus on $k = 2$. The "✓" symbol denotes that the respective property holds. Note that a "✓" in the "k-SSG" column implies a "✓" in the local k-SSG column. The "×" symbol denotes that equilibrium existence is not guaranteed and that an IRC exists, respectively. We denote with "1-regular" almost regular graphs.

	Equilibrium Existence		Finite Improvement Property	
graph classes	k-SSG	local k-SSG	k-SSG	local k-SSG
arbitrary	× ([Aga+21])		× ([Aga+21])	
regular	✓(Thm. 3.4)	✓(Thm. 3.4)	✓(Thm. 3.4)	✓(Thm. 3.4)
1-regular	✓(Thm. 4.4)	✓(Thm. 4.4)	✓(Thm. 4.4)	✓(Thm. 4.4)
trees	× ([Aga+21])	✓(Thm. 4.7)	× ([Aga+21])	
cycles	✓(Thm. 3.4)	✓(Thm. 3.4)	✓(Thm. 3.4)	✓(Thm. 3.4)
paths	✓(Thm. 4.4)	✓(Thm. 4.4)	✓(Thm. 4.4)	✓(Thm. 4.4)
4-grids	✓(Thm. 4.5)	✓(Thm. 4.5)	✓(Thm. 4.5)	✓(Thm. 4.5)
8-grids	✓(Thm. 4.11) $k = 2$	✓(Thm. 4.11) $k = 2$	× (Thm. 4.10)	✓(Thm. 4.9)

Since it was shown in [Aga+21] that the Price of Anarchy can be unbounded for $k \geq 3$, we focus on the Price of Anarchy of the (local) 2-SSG.

We give tight or almost tight bounds to the Price of Anarchy for all mentioned graph classes which in many cases are significant improvements on the $\Theta(n)$ bound proven in [Aga+21]. In particular, for arbitrary graphs, cf. Section 4.3.1, we improve the upper bound for balanced games. This result is obtained as a corollary, cf. Corollary 4.13, of a more general upper bound of $O\left(\frac{b}{o}\right)$ to the Price of Anarchy, see Theorem 4.12, which implies that for instances that do admit swap equilibria, we always have a constant Price of Anarchy whenever none of the two parties forms a clear majority. We also provide an upper bound of $O(\frac{\Delta}{\delta})$ to the Price of Anarchy for general graphs that do admit swap equilibria. This result is obtained by using advanced matching techniques, that are further explored to provide tight bounds to the Local Price of Anarchy for the class of regular and also non-regular graphs. Notably, this result implies non-trivial upper bounds to the Price of Anarchy for graphs with a large minimum degree or bounded degree. We believe that our advanced matching techniques still have the potential to be successfully applied for future refinements to the Price of Anarchy bounds.

Table 4.2: Results overview. For the study of the PoA we focus on $k = 2$. We denote by b and o the number of blue and orange agents, respectively and we assume $o \leq b$. If we use ζ or η in the respective bound, their meaning is defined at the top of the respective column. ϵ is a constant larger than zero.

	Price of Anarchy			
	2-SSG		local 2-SSG	
		$o = 2\zeta + \eta$		$n = 3\zeta + \eta$
arbitrary	∞ ([Aga+21])	$o = 1$	$\left(2n + \frac{8}{n} - 8, 2n - \frac{8}{n}\right)$ (Thm. 4.15)	$o = \frac{n}{2}$
	≤ 3 (Thm. 4.12)	$o = \frac{n}{2}$	$\leq 2\left(1 + \frac{\Delta-1}{\delta-1}\right)$ (Thm. 4.16)	$\delta \geq 2$
	$\leq \frac{no(n-o)-n}{o(o-1)(n-o)}$ (Thm. 4.12)	otherwise	$\left(\frac{\Delta(\Delta-1)}{2} - \epsilon, 2(\Delta^2+1)\right)$ (Thm. 4.17)	$\Delta \leq n-2$
regular	$2 + \frac{1}{\zeta}$ (Cor. 4.20, Thm. 4.21)	$\Delta \in (2\zeta, 2\zeta+1)$	$2 + \frac{1}{\zeta}$ (Cor. 4.20, Thm. 4.21)	$\Delta \in (2\zeta, 2\zeta+1)$
trees	$\left(\frac{\Delta(\Delta-1)}{2} - \epsilon, 2(\Delta^2+1)\right)$ (Cor. 4.18, Thm. 4.17)	$\Delta \leq n-2$	$\left(\frac{\Delta(\Delta-1)}{2} - \epsilon, 2(\Delta^2+1)\right)$ (Cor. 4.18, Thm. 4.17)	$\Delta \leq n-2$
cycles	1 (Thm. 4.22)	$o = 1$	1 (Thm. 4.23)	$o = 1$
	$\frac{n-2}{b+\eta}$ (Thm. 4.22)	otherwise	$\frac{n-2}{b-o}$ (Thm. 4.23)	$o \geq 2, b \geq 2o$
			$\frac{n-2}{\zeta+\eta}$ (Thm. 4.23)	otherwise
paths	∞ (Thm. 4.24)	$n = 3$	∞ (Thm. 4.25)	$n = 3$
	$\frac{2n-2}{2n-5}$ (Thm. 4.24)	$n > 3, o = 1$	$\frac{2n-2}{2n-5}$ (Thm. 4.25)	$n > 3, o = 1$
	$\frac{n-1}{b+1+\eta}$ (Thm. 4.24)	$n > 3, o \geq 2,$ $\eta \leq 2\zeta + 1$	$\frac{n-1}{b-o-1}$ (Thm. 4.25)	$n > 3, o \geq 2, b \geq 2o$
	$\frac{n-1}{b+\eta}$ (Thm. 4.24)	otherwise	$\frac{n-1}{\zeta}$ (Thm. 4.25)	otherwise
4-grids	$\frac{25}{22}$ (Prop. 4.26)	$o = 1$	$(3-\epsilon, 3)$ (Prop. 4.31)	$2 \times h$ grid, $h \geq 3$
	2 (Thm. 4.28, 4.30)	otherwise	$\left(\frac{18}{7} - \epsilon, \frac{18}{7}\right)$ (Prop. 4.32)	$3 \times h$ grid, $h \geq 3$
			$\left(\frac{5}{2} - \epsilon, \frac{5}{2} + \epsilon\right)$ (Thm. 4.33)	$l \times h$ grid, $h, l \geq 8 + \frac{20}{\epsilon}$
8-grids	$\frac{897}{704}$ (Prop. 4.34)	$o = 1$	$\leq \frac{9}{4} + \epsilon$ (Prop. 4.36)	$l \times h$ grid, $h, l \geq 8 + \frac{18}{\epsilon}$
	≤ 4 (Thm. 4.35)	otherwise	≤ 4 (Thm. 4.35)	otherwise

Moreover, we follow up on a proposal by Schelling [Sch71] to restrict the movement of agents locally and study the influence of this restriction. Such local swaps are realistic since people want to stay close to their working place or important facilities like schools. This also holds when considering dynamics where agents repeatedly perform local moves since these dynamics can be understood as a process that happens over a long period and agents adapt to their new neighborhoods over time. Thus, besides analyzing equilibria in the general model of Agarwal et al. [Aga+21], we introduce and analyze a local variant of the model, which, to the best of our knowledge, has not yet been explored for Schelling's model. Our results indicate that the local variant has favorable properties. For instance, equilibria are guaranteed to exist on trees in the local version while in [Aga+21] it was shown that this is not the case for the general model. Moreover, for many cases, we can show that the Price of Anarchy in the local version deteriorates only slightly compared to the global version.

4.1 Model

We consider the Swap Schelling Game with k types in the one-versus-all version with an intolerance parameter $\tau = 1$. The utility of an agent i in $\boldsymbol{\sigma}$ is defined as $\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{|N(\sigma(i)) \cap C_i(\boldsymbol{\sigma})|}{\deg_{\sigma(i)}}$. Furthermore, we restrict strategy changes and investigate local swaps. Remember that the set of local swap equilibria is a superset of the set of swap equilibria. See Example 4.1 for an illustration of the (local) k-SSG.

► **Example 4.1.** Consider Figure 4.1. There are $n = 24$ strategic agents with $k = 3$ types (orange, blue and green) placed on a 4-grid with $l = 4$ rows and $h = 6$ columns. The game is not balanced since $|t_{blue}| = |t_{orange}| = 10$ but $|t_{green}| = 4$. Agent i occupies vertex u and agent j occupies vertex v, hence $1_{ij}(\boldsymbol{\sigma}) = 1$. $\boldsymbol{\sigma}$ is a local swap equilibrium but not a swap equilibrium since agent j, occupying vertex v can swap with agent j' occupying vertex w to increase its utility from $\frac{1}{3}$ to $\frac{1}{2}$, while j' can improve its utility from $\frac{1}{2}$ to $\frac{2}{3}$. ◄

For investigating the dynamic properties, we recall the function $\Phi(\boldsymbol{\sigma}) = \left|\left\{\{u,v\} \in E \mid c(\boldsymbol{\sigma}^{-1}(u)) = c(\boldsymbol{\sigma}^{-1}(v))\right\}\right|$ which counts the number of monochrome edges. However, we will see that potential-preserving profitable swaps exist. For analyzing such swaps, we consider the *extended potential* $\Psi(\boldsymbol{\sigma})$ which essentially

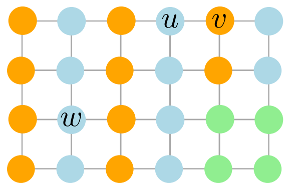


Figure 4.1: Example of a strategy profile $\boldsymbol{\sigma}$ in the (local) k-SSG. Please refer to Example 4.1 for more details.

is $\Phi(\boldsymbol{\sigma})$ augmented with a tie-breaker. It is defined as $\Psi(\boldsymbol{\sigma}) = (\Phi(\boldsymbol{\sigma}), n - z(\boldsymbol{\sigma}))$, where $z(\boldsymbol{\sigma})$ is the number of agents having utility 0 under $\boldsymbol{\sigma}$. We compare Ψ for different strategy profiles $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ lexicographically, i.e., on the one hand we have $\Psi(\boldsymbol{\sigma}) > \Psi(\boldsymbol{\sigma}')$ if $\Phi(\boldsymbol{\sigma}) > \Phi(\boldsymbol{\sigma}')$, or $\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}')$ and $z(\boldsymbol{\sigma}) < z(\boldsymbol{\sigma}')$. On the other hand, we have $\Psi(\boldsymbol{\sigma}) < \Psi(\boldsymbol{\sigma}')$ if $\Phi(\boldsymbol{\sigma}) < \Phi(\boldsymbol{\sigma}')$, or $\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}')$ and $z(\boldsymbol{\sigma}) > z(\boldsymbol{\sigma}')$. Note that any profitable swap which increases (decreases) the potential Φ also increases (decreases) the extended potential Ψ.

4.2 Equilibrium Existence and Dynamics

We start by providing a precise characterization that ties equilibria in both local and general 2-SSGs with the sum of the utilities experienced by any two agents of different colors.

▶ **Lemma 4.2.** A strategy profile $\boldsymbol{\sigma}$ for a (local) 2-SSG is an equilibrium if and only if for any two agents i and j, with $c(i) \neq c(j)$ and $\deg_{\boldsymbol{\sigma}(i)} \leq \deg_{\boldsymbol{\sigma}(j)}$, that are allowed to swap their positions.[5] $\mathrm{U}_i(\boldsymbol{\sigma}) + \mathrm{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(i)}}$. ◀

Proof. Without loss of generality, assume that i is orange and j is blue. Let o_i be the number of orange neighbors of $\boldsymbol{\sigma}(i)$ and b_j be the number of blue neighbors of $\boldsymbol{\sigma}(j)$, respectively.[6] It holds that

$$\mathrm{U}_i(\boldsymbol{\sigma}) = \frac{o_i}{\deg_{\boldsymbol{\sigma}(i)}}, \ \mathrm{U}_j(\boldsymbol{\sigma}) = \frac{b_j}{\deg_{\boldsymbol{\sigma}(j)}}$$

5 This is a restriction only for the local version of the game, where i and j have to be neighboring vertices to perform a local swap.

6 Clearly, in the local version of the game $1_{ij}(\boldsymbol{\sigma}) = 1$.

and

$$\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = \frac{\deg_{\boldsymbol{\sigma}(j)} - b_j - 1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(j)}} = 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(j)}} - \mathsf{U}_j(\boldsymbol{\sigma}),$$

$$\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) = \frac{\deg_{\boldsymbol{\sigma}(i)} - o_i - 1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(i)}} = 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(i)}} - \mathsf{U}_i(\boldsymbol{\sigma}).$$

Consider the case in which there exists a $k \in \{i, j\}$ such that $\mathsf{U}_k(\boldsymbol{\sigma}) \geq \mathsf{U}_k(\boldsymbol{\sigma}_{ij})$. By substituting the formula corresponding to $\mathsf{U}_k(\boldsymbol{\sigma}_{ij})$ and by rearranging the terms, using also the fact that $\deg_{\boldsymbol{\sigma}(k)} \geq \deg_{\boldsymbol{\sigma}(i)}$, we obtain

$$\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(k)}} \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(i)}}.$$

In the complementary case in which $\mathsf{U}_k(\boldsymbol{\sigma}) < \mathsf{U}_k(\boldsymbol{\sigma}_{ij})$ for every $k \in \{i, j\}$, from $\mathsf{U}_i(\boldsymbol{\sigma}) < \mathsf{U}_i(\boldsymbol{\sigma}_{ij})$ we derive

$$\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) < 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(i)}}.$$

Therefore,

$$\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(i)}}$$

iff $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_{ij})$ or $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \mathsf{U}_j(\boldsymbol{\sigma}_{ij})$ holds. As $\boldsymbol{\sigma}$ is an equilibrium iff for all i, j that are allowed to swap $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \mathsf{U}_i(\boldsymbol{\sigma}_{ij})$ or $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \mathsf{U}_j(\boldsymbol{\sigma}_{ij})$ holds, we have that $\boldsymbol{\sigma}$ is an equilibrium iff for all i, j that are allowed to swap $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\deg_{\boldsymbol{\sigma}(i)}}$. ■

By exploiting the potential Φ, we show in Chapter 3 that, for any $k \geq 2$, k-SSGs played on regular graphs have the FIP and that any sequence of profitable swaps has a length of at most m, cf. Theorem 3.4. This result can be extended to β-almost regular graphs for some values of β. First, we need the following technical lemma.

▶ **Lemma 4.3.** Fix a k-SSG $(G, \mathbf{t})$, with $k \geq 2$, a strategy profile $\boldsymbol{\sigma}$ and a profitable swap in $\boldsymbol{\sigma}$ performed by agents i and j with $\deg_{\sigma(i)} \leq \deg_{\sigma(j)}$. If $\deg_{\sigma(j)} - \deg_{\sigma(i)} \leq 1$, then the swap is Φ-increasing. If $\deg_{\sigma(j)} - \deg_{\sigma(i)} = 2$, then the swap is either Φ-increasing or Φ-preserving, with the swap being Φ-preserving only if $\mathsf{U}_j(\boldsymbol{\sigma}) \in \left(\frac{1}{2}, 1\right)$. ◀

Proof. Assume, without loss of generality, that agent i is orange and agent j is blue; moreover, define $\sigma(i) = u$ and $\sigma(j) = v$. Let o_u be the number of orange agents occupying vertices adjacent to u in $\boldsymbol{\sigma}$, let x_u be the number of neither orange nor blue agents occupying vertices adjacent to u in $\boldsymbol{\sigma}$, let b_v be the number of blue agents occupying vertices adjacent to v in $\boldsymbol{\sigma}$ and let x_v be the number of neither orange nor blue agents occupying vertices adjacent to v in $\boldsymbol{\sigma}$. We have $\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{o_u}{\deg_u}$, $\mathsf{U}_j(\boldsymbol{\sigma}) = \frac{b_v}{\deg_v}$ and

$$\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = \frac{\deg_v - b_v - x_v - 1_{ij}(\boldsymbol{\sigma})}{\deg_v}, \quad \mathsf{U}_j(\boldsymbol{\sigma}_{ij}) = \frac{\deg_u - o_u - x_u - 1_{ij}(\boldsymbol{\sigma})}{\deg_u}.$$

As i and j perform a profitable swap in $\boldsymbol{\sigma}$, we have $\mathsf{U}_i(\boldsymbol{\sigma}) < \mathsf{U}_i(\boldsymbol{\sigma}_{ij})$ and $\mathsf{U}_j(\boldsymbol{\sigma}) < \mathsf{U}_j(\boldsymbol{\sigma}_{ij})$ which implies

$$\deg_u b_v + \deg_v o_u + \deg_u x_v + \deg_u 1_{ij}(\boldsymbol{\sigma}) < \deg_u \deg_v \tag{4.1}$$

and

$$\deg_u b_v + \deg_v o_u + \deg_v x_u + \deg_v 1_{ij}(\boldsymbol{\sigma}) < \deg_u \deg_v. \tag{4.2}$$

Moreover, we have

$$\begin{aligned} \Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) &= \deg_u - 1_{ij}(\boldsymbol{\sigma}) - o_u - x_u + \deg_v - 1_{ij}(\boldsymbol{\sigma}) - b_v - x_v - o_u - b_v \\ &= \deg_u + \deg_v - x_u - x_v - 2(o_u + b_v + 1_{ij}(\boldsymbol{\sigma})). \end{aligned}$$

- If $\deg_u = \deg_v := \delta'$, Equation (4.1) implies $o_u + b_v + 1_{ij}(\boldsymbol{\sigma}) + x_v < \delta'$, while Equation (4.2) implies $o_u + b_v + 1_{ij}(\boldsymbol{\sigma}) + x_u < \delta'$ which together yield

$$\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) = 2\delta' - x_u - x_v - 2(o_u + b_v + 1_{ij}(\boldsymbol{\sigma})) > 0.$$

- If $\deg_u = \deg_v - 1$, Equation (4.1) implies

$$o_u + b_v + 1_{ij}(\boldsymbol{\sigma}) + x_v < \deg_v - 1 + \frac{b_v + x_v + 1_{ij}(\boldsymbol{\sigma})}{\deg_v},$$

while Equation (4.2) implies

$$o_u + b_v + 1_{ij}(\boldsymbol{\sigma}) + x_u < \deg_v - 1 + \frac{b_v}{\deg_v}.$$

As $b_v+x_v+1_{ij}(\boldsymbol{\sigma}) \leq \deg_v$ by definition, we get $o_u+b_v+1_{ij}(\boldsymbol{\sigma})+x_v \leq \deg_v-1$ and $o_u + b_v + 1_{ij}(\boldsymbol{\sigma}) + x_u \leq \deg_v - 1$ which together yield

$$\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) = 2\deg_v - 1 - x_u - x_v - 2(o_u + b_v + 1_{ij}(\boldsymbol{\sigma})) > 0.$$

- If $\deg_u = \deg_v - 2$, Equation (4.1) implies

$$o_u + b_v + 1_{ij}(\boldsymbol{\sigma}) + x_v < \deg_v - 2 + \frac{2(b_v + x_v + 1_{ij}(\boldsymbol{\sigma}))}{\deg_v},$$

while Equation (4.2) implies

$$o_u + b_v + 1_{ij}(\boldsymbol{\sigma}) + x_u < \deg_v - 2 + \frac{2b_v}{\deg_v}.$$

As $b_v+x_v+1_{ij}(\boldsymbol{\sigma}) \leq \deg_v$ by definition, we get $o_u+b_v+1_{ij}(\boldsymbol{\sigma})+x_v \leq \deg_v-1$ and $o_u + b_v + 1_{ij}(\boldsymbol{\sigma}) + x_u \leq \deg_v - 1$ which together yield

$$\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) = 2\deg_v - 2 - x_u - x_v - 2(o_u + b_v + 1_{ij}(\boldsymbol{\sigma})) \geq 0.$$

However, note that equality occurs only in the case in which $\frac{2b_v}{\deg_v} > 1$ which requires $b_v > \frac{\deg_v}{2}$, that is, $\mathsf{U}_j(\boldsymbol{\sigma}) > \frac{1}{2}$. Clearly, as j improves after the swap, it must also be $\mathsf{U}_j(\boldsymbol{\sigma}) < 1$. ■

Given the above lemma, the existence and efficient computation of equilibria for k-SSGs played on almost regular graphs can be easily obtained for any $k \geq 2$.

▶ **Theorem 4.4.** For any $k \geq 2$, k-SSGs played on almost regular graphs has the FIP. Moreover, at most m profitable swaps are sufficient to reach an equilibrium starting from any initial strategy profile. ◀

Proof. The first part of the claim comes from Lemma 4.3, as in any almost regular graph G it holds that $\Delta - \delta = 1$. The bound on the number of swaps comes from the fact that for every strategy profile $\boldsymbol{\sigma}$, we have $\Phi(\boldsymbol{\sigma}) \leq m$, and, moreover, $\Phi(\boldsymbol{\sigma})$ is an integer and non-negative. ■

Theorem 4.4 cannot be extended beyond almost regular graphs as Agarwal et al. [Aga+21] provide a 2-SSG played on a 2-almost regular graph, more precisely, a tree, admitting no equilibria. However, in the next theorem, we show that positive results can be still achieved in games played on 2-almost regular graphs obeying some additional properties which are in particular fulfilled by 4-grids.

▶ **Theorem 4.5.** Let G be a 2-almost regular graph such that $\Delta \leq 4$ and every vertex of degree δ is adjacent to at most $\delta - 1$ vertices of degree Δ. Then, for any $k \geq 2$, every k-SSG played on G possesses the FIP. Moreover, at most $O(nm)$ profitable swaps are sufficient to reach an equilibrium starting from any initial strategy profile. ◀

Proof. By Lemma 4.3, we know that any profitable swap occurring in a strategy profile $\boldsymbol{\sigma}$ is Φ-increasing unless it involves an agent i occupying vertex $\sigma(i) = u$, with $\deg_u = \delta$, and an agent j occupying vertex $\sigma(j) = v$, with $\deg_v = \Delta$, and such that $\mathsf{U}_j(\boldsymbol{\sigma}) \in \left(\frac{1}{2}, 1\right)$. As G is connected, we have $\delta \geq 1$, which yields $\Delta \in \{3, 4\}$. This fact, together with $\mathsf{U}_j(\boldsymbol{\sigma}) \in \left(\frac{1}{2}, 1\right)$ implies $\mathsf{U}_j(\boldsymbol{\sigma}) \in \left\{\frac{2}{3}, \frac{3}{4}\right\}$. As $\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) > \mathsf{U}_j(\boldsymbol{\sigma})$, we get $\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) = 1$ which implies that all vertices adjacent to u are occupied by agents of the same color of agent j, which implies $\mathsf{U}_i(\boldsymbol{\sigma}) = 0$. Hence, we can conclude that to have a Φ-preserving profitable swap, we need a profitable swap involving a vertex u of degree δ such that $\mathsf{U}_{\sigma^{-1}(u)}(\boldsymbol{\sigma}) = 0$ and $\mathsf{U}_{\sigma_{ij}^{-1}(u)}(\boldsymbol{\sigma}) = 1$. Thus, for an agent occupying u to perform once again a Φ-preserving profitable swap, all vertices in $N(u)$ need to change their colors, i.e., all agents occupying vertices adjacent to u must perform a profitable swap. By Lemma 4.3, any agent occupying a vertex $v \in N(u)$ can be involved in a Φ-preserving swap only if $\deg_v = \Delta$. By assumption, u has at least a neighbor of degree different from Δ. Thus, between any two consecutive Φ-preserving profitable swaps involving an agent residing at a fixed vertex, a Φ-increasing profitable swap has to occur. This immediately implies that no more than n consecutive Φ-preserving profitable swaps are possible. ■

As 4-grids meet the conditions required by Theorem 4.5, we get the following corollary.

▶ **Corollary 4.6.** For any $k \geq 2$, every k-SSG played on a 4-grid possesses the FIP. Moreover, at most $O(nm)$ profitable swaps are sufficient to reach an equilibrium starting from any initial strategy profile. ◀

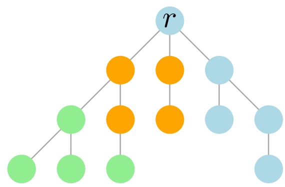


Figure 4.2: Example of the construction yielding an equilibrium on a tree. Let $k = 3$ and assume green $\leq$ orange $\leq$ blue. We root the tree at a vertex r, place the agents bottom-up and ensure for every subtree T' the corresponding root r' is the last vertex in T' to be occupied. See the proof of Theorem 4.7 for more details.

As mentioned before, Agarwal et al. [Aga+21] pointed out that 2-SSGs played on trees are not guaranteed to admit equilibria. We show that this is no longer the case in local k-SSGs for any value of $k \geq 2$. The main reason for this is that the given counter-example in [Aga+21] crucially relies on agents that perform non-local swaps whereas in local k-SSGs such swaps cannot occur.

▶ **Theorem 4.7.** For any $k \geq 2$, every local k-SSG played on a tree has an equilibrium that can be computed in polynomial time. ◀

Proof. Root the tree T at a vertex r. We place the agents color by color, starting with color 1 and ending with color k. Before we place an agent at an inner vertex v all of v's descendants in T have to be occupied. Hence, we place the agents starting from the leaves, and the root r' of every subtree T' is the last vertex in T' which will be occupied. Thus, we ensure that, if the root r' of a subtree T' is occupied by an agent of color $i \in [k]$, T' contains only agents of color $i' \leq i$. Clearly, this construction yields a feasible strategy profile, that we denote by $\boldsymbol{\sigma}$, and can be implemented in polynomial time. See Figure 4.2 for an illustration.

Consider two agents i and j of different colors that occupy two adjacent vertices u and v, respectively. Without loss of generality, we assume that u is the parent of v in T. Since $c(j) < c(i)$, the subtree of T rooted at v contains no vertex of color $c(i)$. As a consequence $\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = 0$. Hence σ is an LSE. ■

Note that, as we move from 4-grids to 8-grids, Corollary 4.6 does not hold anymore. In fact, for the latter class of graphs, we show that for $k = 2$ the FIP is guaranteed to hold only for local games. For this, we first need the following technical lemma which specifies all Φ-decreasing swaps which can occur in

8-grids. Note that by Lemma 4.3 a Φ-decreasing swap can only occur between two agents i and j that occupy vertices that have a difference in their vertex degree of at least 3.

▶ **Lemma 4.8.** Fix a local 2-SSG played on an 8-grid, a strategy profile $\boldsymbol{\sigma}$ and a profitable swap in $\boldsymbol{\sigma}$ performed by agents i and j. It holds that

i) If $\deg_{\sigma(i)} = 3$ and $\deg_{\sigma(j)} = 8$, then the swap is Φ-decreasing by 1 if $\mathrm{U}_i(\boldsymbol{\sigma}) = 0$ and $\mathrm{U}_j(\boldsymbol{\sigma}) = \frac{5}{8}$ otherwise it is a Φ-increasing swap.

ii) If $\deg_{\sigma(i)} = 5$ and $\deg_{\sigma(j)} = 8$, then the swap is Φ-decreasing by 1 if $\mathrm{U}_i(\boldsymbol{\sigma}) = 0$ and $\mathrm{U}_j(\boldsymbol{\sigma}) = \frac{6}{8}$ otherwise it is a Φ-increasing swap.

◀

Proof. Assume, without loss of generality, that agent i is orange and agent j is blue; moreover, define $\sigma(i) = u$ and $\sigma(j) = v$. Let o_u be the number of orange agents occupying vertices adjacent to u in $\boldsymbol{\sigma}$ and b_v be the number of blue agents occupying vertices adjacent to v in $\boldsymbol{\sigma}$.

i) We have $\mathrm{U}_i(\boldsymbol{\sigma}) = \frac{o_u}{3}$, $\mathrm{U}_j(\boldsymbol{\sigma}) = \frac{b_v}{8}$ and $\mathrm{U}_i(\boldsymbol{\sigma}_{ij}) = \frac{7-b_v}{8}$, $\mathrm{U}_j(\boldsymbol{\sigma}_{ij}) = \frac{2-o_u}{3}$. As i and j perform a profitable swap in $\boldsymbol{\sigma}$, we have $\mathrm{U}_i(\boldsymbol{\sigma}) < \mathrm{U}_i(\boldsymbol{\sigma}_{ij})$ and $\mathrm{U}_j(\boldsymbol{\sigma}) < \mathrm{U}_j(\boldsymbol{\sigma}_{ij})$ which imply

$$b_v < \frac{16}{3} - \frac{8}{3}o_u. \tag{4.3}$$

Moreover, we have

$$\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) \quad = \quad 3 - 1 - o_u + 8 - 1 - b_v - o_u - b_v = 9 - 2o_u - 2b_v.$$

From Equation (4.3) it follows that for $o_u = 2$, $b_v < 0$. Therefore, o_u is in the set $\{0, 1\}$, and we have the following cases:

If $o_u = 0$, Equation (4.3) implies $b_v < \frac{16}{3}$ which yields $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) > \frac{-5}{3}$.

If $o_u = 1$, Equation (4.3) implies $b_v < \frac{8}{3}$ which yields $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) > \frac{5}{3}$.

Since $\Phi(\boldsymbol{\sigma})$ is integral, the statement follows.

ii) We have $\mathrm{U}_i(\boldsymbol{\sigma}) = \frac{o_u}{5}$, $\mathrm{U}_j(\boldsymbol{\sigma}) = \frac{b_v}{8}$ and $\mathrm{U}_i(\boldsymbol{\sigma}_{ij}) = \frac{7-b_v}{8}$, $\mathrm{U}_j(\boldsymbol{\sigma}_{ij}) = \frac{4-o_u}{5}$. As i and j perform a profitable swap in $\boldsymbol{\sigma}$, we have $\mathrm{U}_i(\boldsymbol{\sigma}) < \mathrm{U}_i(\boldsymbol{\sigma}_{ij})$ and

$\mathsf{U}_j(\boldsymbol{\sigma}) < \mathsf{U}_j(\boldsymbol{\sigma}_{ij})$ which imply

$$b_v < \frac{32}{5} - \frac{8}{5}o_u. \tag{4.4}$$

Moreover, we have

$$\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) \quad = \quad 5 - 1 - o_u + 8 - 1 - b_v - o_u - b_v = 11 - 2o_u - 2b_v.$$

From Equation (4.4) it follows that for $o_u = 4$, $b_v < 0$. Hence, o_u is in the set $\{0, 1, 2, 3\}$, and we have the following cases:

If $o_u = 0$, Equation (4.4) implies $b_v < \frac{32}{5}$ which yields $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) > \frac{-9}{5}$.

If $o_u = 1$, Equation (4.4) implies $b_v < \frac{24}{5}$ which yields $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) > \frac{-3}{5}$.

If $o_u = 2$, Equation (4.4) implies $b_v < \frac{16}{5}$ which yields $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) > \frac{3}{5}$.

If $o_u = 3$, Equation (4.4) implies $b_v < \frac{8}{5}$ which yields $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) > \frac{9}{5}$.

Since $\Phi(\boldsymbol{\sigma})$ is an integer, we just have to show that, if $o_u = 1$, the swap is in fact not Φ-preserving, but Φ-increasing. Notice that b_v is an integer as well. Hence, since Equation (4.4) implies $b_v < \frac{24}{5}$, it holds that $b_v \leq 4$ which yields $\Phi(\boldsymbol{\sigma}_{ij}) - \Phi(\boldsymbol{\sigma}) \geq 1$. ■

We now show that the FIP is guaranteed to hold for local games played on 8-grids. For this we recall the definition of the function $\Psi(\boldsymbol{\sigma}) = (\Phi(\boldsymbol{\sigma}), n - z(\boldsymbol{\sigma}))$, where $z(\boldsymbol{\sigma})$ is the number of agents having utility 0 under $\boldsymbol{\sigma}$. As shown in Lemma 4.3 and Lemma 4.8, there are only a few local swaps that can preserve or decrease the potential Φ and all of them decrease it by at most 1. We show that after a Φ-preserving or a Φ-decreasing swap, several swaps must happen before at the same pair of vertices another Φ-preserving or Φ-decreasing swap can occur. Hence, we prove that in total the extended potential Ψ increases lexicographically which imply the FIP.

In the following proof, we assume towards a contradiction that an IRC exists and show first, that the IRC must contain at least one Φ-decreasing swap. We then assign necessarily profitable swaps which have to be executed after a Φ-decreasing swap and before a comparable Φ-decreasing swap can again be performed. To this end, we distinguish between the cases of whether another possible Φ-preserving or Φ-decreasing swap can be performed within the neighborhood and if so, how the neighbors are involved in these swaps.

▶ **Theorem 4.9.** Any local 2-SSG played on an 8-grid possesses the FIP. ◀

Proof. The proof is structured as follows: We first show that there are only a few swaps that can preserve or decrease the potential Φ. Then we assume towards a contradiction that an IRC exists. By definition, such an IRC cannot contain only Φ-increasing swaps. Thus, it must contain Φ-preserving or Φ-decreasing swaps. Next, we show that at least one Φ-decreasing swap must occur. Concentrating on such Φ-decreasing swaps, note that we need at least one, assuming, without loss of generality, that an orange agent i with utility 0 occupies some vertex u. We show that reversing the colors of the agents (via swaps) in the neighborhood of u to enable another Φ-decreasing swap involving an agent occupying u entails several Φ-increasing swaps, that contradict the assumed existence of the IRC.

We start by showing that only a few swaps can be non-increasing regarding the potential Φ. By Lemma 4.3, we know that any profitable swap occurring in a strategy profile $\boldsymbol{\sigma}$ is Φ-increasing, and, hence, also Ψ-increasing, unless it involves two agents i and j occupying vertices $\sigma(i) = u$ and $\sigma(j) = v$ with $\deg_u \neq \deg_v$, i.e., with different degrees. We assume, without loss of generality, $\deg_u < \deg_v$ and that agent i is orange and agent j is blue.

First, we note that in a Φ-preserving or a Φ-decreasing swap, for the orange agent i, it must be that $\mathsf{U}_i(\boldsymbol{\sigma}) = 0$. By Lemma 4.8, we know that this is true if $\deg_u = 3$ and $\deg_v = 8$ or $\deg_u = 5$ and $\deg_v = 8$. If $\deg_u = 3$ and $\deg_v = 5$, we know by Lemma 4.3 that we may have a Φ-preserving swap if for the utility of the blue agent it holds that $\mathsf{U}_j(\boldsymbol{\sigma}) \in \left(\frac{1}{2}, 1\right)$. As $\deg_v = 5$ and $\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) > \mathsf{U}_j(\boldsymbol{\sigma}) > \frac{1}{2}$, it must be that $\mathsf{U}_j(\boldsymbol{\sigma}) = \frac{3}{5}$ and $\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) = \frac{2}{3}$, which implies that, in $\boldsymbol{\sigma}$, all vertices adjacent to u are occupied by blue agents, so $\mathsf{U}_i(\boldsymbol{\sigma}) = 0$, cf. Figure 4.3 (a).

We show that after every Ψ-decreasing swap, we can assign corresponding Ψ-increasing swaps such that in total the extended potential Ψ increases lexicographically which imply the FIP. Remember that the extended potential Ψ is simply a more fine-grained version of the potential Φ with the number of agents having utility 0 as a tie-breaker. Thus, for simplicity, in some parts of the proof, we work with Φ instead of Ψ. Since the extended potential Ψ is a vector, we denote the change in Ψ by a profitable swap as (λ, μ) with $\lambda, \mu \in \mathbb{Z}$ where λ denotes the change in Φ and μ denotes the change in $n - z(\cdot)$. Moreover, remember that we consider local games. Hence, two agents i and j are only allowed to swap if $1_{ij}(\sigma) = 1$, i.e., if they are adjacent.

We now assume for the sake of contradiction that an IRC exists. Note that such an IRC contains at least one swap which preserves or decreases the potential Φ.

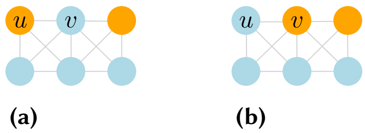


Figure 4.3: The coloring of G in $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_{ij}$ before and after a Φ-preserving swap of the orange agent i and the blue agent j occupying vertices u and v, respectively. The right neighbors of v are occupied by agents of different types, hence, the top vertex can also be occupied by a blue agent if the lower one is occupied by an orange one. Symmetric and equivalent cases are omitted. (a) the strategy profile $\boldsymbol{\sigma}$ before i and j perform a Φ-preserving swap, (b) the strategy profile $\boldsymbol{\sigma}_{ij}$ after i and j perform a Φ-preserving swap. See the proof of Theorem 4.9 for more details.

Hence, assume that there exists an IRC $C = \boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^\ell$. For the sake of brevity, we denote $\boldsymbol{\sigma}^0$ as $\boldsymbol{\sigma}$ in this proof. It holds that $\Psi(\boldsymbol{\sigma}^0) = \Psi(\boldsymbol{\sigma}^\ell)$ and C must contain at least one Φ-decreasing swap since any Φ-preserving swap increases Ψ. This follows from Lemma 4.3 and our above observation that one of the agents involved in a Φ-preserving swap must have utility 0 before the swap and, since the swap is profitable, must have utility greater than 0 after the swap. Hence, the extended potential Ψ increases. As an illustration, consider Figure 4.3. If the agents i and j perform a Φ-preserving swap the number of agents having utility 0 decreases by 1 since no new agent with utility 0 is created.

Therefore, we concentrate on Φ-decreasing swaps. To this end, we need at least one agent with utility 0 in one of the strategy profiles $\boldsymbol{\sigma}^k$ with $0 \leq k \leq \ell - 1$ which is contained in the IRC C. Assume, without loss of generality, that $\boldsymbol{\sigma}^0$ contains at least one agent i with utility 0. Note that since C is a cycle, we can freely define the starting strategy profile $\boldsymbol{\sigma}^0$. Hence, in $\boldsymbol{\sigma}^\ell$ the vertex u has to be occupied by an agent with utility 0 as well.

Recall that, by Lemma 4.8, Φ decreases by at most 1 in any Φ-decreasing swap. Also, we know that we have a Φ-decreasing swap by 1 if and only if we have that for the utility of the orange agent i it holds that $\mathrm{U}_i(\boldsymbol{\sigma}) = 0$ and vertex u has to be a border vertex, i.e., u has degree 3 or 5. This implies that all vertices adjacent to u are occupied by blue agents. Thus, for agent j, occupying vertex u in $\boldsymbol{\sigma}_{ij}$, i.e., after the swap, to be involved once again in a Φ-decreasing profitable swap, all vertices in $N(u) \setminus \{v\}$ must become occupied by orange agents. Hence, we need to reverse the color of the agents in the neighborhood of u. Note, that in the case that an orange agent on vertex u is involved once again in a Φ-decreasing swap without agent j being involved in a Φ-decreasing swap in-between implies an increase in the potential Ψ. In particular, the swap between the agents i and j yields a change in Ψ of at least $(-1, x)$, with $x > 1$, since agent i has utility larger 0 now. Agent j swapping away from vertex u with an agent occupying an adjacent vertex, denoted by w, is Φ-increasing by assumption if $\deg_w \geq 5$, and yields a change in Ψ of at least $(1, -(x-1))$ since it was a profitable swap for both involved agents. Therefore, after the swap, the agent occupying vertex u cannot have utility 0. Or, the swap of the blue agent j is Φ-preserving which implicates $\deg_w = 3$ and a Φ-increasing swap in-between since w was occupied by a blue agent in $\boldsymbol{\sigma}$. Hence, in both cases the swaps, such that agent j swaps away from vertex u, are together Ψ-increasing by at least $(0, 1)$. This contradicts the assumption of an IRC. Note that in the case that agent j performs a Φ-increasing swap with the agent placed on vertex w, w is again occupied by a blue agent, similar to the initial strategy profile $\boldsymbol{\sigma}$. Hence, it will not interfere with the swaps performed on vertex w to negate other decreasing swaps later. Hence, the blue agent j occupying vertex u needs to be involved in a Φ-decreasing profitable swap and needs therefore utility 0. Vertex u has, besides v, at least two further adjacent vertices, say w_1 and w_2. We show in the following that occupying the vertices w_1 and w_2 with orange agents will in total increase the potential Ψ.

Let $\mathrm{dist}(x, y)$ be the number of edges on a shortest path between two vertices x and y and let $N(x)^2 = \{y \in V : \mathrm{dist}(x, y) \leq 2\}$ be the 2-neighborhood of x, i.e., all vertices which are in hop distance at most 2 from x.

We distinguish between two cases: **(1)** We assume that in $N(u)^2$ it holds that there is no agent with utility 0 before the agents occupying w_1 and w_2 swap, and **(2)**, in $N(u)^2$ it holds that there is at least one agent with utility 0 before the agents occupying w_1 and w_2 swap.

In these cases, we consider the direct neighbors of u and show that reversing the colors of the agents occupying these vertices entails several Φ-increasing swaps which we can assign clearly to the Φ-decreasing swap of agent i. This implies that Ψ increases and, hence, contradicts the assumption of the existing IRC.

We start with the case **(1)**, i.e., that in $N(u)^2$ no other agent with utility 0 is around before the agents occupying w_1 and w_2 swap.

Since all neighbors of w_1 and w_2 belong to $N(u)^2$ and have by assumption utility larger 0 and since the agents on w_1 and w_2 have positive utility as well and are, due to locality, restricted to swaps with adjacent agents, two Φ-increasing

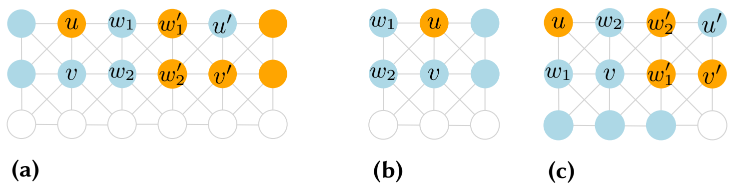


Figure 4.4: The coloring of G where the orange agent i and the blue agent j occupy vertices u and v, respectively. We omitted symmetric and equivalent cases. (a) The strategy profile where the neighbors, w_1 and w_1' and w_2 and w_2', respectively, can perform two Φ-increasing swaps within the neighborhood, (b) starting clockwise from the top left corner, agent i is the first agent with utility 0, (c) the 2-neighborhoods of u and u' overlap. See the proof of Theorem 4.9 for more details.

swaps occur before the agent occupying u can perform once again a Φ-decreasing swap. Thus, in total Φ increases, if we can assign the two Φ-increasing swaps to the Φ-decreasing swap of agent i occupying u under $\boldsymbol{\sigma}$. Note, that this is given if the 2-neighborhoods of vertices which are occupied by agents with utility 0 do not overlap.

Consider Figure 4.4 (a) where the 2-neighborhoods of two such vertices overlap. The agents occupying u and u' can both perform a Φ-decreasing swap, while w_1 and w_1' and w_2 and w_2', respectively, can perform two Φ-increasing swaps, which, in total, is Φ-preserving. However vertex u has, besides vertex v, four neighbors which have to be involved in swaps. To this end, vertex u needs to have clockwise and counter-clockwise along the border overlapping 2-neighborhoods with vertices that are occupied by agents with utility 0. Otherwise, we have a clear assignment of two Φ-increasing swaps to the Φ-decreasing swap of agent i occupying vertex u. In particular, assume, without loss of generality, that vertex u has clockwise along the border no overlapping 2-neighborhoods with vertices which are occupied by agents with utility 0. Then we can assign the two Φ-increasing swaps involving the first two clockwise neighbors of vertex u which are not vertex v, cf., for instance, vertices w_1 and w_2 in Figure 4.4 (b), to the Φ-decreasing swap of the agents i and j. Hence, towards a contradiction to the assumption that an IRC exists, we have to show that there is at least one agent with utility 0 whose neighbors increase Φ (and not only preserve it), and therefore Φ increases in total.

To this end, we consider, starting clockwise from the top left corner, the first agent with utility 0, say agent i. If agent i is not located at the corner vertex, i.e., a vertex with degree 3, cf. Figure 4.4 (b), we already found our agent whose neighbors increase Φ in total since at least one neighbor, vertex w_1 in Figure 4.4 (b), is not involved in a swap with a direct neighbor of another agent with utility 0.

Hence, we assume that agent i is located at the corner vertex, and there is another agent located on a vertex u' with utility 0 with an overlapping 2-neighborhood, cf. Figure 4.4 (c). Note, that since we assume agent i to be involved in a Φ-decreasing swap, vertex v has to be the adjacent vertex with degree 8. Hence, vertex w_1 is not in the 2-neighborhood of the agent occupying vertex u' and therefore, since with the agent occupying vertex w_2 only one direct neighbor of vertex u who is not placed on vertex v can be involved in a swap with an agent occupying vertex w'_1 or vertex w'_2, either the agent occupying vertex w'_1 or vertex w'_2 is not involved in a swap with a direct neighbor of another agent with utility 0 involved in a Φ-decreasing swap. As a result, the potential Φ increases in total since we have two Φ-decreasing swaps involving the vertices u and u', two Φ-increasing swaps involving the vertices w_1 and w_2 with either w'_1 or w'_2, and an additional Φ-increasing swap involving either w'_1 or w'_2.

We now turn our focus to case **(2)**, i.e., that in $N(u)^2$ there is at least one agent with utility 0 before the agents occupying w_1 and w_2 swap.

We first note that we can assume that $\deg_u = 5$. To this end, consider Figure 4.5 and assume that there is no agent with utility 0 occupying a vertex with degree 5 in the IRC C. Furthermore, we consider a $3 \times h$ grid, with $h > 3$, and that in $\boldsymbol{\sigma}$ the agent occupying w'_1, with $\deg_{w'_1} = 3$, has utility 0.

Note that if we consider $\ell \times h$ grids, with $\ell \neq 3$, without an agent with utility 0 occupying a vertex with degree 5, we are in case (1) since due to our assumptions all agents in $N(u)^2$ have positive utility. In this case the swap between agents i and j, cf. Figure 4.5 (a), yields a change in the extended potential Ψ of $(-1, +2)$. To enable another Φ-preserving or decreasing swap involving an agent occupying vertex u, agent i on vertex v needs to perform another profitable swap, yielding a change in Ψ of $(x, -1)$, with $x \geq 1$, since by assumption C does not contain an agent with utility 0 occupying a vertex with degree 5. Therefore, since vertex v is, by the assumption of the grid size, not adjacent to further vertices of degree 3 besides u and w'_1, cf. Figure 4.5 (a), the swap of agent i must be Φ-increasing. Hence, in total the extended potential Ψ increases by at least $(0, 1)$. If the agent occupying vertex w'_1 has utility larger than 0 in $\boldsymbol{\sigma}$, the swap between agents i and j yields a change in Ψ of $(-1, +1)$. However, to create an agent with utility 0

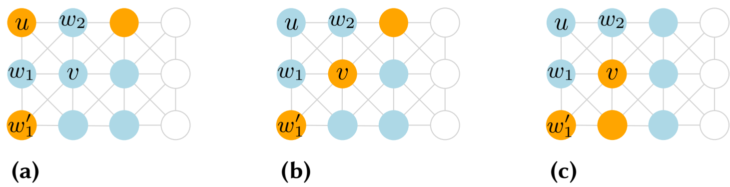


Figure 4.5: The coloring of G in $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_{ij}$ before and after a Φ-decreasing swap of the orange agent i and the blue agent j occupying vertices u and v, respectively. We omitted symmetric and equivalent cases. (a) the strategy profile $\boldsymbol{\sigma}$ before i and j perform a Φ-decreasing swap, (b) the strategy profile $\boldsymbol{\sigma}_{ij}$ after a Φ-decreasing swap of agents i and j occupying vertices u and v when the agent occupying w_1' has utility 0 under $\boldsymbol{\sigma}$. (c) the strategy profile $\boldsymbol{\sigma}_{ij}$ after a Φ-decreasing swap of agents i and j occupying vertices u and v when the agent occupying w_1' has utility larger 0 under $\boldsymbol{\sigma}$. See the proof of Theorem 4.9 for more details.

occupying w_1', at least two Φ-increasing swaps are necessary, cf. Figure 4.5 (c). Note that if vertex w_1' is occupied by a blue agent we are in case (1) since due to our assumptions all agents in $N(u)^2$ have positive utility. Consider Figure 4.6 to check that also a 3×3 grid cannot contain an IRC.

Hence, it holds that $\mathrm{U}_i(\boldsymbol{\sigma}) = 0$, $\deg_u = 5$, and there exists at least one other vertex $w_1' \in N(u)^2$ which is occupied by an agent k with $\mathrm{U}_k(\boldsymbol{\sigma}^q) = 0$, with $q \in [\ell]$, where $\boldsymbol{\sigma}^q$ is a placement before the agents occupying w_1 and w_2 under $\boldsymbol{\sigma}$ performed profitable swaps. Moreover, we assume that $\deg_{w_1'} = 3$ or $\deg_{w_1'} = 5$. Otherwise, by Lemma 4.8, the agent occupying vertex w_1' can only be involved in a Φ-increasing swap, and therefore the agents occupying the direct neighbors of vertices u, w_1, and w_2, can only be involved in a Φ-increasing swap as well which yields that in total the potential Φ increases. Note that we can define a disjoint assignment of the neighbors w_1 and w_2 to their respective vertex u. Since the vertex u has besides v four further neighbors, two with degrees lower than 8, we can, without loss of generality, starting from the top left corner of the grid, clockwise, assign to every border vertex, i.e., a vertex with degree 5, occupied by an agent with utility 0, the first two clockwise vertices which are not vertex v, as distinct w_1 and w_2, respectively, cf. Figure 4.7 (a).

For our case analysis, we consider, without loss of generality, the left neighbors of vertex u, cf. Figure 4.7 (a), with $\deg_{w_1} = 5$ and $\deg_{w_2} = 8$. So in the following, we distinguish between the three following cases: **(a)** vertex w_1 is involved in a Φ-preserving swap, **(b)** vertex w_2 is involved in a Φ-decreasing swap, and **(c)**

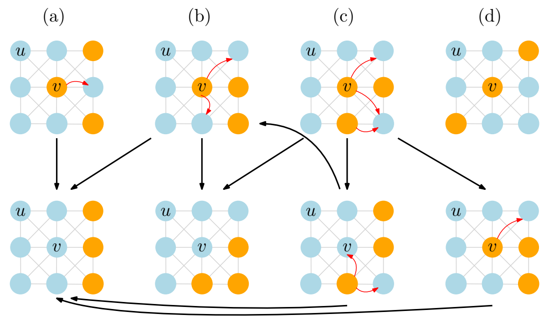


Figure 4.6: The coloring of G in $\boldsymbol{\sigma}_{ij}$ after a Φ-decreasing swap of the orange agent i and the blue agent j occupying vertices u and v, respectively, shown in Figure 4.5 (a) in a 3×3 grid. We omitted symmetric cases. (a) - (d) show the possible strategy profiles of $\boldsymbol{\sigma}_{ij}$. The red arrows show all possible profitable swaps of an orange agent while the black arrows point toward the next possible strategy profile. Strategy profiles without red arrows are swap equilibria. See the proof of Theorem 4.9 for more details.

vertex w_1 is involved in a Φ-preserving swap and vertex w_2 is involved in a Φ-decreasing swap. Note that since $\deg_{w_1} = 5$ the agent occupying vertex w_1 cannot be involved in a Φ-decreasing swap since the agents placed on vertices w_1 and w_2 have positive utility.

(a) We assume that vertex w_1 is involved in a Φ-preserving swap. Note that in this case, it holds that $\deg_{w_1'} = 3$, cf. Figure 4.7 (b). A profitable swap between the agents occupying vertices w_1 and w_1' yields a change in Ψ of $(0, +1)$ since both agents have non-zero utility after the swap, cf. Figure 4.7 (c). However, by assumption, the agent on vertex w_2 must perform a profitable Φ-increasing swap which changes the extended potential Ψ by at least $(+1, -1)$. In total, the extended potential Ψ must change by at least $(0, +1)$, since the swap of the agents i and j yields a change in Ψ of $(0, +1)$ which together with the changes of $(0, +1)$ and $(+1, -1)$ imply a lexicographic increase.

(b) We assume that vertex w_2 is involved in a Φ-decreasing swap. A profitable swap between the agents occupying vertices w_2 and w_1' yields a change in Ψ of $(-1, +1)$, since both agents have non-zero utility after the swap. Recall that we want to reverse the colors of the agents occupying the neighbors of vertex u to enable another Φ-decreasing swap with the agent on vertex u. Now, there are two ways of how vertex w_1 can become occupied by an orange agent, cf. Figure 4.8 (a).

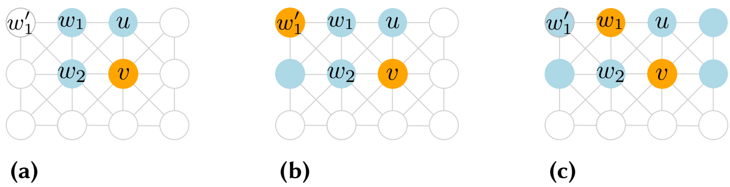


Figure 4.7: The coloring of G in $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_{ij}$ before and after a Φ-decreasing swap of the orange agent i and the blue agent j occupying vertices u and v, respectively. We omitted symmetric and equivalent cases. (a) the coloring of G in $\boldsymbol{\sigma}_{ij}$ after a Φ-decreasing swap by $(-1,+1)$ of agents i and j occupying vertices u and v. (b) the coloring of G before a Φ-preserving swap by $(0,+1)$ of agents occupying vertices w_1 and w_1'. (c) the coloring of G after a Ψ-preserving swap by agents occupying vertices w_1 and w_1'. See the proof of Theorem 4.9 for more details.

First, by swapping with an agent who is in the neighborhood of vertex u, e.g., with the agent occupying vertex w_2. Then, by Lemma 4.8, the extended potential Ψ changes by $(+1,0)$. After this swap, the blue agent who was previously on vertex w_1 has to perform another swap with an orange agent, which changes Ψ again by $(+1,0)$. The second way for vertex w_1 to become occupied by an orange agent is that one of the two vertices which are adjacent to vertex w_1 but not to vertex u is occupied by an orange agent, cf. Figure 4.8 (d), and then this agent swaps with the agent on vertex w_1. To this end, we have to consider two different cases: **(i)** the agent occupying vertex w_2 in $\boldsymbol{\sigma}_{ij}$ swapped with a neighbor of vertex w_1, and **(ii)** the agent occupying vertex w_2 in $\boldsymbol{\sigma}_{ij}$ did not swap with a neighbor of vertex w_1.

Considering case (i) and assuming that the left neighbor of w_1 has degree 3, we note that both vertices which are in the neighborhood of vertex w_1 but not in the neighborhood of vertex u, we denote them by w_1' and w_2', have degree 3 and 5, respectively. Moreover, the agents who are placed on vertices w_1' and w_2' are blue and have utility larger than 0, since one out of the two agents which are placed under $\boldsymbol{\sigma}_{ij}$ on vertices w_1' and w_2' is orange and has utility 0 under $\boldsymbol{\sigma}_{ij}$, cf. Figure 4.8 (e). Remember that we assume that vertex w_2 is involved in a Φ-decreasing swap with one of the agents occupying vertices w_1' and w_2', respectively. Therefore, we have two Ψ-increasing swaps, the swap involving the agent occupying vertex w_1' or vertex w_2' with an orange agent and the swap with the corresponding orange agent and the agent occupying vertex w_1, by at least $(+1,0)$ and $(+1,-1)$, respectively, before an orange agent occupies vertex w_1,

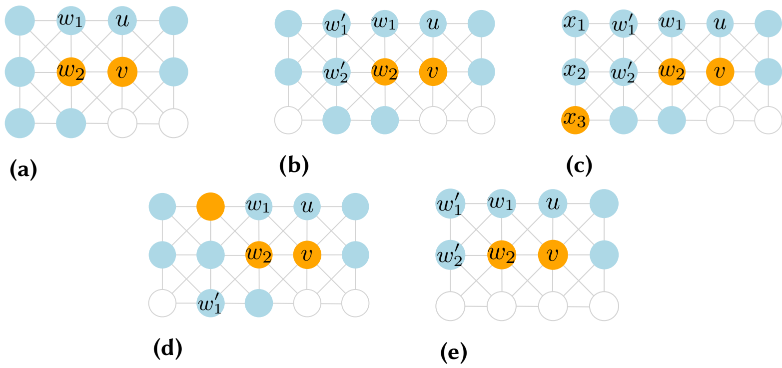


Figure 4.8: The coloring of G in $\boldsymbol{\sigma}_{ij}$ after a Φ-decreasing swap of the orange agent i and the blue agent j occupying vertices u and v, respectively. We omitted symmetric and equivalent cases. (a) and (b) show the coloring of G after a Φ-decreasing swap by $(-1, +1)$ by the agent occupying w_2. (c) shows the coloring of G before the agent occupying w_2' can perform another Φ-decreasing swap by $(-1, +1)$. (d) shows the coloring of G before the agent occupying w_1 can swap with its left orange neighbor. (e) shows the coloring of G after the agent occupying w_2 performed a Φ-decreasing swap with the agent occupying w_1' or w_2'. See the proof of Theorem 4.9 for more details.

which in total yields a lexicographic increase, since we have two Φ-decreasing swaps of $(-1, +1)$ involving the vertices u and v, and w_2, and a neighbor of w_1 plus the two Φ-increasing swaps of $(+1, 0)$ and $(+1, -1)$ for vertex w_1 to be occupied by an orange agent.

Next, we assume that the left neighbor of vertex w_1 has degree 5, cf. Figure 4.8 (b). Note, that all neighbors of vertex w_1' have to be occupied by blue agents, since otherwise, no Φ-decreasing swap with the agent occupying vertex w_2 in $\boldsymbol{\sigma}_{ij}$ is possible. Hence, the agent occupying vertex w_1' has utility larger than 0 and a swap such that vertex w_1' becomes occupied by an orange agent is Ψ-increasing by at least $(+1, 0)$. We denote the remaining neighbor of vertex w_1, that is not adjacent to vertex u, again by w_2', cf. Figure 4.8 (b). Note that it is possible that the agent placed on vertex w_2' swaps with an orange agent via a Φ-decreasing swap, cf. Figure 4.8 (c). However, in this case, the left neighbors of vertex w_2' have to be corner and border vertices, i.e, vertices with degree equal 3 or 5, denoted as x_1, x_2 and x_3 in Figure 4.8 (c). Hence, to end up in an equivalent strategy profile, i.e., having agents with utility 0 placed on vertices w_1' and x_3, the blue agents on x_1, x_2, and w_2' have to leave the neighborhood of vertices w_1' and x_3

(and u), which implies at least three Ψ-increasing swaps by $(+1, 0)$, since all of the blue agents who are occupying the corresponding border and corner vertices have utility larger than 0. In total, the extended potential Ψ lexicographically increases.

Turning our focus to case (ii), i.e., that the agent occupying vertex w_2 in $\boldsymbol{\sigma}_{ij}$ swapped with a non-neighbor of vertex w_1, we note that the only additional case is that the left neighbor of vertex w_1 is occupied by an orange agent since all other cases are already covered by the case (i). Assume that the left neighbor of vertex w_1 has degree 5, cf. Figure 4.8 (d). Note that in this case, i.e., vertex w_1 and the left neighbor of vertex w_1 have the same degree of 5 and the two corresponding agents occupying these two vertices perform a profitable swap, the extended potential Ψ changes by at least $(+2, 0)$, which in total implies a lexicographic increase since we have two Φ-decreasing swaps which change the potential Ψ by $(-1, +1)$, respectively, and one Φ-increasing swap of $(+2, 0)$. In total, this yields a change in Ψ by $(0, +2)$.

Assuming that the left neighbor of vertex w_1 has degree 3, note that a swap of the agent occupying vertex w_1 with its left neighbor changes Ψ by $(+2, 0)$: In case the left neighbor has utility 0, since, by assumption, it is not a Φ-preserving swap, the agent occupying vertex w_1 has a utility of at most $\frac{2}{5}$. If the left neighbor of vertex w_1 has utility $\frac{1}{3}$, the agent occupying vertex w_1 has a utility of at most $\frac{1}{5}$, and if the left neighbor of vertex w_1 has utility $\frac{2}{3}$ no profitable swap between the agent occupying vertex w_1 and its left neighbor is possible. Thus, in total the extended potential Ψ increases lexicographically.

(c) We assume that vertex w_1 is involved in a Φ-preserving swap and vertex w_2 is involved in a Φ-decreasing swap. In this case, we have that there exist two vertices $w_1' \in N(u)^2$ and $w_2' \in N(u)^2$ which are occupied by agents with utility 0. Moreover, it holds that $\deg_{w_1'} = 3$ and $\deg_{w_2'} = 3$ or $\deg_{w_2'} = 5$, cf. Figure 4.9 (a). Let $\deg_{w_2'} = 5$. By Lemma 4.3, a swap by the agents on vertices w_1' and w_1 changes Ψ by $(0, +1)$. Now, note that if such a Φ-preserving swap is possible, it holds that the agent occupying vertex w_1 in $\boldsymbol{\sigma}_{ij}$ has utility $\frac{3}{5}$ and therefore the agent on vertex w_2 has a utility of at most $\frac{5}{8}$, cf. Figure 4.9 (b), and a swap with the agent on vertex w_2' must be Φ-increasing, which in total yields an increase in Ψ. Hence, at least one orange agent in the neighborhood of vertex u has to perform a profitable Φ-increasing swap, which again in total yields an increase in Ψ. Note that a Φ-decreasing swap of the agent occupying vertex w_2 prevents a Φ-preserving swap afterward of the agent occupying vertex w_1, since

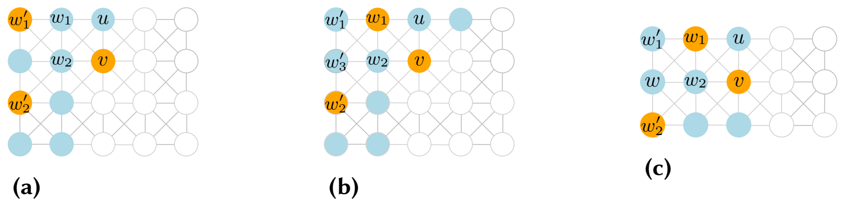
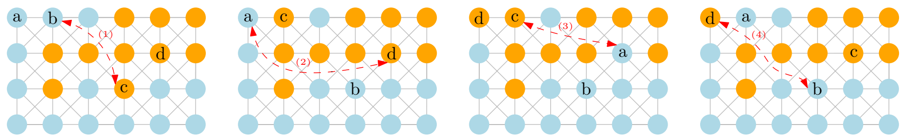


Figure 4.9: The coloring of G in $\boldsymbol{\sigma}_{ij}$ after a Φ-decreasing swap of the orange agent i and the blue agent j occupying vertices u and v, respectively. We omitted symmetric and equivalent cases. (a) the coloring of G in $\boldsymbol{\sigma}_{ij}$ after a Φ-decreasing swap by $(-1, +1)$ of agents i and j occupying vertices u and v. (b) and (c) the coloring of G after a Φ-preserving swap by $(0, +1)$ of agents occupying vertices w_1 and w_1'. See the proof of Theorem 4.9 for more details.

in this case vertex w_2 is occupied by an orange agent and, therefore, the agent occupying vertex w_1' cannot have utility 0, which is a requirement for a non-Φ-increasing swap. Let $\deg_{w_2'} = 3$. By Lemma 4.3, a swap by the agents on vertices w_1' and w_1 changes Ψ by $(0, +1)$. Again, a Φ-decreasing swap of the agent occupying vertex w_2 prevents a Φ-preserving swap afterward of the agent occupying vertex w_1. If the agent on vertex w_2 has a utility of $\frac{5}{8}$ a swap with the agent on vertex w_2' changes Ψ by $(-1, +1)$, cf. Figure 4.9 (c). To end up in an equivalent strategy profile, i.e., that the agents occupying vertices u, w_1' and w_2' are involved in Φ-decreasing and Φ-preserving swaps, the agent occupying vertex w has to perform two Φ-increasing swaps to leave the neighborhood of vertices w_1' and w_2'. In total, the extended potential Ψ must change by at least $(0, +3)$, since together with the Φ-decreasing swap of the agents i and j occupying the vertices u and v, we have two Φ-decreasing swaps of $(-1, +1)$, one Φ-preserving swap of $(0, +1)$, and two Φ-increasing swaps of $(+1, 0)$. This implies a lexicographic increase of the extended potential Ψ.

We have shown that after a Ψ-decreasing profitable local swap involving agents on two vertices u and v additional swaps are necessary before another Ψ-decreasing swap can happen again involving the same vertices. Moreover, we proved that in total these additional swaps increase the extended potential Ψ more than it was decreased by the initial swap. Thus, in total the extended potential Ψ increases. This contradicts the existence of an IRC. ■

Now we see that compared to the local k-SSG, the k-SSG on 8-grids behaves differently. There the FIP does not hold.



(a) Initial strategy profile **(b)** Strategy profile after the first swap **(c)** Strategy profile after the second swap **(d)** Strategy profile after the third swap

Figure 4.10: An improving response cycle for the k-SSG played on an 8-grid. The agent types are marked orange and blue. See the proof of Theorem 4.10 for more details.

▶ **Theorem 4.10.** There cannot exist a potential function for the k-SSG played on an 8-grid, for any $k \geq 2$. ◀

Proof. We prove the statement by providing an improving response cycle $\boldsymbol{\sigma}^0$, $\boldsymbol{\sigma}^1$, $\boldsymbol{\sigma}^2$, $\boldsymbol{\sigma}^3$, $\boldsymbol{\sigma}^4$. The construction is shown in Figure 4.10, where vertices are labeled with the agent occupying them. We have orange and blue agents. Agents with other types can be placed in a grid outside of the depicted cutout.

In the initial strategy profile $\boldsymbol{\sigma}^0$, cf. Figure 4.10 (a), $\mathsf{U}_b(\boldsymbol{\sigma}^0) = \frac{3}{5}$ and $\mathsf{U}_c(\boldsymbol{\sigma}^0) = \frac{3}{8}$. Both agents b and c improve by swapping, since in $\boldsymbol{\sigma}^1 := \boldsymbol{\sigma}^0_{bc}$ we have $\mathsf{U}_b(\boldsymbol{\sigma}^1) = \frac{5}{8}$ and $\mathsf{U}_c(\boldsymbol{\sigma}^1) = \frac{2}{5}$. After the first swap, cf. Figure 4.10 (b), agents a and d can perform a profitable swap, since $\mathsf{U}_a(\boldsymbol{\sigma}^1) = \frac{1}{3}$, $\mathsf{U}_d(\boldsymbol{\sigma}^1) = \frac{5}{8}$ and in $\boldsymbol{\sigma}^2 := \boldsymbol{\sigma}^1_{ad}$ we have $\mathsf{U}_a(\boldsymbol{\sigma}^2) = \frac{3}{8}$ and $\mathsf{U}_d(\boldsymbol{\sigma}^2) = \frac{2}{3}$. Then, cf. Figure 4.10 (c), agents a and c can swap and improve from $\mathsf{U}_a(\boldsymbol{\sigma}^2) = \frac{3}{8}$ and $\mathsf{U}_c(\boldsymbol{\sigma}^2) = \frac{3}{5}$ to $\mathsf{U}_a(\boldsymbol{\sigma}^3) = \frac{2}{5}$ and $\mathsf{U}_c(\boldsymbol{\sigma}^3) = \frac{5}{8}$, respectively, with $\boldsymbol{\sigma}^3 := \boldsymbol{\sigma}^2_{ac}$. Finally, cf. Figure 4.10 (d), agents b and d can improve by swapping, since $\mathsf{U}_b(\boldsymbol{\sigma}^3) = \frac{5}{8}$, $\mathsf{U}_d(\boldsymbol{\sigma}^3) = \frac{1}{3}$ and in $\boldsymbol{\sigma}^4 := \boldsymbol{\sigma}^3_{bd}$ we have $\mathsf{U}_b(\boldsymbol{\sigma}^4) = \frac{2}{3}$ and $\mathsf{U}_d(\boldsymbol{\sigma}^4) = \frac{3}{8}$. Now observe that the coloring induced by $\boldsymbol{\sigma}^4$ is the same as the one induced by $\boldsymbol{\sigma}^0$, see Figure 4.10 (a), where a exchanges position with b and c exchanges position with d. So, the sequence of profitable swaps defined above can be repeated over and over mutatis mutandis. ■

However, even if convergence to an equilibrium is not guaranteed for $k \geq 2$, they are guaranteed to exist for $k = 2$.

▶ **Theorem 4.11.** Every 2-SSG played on an 8-grid has an equilibrium that can be computed in polynomial time. ◀

Proof. Remember that we denote with ℓ the number of rows and with h the number of columns. Assume without loss of generality that the grid is such that

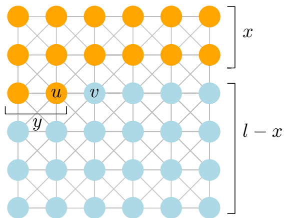


Figure 4.11: The structure of an equilibrium when $o \geq 2h-1$. See the proof of Theorem 4.11 for more details.

$h \leq \ell$. If this is not the case, simply rotate the grid by ninety degrees. We give two different constructions depending on how the number of orange agents compares with the threshold $2h-1$.

If $o \geq 2h-1$, let $\boldsymbol{\sigma}$ be the strategy profile in which orange agents occupy the grid starting from the upper-left corner and proceedings towards the right, filling the grid in increasing order of rows, see Figure 4.11 for a pictorial example. Denote by x the number of entirely orange rows and by y the number of orange vertices in the unique row containing both, orange and blue vertices, if this row exists, otherwise set $y=0$. Moreover, whenever $y \neq 0$, let u be the last orange vertex, i.e., the y-th vertex along the $(x+1)$-th row, and v be the first blue one, i.e., the vertex at the right of u; again, see Figure 4.11 for an example. Observe that, by the assumption $o \geq 2h-1$ and the fact that $o \leq b$, the following two properties hold:

(P.1) $x \geq 1$ and $x=1$ if and only if $y=h-1$;

(P.2) $x \leq \ell-2$ and $x=\ell-2$ only if $y=0$.

Fix an orange agent i. It is easy to see that, by property *(P.1)*, it holds that

$$\mathsf{U}_i(\boldsymbol{\sigma}) \geq \begin{cases} \frac{2}{3} & \text{if } \boldsymbol{\sigma}(i) \text{ is a corner vertex,} \\ \frac{3}{5} & \text{if } \boldsymbol{\sigma}(i) \text{ is a border vertex unless } y=1 \text{ which gives } \mathsf{U}_i(\boldsymbol{\sigma})=\frac{2}{5}, \\ \frac{5}{8} & \text{if } \boldsymbol{\sigma}(i) \text{ is an inner vertex unless } \boldsymbol{\sigma}(i)=u \text{ which gives } \mathsf{U}_i(\boldsymbol{\sigma})=\frac{1}{2}. \end{cases}$$

Fix a blue agent j. It is easy to see that, by property *(P.2)*, it holds that

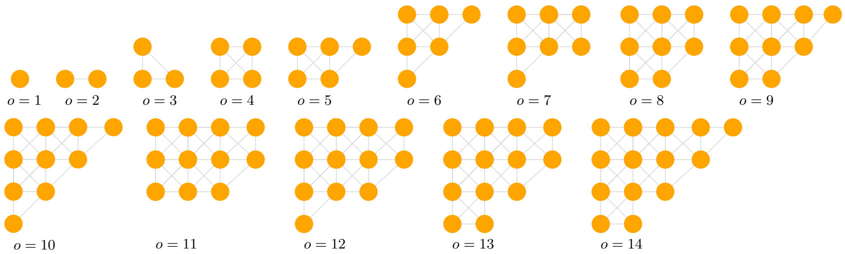


Figure 4.12: The structure of an equilibrium when $o < 2h - 1$ and $o \in [14]$. Only the orange vertices are depicted. See the proof of Theorem 4.11 for more details.

$$\mathsf{U}_j(\boldsymbol{\sigma}) \geq \begin{cases} \frac{2}{3} & \text{if } \boldsymbol{\sigma}(j) \text{ is a corner vertex,} \\ \frac{3}{5} & \text{if } \boldsymbol{\sigma}(j) \text{ is a border vertex unless } y = h-1 \text{ then } \mathsf{U}_j(\boldsymbol{\sigma}) = \frac{2}{5}, \\ \frac{5}{8} & \text{if } \boldsymbol{\sigma}(j) \text{ is an inner vertex unless } \boldsymbol{\sigma}(j) = v \text{ then } \mathsf{U}_j(\boldsymbol{\sigma}) = \frac{1}{2}. \end{cases}$$

As $\frac{2}{5} + \min\{\frac{2}{3}, \frac{3}{5}, \frac{5}{8}\} \geq 1$, it follows by Lemma 4.2 that profitable swaps are possible in $\boldsymbol{\sigma}$ only between an orange agent i and a blue agent j satisfying one of the following three conditions:

(i) $\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{2}{5}$ and $\mathsf{U}_j(\boldsymbol{\sigma}) = \frac{2}{5}$,

(ii) $\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{2}{5}$ and $\mathsf{U}_j(\boldsymbol{\sigma}) = \frac{1}{2}$,

(iii) $\mathsf{U}_i(\boldsymbol{\sigma}) = \frac{1}{2}$ and $\mathsf{U}_j(\boldsymbol{\sigma}) = \frac{2}{5}$.

(i) requires $1 = y = h - 1$ which implies $h = 2$ so that $1_{ij}(\boldsymbol{\sigma}) = 1$. By $\deg_{\boldsymbol{\sigma}(i)} = \deg_{\boldsymbol{\sigma}(j)} = 5$, we get $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{5}$ satisfying the condition of Lemma 4.2.

(ii) requires $y = 1$, which yields $\boldsymbol{\sigma}(i) = u$, and $\boldsymbol{\sigma}(j) = v$ so that $1_{ij}(\boldsymbol{\sigma}) = 1$. By $\deg_{\boldsymbol{\sigma}(i)} = 5$ and $\deg_{\boldsymbol{\sigma}(j)} = 8$, we get $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{5}$ again satisfying the condition of Lemma Lemma 4.2.

(iii) requires $y = h - 1$, which yields $\boldsymbol{\sigma}(j) = v$, and $\boldsymbol{\sigma}(i) = u$ so that $1_{ij}(\boldsymbol{\sigma}) = 1$. By $\deg_{\boldsymbol{\sigma}(j)} = 5$ and $\deg_{\boldsymbol{\sigma}(i)} = 8$, we get $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{5}$ satisfying the condition of Lemma 4.2. Thus, $\boldsymbol{\sigma}$ is an equilibrium and can be constructed in polynomial time.

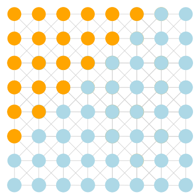


Figure 4.13: The structure of an x-triangle, with $x = 6$. The grid needs to have additional blue rows and columns which are not depicted. See the proof of Theorem 4.11 for more details.

If $o < 2h - 1$, a more involved construction is needed. For any $o \in [14]$, the proposed strategy profile $\boldsymbol{\sigma}$ is depicted in Figure 4.12. We stress that the two assumptions $h \leq \ell$ and $o < 2h - 1$ imply that the grid is large enough to accommodate a coloring implementing $\boldsymbol{\sigma}$. It is not difficult to check by direct inspection that $\boldsymbol{\sigma}$ is an equilibrium. To this aim, it is important to observe that, when $o \geq 7$, there must be at least two blue agents occupying vertices on the first row, otherwise the assumption $o < 2h - 1$ would be contradicted.

Now, for any $15 \leq o < 2h - 1$, we propose a general rule, which can be implemented in polynomial time, to construct an equilibrium profile $\boldsymbol{\sigma}$. First, we define some suitable structures. For an integer $x \geq 5$, an x-*triangle* is a strategy profile obtained as follows: for each $y = x$ down to 1, y orange agents are assigned to the first y vertices of the $(x + 1 - y)$-th row, see Figure 4.13. Thus, a total of $\frac{x(x+1)}{2}$ orange agents are assigned.

For an integer $x \geq 5$, an $(x, 1)$-*almost triangle* is a strategy profile obtained by assigning x orange agents to the first x vertices of the first two rows, $x - 1$ orange agents to the first $x - 1$ vertices of the third row, and then, for each $y = x - 3$ down to 2, y orange agents are assigned to the first y vertices of the $(x + 1 - y)$-th row, see the top-left picture in Figure 4.14. Thus, a total of

$$\sum_{i=2}^{x-3} i + 3x - 1 = \frac{x(x+1)}{2} + 1$$

orange agents are assigned.

For a pair of integers (x, y), with $x \geq 5$ and $2 \leq y \leq x$, we define an (x, y)-*almost triangle* as follows: for $2 \leq y \leq x - 2$, the (x, y)-almost triangle is obtained from the $(x, y - 1)$-one by locating an orange agent to the first non-orange vertex

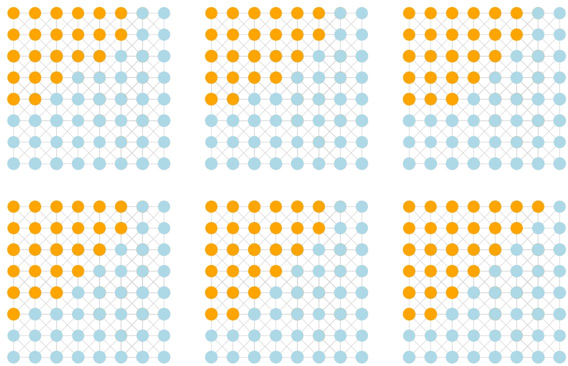


Figure 4.14: The structure of (x, y)-triangles, with $x = 6$ and $y \in [6]$. The grid needs to have additional blue rows and columns which are not depicted. See the proof of Theorem 4.11 for more details.

of the $(y+2)$-th row; the $(x, x-1)$-almost triangle is obtained by locating an orange agent to the first non-orange vertex, i.e., the second, of the x-th row of the $(x, x-2)$-one; the (x, x)-almost triangle is obtained by locating an orange agent to the first non-orange vertex, i.e., the $(x+1)$-th, of the first row of the $(x, x-1)$-one, see Figure 4.14 for a pictorial example.

Now observe that any number $o \geq 15$ can be decomposed as $o = \dfrac{x(x+1)}{2} + y$ for some integers x and y such that $x \geq 5$ and $0 \leq y \leq x$. The strategy profile $\boldsymbol{\sigma}$ is the x-triangle if $y = 0$ and the (x, y)-almost triangle, otherwise. Clearly, $\boldsymbol{\sigma}$ can be constructed in polynomial time. We are left to prove that $\boldsymbol{\sigma}$ is an equilibrium. We shall use Lemma 4.2 in conjunction with the following claims which can be easily verified with the help of Figure 4.13 and Figure 4.14. In any x-triangle $\boldsymbol{\sigma}$ with $x \geq 5$, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \frac{2}{5}$ for any orange agent i and $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \frac{5}{8}$ for any blue agent j. Thus, $\boldsymbol{\sigma}$ is an equilibrium. Now, let us consider (x, y)-almost triangles.

If $y \in [x-3]$, we have $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \frac{1}{2}$ for any orange agent i and $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \frac{1}{2}$ for any blue agent j. So, $\boldsymbol{\sigma}$ is an equilibrium.

If $y = x-2$, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \frac{1}{2}$ for each orange agent i, except for the one occupying the unique orange vertex at the x-th row who gets utility equal to $\frac{2}{5}$; moreover, $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \frac{5}{8}$ for each blue agent j, except for the one occupying the first blue vertex of the x-th row, see the bottom-left picture in Figure 4.14.

Thus, we get

$$\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1_{ij}(\boldsymbol{\sigma})}{\min\{\deg_{\boldsymbol{\sigma}(i)}, \deg_{\boldsymbol{\sigma}(j)}\}}$$

for each orange agent i and blue agent j. So, $\boldsymbol{\sigma}$ is an equilibrium. If $y = x - 1$, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \frac{1}{2}$ for each orange agent i and $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \frac{5}{8}$ for each blue agent j, thus implying that $\boldsymbol{\sigma}$ is an equilibrium, see the bottom-middle picture in Figure 4.14.

Finally, if $y = x$, $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \frac{2}{5}$ for each orange agent i and $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \frac{5}{8}$ for each blue agent j. see the bottom-right picture in Figure 4.14, and so also in this case $\boldsymbol{\sigma}$ is an equilibrium. ∎

4.3 Price of Anarchy for Two Types of Agents

In the following section, we consider the efficiency of equilibrium assignments and bound the Price of Anarchy for different classes of underlying graphs. In particular, besides investigating general graphs, cf. Section 4.3.1, we analyze regular graphs, cf. Section 4.3.2, cycles, paths, cf. Section 4.3.3, 4-grids and 8-grids, cf. Section 4.3.4. Agarwal et al. [Aga+21] already proved that the Price of Anarchy for the 2-SSG is in $\Theta(n)$ on underlying star graphs if there are at least two agents of each type and between $\frac{667}{324}$ and 4 for the balanced version, i.e., $o = \frac{n}{2}$. We improve this result by providing an upper bound of $\frac{8}{3}$ which tends to 2 for n going to infinity. Furthermore, the authors of [Aga+21] showed that the Price of Anarchy can be unbounded for $k \geq 3$ using a cycle topology with additional leaves. Note that topological restrictions could circumvent this non-existence result. Nevertheless, we concentrate on the (local) 2-SSG for several graph classes.

4.3.1 General Graphs

Remember that for a 2-SSG game, we assume that o is the less frequent color.

We significantly improve and generalize the results of [Aga+21] for the case of $o > 1$ by providing a general upper bound of $\frac{no(n-o)-n}{o(o-1)(n-o)}$. For balanced games, it yields an upper bound of $\frac{2(n+2)}{n}$ which shows that the PoA tends to 2 as the number of vertices increases. Moreover, if $\frac{b}{o} \in \mathcal{O}(1)$, the PoA is constant.

With the help of Lemma 4.2, we can now prove our general upper bound for the 2-SSG.

▶ **Theorem 4.12.** The PoA of 2-SSGs with $o > 1$ is at most $\frac{no(n-o)-n}{o(o-1)(n-o)}$. Hence, the PoA $\in \mathcal{O}\left(\frac{b}{o}\right)$. ◀

Proof. Fix a 2-SSG with $o > 1$ orange agents played on a graph G with n vertices. First, we observe that the social welfare of a social optimum is at most

$$n - 2 + \frac{o-1}{o} + \frac{b-1}{b} = n - \frac{1}{o} - \frac{1}{b},$$

as there must be at least one orange vertex that is adjacent to at least one blue vertex, thus getting utility at most $\frac{o-1}{o}$, and at least one blue vertex that is adjacent to at least one orange vertex, thus getting utility at most $\frac{b-1}{b}$.

Given a strategy profile $\boldsymbol{\sigma}'$, a *feasible pair* is a pair of vertices (u, v) such that u and v are occupied by agents of different colors in $\boldsymbol{\sigma}'$ and $\{u, v\} \notin E(G)$, i.e., u and v are not adjacent. Now fix a swap equilibrium $\boldsymbol{\sigma}$ and consider a maximum cardinality matching M of feasible pairs. Clearly $0 \leq |M| \leq o$. Hence, $|M| = o - x$ for some $0 \leq x \leq o$. If $x > 0$, then, there are exactly x orange and at least x blue leftover vertices of V that do not belong to any feasible pair in M. As M has maximum cardinality, each orange leftover vertex has to be adjacent to all leftover blue ones and vice-versa. That is, for each leftover vertex u, we have $\deg_u(G) \geq x$. Let T be a set of pairs of vertices obtained by matching each leftover orange vertex with a leftover blue one. By Lemma 4.2, for each $(u, v) \in M$, it holds that

$$\mathrm{U}_{\boldsymbol{\sigma}^{-1}(u)}(\boldsymbol{\sigma}) + \mathrm{U}_{\boldsymbol{\sigma}^{-1}(v)}(\boldsymbol{\sigma}) \geq 1$$

and for each $(u, v) \in T$, it holds that $\mathrm{U}_{\boldsymbol{\sigma}^{-1}(u)}(\boldsymbol{\sigma}) + \mathrm{U}_{\boldsymbol{\sigma}^{-1}(v)}(\boldsymbol{\sigma}) \geq 1 - \frac{1}{x}$. Thus, the social welfare of $\boldsymbol{\sigma}$ is at least $o - x + x(1 - \frac{1}{x}) = o - 1$. ■

▶ **Corollary 4.13.** The PoA of 2-SSGs is constant if $\frac{b}{o}$ is constant. ◀

We want to emphasize that for the case where both colors are perfectly balanced, the PoA is constant. Although this was already known [Aga+21], we provide an improved upper bound. As for $n = 2$, the 2-SSG is trivial and has a PoA = 1 and for $n = 4$ we can show that the PoA = 1 as well, we get the following corollary.

▶ **Corollary 4.14.** The PoA of balanced 2-SSGs is at most $\min\left\{\frac{8}{3}, \frac{2(n+2)}{n}\right\}$. ◀

Proof. We only have to show that for $n = 4$ the 2-SSG has a PoA = 1. In particular, assume that there are two orange and two blue agents. To show that PoA = 1, it suffices to show that either the underlying graph is a star or that the two orange agents are connected, and the two blue agents are connected. This is enough as the graph is connected and has four vertices.

Observe that it cannot be the case that both blue agents are connected only to orange agents and both orange agents are connected only to blue agents. If this were the case, there would exist an orange-blue pair that would like to swap. So, without loss of generality, the blue agents are connected.

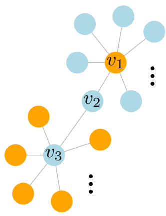
Now, assume that the orange agents are not connected, and thus have utility 0. Observe that it cannot be the case that an orange agent i is connected to both blue agents. If this were the case, consider the swap between i and the blue agent j that is also connected to the other orange agent. Then, i improves its utility by getting connected to the other orange agent, while j remains connected to the other blue agent and, at the same time, decrease the number of orange neighbors. So, every orange agent has only one blue neighbor.

Only two cases are remaining: The two orange agents have the same blue neighbor or they have different blue neighbors. If the orange agents have the same blue neighbor, this implies that the topology is a star with a blue center, hence, the assignment is a swap equilibrium and optimal in terms of social welfare. If the orange agents have different blue neighbors, then, the topology is a line with the two orange agents occupying the outer vertices and the two blue agents occupying the two inner vertices. This is clearly not a swap equilibrium, as, for instance, the left-most blue and the right-most orange want to swap. Hence, the PoA is 1 for $n = 4$. ■

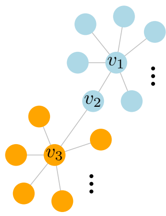
We now show that in contrast to the balanced 2-SSG, the balanced local k-SSG has a much higher LPoA.

▶ **Theorem 4.15.** The LPoA of local balanced 2-SSGs with $o > 1$ is between $2n + \frac{8}{n} - 8$ and $2n - \frac{8}{n}$. ◀

Proof. Fix a 2-SSG with $o > 1$ orange agents played on a graph G with n vertices. First, as derived in the proof of Theorem 4.12, we have that the social welfare of



(a) Worst equilibrium



(b) Social optimum

Figure 4.15: A lower bound for the local balanced 2-SSG. The agent types are marked orange and blue. See the proof of Theorem 4.15 for more details.

a social optimum is at most

$$n - 2 + \frac{o-1}{o} + \frac{n-o-1}{n-o} = n - \frac{n}{o(n-o)},$$

as there must be at least one orange vertex that is adjacent to at least one blue vertex.

Now fix a local swap equilibrium $\boldsymbol{\sigma}$. We show that the social welfare of $\boldsymbol{\sigma}$ is at least $\frac{1}{2}$. First, assume that there is exactly one vertex v with $\deg_v(G) > 1$. Then, G has to be a star and since $o > 1$ there has to be at least one leaf vertex with an agent i with $\mathsf{U}_i(\boldsymbol{\sigma}) = 1$. Therefore, there have to be at least two adjacent vertices v_1 and v_2 with $\deg_{v_i} > 1$ for $i \in \{1, 2\}$. By Lemma 4.2 we know that if v_1 and v_2 are occupied by agents of different types then $\mathsf{U}_{\boldsymbol{\sigma}^{-1}(v_1)} + \mathsf{U}_{\boldsymbol{\sigma}^{-1}(v_2)} \geq \frac{1}{2}$. Hence, assume that there is no such pair v_1 and v_2 and assume, without loss of generality, that all adjacent vertex pairs v_1 and v_2, with $\deg_{v_i} > 1$ for $i \in \{1, 2\}$, are occupied by orange agents. It follows, since G is connected, that all blue agents only occupy leaf vertices. If the social welfare of $\boldsymbol{\sigma}$ is less than $\frac{1}{2}$, all orange agents have to be surrounded by more blue than orange agents. Since one blue agent is only adjacent to one orange agent this contradicts our requirement of a balanced game. Hence, the PoA is upper bounded by $2\left(n - \frac{n}{o(n-o)}\right)$. With $o = \frac{n}{2}$ this is equal to $2n - \frac{8}{n}$.

For the lower bound consider the graph G in Figure 4.15. G consists of two stars which are connected by a common leaf vertex. Let v_1 be the center of the first star, v_3 be the center of the second star and v_2 be the common vertex. We first prove that the configuration shown in Figure 4.15 (a) is an equilibrium. Note, that none of the leaf vertices can perform a profitable swap since the agents on v_1 and v_3, respectively, would receive $\mathsf{U}_{\sigma^{-1}(v_1)} = 0$ and $\mathsf{U}_{\sigma^{-1}(v_2)} = 0$, respectively. So the only possible swap is between the agents placed on v_1 and v_2. However, the orange agent currently located on v_1 would not increase its utility by swapping since it would be surrounded only by two blue agents placed on v_1 and v_2 and therefore would receive a utility equal 0. Hence, no local swap is possible and only the agents placed on v_2 and v_3 receive positive utility. The social welfare is equal to $\frac{1}{2} + \frac{1}{o-1}$ which is for $o = \frac{n}{2}$ equal to $\frac{1}{2} + \frac{2}{n-2}$. The social optimum is shown in Figure 4.15 (b). This is easy to see since we meet the trivial upper bound

$$n - 2 + \frac{o-1}{o} + \frac{n-o-1}{n-o} = n - \frac{n}{o(n-o)}$$

which is for $o = \frac{n}{2}$ equal to $n - \frac{4}{n}$. Hence, the PoA is lower bounded by $\frac{2(n-2)^2}{n} = 2n + \frac{8}{n} - 8$. ■

If the underlying graph G does not contain leaf vertices, i.e., all vertices have at least degree 2, we can prove a smaller LPoA. In particular, if the ratio between the maximum and minimum degree of vertices in G is constant, we achieve a constant LPoA.

▶ **Theorem 4.16.** The LPoA of local 2-SSGs on a graph G with minimum degree $\delta \geq 2$ and maximum degree Δ is at most $2\left(1 + \frac{\Delta+1}{\delta-1}\right)$. ◀

Proof. Fix a local swap equilibrium $\boldsymbol{\sigma}$ on G. Let $\rho := \frac{\delta-1}{2\delta}$ and let o' and b' be the numbers of orange and blue agents that have a utility strictly less than ρ, respectively. Clearly, $o - o'$ and $b - b'$ are the numbers of orange and blue agents that have a utility of at least ρ, respectively. We first prove that $b - b' \geq \frac{\delta o'}{\Delta}$ as well as that $o - o' \geq \frac{\delta b'}{\Delta}$ and show then how these two inequalities imply the theorem statement.

We only prove the inequality $b - b' \geq \frac{\delta o'}{\Delta}$ as the proof of the other inequality is similar. Let i and j, respectively, be a blue agent and an orange agent that occupy two adjacent vertices in G, say $\sigma(i) = u$ and $\sigma(j) = v$, and such that $\mathsf{U}_j(\boldsymbol{\sigma}) < \rho$. By Lemma 4.2, we have that $\mathsf{U}_i(\boldsymbol{\sigma}) + \mathsf{U}_j(\boldsymbol{\sigma}) \geq 1 - \frac{1}{\delta}$, from which we derive

$$\mathsf{U}_i(\boldsymbol{\sigma}) > 1 - \frac{1}{\delta} - \frac{\delta - 1}{2\delta} = \frac{\delta - 1}{2\delta} = \rho.$$

Let G' be the subgraph of G containing all the non-monochrome edges, i.e., each edge of G' connects a vertex occupied by an orange agent with a vertex occupied by a blue agent. Clearly, G' is bipartite. Consider the vertex-induced subgraph H of G' in which we have all the o' orange agents having a utility strictly less than ρ on one side and all the $b - b'$ blue agents having a utility of at least ρ on the other side. Since for each vertex v of H occupied by an orange agent, there are at least $(1-\rho)\deg_v \geq \frac{\delta+1}{2}$ vertices adjacent to u that are occupied by blue agents and each such blue agent has a utility of at least ρ, the degree of v in H is at least $\frac{\delta+1}{2}$. Therefore,

$$|E(H)| \geq \frac{\delta + 1}{2} o'. \tag{4.5}$$

Furthermore, since each edge of H is incident to a blue agent that has a utility of at least ρ, the degree in H of every vertex u that is occupied by a blue agent is at most $(1 - \rho)\deg_u \leq \frac{\delta+1}{2\delta}\Delta$. Therefore,

$$|E(H)| \leq \frac{\Delta(\delta + 1)}{2\delta}(b - b'). \tag{4.6}$$

Plugging Equation (4.5) into Equation (4.6) and simplifying gives $b - b' \geq \frac{\delta}{\Delta} o'$.

Finally, we show how $b - b' \geq \frac{\delta o'}{\Delta}$ and $o - o' \geq \frac{\delta b'}{\Delta}$ imply the theorem statement. The average utility of all the agents in H is at least

$$\frac{\rho(b - b')}{o' + (b - b')} \geq \frac{\rho \frac{\delta}{\Delta}}{1 + \frac{\delta}{\Delta}} = \frac{\delta - 1}{2(\delta + \Delta)}.$$

Similarly, the average utility of the b' blue agents whose utilities are strictly less than ρ and the $o - o'$ orange agents whose utilities are of at least ρ is also at least $\frac{\delta-1}{2(\delta+\Delta)}$. Therefore, the LPoA is at most $\frac{2(\delta+\Delta)}{\delta-1} = 2\left(1 + \frac{\Delta+1}{\delta-1}\right)$. ■

In Argarwal et al. [Aga+21], the authors showed that in the case where agents are unique in their type the PoA can be unbounded. We observe, by using the same instance from [Aga+21], that the LPoA on a graph with minimum degree $\delta = 1$ can be unbounded as well. For this, consider the star graph with Δ leaves and let $\boldsymbol{\sigma}$ be a strategy profile where the unique orange agent occupies the star center, while all the blue agents occupy the leaves. This is clearly a swap equilibrium of 0 social welfare. Any configuration in which a blue agent occupies the star center has strictly positive social welfare.

However, as the following theorem shows, the LPoA can be upper bounded by a function of Δ if we force $n \geq \Delta + 2$, i.e., we avoid the pathological star graph of $\Delta + 1$ vertices.

▶ **Theorem 4.17.** For every $\epsilon > 0$, the LPoA of local 2-SSGs on a graph G with maximum degree $\Delta \leq n - 2$ is between $\frac{\Delta(\Delta-1)}{2} - \epsilon$ and $2(\Delta^2 + 1)$. ◀

Proof. We claim that for every agent i, with $\deg_{\sigma(i)} \geq 2$, there is an agent j, with $\sigma(j) \in N(\sigma(i))$ and $\deg_{\sigma(j)} \geq 2$, such that $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \frac{1}{\Delta}$ or $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \frac{1}{2}$. Indeed, assume that $\mathsf{U}_i(\boldsymbol{\sigma}) < \frac{1}{\Delta}$. This implies that $\mathsf{U}_i(\boldsymbol{\sigma}) = 0$ and, therefore, that every agent occupying a vertex in $N(\sigma(i))$ is of a different type from that of i. Let j be an agent occupying a vertex in $N(\sigma(i))$ and such that $\deg_{\sigma(j)} \geq 2$. By Lemma 4.2 the sum of utilities of agents i and j is of at least $\frac{1}{2}$ and therefore, $\mathsf{U}_j(\boldsymbol{\sigma}) \geq \frac{1}{2}$.

This implies that all the vertices of the graph can be partitioned into two types of sets:

type-1 set: It has a size smaller than or equal to $\Delta + 1$ and contains a vertex u occupied by an agent that has a utility of at least $\frac{1}{\Delta}$ together with a subset of $N(u)$;

type-2 set: It has a size smaller than or equal to $1 + \Delta + \Delta(\Delta - 1) = \Delta^2 + 1$ and contains a vertex u occupied by an agent that has a utility of at least $\frac{1}{2}$ together with a subset of $N(u) \cup \bigcup_{v \in N(u)} N(v)$.

The average utility of all the agents contained in type-1 sets is at least $\frac{1}{\Delta^2+\Delta}$, while the average utility of all the agents contained in type-2 sets is at least $\frac{1}{2(\Delta^2+1)}$. Therefore, as $\Delta \geq 2$, the average utility of an agent is at least

$$\min\left\{\frac{1}{\Delta^2 + \Delta}, \frac{1}{2(\Delta^2 + 1)}\right\} = \frac{1}{2(\Delta^2 + 1)}.$$

The upper bound of the LPoA follows.

For the lower bound of the LPoA, it is enough to consider the instance with o orange agents and $b = (\Delta - 2)o$ blue agents – thus, $n = (\Delta - 1)o$ – consisting of a cycle of length o and whose vertices are all occupied by the orange agents and where each vertex of the cycle is the center of a star of $\Delta - 2$ 1-degree additional vertices that are occupied by the blue agents. Clearly, all the degree-1 vertices are occupied by the blue agents. The utility of an orange agent is equal to $\frac{2}{\Delta}$ while the utility of a blue agent is equal to 0. By Lemma 4.2, we have that the considered strategy profile is a local swap equilibrium. The social welfare of this local swap equilibrium is equal to $\frac{2o}{\Delta} = \frac{2n}{\Delta(\Delta-1)}$.

If we assume that $o = \Delta - 1$ and consider the strategy profile where the $\Delta - 1$ orange agents occupy any vertex of the cycle together with all the $\Delta - 2$ 1-degree vertices appended to it, and the blue agents occupy the remaining vertices, we have that the social welfare of the considered instance is equal to

$$n - 3 + 2\frac{\Delta - 1}{\Delta} + \frac{\Delta - 2}{\Delta} = n - \frac{4}{\Delta}.$$

Therefore, if we choose $n \geq \frac{2(\Delta-1)}{\epsilon}$, we have that the LPoA is lower bounded by

$$\left(n - \frac{4}{\Delta}\right)\frac{\Delta(\Delta - 1)}{2n} = \frac{\Delta(\Delta - 1)}{2} - \frac{2(\Delta - 1)}{n} \geq \frac{\Delta(\Delta - 1)}{2} - \epsilon.$$ ■

If we desist from star graphs, the class of trees meets the conditions required by Theorem 4.17 and we get the following corollary.

▶ **Corollary 4.18.** For every $\epsilon > 0$, the LPoA of local 2-SSGs on a tree graph G with maximum degree $\Delta \leq n - 2$ is at least $\frac{\Delta(\Delta-1)}{2} - \epsilon$. ◀

Proof. Consider the lower bound construction given in Theorem 4.17 in which we remove one edge from the cycle. There is a threshold value $f(\Delta, \epsilon)$ such that for every $n \geq f(\Delta, \epsilon)$, the LPoA is at least $\frac{\Delta(\Delta-1)}{2} - \epsilon$. ■

4.3.2 Regular Graphs

In this section, we provide upper and lower bounds on the LPoA for regular graphs, i.e., for graphs where all vertices have the same degree. The key is the following technical lemma which is later useful also for non-regular graphs.

▶ **Lemma 4.19.** Let σ be a local swap equilibrium, and let $\Delta = 2\zeta + \eta$, with $\zeta \in \mathbf{N}$ and $\eta \in \{0, 1\}$. Let $X \subseteq V$ be a subset of vertices such that $\deg_v = \Delta$ for every $v \in N(X) := \bigcup_{x \in X} N(x)$. Finally, let $Z \subseteq N(X)$ be the set of vertices occupied by the agents that have a utility strictly larger than $\rho := \frac{\zeta}{2\zeta+1}$. Then, the average utility of the agents that occupy the vertices in $X \cup Z$ is at least ρ. ◀

Proof. Let $X_o \subseteq X$ (respectively, $X_b \subseteq X$) be the set of vertices occupied by the orange (respectively, blue) agents that have a utility strictly less than ρ. Similarly, let $Z_o \subseteq N(X)$ (respectively, $Z_b \subseteq N(X)$) be the set of vertices occupied by the orange (respectively, blue) agents that have a utility strictly larger than ρ. We show that the average utility of the agents that occupy the vertices $X_o \cup Z_b$ (respectively, $X_b \cup Z_o$) is at least ρ. Notice that this immediately implies the theorem statement.

In the rest of the proof, without loss of generality, we prove that the average utility of the agents that occupy the vertices in $X_o \cup Z_b$ is at least ρ. First of all, we observe that the utility of each agent in $N(X)$ is in the set $\left\{\frac{\ell}{\Delta} \mid \ell = 0, \ldots, \Delta\right\}$. Let o_ℓ be the number of orange agents that occupy the vertices of X and whose utilities are equal to $\frac{\ell}{\Delta}$. Similarly, let b_ℓ be the number of orange agents that occupy the vertices of $N(X)$ and whose utilities are equal to $\frac{\ell}{\Delta}$. Since we are interested in the orange agents occupying the vertices of X_o, we consider the values o_ℓ such that $\frac{\ell}{\Delta} < \rho$, or, equivalently, $\ell \leq \zeta - 1$. Similarly, since we are interested in the blue agents occupying the vertices of Z_b, we consider the values $b_{\Delta-\ell-1}$ such that $\frac{\Delta-\ell-1}{\Delta} > \rho$, or, equivalently, $\ell \leq \zeta - 1$. We prove that, for every $0 \leq h \leq \zeta - 1$,

$$\sum_{\ell=0}^{h}(\ell+1)b_{\Delta-\ell-1} \geq \sum_{\ell=0}^{h}(\Delta-\ell)o_\ell. \tag{4.7}$$

We observe that if any orange agent i that occupies a vertex $v \in X_o$ has a utility of $\frac{\ell}{\Delta}$, where $0 \leq \ell \leq \zeta - 1$, then, since we are in a local swap equilibrium, any of the $\Delta - \ell$ blue agents that occupy the vertices in $N(v)$ has a utility of at least $\frac{\Delta-\ell-1}{\Delta} > \rho$ by Lemma 4.2.

Let G' be the (bipartite) subgraph of G containing all the non-monochrome edges. Consider the subgraph H of G' that is induced by the vertices in $X_h \subseteq X_o$ that are occupied by agents having a utility of at most $\frac{h}{\Delta}$ and the agents in $Z_h \subseteq Z_b$ having a utility of at least $\frac{\Delta-h-1}{\Delta}$. By construction, the degree of a vertex of X_h occupied by an agent of utility equal to $\frac{\ell}{\Delta}$, with $\ell \leq h$, is equal to $\Delta - \ell$. Therefore, if $\deg_v(H)$ denotes the degree of v in H, we have that

$$|E(H)| = \sum_{v \in X_h} \deg_v(H) = \sum_{\ell=0}^{h}(\Delta-\ell)o_\ell. \tag{4.8}$$

Since the degree in H of each vertex in Z_h that is occupied by a blue agent whose utility is equal to $\frac{\Delta-\ell-1}{\Delta}$, with $\ell \leq h$, is upper bounded by $\ell+1$, we have that

$$|E(H)| \leq \sum_{v \in Z_h} \deg_v(H) = \sum_{\ell=0}^{h} (\ell+1) b_{\Delta-\ell-1}. \tag{4.9}$$

Combining Equation (4.8) with Equation (4.9) gives Equation (4.7). We are now able to compute the average utility concerning the agents occupying the vertices in $X_o \cup Z_b$. The average utility of such agents equals

$$\mathrm{U}_{\mathrm{avg}} := \frac{\sum_{\ell=0}^{\zeta-1}\left(\frac{\Delta-\ell-1}{\Delta} b_{\Delta-\ell-1}\right) + \sum_{\ell=0}^{\zeta-1}\left(\frac{\ell}{\Delta} o_\ell\right)}{\sum_{\ell=0}^{\zeta-1} b_{\Delta-\ell-1} + \sum_{\ell=0}^{\zeta-1} o_\ell}.$$

Now, we prove that $\mathrm{U}_{\mathrm{avg}} \geq \rho$. We assume that the values of all the o_ℓ's are fixed and that there is at least one o_ℓ, with $0 \leq \ell \leq \zeta - 1$, that is strictly greater than 0. Since $\frac{\ell}{\Delta} < \rho$, while $\frac{\Delta-\ell-1}{\Delta} > \rho$, we have that $\mathrm{U}_{\mathrm{avg}}$ is minimized when the values we can assign to the $b_{\Delta-\ell-1}$'s, that must satisfy Equation (4.7) for every $0 \leq h \leq \zeta - 1$, are somehow minimized.

Since, for every $\ell < \ell'$ and every $0 < \epsilon < b_{\Delta-\ell'-1}$,

$$\frac{\Delta-\ell-1}{\Delta} > \frac{\Delta-\ell'-1}{\Delta}$$

as well as

$$(\ell'+1)(b_{\Delta-\ell-1}+\epsilon) + (\ell+1)(b_{\Delta-\ell'-1}-\epsilon) > (\ell'+1)b_{\Delta-\ell-1} + (\ell+1)b_{\Delta-\ell'-1},$$

we have that $\mathrm{U}_{\mathrm{avg}}$ is minimized exactly when $b_{\Delta-\ell-1} = \frac{\Delta-\ell}{\ell+1} o_\ell$.[7] Therefore, if we denote by $\Psi = \{\ell \mid 0 \leq \ell \leq \zeta - 1 \wedge o_\ell > 0\}$, we have that

$$\begin{aligned}\mathrm{U}_{\mathrm{avg}} &\geq \frac{\sum_{\ell\in\Psi}\left(\frac{(\Delta-\ell-1)(\Delta-\ell)}{\Delta(\ell+1)} o_\ell\right) + \sum_{\ell\in\Psi}\left(\frac{\ell}{\Delta} o_\ell\right)}{\sum_{\ell\in\Psi}\left(\frac{\Delta-\ell}{\ell+1} o_\ell\right) + \sum_{\ell\in\Psi} o_\ell} \\ &= \frac{\sum_{\ell\in\Psi} \frac{2\ell^2-2(\Delta-1)\ell+\Delta(\Delta-1)}{\Delta(\ell+1)}}{\sum_{\ell\in\Psi}\frac{\Delta+1}{\ell+1}} \geq \min_{\ell\in\Psi} \frac{2\ell^2-2(\Delta-1)\ell+\Delta(\Delta-1)}{\Delta(\Delta+1)}.\end{aligned}$$

7 We are relaxing the constraint that $b_{\Delta-\ell-1}$ must be an integer.

We complete the proof by showing that

$$\min_{\ell \in \Psi} \frac{2\ell^2 - 2(\Delta - 1)\ell + \Delta(\Delta - 1)}{\Delta(\Delta + 1)} \geq \rho. \tag{4.10}$$

The numerator of the left-hand side of Equation (4.10) is a parabola with respect to the variable ℓ and is therefore minimized when ℓ is chosen as closest as possible to the value $\frac{\Delta-1}{2}$.

As $\left\lfloor \frac{\Delta-1}{2} \right\rfloor \geq \zeta - 1$ and $\ell \leq \zeta - 1$, it follows that the value of ℓ that minimizes Equation (4.10) is $\ell = \zeta - 1$. Therefore,

$$\frac{2(\zeta - 1)^2 - 2(2\zeta - 1)(\zeta - 1) + 2\zeta(2\zeta - 1)}{2\zeta(2\zeta + 1)} = \rho.$$

Hence, $\mathrm{U}_{\mathrm{avg}} \geq \rho$. ■

▶ **Corollary 4.20.** The LPoA of local 2-SSG on a regular graph G with $\Delta = 2\zeta + \eta$, with $\zeta \geq 1$ and $\eta \in \{0, 1\}$ is at most $2 + \frac{1}{\zeta}$. ◀

Proof. The corollary follows from Lemma 4.19 by setting $X = V$. ■

The matching lower bound is provided in the following.

▶ **Theorem 4.21.** The LPoA of local 2-SSG on a regular graph G with $\Delta = 2\zeta + \eta$, with $\zeta \geq 1$ and $\eta \in \{0, 1\}$ is equal to $2 + \frac{1}{\zeta}$. ◀

Proof. For a fixed degree $\Delta \geq 3$[8], we define the Δ-regular graph $G(\Delta) := G$ as follows: There are $q := t(\Delta + 1)$ gadgets $G^1, \ldots, G^q$. For each $i \in [q]$, gadget G^i is obtained from a complete graph of $\Delta + 1$ vertices, denoted as $v_0^1, \ldots, v_\Delta^i$, by removing edge $\{v_0^i, v_\Delta^i\}$. Observe that, by construction, for any $i \in [q]$, each vertex v_j^i, with $1 \leq j \leq \Delta - 1$, has degree Δ, while vertices v_0^i and v_Δ^i have degree $\Delta - 1$. We obtain G by connecting the q gadgets through edges $\{v_\Delta^i, v_0^{i+1}\}$ for each $i \in [q - 1]$ and edge $\{v_\Delta^q, v_0^1\}$. Call these edges *extra-gadget* edges. Thus, G is connected and Δ-regular. Consider now the local 2-SSG played on G in which there are $\lceil \frac{\Delta+1}{2} \rceil q$ blue agents and $\lfloor \frac{\Delta+1}{2} \rfloor q$ orange ones.

On the one hand, the social optimum is at least

$$n - \frac{4}{\Delta} = q(\Delta + 1) - 4\Delta,$$

8 We assume $\Delta \geq 3$ as for $\Delta = 2$ the regular graph G would collapse to a cycle.

as in the strategy profile in which all vertices of the first $\lceil\frac{\Delta+1}{2}\rceil t$ gadgets are colored blue and all vertices of the remaining $\lfloor\frac{\Delta+1}{2}\rfloor t$ gadgets are colored orange, there are $n-4$ vertices getting utility 1, and 4 vertices getting utility $\frac{\Delta-1}{\Delta}$.

On the other hand, the strategy profile $\boldsymbol{\sigma}$ in which the first $\lceil\frac{\Delta+1}{2}\rceil$ vertices of each gadget are colored blue and the remaining ones are colored orange is a swap equilibrium. As extra-gadget edges connect vertices of different colors, every blue vertex is adjacent to $\lceil\frac{\Delta+1}{2}\rceil - 1$ blue ones, while every orange vertex is adjacent to $\lceil\frac{\Delta+1}{2}\rceil$ blue ones. If a blue vertex swaps with an adjacent orange one, it ends up being adjacent to $\lceil\frac{\Delta+1}{2}\rceil - 1$ blue vertices. Thus, no profitable swap exists in $\boldsymbol{\sigma}$.

As the social welfare of $\boldsymbol{\sigma}$ is

$$\begin{aligned}
&\frac{q}{\Delta}\left(\left\lceil\frac{\Delta+1}{2}\right\rceil\left(\left\lceil\frac{\Delta+1}{2}\right\rceil-1\right)+\left\lfloor\frac{\Delta+1}{2}\right\rfloor\left(\left\lfloor\frac{\Delta+1}{2}\right\rfloor-1\right)\right)\\
=&\begin{cases}\frac{q(\Delta^2-1)}{2\Delta} & \text{if } q \text{ is odd,}\\ \frac{q\Delta}{2} & \text{if } q \text{ is even,}\end{cases}
\end{aligned}$$

we get that the LPoA of the game is lower bounded by

$$\frac{2\Delta(q(\Delta+1)-4\Delta)}{q(\Delta^2-1)}$$

when Δ is odd and by $\frac{2(q(\Delta+1)-4\Delta)}{q\Delta}$ when Δ is even. By letting q going to infinity, we get $\frac{2\Delta}{\Delta-1}$ and $\frac{2(\Delta+1)}{\Delta}$, respectively. By using $\Delta = 2\zeta+1$ in the first case, and $\Delta = 2\zeta$ in the second one, we finally obtain the lower bound of $2+\frac{1}{\zeta}$. ■

4.3.3 Paths and Cycles

In this section, we provide upper and lower bounds for the (L)PoA of paths and cycles. We first provide a full characterization of the PoA for cycles.

▶ **Theorem 4.22.** The PoA of 2-SSGs played on cycles with $n \geq 3$ vertices and $o = 2\zeta+\eta$ orange agents, where $\zeta \in \mathbf{N}$, $\eta \in \{0,1\}$, and $b \geq o$, is equal to

$$\text{PoA} = \begin{cases}1 & \text{if } o = 1;\\ \frac{n-2}{b+\eta} & \text{otherwise.}\end{cases}$$

◀

Proof. The social welfare of the social optimum is clearly equal to $n-2$ and is attained when the cycle contains one path whose vertices are all occupied by the b blue agents and another path whose vertices are all occupied by the o orange agents. Now, we prove matching upper and lower bounds for all the cases.

When $o=1$ we clearly have that any strategy profile is a swap equilibrium because the unique orange agent always has a utility of 0, the two blue agents that occupy the vertices adjacent to the vertex occupied by the orange agent have a utility of $\frac{1}{2}$ each, and the remaining $b-2$ blue agents all have a utility of 1. Therefore, the social welfare is equal to $n-2$, and the claim follows.

Let $\boldsymbol{\sigma}$ be a swap equilibrium. Let ℓ be the number of maximal vertex-induced (sub)paths whose vertices are occupied by orange agents only. Clearly, ℓ is also the number of maximal vertex-induced (sub)paths whose vertices are occupied by blue agents only. We claim that $\ell \leq \zeta$ by showing that every agent has a strictly positive utility in $\boldsymbol{\sigma}$, i.e., each of the 2ℓ maximal paths formed by monochrome edges contains 2 or more vertices. Indeed, for the sake of contradiction, assume without loss of generality that there is an orange agent i such that $\mathsf{U}_i(\boldsymbol{\sigma})=0$. This implies that there must be a blue agent j that occupies a vertex v such that v is not adjacent to the vertex occupied by i and v is adjacent to a vertex occupied by an orange agent $i' \neq i$. As a consequence, $\mathsf{U}_j(\boldsymbol{\sigma}) \leq \frac{1}{2}$. In this case, swapping i with j would be an improving move since $u_i(\boldsymbol{\sigma}_{ij}) > 0 = u_i(\boldsymbol{\sigma})$ and $1 = u_j(\boldsymbol{\sigma}_{ij}) > \frac{1}{2} \geq u_j(\boldsymbol{\sigma})$, thus contradicting the fact that $\boldsymbol{\sigma}$ is a swap equilibrium.

As a consequence the utility of 2ℓ orange agents is equal to $\frac{1}{2}$, while the utility of the other $o-2\ell = n-b-2\ell$ orange agents is equal to 1; similarly, the utility of 2ℓ blue agents is equal to $\frac{1}{2}$, while the utility of the other $b-2\ell$ blue agents is equal to 1. Therefore, the social cost is at least

$$\frac{1}{2}(2\ell+2\ell)+(n-b-2\ell)+(b-2\ell) = n-2\ell \geq n-2\zeta = b+\eta.$$

The upper bound to the PoA follows. For the matching lower bound, it is enough to consider the strategy profile in which $\ell=\zeta$, i.e., there are $\zeta-1$ maximal vertex-induced paths occupied by orange (respectively, blue) agents only of length 2 each, and one maximal vertex-induced path occupied by orange (respectively, blue) agents only of length $2+\eta$ (respectively, $b-2\zeta+2$). In this case, the social welfare is exactly equal to

$$\frac{1}{2}2\zeta + \eta + \frac{1}{2}\zeta + (b - 2\zeta) = b + \eta. \qquad \blacksquare$$

The following theorem provides almost tight upper bounds to the LPoA for cycles.

▶ **Theorem 4.23.** The LPoA of local 2-SSGs played on cycles with $n = 3\zeta + \eta$ vertices and b blue agents, where $\zeta \in \mathbf{N}$, $\eta \in \{0, 1, 2\}$, and $b \geq o$, is upper bounded by

$$\text{PoA} \leq \begin{cases} 1 & \text{if } o = 1; \\ \frac{n-2}{b-o} & \text{if } o \geq 2 \text{ and } b \geq 2o; \\ \frac{n-2}{\zeta+\eta} & \text{otherwise (i.e., } o \geq 2 \text{ and } b < 2o). \end{cases}$$

The upper bounds are tight when (i) $o = 1$ and (ii) $o \geq 2$ and $b \geq 2o$. ◀

Proof. The social welfare of the social optimum is equal to $n - 2$. Now, we prove matching upper and lower bounds for all cases.

When $o = 1$, any configuration is a (local) swap equilibrium; therefore the social welfare is equal to $n - 2$ and the claim follows.

Now, we consider the case in which $o \geq 2$. Let o_h and b_h be the numbers of orange and blue agents having a utility equal to $h \in \left\{0, \frac{1}{2}, 1\right\}$, respectively. Every configuration can be decomposed into maximal vertex-induced paths whose vertices are all occupied by agents of the same type. Furthermore, if ℓ is the overall number of these maximal vertex-induced paths whose vertices are all occupied by orange agents, then ℓ is also the overall number of maximal vertex-induced paths whose vertices are all occupied by blue agents. This implies that $o_{\frac{1}{2}} = 2(\ell - o_0)$ and $b_{\frac{1}{2}} = 2(\ell - b_0)$. Therefore,

$$o = o_0 + o_{\frac{1}{2}} + o_1 = 2\ell - o_0 + o_1$$

and

$$b = b_0 + b_{\frac{1}{2}} + b_1 = 2\ell - b_0 + b_1,$$

i.e., $o_1 = o - 2\ell + o_0$ and $b_1 = b - 2\ell + b_0$. As a consequence, using the fact that

$b + o = n$, the social welfare is equal to

$$\sum_{h\in\left\{0,\frac{1}{2},1\right\}} ho_h + \sum_{h\in\left\{0,\frac{1}{2},1\right\}} hb_h = \ell - o_0 + o - 2\ell + o_0 + \ell - b_0 + b - 2\ell + b_0 = n - 2\ell.$$

We observe that each orange agent of utility 0 occupies a vertex that is adjacent to two vertices occupied by blue agents having a utility of $\frac{1}{2}$ each. As a consequence, $b_{\frac{1}{2}} = 2(\ell - b_0) \geq 2o_0$, or, equivalently, $\ell \geq b_0 + o_0$. Therefore, the social welfare is minimized exactly when ℓ is maximized, as shown by the following ILP (where the second and third constraints are of the form $o_0 + o_{\frac{1}{2}} \leq o$ and $b_0 + b_{\frac{1}{2}} \leq b$, respectively):

$$\begin{aligned} \text{maximize} \quad & \ell \\ \text{subject to} \quad & b_0 + o_0 \leq \ell \\ & 2\ell - o_0 \leq o \\ & 2\ell - b_0 \leq b \\ & \ell, b_0, o_0 \in \mathbf{N}. \end{aligned}$$

Combining the first three inequalities, we obtain

$$2\ell + 2\ell \leq o + o_0 + b + b_0 \leq n + \ell,$$

from which we derive $\ell \leq \lfloor \frac{n}{3} \rfloor = \zeta$. Furthermore, since $o_0 \leq \ell$, we have that $\ell \leq 2\ell - o_0 \leq o$. Therefore, the value of an optimum solution is upper bounded by $\ell = \min\{o, \zeta\}$. If $b \geq 2o$, then setting ℓ, $o_0 = o$, and all other variables to 0 is an optimal solution. If $b < 2o$, then setting $\ell = \zeta$, $o_0 = 2\zeta - o$, and $b_0 = 2\zeta - b$ is an optimal solution. The upper bound to the LPoA follows.

For the matching lower bound when $o \geq 2$ and $b \geq 2o$, it is enough to consider the strategy profile in which $\ell = o$, i.e., each orange agent occupies a vertex that is adjacent to vertices occupied by blue agents only. As a consequence, the o orange agents have a utility of 0, the $2o$ blue agents have a utility of $\frac{1}{2}$ each, while the remaining $b - 2o = n - 3o \geq 0$ blue agents have a utility of 1 each. The social welfare in this case is exactly equal to $\frac{1}{2}2o + n - 3o = n - 2o = b - o$. ■

We now prove similar results for paths.

▶ **Theorem 4.24.** The PoA of 2-SSGs played on paths with $n \geq 3$ vertices and

$o = 2\zeta + \eta$ orange agents, where $\zeta \in \mathbb{N}$, $\eta \in \{0, 1\}$, and $b \geq o$, is equal to

$$\text{PoA} = \begin{cases} +\infty & \text{if } n = 3; \\ \frac{2n-2}{2n-5} & \text{if } n > 3 \text{ and } o = 1; \\ \frac{n-1}{b+1+\eta} & \text{if } n > 3,\, o \geq 2 \text{ and } b \leq 2\zeta + 1; \\ \frac{n-1}{b+\eta} & \text{otherwise (i.e., } o \geq 2 \text{ and } b \geq 2\zeta + 2). \end{cases}$$

◀

Proof. For $n \geq 4$, the social welfare of the social optimum is equal to $n - 1$ and is attained when the path contains a subpath whose vertices are all occupied by the b blue agents and one subpath whose vertices are all occupied by the o orange agents. For $n = 3$, the social welfare of the social optimum is equal to $\frac{3}{2}$ and is attained when the orange agent occupies one end vertex of the path. Now, we prove matching upper and lower bounds for all the cases.

When $o = 1$, we clearly have that any strategy profile is a swap equilibrium. The strategy profile with minimum social welfare is when the orange agent occupies a vertex that is adjacent to an end vertex of the path. In this case, the blue agent that occupies such an end vertex has a utility of 0, the orange agent has a utility of 0, the other blue agent that is adjacent to the vertex occupied by the orange agent has a utility of 0, if $n = 3$, and of $\frac{1}{2}$, if $n \geq 4$, while all the other blue agents, if any, have a utility of 1 each. Therefore, for $n = 3$ the social welfare is 0, while for $n \geq 4$, the social welfare is equal to $n - \frac{5}{2}$, and the claim follows.

Therefore, we are only left to prove the bounds to the PoA when $n > 3$ and $o \geq 2$. Let $\boldsymbol{\sigma}$ be a swap equilibrium. We first show that every agent has a strictly positive utility in $\boldsymbol{\sigma}$. Indeed, for the sake of contradiction, assume without loss of generality that there is an orange agent i such that $\mathsf{U}_i(\boldsymbol{\sigma}) = 0$. This implies that there must be a blue agent j that occupies a vertex v such that v is not adjacent to the vertex occupied by i and v is adjacent to a vertex occupied by an orange agent $i' \neq i$. As a consequence, $\mathsf{U}_j(\boldsymbol{\sigma}) \leq \frac{1}{2}$. In this case, swapping i with j would be an improving move since $\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) > 0 = \mathsf{U}_i(\boldsymbol{\sigma})$ and $1 = \mathsf{U}_j(\boldsymbol{\sigma}_{ij}) > \frac{1}{2} \geq \mathsf{U}_j(\boldsymbol{\sigma})$, thus contradicting the fact that $\boldsymbol{\sigma}$ is a local swap equilibrium.

Let ℓ be the number of maximal vertex-induced (sub)paths whose vertices are all occupied by the orange agents. Since every orange agent has strictly positive utility, it follows that $\ell \leq \zeta$. Let x and y be the number of orange and blue agents that occupy the end vertices of the path, respectively. Clearly $x + y = 2$.

Let ℓ' be the number of maximal vertex-induced (sub)paths whose vertices are all occupied by the blue agents. We have that $\ell' \leq \ell + 1$. Furthermore, the utility of $2\ell - x$ orange agents is $\frac{1}{2}$ while the utility of the other $o - 2\ell + x$ orange agents is 1; similarly, the utility of $2\ell' - y$ blue agents is $\frac{1}{2}$, while the utility of the other $b - 2\ell' + y$ blue agents is 1. Therefore, the social welfare is at least

$$\frac{1}{2}(2\ell - x + 2\ell' - y) + (o - 2\ell + x) + (b - 2\ell' + y) = n + \frac{1}{2}(x + y) - \ell - \ell' \geq n + 1 - \ell - \ell'.$$

If $b \leq 2\zeta + 1$, then $\ell' \leq \zeta$ and therefore

$$n + 1 - \ell - \ell' \geq n + 1 - 2\zeta = b + 1 + \eta.$$

If $b \geq 2\zeta + 2$, then $\ell' \leq \ell + 1$ and therefore

$$n + 1 - \ell - \ell' \geq n - 2\zeta = b + \eta.$$

For the matching lower bound, consider the strategy profile that induces $\ell = \zeta$ maximal vertex-induced paths occupied by orange agents only and ℓ' maximal vertex-induced paths that are occupied by blue agents only, where $\ell' = \zeta$ if $b \leq 2\zeta + 1$ and to $\ell' = \ell + 1$ otherwise. In this case, the social welfare is exactly equal to $b + 1 + \eta$ if $b \leq 2\zeta + 1$ and $b + \eta$, otherwise. ■

▶ **Theorem 4.25.** The LPoA of local 2-SSGs played on paths with $n = 3\zeta + \eta$ vertices and b blue agents, where $\zeta \in \mathbb{N}$, $\eta \in \{0, 1, 2\}$, and $b \geq o$, is upper bounded by

$$\mathrm{PoA} \leq \begin{cases} +\infty & \text{if } n = 3; \\ \frac{2n-2}{2n-5} & \text{if } n > 3 \text{ and } o = 1; \\ \frac{n-1}{b-o-1} & \text{if } n > 3,\ o \geq 2,\ b \geq 2o; \\ \frac{n-1}{\zeta} & \text{otherwise (i.e., } n > 3,\ o \geq 2 \text{ and } b < 2o). \end{cases}$$

The upper bounds are tight when (i) $n = 3$, (ii) $n > 3$ and $o = 1$, and (iii) $n > 3$, $o \geq 2$, $b \geq 2o$. ◀

Proof. As shown in Theorem 4.24, the social welfare of the social optimum is equal to $n - 1$. Furthermore, both, the upper and lower bounds to the PoA proved in Theorem 4.24 for $n = 3$ as well as for $n > 3$ and $o = 1$, also hold for the LPoA. Therefore, in the rest of the proof, we assume that $n \geq 4$ and $o \geq 2$.

Let o_r and b_r be the numbers of orange and blue agents having a utility equal to $r \in \left\{0, \frac{1}{2}, 1\right\}$, respectively. Let ℓ (respectively, ℓ') be the overall number of maximal vertex-induced paths whose vertices are all occupied by orange (respectively, blue) agents. We observe that $\ell - 1 \leq \ell' \leq \ell + 1$. Let x_r (respectively, y_r) be the number of orange (respectively, blue) agents that occupy the end vertices of the path and whose utility is equal to $r \in \{0, 1\}$. We have that $x_0 + x_1 + y_0 + y_1 = 2$. Furthermore, we have that $o_{\frac{1}{2}} = 2(\ell - o_0) - x_1$ and $b_{\frac{1}{2}} = 2(\ell' - b_0) - y_1$. Therefore,

$$o = o_0 + o_{\frac{1}{2}} + o_1 = 2\ell - o_0 - x_1 + o_1$$

and

$$b = b_0 + b_{\frac{1}{2}} + b_1 = 2\ell' - b_0 - y_1 + b_1,$$

i.e., $o_1 = o - 2\ell + o_0 + x_1$ and $b_1 = b - 2\ell' + b_0 + y_1$. As a consequence, the social welfare is equal to

$$\begin{aligned}
&\sum_{h \in \{0, \frac{1}{2}, 1\}} h r_h + \sum_{h \in \{0, \frac{1}{2}, 1\}} h b_h \\
&= \ell - o_0 - \frac{1}{2} x_1 + o - 2\ell + o_0 + x_1 + \ell' - b_0 - \frac{1}{2} y_1 + b - 2\ell' + b_0 + y_1 \\
&= n - \ell - \ell' + \frac{1}{2} x_1 + \frac{1}{2} y_1.
\end{aligned}$$

Now, observe that each orange (respectively, blue) agent that has a utility of 0 and occupies neither an end vertex of the path nor its adjacent vertex is adjacent to two blue (respectively, orange) agents of utility equal to $\frac{1}{2}$ each. Therefore $b_{\frac{1}{2}} = 2(\ell' - b_0) - y_1 \geq 2(o_0 - x_0)$ as well as $o_{\frac{1}{2}} = 2(\ell - o_0) - x_1 \geq 2(b_0 - y_0)$, or, equivalently, $\ell' \geq b_0 + o_0 - x_0 + \frac{1}{2} y_1$ as well as $\ell \geq b_0 + o_0 - y_0 + \frac{1}{2} x_1$. Therefore, to minimize social welfare we need to solve the following ILP.

$$\begin{aligned}
\text{maximize} \quad & \ell + \ell' - \frac{1}{2} x_1 - \frac{1}{2} y_1 \\
\text{subject to} \quad & b_0 + o_0 - y_0 + \frac{1}{2} x_1 \leq \ell \\
& b_0 + o_0 - x_0 + \frac{1}{2} y_1 \leq \ell' \\
& 2\ell - o_0 - x_1 \leq o
\end{aligned}$$

$$
\begin{aligned}
&2\ell' - b_0 - y_1 \le b \\
&x_0 + x_1 + y_0 + y_1 = 2 \\
&x_0 \le o_0 \\
&y_0 \le b_0 \\
&\ell' \le \ell + 1 \\
&\ell \le \ell' + 1 \\
&\ell, \ell', x_0, x_1, y_0, y_1, b_0, o_0 \in \mathbf{N}.
\end{aligned}
$$

Combining the first 4 inequalities of the ILP we obtain

$$2\ell + 2\ell' \le o + o_0 + x_1 + b + b_0 + y_1 \le n + \frac{1}{2}\ell + \frac{1}{2}y_0 + \frac{3}{4}y_1 + \frac{1}{2}\ell' + \frac{1}{2}x_0 + \frac{3}{4}x_1,$$

from which we derive

$$\ell + \ell' - \frac{1}{2}(x_1 + y_1) \le \frac{2}{3}n + \frac{1}{3}(x_0 + y_0) = 2\zeta + \frac{2}{3}\eta + \frac{2}{3} - \frac{1}{3}(x_1 + y_1).$$

By considering the constraints $0 \le x_1 + y_1 \le 2$ and the fact that x_1, y_1, ℓ and ℓ' are all non-negative integers, it turns out that the above inequality is maximized exactly when $x_1 + y_1 = 0$ or, equivalently, $x_1 = y_1 = 0$, and therefore, $\ell + \ell' \le \left\lfloor 2\zeta + \frac{2}{3}\eta + \frac{2}{3} \right\rfloor = 2\zeta + \eta$. Furthermore, by combining the seventh inequality of the ILP with the first one, we obtain $o_0 \le \ell$, and therefore, using the third inequality of the ILP, we obtain that $\ell \le o$. Since the eighth inequality implies that $\ell' \le \ell + 1 \le o + 1$, we have the value $\ell + \ell' \le 2o + 1$. As a consequence, the value of an optimum solution is upper bounded by

$$\min\{2o + 1, 2\zeta + \eta\}.$$

We now divide the proof into two cases:

Case 1: $b \ge 2o$. Setting ℓ, $o_0 = o$, $\ell' = o + 1$, $y_0, b_0 = 2$, and all the remaining variables to 0 gives an optimum solution for the ILP and the corresponding value of the objective function matches the upper bound of $2o + 1$. Therefore, the social welfare is at least $n - 2o - 1 = b - o - 1$, and the upper bound to the LPoA follows. Furthermore, this upper bound is tight. Indeed, consider the strategy profile in which each orange agent occupies a vertex that is adjacent to two vertices occupied by blue agents only and two orange agents occupy the second and last but one vertex of the path, i.e., the two vertices adjacent to the path end vertices. Observe that there are exactly $2(o-1)$ blue agents having a utility equal to $\frac{1}{2}$ and 2 agents having a utility of 0, thus $b-2(o-1)-2$ agents having a utility of 1. The social welfare of this configuration is equal to

$$\frac{1}{2}2(o-1)+(b-2(o-1)-2)=o-1+b-2o=b-o-1.$$

Case 2: $b<2o$. The optimum value of the ILP is upper bounded by $2\zeta+\eta$. Hence, the social welfare is at least $n-2\zeta-\eta=\zeta$, and the upper bound to the LPoA follows. ■

4.3.4 Grids

We now turn our focus to grid graphs with 4- and 8-neighbors. Remember that grids are formed by a two-dimensional lattice. Hence, we can partition the vertices of an $l \times h$ grid G into three sets[9]: *corner vertices*, *border vertices* and *middle vertices*, denoted, respectively, as $C(G)$, $B(G)$, and $M(G)$. We have

$$C(G)=\left\{v_{i,j} : i \in \{1,\ell\} \text{ and } j \in \{1,h\}\right\},$$

$$B(G)=\left\{v_{i,j} : i \in \{1,\ell\} \text{ or } j \in \{1,h\}\right\} \setminus C(G)$$

and $M(G)=V(G) \setminus (C(G) \cup B(G))$.

First, we focus on 2-SSGs in 4-grids and start by characterizing the PoA for the case in which one type has a unique representative.

▶ **Proposition 4.26.** The PoA of 2-SSGs played on a 4-grid in which one type has cardinality 1 is equal to $\frac{25}{22}$. ◀

Proof. Assume, without loss of generality, that orange is the type with a unique representative. For this game, any strategy profile $\boldsymbol{\sigma}$ is an equilibrium, since in any profile, the orange agent o gets utility zero, the agents not adjacent to o get utility 1, while all agents adjacent to o get less than 1. Call these last agents the *penalized agents*. Thus, the PoA is maximized by comparing the social welfare of the strategy profile minimizing the overall loss of the penalized agents with one of the strategy profiles maximizing it. It is easy to see that the overall loss of the

9 We assume $\ell, h > 1$ as otherwise, the grid would collapse to a path.

penalized agents is minimized when o is occupying a corner vertex, while it is maximized when o is occupying a border one in a 4-grid with $l = 2$ and $h = 3$. Comparing the two social welfares gives the claimed bound. ■

Clearly, if one type has only one representative, this agent receives utility zero. However, this is not possible in equilibrium assignments when there are at least two agents of each type.

▶ **Lemma 4.27.** In any equilibrium for a 2-SSG played on a 4-grid in which both types have cardinality larger than 1, all agents get positive utility. ◀

Proof. Fix an equilibrium $\boldsymbol{\sigma}$ for a game satisfying the premises of the lemma. Let i be a vertex such that $\mathsf{U}_i(\boldsymbol{\sigma}) = 0$ and assume, without loss of generality, that i is orange. This implies that i is surrounded by blue vertices only.

Pick another orange vertex $j \neq i$ which is adjacent to at least a blue one ℓ. If $\ell \notin N(\sigma(i))$, it follows that i and ℓ can perform a profitable swap contradicting the assumption that $\boldsymbol{\sigma}$ is an equilibrium. Thus, ℓ has to belong to $N(\sigma(i))$. Let us now consider two cases.

If i occupies a corner vertex, ℓ needs to be placed on a border one. So, as ℓ is adjacent to i and j, it holds that $\mathsf{U}_\ell(\boldsymbol{\sigma}) \leq \frac{1}{3}$. Thus, as we have $\mathsf{U}_\ell(\boldsymbol{\sigma}_{i\ell}) = \frac{1}{2}$ and $\mathsf{U}_i(\boldsymbol{\sigma}_{i\ell}) > 0$, i and ℓ can perform a profitable swap contradicting the assumption that $\boldsymbol{\sigma}$ is an equilibrium.

If i is not located on a corner vertex, as ℓ is adjacent to i and j, it holds that $\mathsf{U}_\ell(\boldsymbol{\sigma}) \leq \frac{1}{2}$. Moreover, $|N(\sigma(i))| \geq 3$ which yields

$$\mathsf{U}_\ell(\boldsymbol{\sigma}_{i\ell}) = \frac{|N(\sigma(i))| - 1}{|N(\sigma(i))|} \geq \frac{2}{3}.$$

Thus, also in this case, i and ℓ can perform a profitable swap contradicting the assumption that $\boldsymbol{\sigma}$ is an equilibrium. ■

When no agent gets utility zero, the minimum possible utility is $\frac{1}{4}$. Thus, Proposition 4.26 and Lemma 4.27 together imply an upper bound of 4 on the PoA. However, a much better result can be shown.

▶ **Theorem 4.28.** The PoA of 2-SSGs played on 4-grids is at most 2. ◀

Proof. Without loss of generality, we consider an $l \times h$ grid, with $l \leq h$. By Proposition 4.26, we only need to consider the case in which there are at least

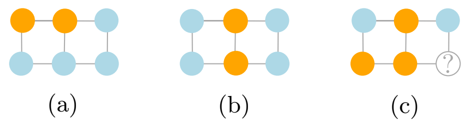


Figure 4.16: The unique swap equilibrium for 2×3 4-grids is shown in (a). Indeed, in (b) the blue agent in $v_{1,1}$ can swap with the orange agent in $v_{2,2}$, while in (c) the blue agent in $v_{1,1}$ can swap with the orange agent in $v_{1,2}$ (the question mark in $v_{2,3}$ means that the vertex can be occupied by an agent of any type). Please refer to Theorem 4.28 for more details.

two agents per type. By Lemma 4.27, we know that, in this case, the utility of each agent is strictly positive. We prove the claim by showing that the average utility of an agent is at least $\frac{1}{2}$. We divide the proof into two cases, depending on the utilities of the *middle* agents, i.e., agents occupying the middle vertices.

Case 1. In the first case, we assume that the utility of every middle agent is at least $\frac{1}{2}$. As corner agents, i.e., agents occupying corner vertices, have a utility of at least $\frac{1}{2}$ each, we only need to prove the claim when there is at least one border agent, i.e., an agent occupying a border vertex, whose utility is equal to $\frac{1}{3}$. This implies that $l + h \geq 5$. Without loss of generality, we assume that there are more orange than blue agents having a utility equal to $\frac{1}{3}$. Let I be the border vertices occupied by the orange (border) agents having a utility of $\frac{1}{3}$. As the overall number of border vertices is

$$2(l-2) + 2(h-2) = 2l + 2h - 8,$$

we have that the number of border agents having a utility greater than or equal to $\frac{2}{3}$ is at least $2l + 2h - 8 - 2|I|$. Therefore, if $|I| = 1$ and $l + h \geq 6$, then

$$2l + 2h - 8 - 2|I| \geq 12 - 8 - 2 = 2;$$

hence, the average utility of an agent is greater than or equal to $\frac{1}{2}$. If $|I| = 1$ and $l + h = 5$, then the only configuration in which a swap equilibrium exists, unless of symmetries, is shown in Figure 4.16 (a).

We observe that, in such a configuration, the average utility of an agent is strictly greater than $\frac{1}{2}$. It remains to prove the case in which $|I| \geq 2$. Since $\boldsymbol{\sigma}$ is a swap equilibrium, the utility of a blue agent that occupies a vertex that is not adjacent to all the vertices in I is at least $\frac{2}{3}$. As each blue agent occupies a vertex that is adjacent to at most 2 vertices in I and because each vertex in I is adjacent to exactly 2 vertices occupied by blue agents, the number of blue agents is at least $2|I|/2 = |I|$. Therefore, if we assume that every blue agent has a utility of at least $\frac{2}{3}$, then the average utility of an agent would be at least $\frac{1}{2}$. We observe that this assumption holds when either (a) $|I| \geq 3$, because no blue agent is occupying a vertex that is adjacent to all the vertices of I, or (b) $|I| = 2$ and the two vertices of I are either at t-hop distance from each other, with $t \geq 2$, or they are at 2-hop distance from each other and the utility of the border agent that occupies the vertex in between is at least $\frac{2}{3}$. For the remaining case in which $|I| = 2$, the two vertices of I are at 2-hop distance from each other, and the agent occupying the border vertex in between is equal to $\frac{1}{3}$ – and thus is of blue type – we simply observe that the overall number of blue agents is at least 4. Indeed, without loss of generality, let $v_{1,x-1}$ and $v_{1,x+1}$ be the two vertices of I. As $v_{1,x}$ is occupied by a blue agent that has strictly positive utility, $v_{2,x}$ is also occupied by a blue agent. Furthermore, either $v_{1,x-2}$ or $v_{2,x-1}$ is occupied by a blue agent. Similarly, either $v_{1,x+2}$ or $v_{2,x+1}$ is occupied by a blue agent. Therefore, there are at least 4 blue agents. Since 3 out of these 4 blue agents have a utility of at least $\frac{2}{3}$, again, the average utility of an agent is at least $\frac{1}{2}$.

Case 2. In the second case, we assume that at least one agent is occupying a middle vertex and whose utility is equal to $\frac{1}{4}$. Without loss of generality, we assume that there are more orange than blue agents having a utility equal to $\frac{1}{4}$. Let I be the vertices of the orange agents having a utility of $\frac{1}{4}$. We prove that

(i) every blue agent has a utility of at least $\frac{1}{2}$;

(ii) the number of blue agents having utility greater than or equal to $\frac{3}{4}$ is at least $|I|$;

(iii) all border and corner agents are of blue type.

This would imply that the average utility of an agent is $\frac{1}{2}$ since the utility of border and corner agents would be at least $\frac{2}{3}$.

Let $v_{x,y}$ be a vertex of I and, without loss of generality, we assume that $v_{x,y-1}, v_{x-1,y}$, and $v_{x,y+1}$ are occupied by blue agents whose utilities are greater than or equal to $\frac{1}{2}$. Similarly, we can prove that the utility of every other blue agent that occupies a vertex that is not adjacent to all vertices in I is at least $\frac{3}{4}$. This implies that at least one vertex between $v_{x-1,y-1}$ and $v_{x-1,y+1}$ is occupied by a blue agent whose utility is greater than or equal to $\frac{3}{4}$; similarly, at least one vertex between $v_{x+1,y-1}$ and $v_{x+1,y+1}$ is occupied by a blue agent whose utility is greater than or equal to $\frac{3}{4}$. Therefore, we have proved (ii) for the case in which $|I| \leq 2$. To prove (ii) when $|I| > 2$, it is enough to observe that all blue agents have a utility greater than or equal to $\frac{3}{4}$ because none of them occupies a vertex that is adjacent to all the vertices in I. But this implies that each blue agent of the utility of at least $\frac{3}{4}$ occupies a vertex that is adjacent to at most one vertex in I. Hence, the overall number of blue agents is at least $|I|$.

We now conclude the proof by proving (iii). First of all, we prove that at least one border or corner vertex is occupied by a blue agent. For the sake of contradiction, we assume that all border and corner vertices are occupied by the orange agents. Let $v_{x,y}$ be the topmost-leftmost vertex occupied by a blue agent, i.e., both $v_{x,y-1}$ and $v_{x-1,y}$ are occupied by orange agents and there is no other vertex $v_{x',y'}$ occupied by a blue agent such that $x' < x$ or $x = x'$ and $y' < y$. We observe that such a vertex always exists because $x, y > 1$ and that $v_{x-1,y-1}$ must be occupied by an orange agent. Furthermore, by the choice of $v_{x,y}$, the utility of the two orange agents that occupy the vertices $v_{x-1,y}$ and $v_{x,y-1}$ must be at least $\frac{1}{2}$. Since the utility of the blue agent occupying the vertex $v_{x,y}$ has to be at least $\frac{1}{2}$, $v_{x+1,y}$ and $v_{x,y+1}$ are occupied by blue agents. As a consequence, $N(v_{x,y}) \cap I = \emptyset$. Therefore, swapping the agent that occupies $v_{x,y}$ with any agent occupying a vertex in I would be an improving move. Now that we know that at least one border or corner agent is of blue type, we prove that all of them must be of blue type. For the sake of contradiction assume that at least one border or corner vertex is occupied by an orange agent. Without loss of generality, let $v_{1,y}$ be a vertex occupied by an orange agent such that $v_{1,y+1}$ is occupied by a blue agent. Since the utility of such a blue agent is at least $\frac{1}{2}$, the unique middle vertex adjacent to $v_{1,y+1}$, i.e., $v_{2,y+1}$, must be occupied by a blue agent. This implies that $v_{1,y+1}$ cannot be adjacent to any vertex in I. As the utility of the agent occupying vertex $v_{1,y+1}$ is at most $\frac{2}{3}$, swapping the agent occupying the vertex $v_{1,y+1}$ and any agent occupying a vertex in I would be an improving move. This completes the proof. ■

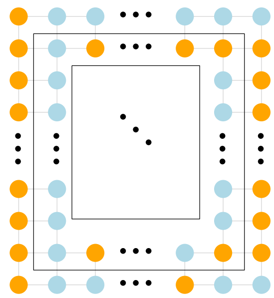


Figure 4.17: Visualization of the first three frames of G with the coloring induced by the strategy profile defined in the proof of Theorem 4.30. Please refer to the proof for more details.

The following lemma gives a sufficient condition for a strategy profile to be an equilibrium.

▶ **Lemma 4.29.** Fix a 2-SSG played on a 4-grid. Any strategy profile in which corner and middle vertices get utility at least $\frac{1}{2}$ and border ones get utility at least $\frac{2}{3}$ is an equilibrium. ◀

Proof. For every two agents i and j of different types we have that the sum of their utilities is at least 1. Therefore, by Lemma 4.2, the considered strategy profile is an equilibrium. ■

We now show a matching lower bound.

▶ **Theorem 4.30.** The PoA of 2-SSGs played on 4-grids is at least 2, even for balanced games. ◀

Proof. Fix a 2-SSG played on an $n \times n$ grid G, with n being an even number. We define a strategy profile $\boldsymbol{\sigma}$ by giving a coloring rule for any frame of G. Clearly, being n an even number, there are $\frac{n}{2}$ frames in G that we number from 1 to $\frac{n}{2}$, with frame 1 corresponding to the outer one, i.e., the biggest. Frame i, whose size is $n_i := n - 2(i-1)$, is colored as follows: all vertices in the right column except for the first and the last and all vertices in the left column are of the *basic color* of i, all other vertices, that are the ones on the upper and lower rows except for the vertices falling along the left column, take the other color. Observe that $n_i + n_i - 2 = 2(n_i - 1)$ vertices take the basic color of i and $2(n_i - 1)$ vertices take the other one so that every frame evenly splits its vertices between the two

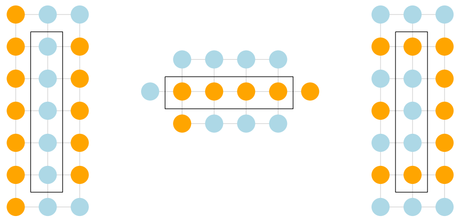


Figure 4.18: Visualization of the neighborhood of vertices belonging to a frame $i > 1$. The target vertices are the ones included in the box. On the left, vertices belong to the left column; on the right, vertices belong to the right column; on the center, vertices belong to a row but not to a column. See the proof of Theorem 4.30 for more details.

colors. Thus, σ is a well-defined strategy profile for a 2-SSG with both types having the same cardinality. The basic color of frame i is orange if i is odd and blue otherwise, see Figure 4.17 for a pictorial example. To show that σ is an equilibrium, it suffices to prove that it satisfies the premises of Lemma 4.29.

To address corner and border vertices, consider frame 1, see again Figure 4.17. It comes by construction that every corner vertices gets utility $\frac{1}{2}$ and that every border vertices gets utility at least $\frac{2}{3}$, except for vertices $(1, 2)$, $(2, n)$, $(n-1, n)$ and $(n, 2)$ for which further investigation is needed. In particular, they get utility $\frac{2}{3}$ if and only if the following coloring holds: $(2, 2)$ is blue, $(2, n-1)$ is orange, $(n-1, n-1)$ is orange and $(n-1, 2)$ is blue. This holds by construction and can be verified by a direct inspection of Figure 4.17.

To address middle vertices, it suffices to prove that, any vertex belonging to frame $i > 1$ has two orange and two blue neighbors. Let c denote the basic color of frame i and $\bar{c}$ be the other color. Consider a generic vertex v belonging to frame i. By inspecting all possible positions of v within the frame as shown in Figure 4.18, it can be easily verified that the desired property holds. By Lemma 4.29, σ is an equilibrium. ■

We now show matching upper and lower bounds on the LPoA for local 2-SSGs played on grids. By inspecting all the possibilities, the LPoA of local 2-SSGs played on 2×2 grids is 1. Indeed, assuming $b \geq o$, for $o = 1$, all the configurations are isomorphic to each other, while, for $o = 2$, the unique (local) swap equilibrium – up to isomorphisms – is $\left[\begin{smallmatrix} o & b \\ o & b \end{smallmatrix}\right]$.

▶ **Proposition 4.31.** The LPoA of local 2-SSGs played on $2 \times h$ 4-grids, with

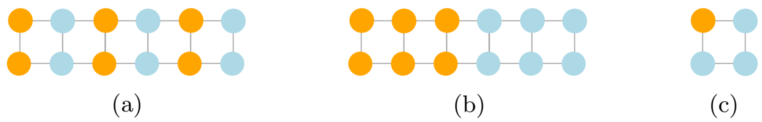


Figure 4.19: The local swap equilibrium with the lowest social welfare is shown in (a) and the social optimum is shown in (b). (c) shows the unique local swap equilibrium which contains an agent with utility 0. See the proof of Proposition 4.31 for more details.

$h \geq 2$ is 3. Furthermore, for every $\epsilon > 0$, there is a value h_0 such that, for every $h \geq h_0$, the PoA of $2 \times h$ 4-grid is at least $3 - \epsilon$. ◀

Proof. For the lower bound consider the strategy profile in which h is a multiple of 6, $o = b$, odd columns are filled with orange agents, and even columns are filled with blue agents, see Figure 4.19 (a) for an example on a 2×6 4-grid. The strategy profile is a local swap equilibrium and the corresponding social welfare is equal to $\frac{1}{3}(n-4)+2 = \frac{n+2}{3}$. A social optimum having social welfare of $n - \frac{4}{3} = \frac{3n-4}{3}$ is the strategy profile in which all the orange agents occupy the first $\frac{h}{2}$ columns, and the blue agents occupy the last $\frac{h}{2}$ columns, see Figure 4.19 (b) for an example on a 2×6 4-grid. Therefore, for every $h \geq \frac{5-\epsilon}{\epsilon}$, we have that the following formula is a lower bound to the LPoA

$$\frac{3n-4}{n+2} = 3 - \frac{10}{n+2} = 3 - \frac{5}{h+1} \geq 3 - \epsilon.$$

To prove the upper bound of 3, we show that the average utility of an agent is at least $\frac{1}{3}$. We consider only the agents that have a utility of 0 since all the other agents have a utility of at least $\frac{1}{3}$ each. When h is equal to 2, the unique strategy profile (unless of symmetries) that is in local swap equilibrium and contains at least one agent that has 0 utility is depicted in Figure 4.19 (c). However, it is easy to check that the average utility of an agent is $\frac{1}{2}$. Therefore, we only need to prove the claim for $h \geq 3$. We prove that if x is the number of agents whose utilities are equal to 0, then there are at least x agents that have a utility of at least $\frac{2}{3}$ each. Indeed, let i be any agent that has a utility equal to 0. Since $\boldsymbol{\sigma}$ is a local swap equilibrium and $h \geq 3$, we have that there is an agent j such that (i) $\sigma(j) \in N(\sigma(i))$, (ii) the type of i is different from the type of j, and (iii) $\mathrm{U}_j(\boldsymbol{\sigma}) \geq \frac{2}{3}$. Indeed, if i occupies a corner vertex, say v, then we can swap i with the agent occupying the unique border vertex adjacent to v, say u. Furthermore,

Figure 4.20: The strategy profile inducing an average agent's utility that can be made arbitrarily close to $\frac{7}{18}$ is shown on the left side via a small example (3×9 4-grid). On the right side, it is shown a strategy profile inducing an average agent's utility arbitrarily close to 1. See the proof of Proposition 4.32 for more details.

since by Lemma 4.2 the utility of the agent occupying the corner vertex adjacent to v, say u', has a strictly positive utility, we have that the border agent adjacent to u is occupied by an agent of the same type of the two ones that occupy u and u'. If i occupies a border vertex of the first (respectively, second) row, say v, then we can swap i with the agent j occupying the unique border vertex adjacent to the second (respectively, first) row that is adjacent to v. In either case, we are uniquely assigning an agent j that has a utility of at least $\frac{2}{3}$ to every agent i that has a utility of 0. The claim follows. ■

▶ **Proposition 4.32.** The LPoA of local 2-SSG played on $3 \times h$ 4-grids, with $h \geq 3$ is $\frac{18}{7}$. Furthermore, for every $\epsilon > 0$, there is a value h_0 such that, for every $h \geq h_0$, the PoA of $3 \times h$ 4-grid is at least $\frac{18}{7} - \epsilon$. ◀

Proof. For the lower bound of $\frac{18}{7} - \epsilon$ consider the strategy profile in Figure 4.20. The average utility of the agents that occupy any column from 2 to $h-1$ is equal to $\frac{7}{18}$.

Now, we prove the upper bound of $\frac{18}{7}$. In the remainder of the proof, by utility of the r-th column we mean the overall utility of the agents that occupy the vertices of the r-th column. We show that the utility of the first (respectively, last) column is of at least $\frac{5}{6}$ and we show that the average utility of the other columns is at least $\frac{7}{6}$.

First of all, using Lemma 4.2, we have that at most one of the agents that occupy the vertices of the r-th column can have a utility of 0. This observation implies that the utility of the r-th column, with $r = 1, h$, is lower bounded by $\frac{5}{6}$.

Now, we show that on average the utility of the r-th column, with $2 \leq r \leq h-1$, is of at least $\frac{7}{6}$. We divide the proof into cases:

Case 1. We assume that the middle agent has a utility of 0. By Lemma 4.2,

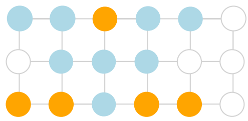


Figure 4.21: Case 2: In case the r-th column has utility $\frac{13}{12}$, and therefore, a border agent has utility 0, without loss of generality, the upper orange one in the r-th column, the middle agent utility $\frac{3}{4}$ and the other border agent utility $\frac{1}{3}$. Applying Lemma 4.2 yields that the right (respectively left) neighbor of the orange border agent with utility 0 has a utility of at least $\frac{2}{3}$ and is blue. Therefore, the right (left) neighbor of the middle agent has a utility of at least $\frac{1}{2}$. Again applying Lemma 4.2 yields that the right (respectively left) neighbor of the lower blue border agent has a utility of at least $\frac{1}{3}$. Summing up all utilities yields a utility of $\frac{3}{2}$ for the $r+t$-th (respectively $r-1$-th) column. Please refer to Case 2 in the proof of Proposition 4.32 for more details.

both border agents of the column have a utility of at least $\frac{2}{3}$ and therefore, the utility of the r-th column is at least $\frac{4}{3} \geq \frac{7}{6}$.

Case 2. We assume that a border agent has a utility of 0. This implies that the middle agent has a utility of $\frac{3}{4}$ and the other border agent has a utility of at least $\frac{1}{3}$. Therefore, the utility of the r-th column is at least $\frac{13}{12}$. In case the column has utility $\frac{13}{12}$, then the next column has utility at least $\frac{3}{2}$ by applying Lemma 4.2, cf. Figure 4.21. As a consequence, for column with utility $\frac{13}{12}$ there is another column with utility $\frac{3}{2}$, and the average utility is at least $\frac{31}{24} \geq \frac{7}{6}$. Otherwise, if the column has a utility of at least $\frac{17}{12} \geq \frac{7}{6}$.

Case 3. We assume that all agents that occupy the vertices of the r-th column have a strictly positive utility. We observe that the only interesting case to look at is when the border agents both have a utility of $\frac{1}{3}$ and the middle agent has a utility of $\frac{1}{4}$, as in all the other cases, the utility of the r-th column would be greater than or equal to $\frac{7}{6}$. However, by Lemma 4.2 this case cannot occur since at least one border agent has the opposite color than the middle agent, who has utility $\frac{1}{4}$, and they swap.

This completes the proof. ■

▶ **Theorem 4.33.** For every $\epsilon > 0$, the LPoA of local 2-SSG played on $l \times h$ 4-grids, with $\ell, h \geq 8 + \frac{20}{\epsilon}$ is in the interval $\left(\frac{5}{2} - \epsilon, \frac{5}{2} + \epsilon\right]$. ◀

Proof. Let X be the set of middle vertices that are adjacent neither to the border nor to corner vertices. Clearly, $N(X) = \bigcup_{v \in X} N(v)$ is the set of all the middle vertices. Therefore, the degree of each vertex $v \in N(X)$ is equal to 4. Let $Z \subseteq N(X)$ be the set of vertices occupied by agents that have a utility strictly greater than $\frac{2}{5}$. From Lemma 4.19, we have that the average utility of the agents in $X \cup Z$ is at least $\frac{2}{5}$. As a consequence, the social welfare is lower bounded by

$$\frac{2}{5}|X \cup Z| \geq \frac{2}{5}(l-4)(h-4) > \frac{2}{5}lh - \frac{8}{5}(l+h).$$

Therefore, the LPoA can be upper bounded by

$$\frac{lh}{\frac{2}{5}lh - \frac{8}{5}(l+h)} = \frac{1}{\frac{2}{5} - \frac{8}{5}\frac{l+h}{lh}} \leq \frac{1}{\frac{2}{5} - \frac{8}{5}\frac{2(8+20/\epsilon)}{(8+20/\epsilon)^2}} = \frac{5}{2} + \epsilon.$$

For the lower bound, consider the $l \times h$ grid, with $l = 5l' + 1$ and $h = 5h'$, that is filled as shown in Figure 4.22. The social welfare for arbitrarily large values of l' and h', i.e., l and h, can be made arbitrarily close to the average utility of the agents that occupy the vertices of the tiles labeled with T. Observe that $\frac{2}{5}$ is the average utility of the agents that occupy all the vertices of any tile labeled with T. As the ratio between blue and orange agents can be made arbitrarily close to $\frac{3}{2}$, the maximum average utility of an agent is arbitrarily close to 1 by placing the orange agents over the vertices of the first $\frac{2}{5}h$ columns and the blue agents in the remaining $\frac{3}{5}h$ columns. Therefore, the LPoA is lower bounded by $\frac{5}{2} - \epsilon$. ■

We now turn our focus to the 8-grid and prove upper bounds to the LPoA.

▶ **Proposition 4.34.** The PoA of 2-SSGs played on an 8-grid in which one type has cardinality 1 is equal to $\frac{897}{704}$. ◀

Proof. Assume, without loss of generality, that orange is the type with a unique representative. For this game, any strategy profile $\boldsymbol{\sigma}$ is an equilibrium, since in any profile $\boldsymbol{\sigma}$ the orange agent o gets utility zero, the agents not adjacent get utility 1, while all agents adjacent to o get strictly less than 1. Call these last agents the *penalized agents*. Thus, the PoA is maximized by comparing the social welfare of the strategy profile minimizing the overall loss of the penalized agents with one of the strategy profiles maximizing it. The overall loss of the penalized agents is minimized when o is occupying a corner vertex, while it is maximized

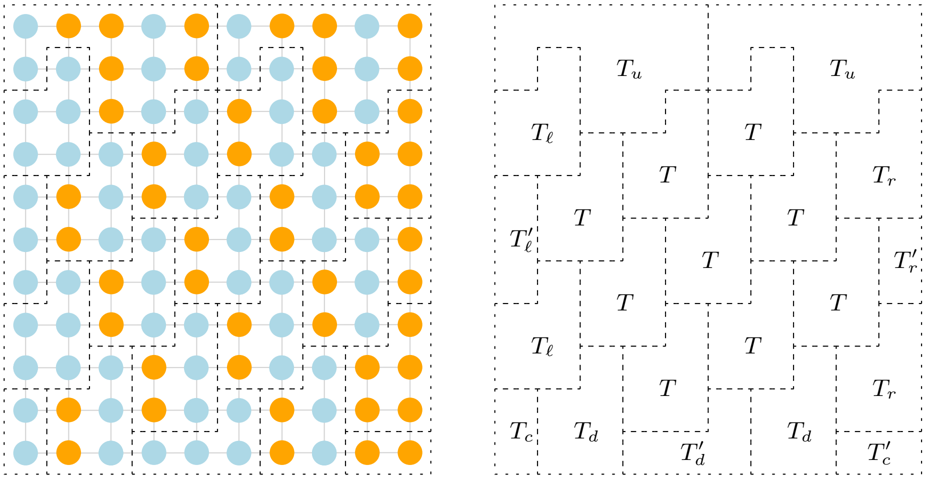


Figure 4.22: The strategy profile inducing an average agent's utility arbitrarily close to $\frac{2}{5}$ is shown on the left side via a small example over an 11×10 4-grid. On the right side, the tiling shows the pattern we have used for building the instance. The tiles T_c and T'_c are only used to fill the bottom-left and bottom-right corners of the 4-grid. Observe that using exactly the same tiles, one can build arbitrarily large instances. Moreover, for arbitrarily large instances, the average utility of an agent is determined by the average utility of the agents that occupy the vertices of any tile T, i.e., $\frac{2}{5}$. See the proof of Theorem 4.33 for more details.

when o is a middle one on an 8-grid with $l = h = 3$. Comparing the two social welfares gives the claimed bound. ∎

▶ **Theorem 4.35.** The LPoA of 2-SSGs played on an 8-grid is at most 4. ◀

Proof. Let i be an agent with utility strictly less than $\frac{1}{4}$ and let j be an agent of type different from the one of i that occupies a vertex, say v, that is adjacent to the one occupied by i, say u. By Lemma 4.2 the sum of the utilities of agent i and j is at least $1 - \frac{1}{3} = \frac{2}{3}$ if either u or v is a corner vertex and at least $1 - \frac{1}{5} = \frac{4}{5}$ in any other case.

Now observe that $N(v) \setminus \{u\}$ is occupied by at most one agent of the same type of i, say i', but only if neither u nor v is a corner vertex; in any other case, $N(v) \setminus \{u\}$ is occupied by agents of the same type of j except for the unique vertex of $N(v)$ occupied by i. As a consequence, if either u or v is a corner vertex then the average utility of i and j is greater than or equal to $\frac{1}{3}$; in the other cases the average utility of i, j, and the potential agent i' of the same type of i that occupies a vertex in $N(v) \setminus \{u\}$ is at least $\frac{4}{15} > \frac{1}{4}$. In either case, the average utility of the considered agents is at least $\frac{1}{4}$. As we are assigning j to the unique agents of different types of j that occupy vertices in $N(v)$, we have that the average utility of an agent is greater than or equal to $\frac{1}{4}$. This completes the proof. ■

We conclude by proving a much better bound for the (L)PoA if the 8-grid is large enough.

▶ **Proposition 4.36.** For every $\epsilon > 0$, the LPoA of local 2-SSGs played on an $l \times h$ 8-grid, with $l, h \geq 8 + \frac{18}{\epsilon}$ is at most $\frac{9}{4} + \epsilon$. ◀

Proof. Let X be the set of middle vertices that are adjacent neither to the border nor to corner vertices. Clearly, $N(X) = \bigcup_{v \in X} N(v)$ is the set of all the middle vertices. Therefore, the degree of each vertex $v \in N(X)$ is equal to 8. Let $Z \subseteq N(X)$ be the set of vertices occupied by agents that have a utility strictly greater than $\frac{4}{9}$. From Lemma 4.19, we have that the average utility of the agents in $X \cup Z$ is at least $\frac{4}{9}$. As a consequence, the social welfare is lower bounded by

$$\frac{4}{9}|X \cup Z| \geq \frac{4}{9}(l-4)(h-4) > \frac{4}{9}lh - \frac{16}{9}(l+h).$$

Therefore, the LPoA is at most

$$\frac{lh}{\frac{4}{9}lh - \frac{16}{9}(l+h)} = \frac{1}{\frac{4}{9} - \frac{16}{9}\frac{l+h}{lh}} \leq \frac{1}{\frac{4}{9} - \frac{16}{9}\frac{2(8+18/\epsilon)}{(8+18/\epsilon)^2}} = \frac{9}{4} + \epsilon.$$ ■

4.4 Price of Stability

Although our work is mainly devoted to the characterization of the Price of Anarchy in (local) Swap Schelling Games, some results for the Price of Stability can be derived as a by-product of our analysis. The characterization of the (L)PoS is much more challenging than that of the (L)PoA, and very few results are known in this setting within the realm of Swap Schelling Games. In particular, Agarwal et al. [Aga+21] show that $\mathrm{PoS}(\mathcal{G}, 2) \geq \frac{4}{3}$ when $\mathcal{G}$ is the class of trees and $\mathrm{PoS}(\mathcal{G}, k) = 1$, for any $k \geq 2$, when $\mathcal{G}$ is the class of regular graphs. The last result is shown by employing the potential method, which leverages the existence of a potential function for games played on regular graphs. In the same spirit, we can exploit Theorem 4.4 to obtain a significant upper bound on the PoS for games played on almost-regular graphs.

▶ **Theorem 4.37.** For any $k \geq 2$, $\mathrm{PoS}(\mathcal{G}, k) \leq \frac{\Delta}{\delta} = \frac{\delta+1}{\delta}$ when $\mathcal{G}$ is the class of almost regular graphs. ◀

Proof. For any $k \geq 2$, fix a k-*SSG* (G, t) defined on an almost-regular graph of minimum degree δ, so that, the maximum degree is $\Delta = \delta + 1$. Observe that, for any feasible strategy profile $\boldsymbol{\sigma}$, it holds that

$$\mathsf{U}(\boldsymbol{\sigma}) \geq \frac{2\Phi(\boldsymbol{\sigma})}{\delta + 1} \tag{4.11}$$

and

$$\mathsf{U}(\boldsymbol{\sigma}) \leq \frac{2\Phi(\boldsymbol{\sigma})}{\delta}. \tag{4.12}$$

Let $\overline{\boldsymbol{\sigma}}$ be a feasible strategy profile maximizing Φ. By Theorem 4.4, we know that Φ is a potential for (G, t). This implies that $\overline{\boldsymbol{\sigma}}$ is a swap equilibrium and

$$\Phi(\boldsymbol{\sigma}^*) \leq \Phi(\overline{\boldsymbol{\sigma}}), \tag{4.13}$$

where $\boldsymbol{\sigma}^*$ is a short-hand for $\boldsymbol{\sigma}^*(G, t)$ Thus, the $\mathrm{PoS}(G, t)$ is upper bounded by $\frac{\mathsf{U}(\boldsymbol{\sigma}^*)}{\mathsf{U}(\overline{\boldsymbol{\sigma}})}$. Putting everything together, we get

$$\mathrm{PoS}(G, t) \leq \frac{\mathsf{U}(\boldsymbol{\sigma}^*)}{\mathsf{U}(\overline{\boldsymbol{\sigma}})} \leq \frac{\frac{2\Phi(\boldsymbol{\sigma}^*)}{\delta}}{\frac{2\Phi(\overline{\boldsymbol{\sigma}})}{\delta+1}} \leq \frac{\frac{2\Phi(\overline{\boldsymbol{\sigma}})}{\delta}}{\frac{2\Phi(\overline{\boldsymbol{\sigma}})}{\delta+1}} = \frac{\delta + 1}{\delta},$$

where the second inequality comes from both Equation (4.11) and Equation (4.12) and the third inequality comes from Equation (4.13). ■

We observe that the proof of Theorem 4.37 can be generalized to any game for which the global maximum of Φ is a swap equilibrium, to produce an upper bound of $\frac{\Delta}{\delta}$ on the PoS. As Corollary 4.6, claiming that games played on 4-grids possess the FIP, is proved by showing that a global maximum of Φ is a swap equilibrium, we immediately get an upper bound of 2 on the PoS which, however, does not improve on the upper bound on the PoA shown in Theorem 4.28. However, by refining the proof of Theorem 4.37, an upper bound of $\frac{3}{2}$ can be derived.

▶ **Theorem 4.38.** For any $k \geq 2$, $\mathrm{PoS}(\mathcal{G}, k) \leq \frac{3}{2}$ when $\mathcal{G}$ is the class of 4-grids. ◀

Proof. Fix a 4-grid $G = (V, E)$. Let $E_1 \subseteq E$ be the set of edges that are incident to a corner vertex of G, that is, to one of the 4 vertices of degree 2. Concerning partition $(E_1, E \setminus E_1)$ of E, we can refine the definition of Φ to be equal to the number of monochrome edges in E_1 plus the number of monochrome edges in $E \setminus E_1$. So, for each $\boldsymbol{\sigma}$, define $\Phi(\boldsymbol{\sigma}) := \Phi_{E_1}(\boldsymbol{\sigma}) + \Phi_{E \setminus E_i}(\boldsymbol{\sigma})$. Observing that the degree of every vertex incident to an edge in E_1 is either 2 or 3 and the degree of every vertex incident to an edge in $E \setminus E_1$ is either 3 or 4, inequalities Equation (4.11) and Equation (4.12) rewrite as

$$\mathsf{U}(\boldsymbol{\sigma}) \geq \frac{2\Phi_{E_1}(\boldsymbol{\sigma})}{3} + \frac{2\Phi_{E \setminus E_1}(\boldsymbol{\sigma})}{4} \tag{4.14}$$

and

$$\mathsf{U}(\boldsymbol{\sigma}) \leq \frac{2\Phi_{E_1}(\boldsymbol{\sigma})}{2} + \frac{2\Phi_{E \setminus E_1}(\boldsymbol{\sigma})}{3}. \tag{4.15}$$

By using these inequalities in place of Equation (4.11) and Equation (4.12) within the final derivation in the proof of Theorem 4.37, we get the desired bound. ■

In Theorem 4.9, we show that local games with two types played on 8-grids have the FIP. This is achieved by proving that function Ψ is a potential for these games, which implies that the global maximum of Φ is a local swap equilibrium. Hence, the same approach of Theorem 4.38 can be adapted to obtain an upper bound of $\frac{5}{3}$ on the local Price of Stability.

▶ **Proposition 4.39.** $\mathrm{LPoS}(\mathcal{G}, 2) \leq \frac{5}{3}$ when $\mathcal{G}$ is the class of 8-grids. ◀

Moreover, by using the algorithmic construction used to show Theorem 4.11, we can derive an upper bound of $\frac{5}{2}$ which holds even for the Price of Stability and rapidly approaches 1 as both dimensions of the grid increase.

▶ **Theorem 4.40.** $\mathrm{PoS}(\mathcal{G}, 2) \leq \frac{5}{2}$ when $\mathcal{G}$ is the class of 8-grids. ◀

Proof. Observe that, when $o \geq 2h-1$, we show in the proof of Theorem 4.11 that the computed swap equilibrium $\boldsymbol{\sigma}$ is such that, for each $i \in [n]$, either $\mathsf{U}_i(\boldsymbol{\sigma}) = 1$ or $\mathsf{U}_i(\boldsymbol{\sigma}) \geq \frac{2}{5}$. This immediately implies an upper bound of $\frac{5}{2}$ of the PoS. Similarly, for the case in which $o < 2h - 1$, the computed swap equilibrium $\boldsymbol{\sigma}$ is such that the minimum utility of any player is at least $\frac{2}{5}$, see Figure 4.14, which gives an upper bound of $\frac{5}{2}$ on the PoS also in this case. ■

▶ **Corollary 4.41.** For any game with 2 types played on an 8-grid, the PoS , and so also the LPoS, approaches 1 as both dimensions of the grid increase. ◀

Proof. When $o \geq 2h-1$, observe that the number of agents whose utility is not 1 is at most $2(h+1)$, see Figure 4.11. As there are $n = \ell h$ agents in total, and at most $2(h+1)$ of them lose at most $\frac{3}{5}$ over their possible maximum utility, which equals 1, it follows that the PoS is upper bounded by $\frac{\ell h}{\ell h - \frac{3}{5} \cdot 2(h+1)}$. This function approaches 1 when both h and ℓ increase. When $o < 2h-1$, the number of agents whose utility is not 1 can be upper bounded by $2(\ell + h)$, see Figure 4.14. Thus, the PoS is upper bounded by

$$\frac{\ell h}{\ell h - \frac{3}{5} \cdot 2(h+\ell)}.$$

Also this function approaches 1 when both h and ℓ increase. ■

Finally, it can be tempting to use the local swap equilibrium computed in Theorem 4.7 to upper bound the LPoS in games played on trees. However, it is easy to show that the constructed local swap equilibrium may have an arbitrarily bad performance. Consider a game with 3 types such that $t = (t_1, t_2, 1)$ played on a tree whose root r has t_2 children and one of these children, say u, has t_1 children. Our algorithm assigns all agents of type 1 to the children of u, all agents of type 2 to the children of r, and the unique agent of type 3 to r. This is a local swap equilibrium $\boldsymbol{\sigma}$ such that $\mathrm{U}(\boldsymbol{\sigma}) = 0$. As there is a feasible strategy profile $\boldsymbol{\sigma}^*$ such that $\mathrm{U}(\boldsymbol{\sigma}^*) > 0$, the ratio between $\mathrm{U}(\boldsymbol{\sigma}^*)$ and $\mathrm{U}(\boldsymbol{\sigma})$ is unbounded.

4.5 Conclusion and Open Problems

We have shed light on the influence of the underlying graph topology on the existence of equilibria, the game dynamics, and the Price of Anarchy in Swap Schelling Games on graphs. Moreover, we have studied the impact of restricting agents to local swaps. We present tight or almost tight bounds for a variety of graph classes and for both the Swap Schelling Game and its local variant, where only swaps between neighboring agents are allowed.

As the main takeaway from this chapter, we find that both the specific structure of the underlying graph and the restriction to only local swaps strongly influence the existence and the quality of equilibria. Regarding the existence

Table 4.3: Asymptotic Price of Anarchy results. For the study of the PoA we focus on $k = 2$ types. Remember that n is the cardinality of V and Δ denotes the maximum degree of vertices in $G = (V, E)$. We denote by b and o the number of blue and orange agents, respectively, and we assume $o \leq b$.

	Price of Anarchy			
	2-SSG		local 2-SSG	
arbitrary	∞	$o = 1$	$\Theta(2n)$	$o = \frac{n}{2}$
	$O(\frac{b}{o})$	$o = \frac{n}{2}$	$O(\frac{\Delta}{\delta})$	$\delta \geq 2$
			$\Theta(\Delta^2)$	$\Delta \leq n - 2$
regular	$O(1)$		$O(1)$	
trees	$\Theta(\Delta^2)$		$\Theta(\Delta^2)$	$\Delta \leq n - 2$
cycles	$\Theta(\frac{n}{b})$		$\Theta(\frac{n}{b-o})$	$o \geq 2, b \geq 2o$
			$O(1)$	otherwise
paths	∞	$n = 3$	∞	$n = 3$
	$O(\frac{n}{b})$	$n > 3, o \geq 2,$	$\Theta(\frac{n}{b-o})$	$n > 3, o \geq 2, b \geq 2o$
			$O(1)$	otherwise
4-grids	$O(1)$		$O(1)$	
8-grids	$O(1)$		$O(1)$	

of equilibria, we find that for the Swap Schelling Game existence is guaranteed on all investigated graph classes, except for trees, as proven earlier by Agarwal et al. [Aga+21]. Interestingly, by enforcing only local swaps, and thereby strictly enlarging the set of equilibria, we also have equilibrium existence on trees. Moreover, as our bounds on the Price of Anarchy indicate, see Table 4.3 for a condensed overview of the asymptotic bounds, the quality of the equilibrium states deteriorates only slightly when enforcing local swaps. For deriving these bounds in the Price of Anarchy, we introduce novel techniques that are based on matchings. We believe that this approach might be advantageous for future research on the quality of equilibria in Schelling games.

Clearly, improving on the non-tight bounds is an interesting challenge for future work. Regarding the local Swap Schelling Game, we leave some interesting problems open. Among them is the question of whether local swap equilibria are guaranteed to exist in general and whether the local k-SSG always has the finite improvement property. So far, we are not aware of any counter-examples for both questions and extensive agent-based simulations indicate that both equilibrium existence and guaranteed convergence of improving response dynamics may hold. Another open problem is of understanding whether the finite improvement property holds for tree instances when we consider local swap equilibria. This result would create a sharp contrast between the concepts of swap equilibrium and local swap equilibrium as we know of the existence of a tree instance that does not admit a swap equilibrium, and thus, cannot satisfy the finite improvement property, [Aga+21].

Another interesting line of study is to analyze the Jump Schelling Game concerning varying underlying graphs and locality.

5 Single-Peaked Swap Schelling Games

This chapter is based on joint work with Davide Bilò, Vittorio Bilò, and Pascal Lenzner [Bil+22a].

The game-theoretic variants of Schelling's model that we investigated in Chapter 3 and Chapter 4 and which were introduced and studied very recently in the literature [Aga+21; BSV21; KKV21], incorporate utility functions that are monotone in the fraction of same-type neighbors, i.e., the utility of an agent is proportional to the fraction of same-type neighbors in its neighborhood. In Chapter 3 agents are equipped with a utility function as shown in Figure 5.1 (left); in Chapter 4 we investigate a simplified model with $\tau = 1$.

However, non-monotone utility functions are well-justified by real-world data and hence might be more suitable for modeling real-world segregation. Representative sociological polls, in particular data from the General Social Survey[10] (GSS) [Smi+19], indicate that this assumption of monotone utility functions should be challenged. For example, in 1982 all black respondents were asked *"If you could find the housing that you would want and like, would you rather live in a neighborhood that is all black; mostly black; half black, half white; or mostly white?"* and 54% responded with *"half black, half white"* while only 14% chose *"all black"*. Later, starting from 1988 until 2018 all respondents, of whom on average 78% were white, were asked what they think of *"Living in a neighborhood where half of your neighbors were blacks?"* a clear majority[11] responded *"strongly favor"*, *"favor"* or *"neither favor nor oppose"*. This shows that the maximum utility should not be attained in a homogeneous neighborhood.

Based on these real-world empirical observations, this chapter sets out to explore a game-theoretic variant of Schelling's model with non-monotone utility functions. In particular, we focus on single-peaked utility functions with maximum utility at a Λ-fraction of same-type neighbors, see Figure 5.1 (mid-

10 Since 50 years the GSS is regularly conducted in the US and it is a valuable and widely used data set for social scientists.

11 In numbers: 1988: 57%, 1998: 70%, 2008: 79%, 2018: 82%. In 2018 33% answered with "favor" or "strongly favor".

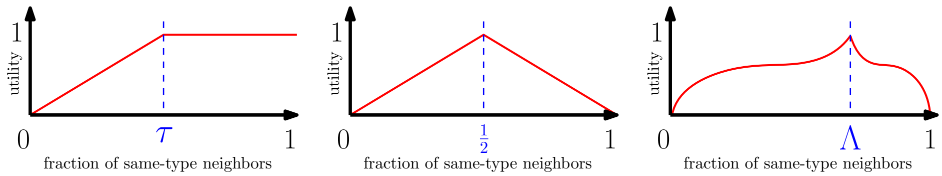


Figure 5.1: Left: example of the monotone utility functions employed in recent related work. Middle and right: example of a single-peaked utility function considered in this chapter.

dle and right), with $\Lambda \in (0, 1)$, satisfying mild assumptions. More precisely, we only require a function $p(x)$ to be zero-valued at $x = 0$ and $x = 1$, to be strictly increasing in the interval $[0, \Lambda]$ and to be such that $p(x) = p\left(\frac{\Lambda(1-x)}{1-\Lambda}\right)$ for each $x \in [\Lambda, 1]$, that is, both sides of p approach the peak, one from the left and the other from the right, in the same way, up to a re-scaling due to the width of their domains, $[0, \Lambda]$, vs. $[\Lambda, 1]$. Our main findings shed light on the existence of equilibrium states and their quality in terms of the recently defined Degree of Integration [Aga+21] that measures the number of agents that live in a heterogeneous neighborhood.

Zhang [Zha04] proposed a model that is similar to our model. There, agents on a toroidal grid graph with degree 4 also have a non-monotone single-peaked utility function. However, in contrast to our model, random noise is added to the utilities and transferable utilities are assumed. Zhang analyzed the Markov process of random swaps and showed that this process converges with a high probability to segregated states. For the jump version, Pancs & Vriend [PV07] investigated empirically different individual preferences, especially a single-peaked utility function where agents strictly prefer to live in a perfectly integrated neighborhood and any deviation regardless of the direction is equally bad, cf. Figure 5.1 (middle). This is a special case of our proposed utility function and corresponds to $\Lambda = \frac{1}{2}$. They showed for paths, cycles, grids, and torus that even if all individual agents have a strict preference for perfect integration, best response dynamics lead to segregation, although complete segregation seems to be avoided.

Recently, a model was introduced where the agent itself is included in the set of its neighbors [KKV21]. We adapt this modified version in our model. Since the agent itself contributes to the diversity in its neighborhood, we include for the computation of the fraction of same-type neighbors the agent itself in the set of its neighbors but also in the set of same-type neighbors.

Our main focus is on a broad class of non-monotone utility functions well-known in economics and algorithmic game theory: single-peaked utilities. This can be understood as single-peaked preferences, which date back to Black [Bla48] and are a common theme in the economics and game theory literature. In particular, such preferences yield favorable behavior in the hedonic diversity games [BE20; BEI19] and the realm of voting and social choice [BSU13; Bra+15; EFS20; Wal07; YCE13]. We emphasize that our results hold for all such functions that satisfy our mild assumptions. See Table 5.1 for a detailed result overview.

Concerning the existence of equilibria, cf. Section 5.3, for games with integration oriented agents, i.e., $\Lambda \leq \frac{1}{2}$, we show that swap equilibria exist on almost regular graphs and that improving response dynamics are guaranteed to converge to such stable states, cf. Theorem 5.6. Moreover, for $\Lambda = \frac{1}{2}$ swap equilibria exist on the broad class of graphs that admit an independent set that is large enough to accommodate the minority type agents, cf. Theorem 5.9. In particular, this implies equilibrium existence and efficient computability on bipartite graphs, including trees, cf. Corollary 5.10, which is in contrast to the non-existence results by [Aga+21].

Another contrast are our bounds on the Price of Anarchy, cf. Section 5.4. On general graphs, we prove a tight bound on the Price of Anarchy that depends on o, the number of agents of the minority color, and we give a bound of Δ on all graphs G, cf. Theorem 5.13, that is asymptotically tight on Δ-regular graphs, cf. Theorem 5.14 and Theorem 5.15. Also for the Price of Stability, cf. Section 5.5, we get stronger positive results compared to [Aga+21]. For $\Lambda = \frac{1}{2}$ we give a tight Price of Stability bound of 2 on bipartite graphs, cf. Theorem 5.18 and Theorem 5.19, and show that the Price of Stability is 1 on almost regular graphs with maximum degree 3, cf. Theorem 5.20, or if the size of the maximum independent set of the graph is at most o, cf. Theorem 5.21. The latter implies a Price of Stability of 1 on regular graphs for balanced games, i.e., if there are equally many agents of both colors, cf. Corollary 5.22. Even more general, for constant $\Lambda \leq \frac{1}{2}$ we prove a constant Price of Stability on almost regular graphs via a sophisticated proof technique that relies on the greedy algorithm for the κ-Max-Cut problem, cf. Theorem 5.25 and Corollary 5.26.

Moreover, we investigate the influence of the underlying graph on computational complexity aspects, cf. Section 5.6, and provide hardness results for

Table 5.1: Overview of our results. We investigate the existence of equilibria, the finite improvement property, the PoA, and the PoS. The "✓" symbol denotes that the respective property holds, and the "×" means the opposite. The respective conditions are stated next to the result. ϵ is a constant larger than zero. "1-regular" stands for almost regular graphs. Note, PoS results for almost regular graphs hold for regular graphs as well. For the PoA the stated lower bounds of other graph classes hold for arbitrary graphs as well.

graph classes	Equilibrium Existence		Finite Improvement Property	
arbitrary	× (Thm. 5.7)	$\Lambda > \frac{1}{2}$	× (Thm. 5.7, 5.8)	$\Lambda \geq \frac{1}{2}$
	✓(Thm. 5.9)	$\frac{1}{\delta+1} \leq \Lambda \leq \frac{1}{2}, \alpha + 1 \geq o$		
bipartite	✓(Cor. 5.10)	$\Lambda = \frac{1}{2}$		
1-regular	✓(Thm. 5.6)	$\Lambda \leq \frac{1}{2}$	✓(Thm. 5.6)	$\Lambda \leq \frac{1}{2}$
2-regular	× (Thm. 5.7)	$\Lambda > \frac{1}{2}$	× (Thm. 5.7)	$\Lambda > \frac{1}{2}$
	Price of Anarchy		Price of Stability	
arbitrary	$\leq \min\{\Delta, \frac{n}{o+1}, \frac{(\Delta+1)o}{o+1}\}$ (Thm. 5.13)		$\geq \Omega\left(\sqrt{n\Lambda}\right)$ (Thm. 5.17)	
bipartite	$\geq \frac{n-1}{3}$ (Thm. 5.13)	$o = 1$	2 (Thm. 5.18, 5.19)	$\Lambda = \frac{1}{2}$
	$\geq \frac{n}{o+1}$ (Thm. 5.13)	$o > 1$		
regular	$\leq \min\{\frac{\Delta+1}{2}, \frac{n}{2o}\}$ (Thm. 5.14)	$\Lambda < \frac{1}{\Delta}$		
	$\geq \frac{\Delta+1}{2} - \frac{\Delta+1}{4\Delta+2}$ (Thm. 5.15)	$\Lambda \leq \frac{1}{2}, \Delta \geq 2$		
1-regular			1 (Thm. 5.20)	$\Lambda \leq \frac{1}{2}, \Delta \leq 3$ or
			(Thm. 5.21)	$\Lambda \in \left[\frac{1}{\delta+1}, \frac{1}{2}\right], o \geq \alpha(G)$
			$\min\{\Delta + 1, O(\frac{1}{\Lambda})\}$ (Thm. 5.25)	$\Lambda \leq \frac{1}{2}, o < \alpha$
			$O(1)$ (Cor. 5.26)	$\Lambda \leq \frac{1}{2}$
ring	$> 2 - \epsilon$ (Thm. 5.16)			
	$\frac{3}{2} - \epsilon$ (Thm. 5.16)	$\Lambda < \frac{1}{2}$		

computing a strategy profile or a swap equilibrium that maximize the Degree of Integration on cubic graphs or on bipartite graphs. We establish that all these problems are NP-hard and that they cannot be approximated to within a certain constant factor in polynomial time, unless P = NP.

5.1 Model

We consider the Single-Peaked Swap Schelling Game with 2 types. Thus, any strategy profile $\boldsymbol{\sigma}$ corresponds to a bi-coloring of G in which exactly o vertices of G are colored orange and $n - o$ are colored blue.

The utility of an agent i in $\boldsymbol{\sigma}$ is defined as $\mathsf{U}_i(\boldsymbol{\sigma}) = p\left(\frac{|N[\boldsymbol{\sigma}(i)] \cap C_i(\boldsymbol{\sigma})|}{|N[\boldsymbol{\sigma}(i)]|}\right)$. Note that we consider the closed neighborhood, that is, the agent itself is included. The function p is a single-peaked function with the peak at Λ. For the exact definition of p, we refer to Definition 2.4. We say an agent i is *below the peak* when $f_i(\boldsymbol{\sigma}) < \Lambda$, *above the peak* when $f_i(\boldsymbol{\sigma}) > \Lambda$, at *the peak* when $f_i(\boldsymbol{\sigma}) = \Lambda$, and *segregated* when $f_i(\boldsymbol{\sigma}) = 1$.

Note that the SP-2-SSG (G, o, Λ) depends on the choice of p. However, as all our results are independent of p, we remove it from the notation for the sake of simplicity. Remember that we denote by α the *independence number* of G. Moreover, we measure the quality of a strategy profile $\boldsymbol{\sigma}$ via the Degree of Integration. We prefer it to the standard utilitarian welfare since it measures segregation independently of the value of Λ. For investigating the dynamic properties, we mainly use the potential function $\Phi(\boldsymbol{\sigma}) = \left|\left\{\{u, v\} \in E \mid c(\boldsymbol{\sigma}^{-1}(u)) = c(\boldsymbol{\sigma}^{-1}(v))\right\}\right|$.

5.2 Preliminaries

In this section, we provide some facts and lemmas that will be widely exploited throughout this chapter. We start by observing the following fundamental relationship occurring between $f_i(\boldsymbol{\sigma})$ and $f_j(\boldsymbol{\sigma}_{ij})$ for two swapping agents i and j:

$$\text{if } f_i(\boldsymbol{\sigma}) = \frac{x}{y},^{12} \text{ then } f_j(\boldsymbol{\sigma}_{ij}) = \frac{y + 1 - x - 1_{ij}(\boldsymbol{\sigma})}{y}. \tag{5.1}$$

12 For the sake of conciseness, from now on, whenever we write $f_i(\boldsymbol{\sigma}) = \frac{x}{y}$ for some agent i, we implicitly mean that $x := |N[\boldsymbol{\sigma}(i)] \cap C_i(\boldsymbol{\sigma})|$ and $y := |N[\boldsymbol{\sigma}(i)]|$. Observe that, under this assumption, $f_i(\boldsymbol{\sigma}) = \frac{3}{6}$ is different than $f_i(\boldsymbol{\sigma}) = \frac{1}{2}$.

Using property (1), we claim the following observation.

▶ **Observation 5.1.** If $f_i(\boldsymbol{\sigma}) = \frac{x}{y} < \frac{1}{2}$, then $f_j(\boldsymbol{\sigma}_{ij}) > \frac{1}{2}$. If $f_i(\boldsymbol{\sigma}) = \frac{x}{y} > \frac{1}{2}$, then $f_j(\boldsymbol{\sigma}_{ij}) \leq \frac{1}{2}$, unless $y = 2x - 1$ and $1_{ij}(\boldsymbol{\sigma}) = 0$ for which $f_j(\boldsymbol{\sigma}_{ij}) = f_i(\boldsymbol{\sigma}) = \frac{x}{y} > \frac{1}{2}$. ◀

Proof. If $f_i(\boldsymbol{\sigma}) = \frac{x}{y} < \frac{1}{2}$, by Equation (5.1) we have

$$f_j(\boldsymbol{\sigma}_{ij}) = \frac{y + 1 - x - 1_{ij}(\boldsymbol{\sigma})}{y} = 1 - \frac{x}{y} + \frac{1 - 1_{ij}(\boldsymbol{\sigma})}{y} > \frac{1}{2}.$$

If $f_i(\boldsymbol{\sigma}) = \frac{x}{y} > \frac{1}{2}$, we distinguish among different cases. If $1_{ij}(\boldsymbol{\sigma}) = 1$, by Equation (5.1) we get

$$f_j(\boldsymbol{\sigma}_{ij}) = \frac{y + 1 - x - 1_{ij}(\boldsymbol{\sigma})}{y} = 1 - \frac{x}{y} < \frac{1}{2},$$

while, if $y < 2x - 1$, which implies $x \geq \frac{y+2}{2}$, it follows

$$f_j(\boldsymbol{\sigma}_{ij}) = \frac{y + 1 - x - 1_{ij}(\boldsymbol{\sigma})}{y} = 1 - \frac{x}{y} + \frac{1 - 1_{ij}(\boldsymbol{\sigma})}{y} \leq \frac{1}{2} - \frac{1_{ij}(\boldsymbol{\sigma})}{y} \leq \frac{1}{2}.$$

Finally, for $y = 2x - 1$ and $1_{ij}(\boldsymbol{\sigma}) = 0$, by Equation (5.1) we get

$$f_j(\boldsymbol{\sigma}_{ij}) = \frac{y + 1 - x - 1_{ij}(\boldsymbol{\sigma})}{y} = \frac{x}{y} > \frac{1}{2}.$$ ■

The following series of lemmas characterizes the conditions under which a profitable swap can take place.

▶ **Lemma 5.2.** For any $\Lambda \leq \frac{1}{2}$, no profitable swaps can occur between agents below the peak. ◀

Proof. Fix a strategy profile $\boldsymbol{\sigma}$ and two agents i and j, below the peak, who can perform a profitable swap in $\boldsymbol{\sigma}$. By Observation 5.1, both i and j are above the peak in $\boldsymbol{\sigma}_{ij}$. Assume, without loss of generality, that $f_i(\boldsymbol{\sigma}) = \frac{x}{y} < \Lambda$ which by Equation (5.1) yields

$$f_j(\boldsymbol{\sigma}_{ij}) = \frac{y + 1 - x - 1_{ij}(\boldsymbol{\sigma})}{y} > \Lambda.$$

We claim that $\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) \leq \mathsf{U}_i(\boldsymbol{\sigma})$. By the definition of p, this holds whenever

$$\frac{x}{y} \geq \frac{\Lambda}{1-\Lambda}\left(1 - \frac{y+1-x-1_{ij}(\boldsymbol{\sigma})}{y}\right) = \frac{\Lambda}{1-\Lambda}\left(\frac{x}{y} - \frac{1-1_{ij}(\boldsymbol{\sigma})}{y}\right)$$

which holds true as $\frac{\Lambda}{1-\Lambda} \leq 1$ and $1 - 1_{ij}(\boldsymbol{\sigma}) \geq 0$.

By applying the same argument, with i and j swapped, we also get $\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) \leq \mathsf{U}_j(\boldsymbol{\sigma})$. As the swap is profitable, we have $\mathsf{U}_i(\boldsymbol{\sigma}) < \mathsf{U}_i(\boldsymbol{\sigma}_{ij})$ and $\mathsf{U}_j(\boldsymbol{\sigma}) < \mathsf{U}_j(\boldsymbol{\sigma}_{ij})$. Putting all these inequalities together, we conclude that

$$\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) \leq \mathsf{U}_i(\boldsymbol{\sigma}) < \mathsf{U}_i(\boldsymbol{\sigma}_{ij}) \leq \mathsf{U}_j(\boldsymbol{\sigma}) < \mathsf{U}_j(\boldsymbol{\sigma}_{ij}),$$

which yields a contradiction. ∎

▶ **Lemma 5.3.** For any $\Lambda \leq \frac{1}{2}$, no profitable swaps can occur between adjacent agents at different sides of the peak. ◀

Proof. Assume towards a contradiction, that i and j can perform a profitable swap in $\boldsymbol{\sigma}$, and, without loss of generality, that $f_i(\boldsymbol{\sigma}) = \frac{x}{y} < \Lambda$ and $f_j(\boldsymbol{\sigma}) = \frac{x'}{y'} > \Lambda$. By Observation 5.1, j ends up above the peak in $\boldsymbol{\sigma}_{ij}$. As j improves after the swap, we have

$$\mathsf{U}_j(\boldsymbol{\sigma}_{ij}) = p\left(1 - \frac{x}{y}\right) > \mathsf{U}_j(\boldsymbol{\sigma}) = p\left(\frac{x'}{y'}\right)$$

which, given that $1 - \frac{x}{y} > \Lambda$ and $\frac{x'}{y'} > \Lambda$, yields $1 - \frac{x}{y} < \frac{x'}{y'}$. This implies that

$$f_i(\boldsymbol{\sigma}_{ij}) = 1 - \frac{x'}{y'} < 1 - 1 + \frac{x}{y} = \frac{x}{y} = f_i(\boldsymbol{\sigma})$$

which, given that $f_i(\boldsymbol{\sigma}) < \Lambda$, contradicts the fact that i improves after the swap. ∎

In the following, we present a technical result that will help to prove Lemma 5.5.

▶ **Lemma 5.4.** For any $\Lambda \leq \frac{1}{2}$, any profitable swap occurring between two agents i and j in a strategy profile $\boldsymbol{\sigma}$, with $f_i(\boldsymbol{\sigma}) < \Lambda$ and $f_j(\boldsymbol{\sigma}) > \Lambda$, requires $\deg(\boldsymbol{\sigma}(i)) > \deg(\boldsymbol{\sigma}(j))$. ◀

Proof. Assume towards a contradiction, that $\deg(\boldsymbol{\sigma}(i)) \leq \deg(\boldsymbol{\sigma}(j))$ and i and j can perform a profitable swap in $\boldsymbol{\sigma}$, and, without loss of generality, that $f_i(\boldsymbol{\sigma}) = \frac{x}{y} < \Lambda$ and $f_j(\boldsymbol{\sigma}) = \frac{x'}{y'} > \Lambda$.

By Lemma 5.3, it must be $1_{ij}(\boldsymbol{\sigma}) = 0$. By Observation 5.1, j ends up above the peak in $\boldsymbol{\sigma}_{ij}$. As j improves after the swap, we have

$$\mathrm{U}_j(\boldsymbol{\sigma}_{ij}) = p(1 - \frac{x}{y} + \frac{1}{y}) > \mathrm{U}_j(\boldsymbol{\sigma}) = p\left(\frac{x'}{y'}\right)$$

which, given that $1 - \frac{x}{y} + \frac{1}{y} > \Lambda$ and $\frac{x'}{y'} > \Lambda$, yields $1 - \frac{x}{y}y + \frac{1}{y} < \frac{x'}{y'}$. This implies that

$$f_i(\boldsymbol{\sigma}_{ij}) = 1 - \frac{x'}{y'} + \frac{1}{y'} < 1 - 1 + \frac{x}{y} - \frac{1}{y} + \frac{1}{y'} = \frac{x}{y} + \frac{1}{y'} - \frac{1}{y}.$$

Now, as the hypothesis $\deg(\boldsymbol{\sigma}(i)) \leq \deg(\boldsymbol{\sigma}(j))$ can be restated as $y \leq y'$, we derive

$$f_i(\boldsymbol{\sigma}_{ij}) < \frac{x}{y} + \frac{1}{y'} - \frac{1}{y} \leq \frac{x}{y} = f_i(\boldsymbol{\sigma}),$$

which, given that $f_i(\boldsymbol{\sigma}) < \Lambda$, contradicts the fact that i improves after the swap. ■

▶ **Lemma 5.5.** For any $\Lambda \leq \frac{1}{2}$, no profitable swaps can occur between agents at different sides of the peak in SP-2-SSGs on almost regular graphs. ◀

Proof. Fix a strategy profile $\boldsymbol{\sigma}$ and two agents i and j at different sides of the peak admitting a profitable swap in $\boldsymbol{\sigma}$. As the game is played on an almost regular graph, by Lemma 5.4, it must be $f_i(\boldsymbol{\sigma}) = \frac{x}{y+1} < \Lambda$, $f_j(\boldsymbol{\sigma}) = \frac{x'}{y} > \Lambda$. Moreover, by Lemma 5.3, we have $1_{ij}(\boldsymbol{\sigma}) = 0$.

Since j improves after the swap, we have

$$\mathrm{U}_j(\boldsymbol{\sigma}_{ij}) = p\left(\frac{y - x + 2}{y + 1}\right) > \mathrm{U}_j(\boldsymbol{\sigma}) = p\left(\frac{x'}{y}\right)$$

which, given that $\frac{y-x+2}{y+1} > \Lambda$ and $\frac{x'}{y} > \Lambda$, yields $\frac{x'}{y} > \frac{y-x+2}{y+1}$. We derive $x'(y+1) > y(y - x + 2)$, which, given that both sides of the inequality are integers, yields

$$x'(y + 1) \geq y(y - x + 2) + 1. \tag{5.2}$$

Since i improves after the swap, we have

$$\mathrm{U}_i(\boldsymbol{\sigma}_{ij}) = p\left(\frac{y - x' + 1}{y}\right) > \mathrm{U}_i(\boldsymbol{\sigma}) = p\left(\frac{x}{y + 1}\right).$$

We now distinguish between two possible cases: Since i improves after the swap, we have

$$\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = p\left(\frac{y - x' + 1}{y}\right) > \mathsf{U}_i(\boldsymbol{\sigma}) = p\left(\frac{x}{y + t}\right).$$

We now distinguish between two possible cases:

(i) $\frac{y-x'+1}{y} \leq \Lambda$ and

(ii) $\frac{y-x'+1}{y} > \Lambda$.

If case *(i)* occurs, it must be $\frac{x}{y+1} < \frac{y-x'+1}{y}$ which is equivalent to $x'(y+1) < (y+1)^2 - xy$. Together with Inequality 5.2, this yields $y(y-x+2)+1 < (y+1)^2 - xy$ which yields a contradiction.

If case *(ii)* occurs, from $\frac{y-x'+1}{y} > \Lambda$, we get

$$x' < y(1 - \Lambda) + 1. \tag{5.3}$$

Since

$$\mathsf{U}_i(\boldsymbol{\sigma}_{ij}) = p\left(\frac{y - x' + 1}{y}\right) > \mathsf{U}_i(\boldsymbol{\sigma}) = p\left(\frac{x}{y + 1}\right), \ \frac{x}{y+1} < \Lambda$$

and $\frac{y-x'+1}{y} > \Lambda$, by the definition of p, we derive

$$\frac{x}{y+1} < \frac{\Lambda}{1-\Lambda}\left(1 - \frac{y - x' + 1}{y}\right) = \frac{\Lambda}{1-\Lambda}\frac{x'-1}{y},$$

by which we get $x < \frac{\Lambda(x'-1)(1+y)}{(1-\Lambda)y}$. Together with Inequality 5.3, this yields

$$x < \Lambda(y + 1). \tag{5.4}$$

By summing up Inequality 5.3 and Inequality 5.4, we get $x + x' < y + 1 + \Lambda$. As x, x', y are integers and $\Lambda \in [0, \frac{1}{2}]$, we derive

$$x + x' \leq y + 1. \tag{5.5}$$

Starting from Inequality 5.2 and then using Inequality 5.5, we derive

$$x' \geq y(y - x - x' + 2) + 1 \geq y + 1,$$

which, given that $x' \leq y$, yields a contradiction. ■

5.3 Equilibrium Existence and Dynamics

In this section, we provide existential results for SP-2-SSGs played on some specific graph topologies. We start by showing that games on almost regular graphs enjoy the FIP property and converge to a swap equilibrium in at most m swaps in any game in which the peak does not exceed $\frac{1}{2}$. This result does not hold when the peak exceeds $\frac{1}{2}$, as we prove the existence of a game played on a 2-regular graph (i.e., a ring) admitting no swap equilibrium.

▶ **Theorem 5.6.** For any $\Lambda \leq \frac{1}{2}$, fix a SP-2-SSG (G, o, Λ) on an almost regular graph G and a strategy profile $\boldsymbol{\sigma}$. Any sequence of profitable swaps starting from $\boldsymbol{\sigma}$ ends in a swap equilibrium after at most m swaps. ◀

Proof. We show that, after a profitable swap, the potential function Φ decreases by at least 1. Consider a profitable swap performed by agents i and j such that $f_i(\boldsymbol{\sigma}) = \frac{x}{y}$ and $f_j(\boldsymbol{\sigma}) = \frac{x'}{y+t)}$, with $t \in \{0, 1\}$ since G is almost regular. By Lemma 5.2 and Lemma 5.5, we have that both, i and j, are above the peak, i.e., $\frac{x}{y} > \Lambda$ and $\frac{x'}{y+t} > \Lambda$. By Observation 5.1, after the swap, both i and j do not go below the peak. Thus, a necessary condition for the swap to be profitable is that $f_i(\boldsymbol{\sigma}_{ij}) < f_i(\boldsymbol{\sigma})$ and $f_j(\boldsymbol{\sigma}_{ij}) < f_j(\boldsymbol{\sigma})$. Again, by Observation 5.1, the latter yields

$$\frac{x'}{y+t} > 1 - \frac{x}{y} + \frac{1 - 1_{ij}(\boldsymbol{\sigma})}{y},$$

which gives

$$x' > y - x + 1 - 1_{ij}(\boldsymbol{\sigma}) + t\left(1 - \frac{x}{y} + \frac{1 - 1_{ij}(\boldsymbol{\sigma})}{y}\right) \geq y - x + 1 - 1_{ij}(\boldsymbol{\sigma}).$$

Since x, x', y and $1_{ij}(\boldsymbol{\sigma})$ are integers, we derive $x' \geq y - x + 2 - 1_{ij}(\boldsymbol{\sigma})$. As it holds that $\Phi(\boldsymbol{\sigma}) - \Phi(\boldsymbol{\sigma}_{ij})$ equals

$$x - 1 + x' - 1 - \big(y - x - 1_{ij}(\boldsymbol{\sigma}) + y + t - x' - 1_{ij}(\boldsymbol{\sigma})\big) = 2(x + x' - 1 + 1_{ij}(\boldsymbol{\sigma})) - 2y - t,$$

we get $\Phi(\boldsymbol{\sigma}) - \Phi(\boldsymbol{\sigma}_{ij}) \geq 1$. ■

This result does not hold when the peak exceeds $\frac{1}{2}$.

▶ **Theorem 5.7.** For any $\Lambda > \frac{1}{2}$, there exists a SP-2-SSG played on a 2-regular graph admitting no swap equilibrium. ◀

Proof. Consider an instance of a game played on a ring with 6 vertices, where $o = b = 3$. Only the following two complementary cases may occur:

- The orange agents occupy vertices that induce a path of length 2. In this case, there are two segregated agents of different colors, both with utility 0. As $p(0) = 0$ and $p(x) > 0$ for $x \in (0, 1)$, the two agents swap their positions.

- There are two neighboring agents i and j of different colors both being below the peak. In this case, as $p\left(\frac{1}{3}\right) < p\left(\frac{2}{3}\right)$, both i and j prefer to swap their positions. ∎

A fundamental question is whether a swap equilibrium always exists in SP-2-SSGs with tolerant agents, i.e., for $\Lambda \leq \frac{1}{2}$. The next result shows that Theorem 5.6 cannot be generalized to all graphs.

▶ **Theorem 5.8.** There cannot exist a generalized ordinal potential function in SP-2-SSGs on arbitrary graphs for $\Lambda = \frac{1}{2}$. ◀

Proof. We prove the statement by providing an improving response cycle where in every step a profitable swap is possible. The construction and arising strategy profiles are shown in Figure 5.2. For the sake of simplicity we assume $p(x) = x$ for $x \in [0, \frac{1}{2}]$. However, our result is independent of the choice of p. Remember that, since $\Lambda = \frac{1}{2}$, $p(x) = p(1 - x)$ for $x \in [\frac{1}{2}, 1]$.

In the initial feasible strategy profile, cf. Figure 5.2 (a), agents a and b can swap. By swapping their positions, agent a increases its utility from $1 - \frac{5}{7} = \frac{2}{7}$ to $1 - \frac{9}{13} = \frac{4}{13}$ and agent b increases its utility from $\frac{5}{13}$ to $\frac{3}{7}$.

Next, agents c and d can swap, cf. Figure 5.2 (b). Swapping with agent d increases agent c's utility from $1 - \frac{4}{7} = \frac{3}{7}$ to $\frac{6}{13}$, and agent d increases its utility from $1 - \frac{8}{13} = \frac{5}{13}$ to $1 - \frac{4}{7} = \frac{3}{7}$.

After this, cf. Figure 5.2 (c), agents e and f, and agents g and h, respectively, have the opportunity to swap and increase their utility. Agent e increases its utility from $\frac{3}{7}$ to $\frac{1}{2}$ while agent f increases its utility from $1 - \frac{2}{2} = 0$ to $1 - \frac{5}{7} = \frac{2}{7}$. Agent g improves its utility from $1 - \frac{2}{2} = 0$ to $1 - \frac{3}{5} = \frac{2}{5}$ and agent h increases its utility from $1 - \frac{3}{5} = \frac{2}{5}$ to $\frac{1}{2}$.

Next, cf. Figure 5.2 (d), swaps between agents i and j, and k and l, respectively, are possible. Agent i increases its utility from $1 - \frac{2}{2} = 0$ to $1 - \frac{4}{5} = \frac{1}{5}$ and j increases its utility from $\frac{2}{5}$ to $\frac{1}{2}$. Agent k improves its utility from $1 - \frac{2}{2} = 0$ to $1 - \frac{3}{5} = \frac{2}{5}$ and agent l increases its utility from $1 - \frac{3}{5} = \frac{2}{5}$ to $\frac{1}{2}$.

In the next step, agents e and m can swap. Agent e increases its utility from $1 - \frac{2}{2} = 0$ to $1 - \frac{3}{5} = \frac{2}{5}$ and m increases its utility from $1 - \frac{3}{5} = \frac{2}{5}$ to $\frac{1}{2}$.

In the final step, agents n and o swap. Swapping with agent o increases agent n's utility from $1 - \frac{2}{2} = 0$ to $1 - \frac{8}{13} = \frac{5}{13}$ and agent o increases its utility from $\frac{6}{13}$ to $\frac{1}{2}$. Now the reached feasible strategy profile, cf. Figure 5.2 (g), is equivalent to the initial feasible strategy profile, cf. Figure 5.2 (a).

However, note that although convergence is not guaranteed there still exists a stable state, cf. Figure 5.2 (h). ■

For the special case of $\Lambda = \frac{1}{2}$, however, the existence of a swap equilibrium is guaranteed in any graph whose independence number is at least the number of orange agents.

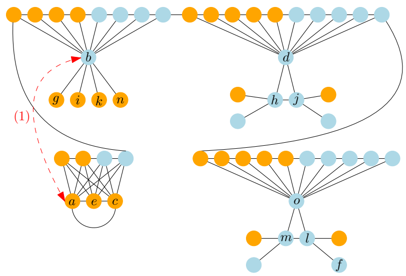
▶ **Theorem 5.9.** Fix a SP-2-SSG (G, o, Λ) with $\frac{1}{\delta+1} \leq \Lambda \leq \frac{1}{2}$. Any strategy profile in which all agents of the same color are located on an independent set of G is a swap equilibrium. ◀

Proof. Let $\boldsymbol{\sigma}$ be a strategy profile in which all agents of the same color are located on an independent set of G. Assume, without loss of generality, that all orange agents are assigned to the vertices of an independent set of G and consider a profitable swap performed by an orange agent i and a blue agent j. If $1_{ij}(\boldsymbol{\sigma}) = 0$, since i is only adjacent to blue agents other than j, it holds that $f_j(\boldsymbol{\sigma}_{ij}) = 1$, which gives $\mathrm{U}_j(\boldsymbol{\sigma}_{ij}) = 0$, thus contradicting the fact that j performs a profitable swap. If $1_{ij}(\boldsymbol{\sigma}) = 1$, instead, we obtain

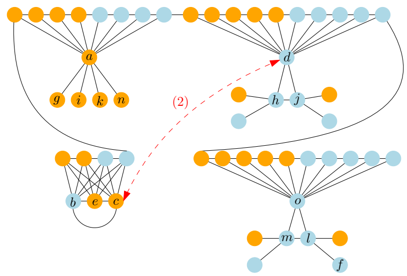
$$f_i(\boldsymbol{\sigma}) = \frac{1}{\deg(\boldsymbol{\sigma}(i)) + 1} \leq \frac{1}{\delta + 1} \leq \Lambda.$$

The numerator comes from the fact that i is only adjacent to blue agents. Knowing that i cannot be at the peak, we conclude that it is below the peak. If j is also below the peak, Lemma 5.2 contradicts the fact that the swap is profitable, while, if j is above the peak, the contradiction comes from Lemma 5.3. ■

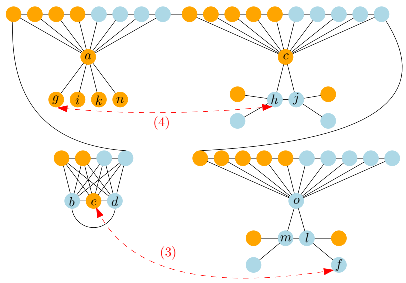
▶ **Corollary 5.10.** For $\Lambda = \frac{1}{2}$, SP-2-SSGs played on bipartite graphs always admit a swap equilibrium which can be efficiently computed. ◀



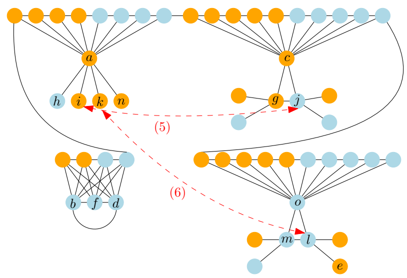
(a) Initial feasible strategy profile



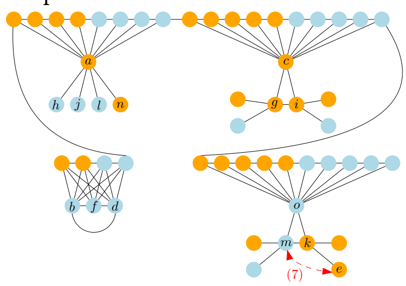
(b) Feasible strategy profile after the first swap of a and b



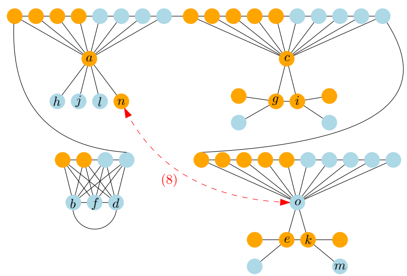
(c) Feasible strategy profile after the second swap of c and d



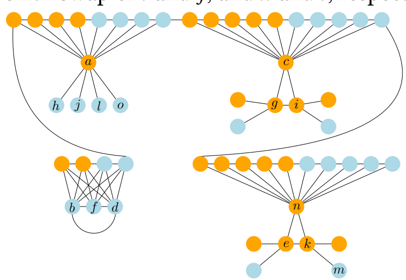
(d) Feasible strategy profile after the third and forth swap of e and f, and g and h, respectively.



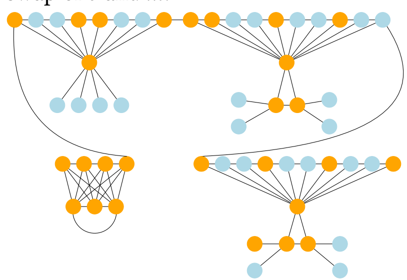
(e) Feasible strategy profile after the fifth and sixth swap of i and j, and k and l, respectively.



(f) Feasible strategy profile after the seventh swap of e and m.



(g) Feasible strategy profile after the last swap of n and o.



(h) The swap equilibrium for the same instance.

Figure 5.2: An IRC and the swap equilibrium for a SP-2-SSG $(G, o, \frac{1}{2})$. See the proof of Theorem 5.8 for more details.

5.4 Price of Anarchy

In this section, we give bounds on the PoA for SP-2-SSGs played on different topologies, even in those cases for which the existence of a swap equilibrium is not guaranteed.

5.4.1 General Graphs

The next lemma provides a necessary condition that needs to be satisfied by any swap equilibrium and an upper bound on the value of the social optimum, respectively.

▶ **Lemma 5.11.** In a swap equilibrium for any SP-2-SSG (G, o, Λ), no agents of different colors can be segregated. ◀

Proof. Fix a strategy profile $\boldsymbol{\sigma}$. If there exist two agents i and j such that $f_i(\boldsymbol{\sigma}) = f_j(\boldsymbol{\sigma}) = 1$, they can perform a profitable swap, as $f_i(\boldsymbol{\sigma}) = f_j(\boldsymbol{\sigma}) = 1$ and $f_i(\boldsymbol{\sigma}_{ij}) = f_j(\boldsymbol{\sigma}_{ij}) \notin \{0, 1\}$. So, $\boldsymbol{\sigma}$ cannot be a swap equilibrium for (G, o, Λ). ■

This directly gives an upper bound for the Degree of Integration.

▶ **Lemma 5.12.** For any SP-2-SSG (G, o, Λ), we have

$$\mathsf{DoI}(\boldsymbol{\sigma}^*) \leq \min\{(\Delta + 1)o, n\}.$$

◀

Proof. As an orange vertex can be adjacent to at most Δ blue ones, it follows that, in any strategy profile, there cannot be more than $(\Delta + 1)o$ non-segregated agents, so that $\mathsf{DoI}(\boldsymbol{\sigma}^*) \leq \min\{(\Delta + 1)o, n\}$. ■

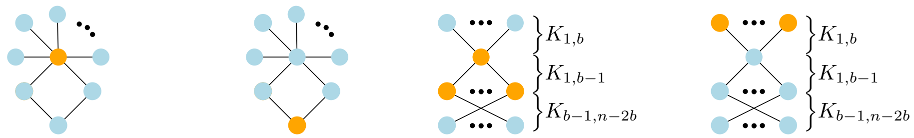
We now give (almost) tight bounds on the PoA for general graphs.

▶ **Theorem 5.13.** For any SP-2-SSG (G, o, Λ),

$$\mathsf{PoA}_{\mathsf{DoI}}(G, o, \Lambda) \leq \min\left\{\Delta, \frac{n}{o+1}, \frac{(\Delta + 1)o}{o+1}\right\}.$$

Moreover, there exists a SP-2-SSG on a bipartite graph such that

$$\mathsf{PoA}_{\mathsf{DoI}}(G, o, \Lambda) \geq \frac{n}{o+1}$$



(a) An instance with $o = 1$ orange agents. Left: $\boldsymbol{\sigma}^*$ with $\mathsf{DoI}(\boldsymbol{\sigma}^*) = n - 1$. Right: a swap equilibrium $\boldsymbol{\sigma}$ with $\mathsf{DoI}(\boldsymbol{\sigma}) = 3$.

(b) An instance with $o \geq 2$ orange agents. **(a)** $\boldsymbol{\sigma}^*$ with $\mathsf{DoI}(\boldsymbol{\sigma}^*) = n$. **(b)** a swap equilibrium $\boldsymbol{\sigma}$ with $\mathsf{DoI}(\boldsymbol{\sigma}) = o + 1$.

Figure 5.3: Lower bounds for $\mathsf{PoA}_{\mathsf{DoI}}(G, o, \Lambda)$ when (a) $o = 1$, and (b) $o > 1$. Left: the socially optimal feasible strategy profile $\boldsymbol{\sigma}^*$. Right: the swap equilibrium $\boldsymbol{\sigma}$ with minimum social welfare. See the proof of Theorem 5.13 for more details.

when $o > 1$ and when $o = 1$

$$\mathsf{PoA}_{\mathsf{DoI}}(G, o, \Lambda) \geq \frac{n-1}{3}.$$

◀

Proof. For the upper bound, fix a game (G, o, Λ) and a swap equilibrium $\boldsymbol{\sigma}$. By Lemma 5.11, only agents of one color, say c, can be segregated in $\boldsymbol{\sigma}$. Thus, we get $\mathsf{DoI}(\boldsymbol{\sigma}) \geq o + 1$. Let V' be the set of vertices of color $c' \neq c$. Every vertex in V' has to be adjacent to a vertex of color c. So, there are at least $|V'| \geq o$ non-monochrome edges in the coloring induced by $\boldsymbol{\sigma}$. As every vertex of color c can be adjacent to at most Δ vertices of color c', there must be at least $\lceil \frac{o}{\Delta} \rceil$ vertices of color c incident to a non-monochrome edge, that is, being non-segregated in $\boldsymbol{\sigma}$. Thus, we get

$$\mathsf{DoI}(\boldsymbol{\sigma}) \geq \frac{(\Delta + 1)o}{\Delta}.$$

We conclude that

$$\mathsf{DoI}(\boldsymbol{\sigma}) \geq \max\left\{\frac{(\Delta + 1)o}{\Delta}, o + 1\right\}.$$

The upper bounds follow from Lemma 5.12. For the lower bounds, consider the SP-2-SSGs defined in Figure 5.3. ∎

5.4.2 Regular Graphs

For Δ-regular graphs, we derive an upper bound of Δ on the PoA from Theorem 5.13. A better result is possible when Λ is sufficiently small.

▶ **Theorem 5.14.** For any SP-2-SSG (G, o, Λ) on a Δ-regular graph G with $\Lambda < \frac{1}{\Delta}$, $\mathsf{PoA}_{\mathsf{DoI}}(G, o, \Lambda) \leq \min\{\frac{\Delta+1}{2}, \frac{n}{2o}\}$. ◀

Proof. Fix a swap equilibrium $\boldsymbol{\sigma}$. By Lemma 5.11, only agents of a unique color, say c, can be segregated in $\boldsymbol{\sigma}$. Let V be the set of vertices of color $c' \neq c$. As $\Lambda < \frac{1}{\Delta}$, every vertex in V has to be adjacent to vertices of color c only. Otherwise, any agent in V that is adjacent to an agent of color c' can perform a profitable swap with a segregated agent of color c. Thus, there are at least $\Delta|V| \geq \Delta o$ non-monochrome edges in the coloring induced by $\boldsymbol{\sigma}$. As every agent of one color can be adjacent to at most Δ agents of the other one, there are at least o non-segregated agents of color c. Together with the at least o agents of color c', this gives $\mathsf{DoI}(\boldsymbol{\sigma}) \geq 2o$ which, together with Lemma 5.12, yields the claim. ■

As a lower bound, we have the following.

▶ **Theorem 5.15.** For every $\Delta \geq 2$ and $\Lambda \leq \frac{1}{2}$, there exists a SP-2-SSG (G, o, Λ) on a Δ-regular graph such that

$$\mathsf{PoA}_{\mathsf{DoI}}(G, o, \Lambda) \geq \frac{\Delta(\Delta+1)}{2\Delta+1} = \frac{\Delta+1}{2} - \frac{\Delta+1}{4\Delta+2}.$$

◀

Proof. Consider graph G shown in Figure 5.4. G consists of three combined gadgets that we call the *left gadget*, the *upper right gadget* and the *lower right gadget*. The left gadget is essentially the complete bipartite graph $K_{\Delta,\Delta}$ with a missing edge: the one connecting the last two vertices of the respective partitions. These vertices are connected to the upper right gadget and the lower right one, respectively. So each vertex in the left gadget has degree Δ. The upper right gadget consists of a clique $K_{\Delta-1}$ whose vertices are all connected to two special vertices, one on the left of $K_{\Delta-1}$ and one on the right. The vertex on the left is the one adjacent to the vertex from the left gadget, while the vertex on the right connects the gadget with the lower right one. Thus, every vertex in this gadget has degree Δ. Finally, the lower right gadget is any Δ-regular graph with n^* vertices with a missing edge: the one connecting the vertex incident to the edge

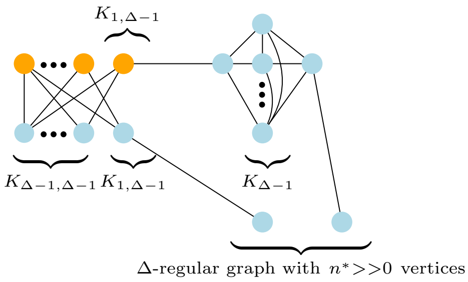


Figure 5.4: Lower bound construction for the PoA on Δ-regular graphs. The swap equilibrium $\boldsymbol{\sigma}$ has $\mathsf{DoI}(\boldsymbol{\sigma}) = 2\Delta + 1$. See the proof of Theorem 5.15 for more details.

coming from the left gadget with the vertex incident to the edge coming from the upper right gadget. So, every vertex in this gadget has a degree equal to Δ and we conclude that G is Δ-regular.

Now set $o = \Delta$. We claim that the strategy profile $\boldsymbol{\sigma}$ depicted in Figure 5.4 is a swap equilibrium. If a blue agent i swaps with a non-adjacent orange agent j, we have $f_i(\boldsymbol{\sigma}_{ij}) = 1$ which results in a non-profitable swap. So a blue agent can profitably swap only with an adjacent orange agent. Any orange agent j has $f_j(\boldsymbol{\sigma}) = \frac{1}{\Delta+1}$. By swapping with an adjacent blue agent i, we have that either $f_j(\boldsymbol{\sigma}_{ij}) = \frac{1}{\Delta+1}$ or $f_j(\boldsymbol{\sigma}_{ij}) = \frac{\Delta}{\Delta+1}$ which never yields an improvement when $\Lambda \leq \frac{1}{2}$ or $f_j(\boldsymbol{\sigma}_{ij}) = \frac{\Delta-1}{\Delta+1}$ when swapping one orange agent j with the rightmost blue one of the left gadget. In this case, it is blue agent i that has no improvement in swapping since $f_i(\boldsymbol{\sigma}) = \frac{2}{\Delta+1}$ and $f_i(\boldsymbol{\sigma}_{ij}) = \frac{\Delta}{\Delta+1}$ which never yields an improvement when $\Lambda \leq \frac{1}{2}$. So, $\boldsymbol{\sigma}$ is a swap equilibrium such that $\mathsf{DoI}(\boldsymbol{\sigma}) = 2\Delta + 1$.

By letting n^* go to infinity, it is always possible to select $o = \Delta$ vertices in the lower right gadget such that their closed neighborhoods are pairwise disjoint, which yields $\mathsf{DoI}(\boldsymbol{\sigma}^*) = \Delta(\Delta + 1)$ and thus the desired lower bound. ■

The lower bound given in Theorem 5.15 holds for all values of Δ. It may be the case then that, for fixed values of Δ, better lower bounds are possible. For $\Delta = 2$ indeed, lower bounds matching the upper bounds given in Theorem 5.13 and Theorem 5.14 can be derived.

▶ **Theorem 5.16.** For any $\epsilon > 0$, there exists a SP-2-SSG (G, o, Λ) on a ring such that $\mathrm{PoA}_{\mathsf{DoI}}(G, o, \frac{1}{2}) > 2 - \epsilon$ and $\mathrm{PoA}_{\mathsf{DoI}}(G, o, \Lambda) > \frac{3}{2} - \epsilon$ for $\Lambda < \frac{1}{2}$. ◀

Proof. For any even $o \geq 2$, let G be a ring defined by the sequence of vertices $v_1, v_2, \ldots, v_n$, with $n = 3o$.

Assume $\Lambda = \frac{1}{2}$. In this case, $p\left(\frac{1}{3}\right) = p\left(\frac{2}{3}\right) > p(0) = p(1)$. Let $\boldsymbol{\sigma}$ be the strategy profile obtained as follows: starting from v_1, assign two orange agents followed by a blue one as long as this is possible. At this time, vertices up to v_x, with $x = \frac{3o}{2}$, have been assigned an agent. All the remaining vertices are assigned to the remaining blue agents. Since for any orange agent i we have $f_i(\boldsymbol{\sigma}) = \frac{2}{3}$, all orange agents are getting the largest possible utility and are not interested in swapping. Thus, $\boldsymbol{\sigma}$ is a swap equilibrium. As all agents residing at vertices $v_n, v_1, \ldots, v_x$ are not segregated, we have $\mathsf{DoI}(\boldsymbol{\sigma}) = \frac{3o}{2} + 1$.

Now assume $\Lambda < \frac{1}{2}$. In this case, $p\left(\frac{1}{3}\right) > p\left(\frac{2}{3}\right) > p(0) = p(1)$. Let $\boldsymbol{\sigma}$ be the strategy profile obtained by alternating orange and blue agents for as much as possible. Since for any orange agent i we have $f_i(\boldsymbol{\sigma}) = \frac{1}{3}$, all orange agents are getting the largest possible utility and are not interested in swapping. Thus, $\boldsymbol{\sigma}$ is a swap equilibrium. As all agents residing at vertices $v_n, v_1, \ldots, v_{2o}$ are not segregated, we have $\mathsf{DoI}(\boldsymbol{\sigma}) = 2o + 1$.

A strategy profile of social value $3o$ can be obtained by sequencing triplets made of two blue agents with an orange one in between. Both claims follow by choosing o sufficiently large. ■

5.5 Price of Stability

In this section, we give bounds on the PoS for SP-2-SSGs played on different topologies.

5.5.1 General Graphs

We give a lower bound on the Price of Stability on general graphs which asymptotically matches the upper bound on the Price of Anarchy when $o = \Theta(\sqrt{n})$ and Λ is a constant with respect to n.

▶ **Theorem 5.17.** For every Λ, there is a SP-2-SSG (G, o, Λ) such that

$$\mathsf{PoS}_{\mathsf{DoI}}(G, o) = \Omega(\sqrt{n\Lambda}).$$

◀

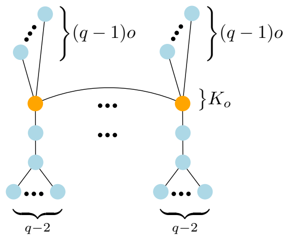


Figure 5.5: The instance used in the proof of Theorem 5.17. Shown is the socially optimal feasible strategy profile $\boldsymbol{\sigma}^*$.

Proof. Let $q \geq 2$ be an integer such that $\frac{1}{q} \leq \Lambda < \frac{1}{q-1}$. Consider the instance in Figure 5.5 in which there is a clique of o vertices such that every vertex in the clique is additionally adjacent to $(q-1)o$ leaves (depicted to the up) and to the leaf of a star with q vertices (depicted at the bottom). It is easy to check that $n = (q-1)o^2 + o + oq$. Letting o go to infinity, we get $n = \Theta((q-1)o^2)$, by which

$$o = \Omega(\sqrt{\frac{n}{q-1}}) = \Omega(\sqrt{n\Lambda}).$$

The socially optimal feasible strategy profile $\boldsymbol{\sigma}^*$, depicted in Figure 5.5 has all orange agents on the vertices of the clique and thereby achieves

$$\mathsf{DoI}(\boldsymbol{\sigma}^*) = (q-1)o^2 + 2o = \Omega((q-1)o^2).$$

We claim that, in contrast, any swap equilibrium $\boldsymbol{\sigma}$ can have at most one orange agent in the clique. Assume, by way of contradiction, that there are two or more orange agents in the clique, and let i be one of these agents. Then, there is at least one center of a star of q vertices that is occupied by a segregated blue agent j. If i and j swap, we have $f_j(\boldsymbol{\sigma}) = 1$ and $f_j(\boldsymbol{\sigma}_{ij}) \notin \{0, 1\}$ so that j improves, and

$$f_i(\boldsymbol{\sigma}) \leq \frac{o}{qo+1}$$

and

$$f_i(\boldsymbol{\sigma}_{ij}) = \frac{1}{q} \in \left(\frac{o}{qo+1}, \Lambda\right],$$

so that i improves too. This contradicts that $\boldsymbol{\sigma}$ is a swap equilibrium. Hence, for any swap equilibrium $\boldsymbol{\sigma}$, it holds that

$$\mathsf{DoI}(\boldsymbol{\sigma}) \leq (o-1)q + qo + 1 = O(2qo),$$

the fact that a swap equilibrium exists can be easily checked by considering the strategy profile obtained by placing one orange agent in a vertex v of the clique and all the remaining $o-1$ ones to the $o-1$ centers of a star of q vertices that are not appended to v. It follows that the Price of Stability is

$$\frac{\Omega((q-1)o^2)}{O(2qo)} = \Omega(o) = \Omega(\sqrt{n\Lambda}).$$ ■

5.5.2 Bipartite Graphs

For bipartite graphs, we provide a tight bound of 2 for the PoS of SP-2-SSGs for which the peak is at $\frac{1}{2}$. We start with the upper bound.

▶ **Theorem 5.18.** For any SP-2-SSG $(G, o, \frac{1}{2})$ on a bipartite graph G, we have $\mathsf{PoS}_{\mathsf{DoI}}(G, o, \frac{1}{2}) \leq 2$. ◀

Proof. Let (V_1, V_2), with $|V_1| \leq |V_2|$, be the bipartition of the vertices of G. For a fixed optimal profile $\boldsymbol{\sigma}^*$, denote by O_1 (respectively O_2) the set of vertices of V_1 (respectively V_2) occupied by an orange agent in $\boldsymbol{\sigma}^*$. Moreover, denote by B_1 (respectively B_2) the set of vertices occupied by a blue agent in $\boldsymbol{\sigma}^*$ falling in the neighborhood of some vertex in O_2 (respectively O_1). Clearly, we have $\mathsf{DoI}(\boldsymbol{\sigma}^*) \leq o + |B_1| + |B_2|$. We shall prove the existence of two swap equilibria, namely $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, whose performance compares nicely with that of $\boldsymbol{\sigma}^*$.

To construct $\boldsymbol{\sigma}_1$, start from $\boldsymbol{\sigma}^*$ and swap all orange agents in O_2 with blue agents in V_1 as long as this is possible. If all orange agents end up in V_1, we have that all orange agents occupy the vertices of an independent set of G and so, by Theorem 5.9, $\boldsymbol{\sigma}_1$ is a swap equilibrium. If some orange agents are left out from V_1, then all blue agents are located in V_2. So, we have that all blue agents occupy the vertices of an independent set of G and, by Theorem 5.9, $\boldsymbol{\sigma}_1$ is a swap equilibrium also in this case. As the set of vertices in O_1 are orange in both $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}_1$, we obtain that $\mathsf{DoI}(\boldsymbol{\sigma}_1) \geq |B_2|$.

Equilibrium $\boldsymbol{\sigma}_2$ is obtained symmetrically by swapping all orange agents in O_1 with blue agents in V_2 as long as this is possible. In this case, as $o \leq \frac{n}{2} \leq |V_2|$, all orange agents end up in V_2 and, by Theorem 5.9, $\boldsymbol{\sigma}_2$ is a swap equilibrium. As

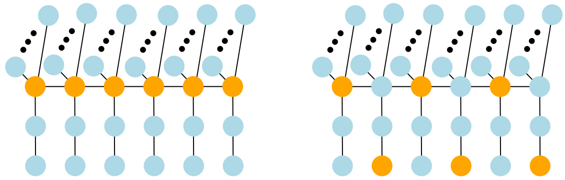


Figure 5.6: Left: a SP-2-SSG with its socially optimal strategy profile $\boldsymbol{\sigma}^*$ shown. Right: the swap equilibrium with maximum social welfare for the same instance. Please refer to Theorem 5.19 for more details.

the set of vertices in O_2 are orange in both $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}_2$ and all orange agents are adjacent to some blue agent in $\boldsymbol{\sigma}_2$, we obtain that $\mathsf{DoI}(\boldsymbol{\sigma}_2) \geq o + |B_1|$. Thus, we conclude that

$$\mathsf{PoS}_{\mathsf{DoI}}(G, o, \frac{1}{2}) \leq \frac{\mathsf{DoI}(\boldsymbol{\sigma}^*)}{\max\{\mathsf{DoI}(\boldsymbol{\sigma}_1), \mathsf{DoI}(\boldsymbol{\sigma}_2)\}} \leq \frac{o + |B_1| + |B_2|}{\max\{o + |B_1|, |B_2|\}} \leq 2. \quad \blacksquare$$

We now give the matching lower bound.

▶ **Theorem 5.19.** There exists a SP-2-SSG $(G, o, \frac{1}{2})$ on a bipartite graph such that $\mathsf{PoS}_{\mathsf{DoI}}(G, o, \frac{1}{2}) \geq 2$. ◀

Proof. Consider the instance $(G, o, \frac{1}{2})$ defined in Figure 5.6. G consists of a path of o vertices, that we call the base of the graph. Any vertex in the base of the graph is additionally connected to $2(o-1)$ leaves (depicted on the top) and to a 2-vertex path (depicted on the bottom). For the socially optimal profile $\boldsymbol{\sigma}^*$, we get $\mathsf{DoI}(\boldsymbol{\sigma}^*) = 2(o-1)o + 2o$. However, this is not a swap equilibrium. In any strategy profile $\boldsymbol{\sigma}$ in which two orange agents are adjacent in the base of the graph, like in the socially optimal profile $\boldsymbol{\sigma}^*$, one of them, denote this agent by i, can swap with a segregated blue agent, denoted by j, placed on a leaf vertex in the lower row. Observe that agent j is always guaranteed to exists. Since we have $f_i(\boldsymbol{\sigma}) \leq \frac{o}{2o+1} < \frac{1}{2}$ and $f_i(\boldsymbol{\sigma}_{ij}) = \frac{1}{2}$, agent i improves its utility. For agent j, we have $f_j(\boldsymbol{\sigma}) = 1$ and $f_j(\boldsymbol{\sigma}_{ij}) \notin \{0, 1\}$, so the swap is profitable and $\boldsymbol{\sigma}$ cannot be a swap equilibrium. The maximum number of agents with non-zero utility that can be obtained by respecting this necessary constraint is $\mathsf{DoI}(\boldsymbol{\sigma}) = o(o-1) + \frac{5o}{2}$, achieved by the swap equilibrium $\boldsymbol{\sigma}$ depicted in Figure 5.6 (right). The claim follows by letting o go to infinity. ■

5.5.3 Almost Regular Graphs

We provide upper bounds to the PoS for SP-2-SSGs played on almost regular graphs. We start by considering the case of graphs with a small degree.

▶ **Theorem 5.20.** For any SP-2-SSG (G, o, Λ) on an almost regular graph with $\Delta \leq 3$ and $\Lambda \leq \frac{1}{2}$, $\mathsf{PoS}_{\mathsf{DoI}}(G, o, \Lambda) = 1$. ◀

Proof. Let $\boldsymbol{\sigma}^*$ be a socially optimal profile. Using Lemma 5.28 from the next section, we have that there exists a swap equilibrium $\boldsymbol{\sigma}$ satisfying $\mathsf{DoI}(\boldsymbol{\sigma}) \geq \mathsf{DoI}(\boldsymbol{\sigma}^*)$. ■

An analogous result holds for the case in which $o \geq \alpha$.

▶ **Theorem 5.21.** For any SP-2-SSG (G, o, Λ) on an almost regular graph with $o \geq \alpha$ and $\frac{1}{\delta+1} \leq \Lambda \leq \frac{1}{2}$, we have $\mathsf{PoS}_{\mathsf{DoI}}(G, o, \Lambda) = 1$. ◀

Proof. We prove the claim by showing the existence of a swap equilibrium $\boldsymbol{\sigma}$ such that $\mathsf{DoI}(\boldsymbol{\sigma}) = n$. Clearly, if $o = \alpha$, then the strategy profile $\boldsymbol{\sigma}$ in which the orange agents occupy all the vertices of a maximum independent set of G is a swap equilibrium by Theorem 5.9, and thus, $\mathsf{DoI}(\boldsymbol{\sigma}) = n$, and the statement follows. Therefore, in the following, we assume that $o, b > \alpha$.

Let $\boldsymbol{\sigma}$ be a strategy profile minimizing the value $\Phi(\boldsymbol{\sigma})$ (ties are arbitrarily broken). By Theorem 5.6, $\boldsymbol{\sigma}$ is a swap equilibrium. We now prove that $\mathsf{DoI}(\boldsymbol{\sigma}) = n$. For the sake of contradiction, assume that $\mathsf{DoI}(\boldsymbol{\sigma}) < n$, i.e., there is at least a segregated agent, say i, in $\boldsymbol{\sigma}$. Assume without loss of generality that i is orange. We claim that all blue agents are placed on vertices that form an independent set, i.e., $b \leq \alpha$. This allows us to obtain the desired contradiction as $b > \alpha$. If the blue agents are not placed on vertices that form an independent set, then there exists a blue agent, say j, having at least a blue neighbor in $\boldsymbol{\sigma}$. The strategy profile $\boldsymbol{\sigma}_{ij}$ satisfies

$$\Phi(\boldsymbol{\sigma}) - \Phi(\boldsymbol{\sigma}_{ij}) \geq \deg(\boldsymbol{\sigma}(i)) + 1 - (\deg(\boldsymbol{\sigma}(j)) - 1) \geq 1,$$

since $|\deg(\boldsymbol{\sigma}(i)) - \deg(\boldsymbol{\sigma}(j))| \leq 1$. Therefore, $\Phi(\boldsymbol{\sigma}_{ij}) < \Phi(\boldsymbol{\sigma})$, thus contradicting the fact that $\boldsymbol{\sigma}$ minimizes Φ. ■

Recall that a SP-2-SSG (G, o, Λ) is *balanced* if $o = \left\lfloor \frac{n}{2} \right\rfloor$. Using Theorem 5.21, we show that the PoS is 1 in balanced games on regular graphs.

▶ **Corollary 5.22.** For any balanced SP-2-SSG (G, o, Λ) on a Δ-regular graph G and $\frac{1}{\Delta+1} \leq \Lambda \leq \frac{1}{2}$, we have $\mathrm{PoS}_{\mathrm{DoI}}(G, o, \Lambda) = 1$. ◀

Proof. We have that $o = \lfloor \frac{n}{2} \rfloor$. We show that $\alpha \leq \lfloor \frac{n}{2} \rfloor$ using a simple counting argument. This allows us to use Theorem 5.21 to prove the claim.

To show the upper bound on α, we count all the edges that are incident to the vertices of a fixed maximum independent set of G and bound this value from above by the number of edges of the graph, thus obtaining the following inequality $\Delta\alpha \leq \frac{\Delta}{2} n$, i.e., $\alpha \leq \frac{n}{2}$. Using the fact that α is an integer value, we derive $\alpha \leq \lfloor \frac{n}{2} \rfloor$. ∎

To bound the Price of Stability when $o < \alpha$, we need to introduce some new definitions and additional technical lemmas based on some well-known optimization cut problems. For a given graph G and a subset of vertices of V, we denote by $G[U]$ the sub-graph of G *induced* by U. More precisely, the vertex set of $G[U]$ is U and, for every $u, v \in U$, $G[U]$ contains the edge (u, v) if and only if G contains the edge (u, v).

The k-Max-Cut problem is an optimization problem in which, given a graph G as input, we want to compute a k-partition $\{V_1, \ldots, V_k\}$ of the vertices of G that maximizes the number of edges that cross the cut induced by the k-partition, that is, the number of edges $\{u, v\}$ such that $u \in V_t$, $v \in V_h$, and $h \neq t$. It is well-known that the greedy algorithm for the k-Max-Cut problem computes a k-partition $\{V_1, \ldots, V_k\}$ of the vertices of a graph such that, for every vertex v, the number of edges incident to v that cross the cut induced by the k-partition is at least $\lceil (1 - \frac{1}{k}) \deg(v) \rceil$ [Vaz13]. Using this folklore result, we can derive the following useful lemma.

▶ **Observation 5.23.** Let G be a graph and $U \subseteq V$ such that $|U| \geq k$. There exists a k-partition $\{V_1, \ldots, V_k\}$ of $G[U]$ such that, for every $t \in \{1, \ldots, k\}$, the degree of each vertex $v \in V_t$ in $G[V_t]$ is at most $\lfloor \frac{\deg(v)}{k} \rfloor$. ◀

Proof. The greedy algorithm for the k-Max-Cut problem computes a k-partition $\{V_1, \ldots, V_k\}$ of $G[U]$ such that, for every vertex v of U, the number of edges incident to v that cross the cut induced by the k-partition is at least $\lceil (1 - \frac{1}{k}) \deg(v) \rceil$. As a consequence, for any $v \in V_t$, with $t \in \{1, \ldots, k\}$, the number of edges that are incident to v in $G[V_t]$ is at most

$$\deg(v) - \left\lceil \left(1 - \frac{1}{k}\right) \deg(v) \right\rceil = \left\lfloor \frac{\deg(v)}{k} \right\rfloor. \qquad ∎$$

The Balanced k-Max-Cut problem is a k-Max-Cut in which we additionally require the k-partition $\{V_1, \dots, V_k\}$ to be *balanced*, i.e., for every $t \in \{1, \dots, k\}$, $|V_t| \geq \lfloor \frac{n}{k} \rfloor$.[13] For the Balanced k-Max-Cut problem we can prove a useful lemma that is analogous to Observation 5.23.

▶ **Lemma 5.24.** Let G be a graph and $U \subseteq V$ such that $|U| \geq 2$. There exists a balanced k-partition $\{V_1, \dots, V_k\}$ of U such that, for at least one $t \in \{1, \dots, k\}$, $|V_t| \geq 1$ and the degree of every $v \in V_t$ in $G[V_t]$ is at most $\lfloor \frac{\deg(v)}{k} \rfloor$. ◀

Proof. In the remainder of this proof, for a given k-partition $\{V_1, \dots, V_k\}$ of U and a bijective function $\rho : \{v_1, \dots, v_k\} \to \{v_1, \dots, v_k\}$, with $v_t \in V_t$ for every $t \in \{1, \dots, k\}$, we define the ρ-swap as the k-partition $\{V_1', \dots, V_k'\}$ in which, for every $t \in \{1, \dots, k\}$, the set V_t' is obtained from V_t by replacing v_t with $\rho(v_t)$ (it may happen that $\rho(v_t) = v_t$). We say that the ρ-swap is profitable if the number of edges crossing the cut induced by $\{V_1', \dots, V_k'\}$ is strictly larger than the number of edges crossing the cut induced by $\{V_1, \dots, V_k\}$. A balanced k-partition $\{V_1, \dots, V_k\}$ is stable if there is no profitable ρ-swap.

Let $\{V_1, \dots, V_k\}$ be a balanced k-partition of U that maximizes the number of edges in the cut, ties can be arbitrarily broken. We claim that $\{V_1, \dots, V_k\}$ satisfies all the properties of the lemma statement. This is true when $|U| \leq k$ as all edges of G are in the cut, for this, consider, for instance, the solution in which each set of the k-partition contains at most one vertex. Therefore, we only need to prove the claim when $|U| > k$. This implies that $|V_t| > 0$ for every $t \in \{1, \dots, k\}$. As a consequence, we only need to prove that there exist $t \in \{1, \dots, k\}$ such that, for every vertex $v \in V_t$, the degree of v in $G[V_t]$ is at most $\lfloor \frac{\deg(v)}{k} \rfloor$.

For the sake of contradiction, assume that there are k vertices $v_1, \dots, v_k$, with $v_t \in V_t$, such that, for every $t \in \{1, \dots, k\}$, the degree of v_t in $G[V_t]$ is strictly larger than $\lfloor \frac{\deg(v_t)}{k} \rfloor$. Consider the graph H on the k vertices $v_1, \dots, v_k$, where we add the direct edge (v_t, v_h) between v_t and v_h, with $t \neq h$, if and only if the number of edges incident to v_t whose other endpoints are in V_h is at most $\lfloor \frac{\deg(v_t)}{k} \rfloor$. We observe that the out-degree of each vertex in H is at least 1. As a consequence, H contains at least one directed cycle. Let C be any fixed directed cycle in H. We define $\rho : \{v_1, \dots, v_k\} \to \{v_1, \dots, v_k\}$ as follows. For each edge (v_t, v_h) of C, we define $\rho(v_t) = v_h$, while $\rho(v_\ell) = v_\ell$ for every $\ell \in \{1, \dots, k\}$ such

13 When $n < k$, we might have empty sets in the k-partition.

that v_ℓ is not contained in C. Clearly, ρ is a bijective function. Furthermore, the ρ-swap is profitable as each vertex v_t that is moved from V_t to V'_h, with $h \neq t$, contributes with at least one more edge in the cut. This contradicts the fact that $\{V_1, \ldots, V_k\}$ maximizes the number of edges in the cut. ■

We are now ready to prove the upper bound on the PoS for SP-2-SSGs played on almost regular graphs when $o < \alpha$.

▶ **Theorem 5.25.** For any SP-2-SSG (G, o, Λ) on an almost regular graph G with $o < \alpha$ and $\Lambda \leq \frac{1}{2}$, we have $\mathsf{PoS}_{\mathsf{DoI}}(G, o, \Lambda) = \min\{\Delta + 1, \mathrm{O}(\frac{1}{\Lambda})\}$. ◀

Proof. Let $\boldsymbol{\sigma}^*$ be a strategy profile that maximizes $\mathsf{DoI}(\boldsymbol{\sigma}^*)$ for (G, o, Λ). Let O and B be the vertices occupied by the non-segregated orange and blue agents in $\boldsymbol{\sigma}^*$, respectively. Clearly, $\mathsf{DoI}(\boldsymbol{\sigma}^*) = |O| + |B|$. Moreover, $|O| \geq 1$ if and only if $|B| \geq 1$. We prove the claim by showing the existence of a strategy profile $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}$ is a swap equilibrium and $\mathsf{DoI}(\boldsymbol{\sigma}) = \Omega(\Lambda(|O| + |B|))$.

We first rule out the case in which $o \leq \lfloor \Lambda(\Delta - 1) \rfloor - 1$. In fact, in this case, $G[O]$ has a maximum degree of at most $\lfloor \Lambda(\Delta - 1) \rfloor - 1$, which implies that every orange agent is below the peak in $\boldsymbol{\sigma}^*$. As a consequence, from Lemma 5.2 and Lemma 5.5, $\boldsymbol{\sigma}^*$ is also a swap equilibrium. Therefore, in the following, we assume that $o > \lfloor \Lambda(\Delta - 1) \rfloor - 1$. As o is an integer, we have that $o \geq \Lambda\Delta - \Lambda - 1$, from which we derive

$$\Delta \leq \frac{o}{\Lambda} + 1 + \frac{1}{\Lambda}. \tag{5.6}$$

Next, we rule out the case in which $|O| + |B| = \mathrm{O}(\frac{o}{\Lambda})$, i.e., $o = \Omega(\Lambda(|O| + |B|))$. In fact, in this case, let $\boldsymbol{\sigma}$ be any swap equilibrium, that we know to exist by Theorem 5.6. By Lemma 5.11, there is a color for which all agents of that color are not segregated in $\boldsymbol{\sigma}$. Therefore

$$\mathsf{DoI}(\boldsymbol{\sigma}) \geq \min\{o, r\} = o = \omega(\Lambda(|O| + |B|)).$$

Hence, for the rest of the proof, we assume that $|O| + |B| = \omega(\frac{o}{\Lambda})$. As $|O| \leq o$, it holds that

$$|B| = \omega(\frac{o}{\Lambda}). \tag{5.7}$$

Finally, we rule out the case in which $\Delta = \mathrm{O}(\frac{o}{\Lambda})$. By Theorem 5.13, $\mathsf{PoS}_{\mathsf{DoI}}(G, o, \Lambda) \leq \Delta + 1$. In the following, for any subset O' of O, we denote by $\mathcal{B}(O')$ the subset of vertices of B that are *dominated* by O', i.e., $\mathcal{B}(O')$ contains all vertices $u \in B$ for which there exists a vertex v in O' such that (u, v) is an edge of G.

We iteratively use Observation 5.23 to compute hierarchical k-partitions of O, with $k = \lceil \frac{\Delta-1}{\Lambda\Delta-1} \rceil$. We observe that $k \geq 1$ as $\Lambda > \frac{1}{\Delta+1}$.

Starting from O, we compute the k-partition $\{O_0^1, \ldots, O_0^k\}$ of O that satisfies the premises of Observation 5.23. This is called the k-partition of *level* 0. For the remaining part of this proof, we use the shortcut B_0^t to denote $\mathcal{B}(O_0^t)$. Without loss of generality, we assume that $|B_0^1| \geq \max_t |B_0^t|$. Now, given the k-partition $\{O_h^1, \ldots, O_h^k\}$ of level h such that $|O_h^1| \geq 2$ and $|B_h^1| \geq \max_t |B_h^t|$, we compute a k-partition $\{O_{h+1}^1, \ldots, O_{h+1}^k\}$ of O_h^1 that satisfies the premises of Observation 5.23. This is the k-partition of *level* $h+1$ where, again, we use the shortcut B_{h+1}^t to denote $\mathcal{B}\left(O_{h+1}^t\right)$ and, without loss of generality, we assume that $|B_{h+1}^1| \geq \max_t |B_{h+1}^t|$. We stop the process at level L, where we compute a k-partition $\{O_L^1, \ldots, O_L^k\}$ such that $|O_L^1| = 1$ and $|B_L^1| \leq \Delta$. We observe that

$$|B| \geq |B_0^1| \geq |B_1^1| \geq \cdots \geq |B_L^1|$$

by construction.

Let ℓ be the minimum index such that $|B \setminus B_\ell^1| \geq k(o-1)$. Such an index always exists because, by Inequality 5.7, Equation (5.6), and the fact that $\Delta = \omega\left(\frac{1}{\Lambda}\right)$, we have

$$|B \setminus B_L^1| = |B| - |B_L^1| \geq |B| - \Delta = \omega\left(\frac{o}{\Lambda}\right) \geq ko.$$

Let $\{B^1, \ldots, B^k\}$ be a balanced k-partition of $B \setminus B_\ell^1$ that satisfies the premises of Lemma 5.24. We have that $|B^t| \geq o$ for every $t \in \{1, \ldots, k\}$. Without loss of generality, we assume that the degree of each vertex $v \in B^1$ in $G[B^1]$ is at most $\lfloor \frac{\deg(v)}{k} \rfloor$. By construction, also each vertex $v \in O_\ell^1$ has a degree in $G[O_\ell^1]$ of at most $\lfloor \frac{\deg(v)}{k} \rfloor$. Since there is no edge (u, v) of G such that $u \in O_\ell^1$ and $v \in B \setminus B_\ell^1$, it follows that the degree of each vertex $v \in O_\ell^1 \cup B^1$ in $G[O_\ell^1 \cup B^1]$ is also upper bounded by $\lfloor \frac{\deg(v)}{k} \rfloor$.

Let $\boldsymbol{\sigma}$ be the strategy profile in which exactly $|O_\ell^1|$ orange agents are placed on the vertices of O_ℓ^1 and the remaining $o - |O_\ell^1|$ orange agents are placed on a subset of vertices of B^1 (ties among vertices of B^1 are arbitrarily broken). Let i be any orange agent and let v be the vertex occupied by i in $\boldsymbol{\sigma}$. By the choice of k we have $k \geq \lceil \frac{\deg(v)}{\Lambda(\deg(v)+1)-1} \rceil$. This implies that

$$f_i(\boldsymbol{\sigma}) \leq \frac{\lfloor \frac{\deg(v)}{k} \rfloor + 1}{\deg(v) + 1} \leq \frac{\frac{\deg(v)}{k} + 1}{\deg(v) + 1} \leq \Lambda.$$

Therefore, the orange agent i is below the peak in $\boldsymbol{\sigma}$. Since every orange agent is below the peak in $\boldsymbol{\sigma}$, from Lemma 5.2 and Lemma 5.5, $\boldsymbol{\sigma}$ is a swap equilibrium.

We conclude the proof by showing that $\mathsf{DoI}(\boldsymbol{\sigma}) = \Omega(\Lambda(|O| + |B|))$. By construction, each orange agent occupies a vertex that is adjacent to at least one other vertex occupied by a blue agent. Moreover, each blue agent that occupies a vertex of B_ℓ^1 is in the neighborhood of at least one vertex of O_ℓ^1 that is occupied by an orange agent. Therefore,

$$\mathsf{DoI}(\boldsymbol{\sigma}) \geq o + |B_\ell^1|. \tag{5.8}$$

For proof convenience, let us denote by B_{-1}^1 the set B. By the choice of ℓ we know that $|B| - |B_{\ell-1}^1| = |B \setminus B_{\ell-1}^1| < ko$, which implies that $|B_{\ell-1}^1| > |B| - ko$. As a consequence, since

$$\sum_{i=1}^{k} |B_\ell^i| \geq |B_{\ell-1}^1| > |B| - kb$$

and $|B_\ell^1| \geq \max_i |B_\ell^i|$, we obtain

$$|B_\ell^1| > \frac{|B|}{k} - o. \tag{5.9}$$

Combining Inequality 5.8 and Inequality 5.9 we obtain $\mathsf{DoI}(\boldsymbol{\sigma}) > \frac{|B|}{k}$. Using Inequality 5.7 and the fact that $k = O\left(\frac{1}{\Lambda}\right)$ we finally obtain $\mathsf{DoI}(\boldsymbol{\sigma}) = \Omega(\Lambda(|O| + |B|))$, as desired. ■

We can derive the following upper bound to the Price of Stability.

▶ **Corollary 5.26.** For any SP-2-SSG (G, o, Λ) on an almost regular graph with a constant value of $\Lambda \leq \frac{1}{2}$, we have $\mathrm{PoS}_{\mathsf{DoI}}(G, o, \Lambda) = \mathrm{O}(1)$. ◀

Proof. By Theorem 5.13, the $\mathrm{PoS}_{\mathsf{DoI}}$ is constant if Δ is constant. The result when Δ is not constant is divided into two cases. For the case $o \geq \alpha$ the claim immediately follows from Theorem 5.21. For the case $o < \alpha$ the claim follows from Theorem 5.25 and the fact that Λ is constant by assumption. ■

5.6 Computational Complexity

In this section, we analyze the computational complexity aspects of the SP-2-SSG played on both, bipartite and regular graphs. More precisely, we provide hardness results for the two problems of computing a social optimum and a swap equilibrium $\boldsymbol{\sigma}$ that maximizes the value $\mathsf{DoI}(\boldsymbol{\sigma})$, respectively.

▶ **Theorem 5.27.** There is a constant $c > 1$ such that, given a SP-2-SSG (G, o, Λ) played on a cubic graph G with $\Lambda \in (0, 1)$, the problem of computing a strategy profile σ that maximizes $\mathsf{DoI}(\boldsymbol{\sigma})$ is not c-approximable in polynomial time, unless $\mathsf{P} = \mathsf{NP}$. ◀

Proof. The reduction is from the Minimum Dominating Set problem on cubic graphs, an optimization problem where the goal is to compute a minimum-size set D of vertices of a given cubic graph G' that *dominates* $V(G')$, i.e., for every vertex $v \in V(G')$, $v \in D$ or there is an edge $\{u, v\} \in E(G')$ such that $u \in D$. It is known that a minimum dominating set on cubic graphs is not approximable within some constant $c' > 1$, unless $\mathsf{P} = \mathsf{NP}$, see [AK97].

Let G be a cubic graph of n vertices that has a minimum dominating set of size k^* and let $b = k^*$.[14] We claim that a strategy profile $\boldsymbol{\sigma}^*$ satisfies $\mathsf{DoI}(\boldsymbol{\sigma}^*) = n$ if and only if the o orange agents are placed on the vertices that form a minimum dominating set of G. Indeed, a blue agent placed on a vertex v is not segregated in $\boldsymbol{\sigma}^*$ if and only if there is an orange agent placed on a vertex that dominates v. Furthermore, an orange agent placed on a vertex u is never segregated in $\boldsymbol{\sigma}^*$ because of the minimality of the dominating set, i.e., each vertex of a minimum dominating set D must dominate a vertex of the graph that is not in D.

Let $c = \frac{4}{5-c'}$. Since all vertices of the graph trivially form a dominating set of size $n \leq 4k^*$, each vertex of the dominating set dominates 4 vertices, we have that $c' < 4$ and therefore, $c > 1$.

14 Without loss of generality, we can *guess* the value of k^*. Indeed, when k^* is unknown, it is enough to generate all the possible n instances of the problem where, in the i-th instance, we set the number of orange agents o to be equal to i. Any c-approximation algorithm for all the n instances is obviously a c-approximation algorithm for the instance in which $o = k^*$. This explains why it is enough to consider only the instance for which $o = k^*$ in the rest of the proof.

We complete the proof by showing that, if we were able to compute, in polynomial time, a strategy profile $\boldsymbol{\sigma}$ such that $\frac{\mathsf{DoI}(\boldsymbol{\sigma}^*)}{\mathsf{DoI}(\boldsymbol{\sigma})} \leq c$, then we could compute, in polynomial time, a c'-approximate dominating set of G.

Let $\boldsymbol{\sigma}$ be a strategy profile such that $\frac{\mathsf{DoI}(\boldsymbol{\sigma}^*)}{\mathsf{DoI}(\boldsymbol{\sigma})} \leq c$ and let D be the set of vertices that are occupied by the orange agents in $\boldsymbol{\sigma}$. Let n^* be the vertices of G that are not dominated by D. We have that $\frac{n}{n-^*} \leq c$, from which we derive that

$$n^* \leq \frac{c-1}{c} n = \frac{\frac{4}{5-c'} - 1}{\frac{4}{5-c'}} n \leq \frac{c'-1}{4} 4k^* = (c'-1)k^*.$$

We now compute, in polynomial time, a dominating set D' of G whose size is at most $k^* + (c'-1)k^* = c'k^*$. D' contains D and all the * vertices of G that are not dominated by D. Clearly, D' is a dominating set of G that approximates the value k^* within a factor of c'. This completes the proof. ■

The following lemma allows us to convert any strategy profile into a swap equilibrium without increasing the number of segregated agents.

► **Lemma 5.28.** Given a SP-2-SSG (G, o, Λ) on an almost regular graph, with $\Delta \leq 3$ and $\Lambda \leq \frac{1}{2}$, and given a strategy profile $\boldsymbol{\sigma}$, we can compute a swap equilibrium $\boldsymbol{\sigma}'$ such that $\mathsf{DoI}(\boldsymbol{\sigma}') \geq \mathsf{DoI}(\boldsymbol{\sigma})$ in polynomial time. ◄

Proof. We prove the following claim: if a feasible strategy profile $\boldsymbol{\sigma}$ is not a swap equilibrium, then there exists a (not necessarily profitable) swap decreasing the potential function Φ and not creating new segregated agents. This implies that after a sequence of at most $m = |E(G)|$ swaps of this type, we obtain a swap equilibrium $\boldsymbol{\sigma}'$ such that $\mathsf{DoI}(\boldsymbol{\sigma}') \geq \mathsf{DoI}(\boldsymbol{\sigma})$. Therefore, given $\boldsymbol{\sigma}$, we have that $\boldsymbol{\sigma}'$ can be computed in polynomial time.

It remains to prove the existence of a (not necessarily profitable) swap $\boldsymbol{\sigma}'$ such that $\Phi(\boldsymbol{\sigma}') < \Phi(\boldsymbol{\sigma})$ and not creating new segregated agents. Towards this end, fix a non-equilibrium feasible strategy profile $\boldsymbol{\sigma}$ and consider an orange agent i and a blue agent j possessing a profitable swap in $\boldsymbol{\sigma}$. If no segregated agents are created in $\boldsymbol{\sigma}_{ij}$, then the claim holds. So assume that a segregated agent k is created in $\boldsymbol{\sigma}_{ij}$. Clearly, by definition of profitable swaps, it must be $k \notin \{i, j\}$. Assume, without loss of generality, that k is blue. Then, since we have

$$f_k(\boldsymbol{\sigma}_{ij}) = \frac{\deg(\boldsymbol{\sigma}(k)) + 1}{\deg(\boldsymbol{\sigma}(k)) + 1} = 1,$$

k needs be adjacent to i in $\boldsymbol{\sigma}$, i.e. $1_{ik}(\boldsymbol{\sigma}) = 1$, and

$$f_k(\boldsymbol{\sigma}) = \frac{\deg(\boldsymbol{\sigma}(k))}{\deg(\boldsymbol{\sigma}(k)) + 1}.$$

Let x_o (respectively x_b) be the number of orange agents (respectively blue agents other than k) adjacent to i in $\boldsymbol{\sigma}$. Since profitable swaps in almost regular graphs can only occur between agents above the peak, we have $\frac{x_o+1}{x_o+x_b+2} > \Lambda$ which implies $x_b < x_o$ as $\Lambda \leq \frac{1}{2}$. By swapping i and k, we get

$$\Phi(\boldsymbol{\sigma}) - \Phi(\boldsymbol{\sigma}_{ik}) = \deg(\boldsymbol{\sigma}(k)) - 1 + x_o - x_b > 0.$$

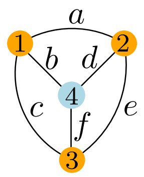
Therefore $\boldsymbol{\sigma}_{ik}$ is a swap such that $\Phi(\boldsymbol{\sigma}_{ik}) < \Phi(\boldsymbol{\sigma})$.

We are left to prove that no segregated agents are created in $\boldsymbol{\sigma}_{ik}$. The neighborhood of vertex $\boldsymbol{\sigma}(k)$ in $\boldsymbol{\sigma}$ is composed of vertex $\boldsymbol{\sigma}(i)$ and a remaining set of blue vertices. Thus, when $\boldsymbol{\sigma}(k)$ and $\boldsymbol{\sigma}(i)$ exchange their colors in $\boldsymbol{\sigma}_{ik}$, no segregated agents are created in the closed neighborhood of $\boldsymbol{\sigma}(k)$. The neighborhood of vertex $\boldsymbol{\sigma}(i)$ in $\boldsymbol{\sigma}$ is composed of vertex $\boldsymbol{\sigma}(k)$ and a remaining set of x_o orange vertices and x_b blue vertices, not counting $\boldsymbol{\sigma}(k)$, with $x_o > x_b$. As the maximum degree of G is at most 3 and $\boldsymbol{\sigma}(i)$ is adjacent to $\boldsymbol{\sigma}(k)$, we have $x_o + x_b \leq 2$, which, since $x_o > x_b$, implies $x_b = 0$. Thus, when $\boldsymbol{\sigma}(k)$ and $\boldsymbol{\sigma}(i)$ exchange their colors in $\boldsymbol{\sigma}_{ik}$, no segregated agents are created in the closed neighborhood of $\boldsymbol{\sigma}(i)$. No other vertices are affected by the swap, thus no segregated agents are created. ■

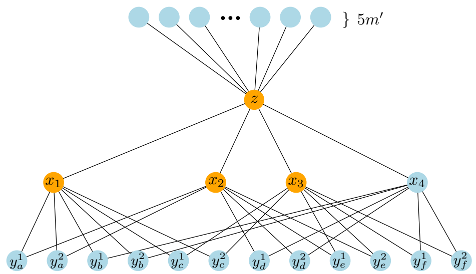
▶ **Corollary 5.29.** There is a constant $c > 1$ such that, given a SP-2-SSG (G, o, Λ) on a cubic graph G, with $\Lambda \leq \frac{1}{2}$, the problem of computing a swap equilibrium $\boldsymbol{\sigma}$ that maximizes $\mathsf{DoI}(\boldsymbol{\sigma})$ is not c-approximable in polynomial time, unless P = NP. ◀

Proof. Let $\boldsymbol{\sigma}^*$ be a strategy profile that maximizes the value $\mathsf{DoI}(\boldsymbol{\sigma}^*)$. Thanks to Lemma 5.28, we know that there is a swap equilibrium $\boldsymbol{\sigma}$ such that $\mathsf{DoI}(\boldsymbol{\sigma}) \geq \mathsf{DoI}(\boldsymbol{\sigma}^*)$. As a consequence, any swap equilibrium that approximates $\mathsf{DoI}(\boldsymbol{\sigma})$ within a factor of c would also approximate $\mathsf{DoI}(\boldsymbol{\sigma}^*)$ within a factor of c. The claim now follows from Theorem 5.27. ■

We now provide analogous results for bipartite graphs.



(a) A cubic graph G' with $n^* = 4$, $m^* = 6$, with a minimum vertex cover $k^* = 3$.



(b) An instance (G, o) of a SP-2-SSG constructed from G' with $n = n^* + 7m^* + 1$ vertices and $o = k^* + 1$ orange agents.

Figure 5.7: An example instance of the reduction from Vertex Cover shown in Theorem 5.30.

▶ **Theorem 5.30.** There is a constant $c > 1$ such that, given a SP-2-SSG (G, o, Λ) on a bipartite graph G with $\Lambda \in (0, 1)$, the problem of computing a strategy profile that maximizes $\mathsf{DoI}(\boldsymbol{\sigma})$ is not c-approximable in polynomial time, unless $\mathsf{P} = \mathsf{NP}$. ◀

Proof. The reduction is from the Minimum Vertex Cover problem on cubic graphs, an optimization problem in which the goal is to compute a minimum-size set C of vertices of a given cubic graph G' such that every edge $\{u, v\}$ of G' is *covered* by C, i.e., $\{u, v\} \cap C \neq \emptyset$. It is known that a minimum vertex cover on cubic graphs is not approximable within some constant $c' > 1$, unless $\mathsf{P} = \mathsf{NP}$, see [AK97].

Let us assume that n^* and $m^* = \frac{3}{2}n^*$ are the number of vertices and edges of the input graph G', respectively. We construct a graph G as follows, see Figure 5.7 for an example. G contains $n = n^* + 7m^* + 1$ vertices. More precisely, each vertex v of G' is modeled by a vertex x_v in G, while each edge e of G' is modeled by two vertices y_e^1 and y_e^2 in G. G also contains a special vertex z and $5m^*$ additional dummy vertices. The special vertex z is connected by an edge to each of the $5m^*$ dummy vertices and the n^* vertices x_v, with v being a vertex of G'. Finally, G contains the two edges connecting x_v with y_e^1 and y_e^2 if and only if v is an endpoint of the edge e in G'.

By construction, we have that G is a bipartite graph. Let k^* denote the size of a minimum vertex cover of G'.[15] We consider the SP-2-SSG (G, o) played on the constructed graph G, where $o = k^* + 1$.

We claim that a social optimum $\boldsymbol{\sigma}^*$ has a $\mathsf{DoI}(\boldsymbol{\sigma}^*) = n$ if and only if G' admits a vertex cover of size k^*.

($\Leftarrow$) Let C^* be a vertex cover of G' of size k^*. Consider the strategy profile $\boldsymbol{\sigma}^*$ in which one orange agent is placed on the special vertex z, while the remaining k^* orange agents are placed on the vertices x_v, with $v \in C^*$. Clearly, the blue agents are placed on the remaining vertices of the graph. By construction, one can check that no agent in G is segregated, see Figure 5.7 for an example. Therefore, $\mathsf{DoI}(\boldsymbol{\sigma}^*) = n$.

($\Rightarrow$) Let $\boldsymbol{\sigma}^*$ be a strategy profile such that $\mathsf{DoI}(\boldsymbol{\sigma}^*) = n$. First of all, as $k^* + 1 \leq n^* + 1 < 5m^*$, we have that no dummy vertex can be occupied by an orange agent. This is because all edges that connect a dummy vertex with the special vertex z must be non-monochrome and the number of orange agents is not sufficient to cover all the dummy vertices. Therefore, all dummy vertices must be occupied by blue agents and, as a consequence, the special vertex z is occupied by an orange agent. We claim that the set

$$C(\boldsymbol{\sigma}^*) := \{v \in V(G') \mid x_v \text{ is occupied by an orange agent}\}$$

has size k^* and forms a vertex cover of G'. We observe that, by construction, it is enough to prove that $C(\boldsymbol{\sigma}^*)$ has size k^* as each vertex y_e^i, with $i \in \{1, 2\}$, is adjacent to the vertices x_v such that v covers e in G'. For the sake of contradiction, assume that $|C(\boldsymbol{\sigma}^*)| < k$. We show the existence of a vertex cover of G' of size strictly smaller than k^*. Let E' be the subset of the edges of G' such that, for each $e \in E'$, y_e^1 and y_e^2 are both occupied by orange agents. Let C be a set of vertices of G' that contains $C(\boldsymbol{\sigma}^*)$ plus any of the two end vertices of e, for each $e \in E'$. We now show that $|C| < k^*$. As $\mathsf{DoI}(\boldsymbol{\sigma}^*) = n$, each vertex y_e^i that is occupied by a blue agent should be adjacent to a vertex x_v occupied by an orange agent. By construction v covers e and $v \in C(\boldsymbol{\sigma}^*)$. Therefore, $C(\boldsymbol{\sigma}^*)$ covers all the edges of $E(G) \setminus E'$. Hence, C is a vertex cover of G' of size strictly smaller than k^*.

We complete the proof by showing that there is a constant $c > 1$ such that the

15 Similar to the proof of Theorem 5.27, we can guess the value k^*.

problem of computing, in polynomial time, a strategy profile $\boldsymbol{\sigma}$ that approximates the social optimum $\boldsymbol{\sigma}^*$ is not approximable within c, unless $\mathsf{P} = \mathsf{NP}$. Let $c' > 1$ be the constant such that the Minimum Vertex Cover problem on cubic graphs is not approximable within a factor of c' in polynomial time. As each vertex of G' covers 3 edges, we have that $k^* \geq \frac{m^*}{3} = \frac{n^*}{2}$. This implies that the Minimum Vertex Cover problem on cubic graphs is approximable within a factor of 2, and all vertices of the graph suffice. Therefore, $c' < 2$. We set $c = \frac{13}{(14-c')}$. Observe that $c > 1$ as $1 < c' < 2$. We now prove that if we were able to compute, in polynomial time, a strategy profile $\boldsymbol{\sigma}$ such that $\frac{\mathsf{DoI}(\boldsymbol{\sigma}^*)}{\mathsf{DoI}(\boldsymbol{\sigma})} \leq c$, then we could c'-approximate the Minimum Vertex Cover problem on cubic graphs in polynomial time.

For the sake of contradiction, let $\boldsymbol{\sigma}$ be a strategy profile that c-approximates $\mathsf{DoI}(\boldsymbol{\sigma}^*)$ and assume that $\boldsymbol{\sigma}$ can be computed in polynomial time. We use $\boldsymbol{\sigma}$ to define a new strategy profile $\boldsymbol{\sigma}'$ such that (i) $\mathsf{DoI}(\boldsymbol{\sigma}') \geq \mathsf{DoI}(\boldsymbol{\sigma})$, (ii) one orange agent is placed on the special vertex z, and (iii) all the other orange agents are placed on a subset of vertices x_v with $v \in V(G')$.

First of all, we show that the special vertex z is occupied by an orange agent in $\boldsymbol{\sigma}$. If not, there would be at least $5m^* - k^* - 1 \geq 4m^*$ dummy vertices occupied by blue agents and therefore $\mathsf{DoI}(\boldsymbol{\sigma}) \leq n - 4m^*$. As a consequence, using also the fact that $m^* = \frac{3}{2}n^*$ and $m^* \geq 1$, we would obtain $\frac{\mathsf{DoI}(\boldsymbol{\sigma}^*)}{\mathsf{DoI}(\boldsymbol{\sigma})} \geq$

$$\frac{n}{n-4m^*} = 1 + \frac{4m^*}{n-4m^*} = 1 + \frac{4m^*}{n^* + 3m^* + 1} \geq 1 + \frac{3m^*}{\frac{2}{3}m^* + 3m^* + m^*} = \frac{23}{14} > c,$$

thus contradicting the fact that our solution is c-approximate.

The strategy profile $\boldsymbol{\sigma}'$ is obtained by modifying $\boldsymbol{\sigma}$ as follows. Blue agents that occupy dummy vertices exchange their position with blue agents occupying vertices of the form x_v, with $v \in V(G')$. At the same time, every orange agent that occupies a vertex y_e^i, with $e \in E(G')$ and $i \in \{1, 2\}$, exchanges its position with a blue agent occupying a vertex x_v such that $v \in V(G')$, where we give priority to vertices that cover e. Clearly, given $\boldsymbol{\sigma}$, strategy profile $\boldsymbol{\sigma}'$ can be computed in polynomial time and we have $\mathsf{DoI}(\boldsymbol{\sigma}') \geq \mathsf{DoI}(\boldsymbol{\sigma})$.

Let m' be the number of edges of G' that are not covered by

$$C(\boldsymbol{\sigma}') := \{v \in V(G') \mid x_v \text{ is occupied by an orange agent}\}.$$

We show that $m' \leq (c' - 1)k^*$. First of all, we observe all the $5m^*$ dummy vertices, the special vertex z, and all the n^* vertices x_v corresponding to the vertices $v \in V(G')$ are not segregated in $\boldsymbol{\sigma}'$. Therefore, the number of uncovered edges of G' equals twice the number of segregated blue agents in G, two blue agents per uncovered edge e of G' that occupy the vertices y_e^1 and y_e^2. Therefore $\mathsf{DoI}(\boldsymbol{\sigma}') = n - 2m'$. As a consequence

$$c > \frac{\mathsf{DoI}(\boldsymbol{\sigma}^*)}{\mathsf{DoI}(\boldsymbol{\sigma}')} \geq \frac{n}{n-2m'},$$

from which we derive

$$\begin{aligned} m' &\leq \frac{c-1}{2c} n = \frac{\frac{13}{14-c'} - 1}{2\frac{13}{14-c'}} n = \frac{c'-1}{26}(n^* + 7m^* + 1) \\ &\leq \frac{c'-1}{26}\left(\frac{2}{3}m^* + 7m^* + m^*\right) = \frac{c'-1}{3} m^* \leq (c'-1)k^*, \end{aligned}$$

where we use the facts that $m^* = \frac{3}{2}n^*$, i.e., G' is cubic, $1 \leq m^*$, and $k^* \geq \frac{m^*}{3}$, each vertex of G' covers 3 edges.

To complete the proof, let C be a vertex cover of G' that contains $C(\boldsymbol{\sigma}')$ and a vertex that covers each of the edges of G' that are not covered by $C(\boldsymbol{\sigma}')$. Clearly, given $\boldsymbol{\sigma}'$, C can be computed in polynomial time. The size of C is upper bounded by the size of $C(\boldsymbol{\sigma}')$ plus the number of uncovered edges, i.e., $|C| \leq k^* + m' \leq c'k^*$. Hence, C is a c'-approximate vertex cover of G'. This completes the proof. ■

▶ **Theorem 5.31.** There is a constant $c > 1$ such that, given a SP-2-SSG $(G, o, \frac{1}{2})$ on a bipartite graph G, the problem of computing a swap equilibrium $\boldsymbol{\sigma}$ that maximizes $\mathsf{DoI}(\boldsymbol{\sigma})$ is not c-approximable in polynomial time, unless P = NP. ◀

Proof. We consider the same reduction that we used in the proof of Theorem 5.30 and show the existence of a strategy profile $\boldsymbol{\sigma}^*$ that maximizes $\mathsf{DoI}(\boldsymbol{\sigma}^*)$ which is also a swap equilibrium. Observe that, once we prove that $\boldsymbol{\sigma}^*$ is a swap equilibrium, the rest of the proof can be derived from Theorem 5.30.

Consider the strategy profile $\boldsymbol{\sigma}^*$ in which an orange agent occupies the special vertex z, while the remaining k^* orange agents are placed on vertices of the form x_v such that $v \in C^*$ and C^* is an optimal vertex cover of G', see also Figure 5.7. In the proof of Theorem 5.30 we already showed that $\mathsf{DoI}(\boldsymbol{\sigma}^*) = n$. In the following, we show that $\boldsymbol{\sigma}^*$ is also a swap equilibrium.

The blue agents on the dummy vertices have maximum utility, so they never swap. Let j be a blue agent that is placed on a vertex of the form x_v, with $v \in V(G')$. The strategy $\boldsymbol{\sigma}_{ij}$ where i is an orange agent placed on a vertex x_v, with $v \in V(G')$, is not a profitable swap by Lemma 5.4. The strategy $\boldsymbol{\sigma}_{ij}$ where i is the orange agent placed on the special vertex z is not a profitable swap by Lemma 5.3.

Finally, consider any blue agent j that is placed on a vertex of the form y_e^ℓ with $e \in E(G')$ and $\ell \in \{1,2\}$. This agent has a utility of either $p\left(\frac{1}{3}\right)$ or $p\left(\frac{2}{3}\right)$. But $p\left(\frac{1}{3}\right) = p\left(\frac{2}{3}\right)$ whenever $\Lambda \leq \frac{1}{2}$. The strategy $\boldsymbol{\sigma}_{ij}$, where i is an orange agent placed on a vertex x_v with $v \in V(G')$, is not a profitable swap either by Lemma 5.3, when $\sigma(i)$ is adjacent to $\sigma(j)$, or simply because the utility of j in $\boldsymbol{\sigma}_{ij}$ is $p\left(\frac{7}{8}\right) < p\left(\frac{2}{3}\right)$. The orange agent i on the special vertex z has a utility that is strictly smaller than $p\left(\frac{1}{6}\right)$ as

$$\frac{k^* + 1}{5m^* + n^* + 1} \leq \frac{n^*}{5n^* + n^* + 1} < \frac{1}{6}.$$

Therefore, $\boldsymbol{\sigma}_{ij}$ is not a profitable swap because the utility of j in $\boldsymbol{\sigma}_{ij}$ is upper bounded by $p\left(\frac{5}{6}\right) < p\left(\frac{2}{3}\right)$. ■

5.7 Conclusion and Open Problems

We studied game-theoretic residential segregation with integration-oriented agents and thereby opened up the novel research direction of considering non-monotone utility functions. Our results clearly show that moving from monotone to non-monotone utilities yields novel structural properties and different results in terms of equilibrium existence and quality. We have equilibrium existence for a larger class of graphs, compared to [Aga+21], and it is an important open problem to prove or disprove if swap equilibria for our model with $\Lambda \leq \frac{1}{2}$ are guaranteed to exist on any graph.

So far we considered single-peaked utilities that are supported by data from real-world sociological polls. However, also other natural types of non-monotone utilities could be studied. Ties in the utility function could be resolved by breaking them consistently towards favoring being in the minority or being in the majority. The non-existence example of swap equilibria used in the proof of Theorem 5.7 also applies to the case with $\Lambda = \frac{1}{2}$ and breaking ties towards being in the majority. Interestingly, by breaking ties the other way we get the same existence results as without tie-breaking and also our other results hold in this case. This is another indication that tolerance helps with stability. Moreover, all our existence results also hold for utility functions having a symmetric plateau shape around Λ. Investigating the Price of Anarchy for these utility functions is open.

Regarding the quality of equilibria, we analyzed the Degree of Integration as social welfare function, as this is in line with considering integration-oriented agents. Of course, studying the quality of the equilibria in terms of the standard utilitarian social welfare, i.e., $\mathsf{SUM}(\boldsymbol{\sigma}) = \sum_{i=1}^{n} \mathsf{U}_i(\boldsymbol{\sigma})$, would also be interesting. We note that in passing that on ring topologies the Price of Anarchy and the Price of Stability concerning both social welfare functions coincide.

So far, we only considered single-peaked utilities for two types of agents. However, it is not obvious at all how to generalize the single-peaked model to more than two agent types. As discussed in Chapter 3, this is already non-trivial for the model with monotone utility functions. The simplest setting would be the "1-versus-all" variant, where the utility only depends on the numbers of same-type and other-type neighbors. But, as shown in Chapter 3, even in this simple setting, the behavior of Schelling games changes drastically. We expect similarly drastic changes for this model. Moreover, we are not convinced that "1-versus-all" captures realistic agent behavior. Ideally, in a setting with more than two types, a diverse neighborhood should contain agents of many different types and it should be balanced such that no subgroup dominates the neighborhood. This would rather suggest the "1-versus-1" variant.

6 The Impact of Geometry in the Flip Schelling Process

This chapter is based on joint work with Thomas Bläsius, Tobias Friedrich, and Martin S. Krejca [Blä+23]. Moreover, I want to thank Thomas Sauerwald for the discussions on random walks.

In the following chapter, we initiate the study of the Flip Schelling Process (FSP), which can be understood as the Schelling model in a saturated open city. Starting from an initial configuration where the type of each agent is chosen uniformly at random, we investigate a simultaneous-move, one-shot process and bound the number of monochrome edges, which is a popular measurement for segregation strength [CR15; Fre78].

Saturated city models are also known as voter models [DS93; Lig94; Lig99]. In general, in voter models, two types of agents are placed on a graph. Agents examine their neighbors and, if a certain threshold is of another type, they change their types. Also in this model, segregation is visible. There is a line of work, mainly in physics, that studies the voting dynamics on several types of graphs [Bal+10; COM03; LSS08; PM05; WH10]. Related to voter models, Granovetter [Gra78] proposed another threshold model treating binary decisions and spurred a number of research, which studied and motivated variants of the model, see [BKH21; KKT03; Mac91; Poi21].

Close to the FSP is the work by Omidvar and Franceschetti [OF18a; OF18b], who initiated an analysis of the size of monochrome regions in the so called *Schelling Spin Systems*. Agents of two different types are placed on a grid [OF18a] and a geometric graph [OF18b], respectively. Then independent and identical Poisson clocks are assigned to all agents and, every time a clock rings, the state of the corresponding agent is flipped if and only if the agent is discontent concerning a certain intolerance threshold τ regarding the neighborhood size. The model corresponds to the Ising model with zero temperature with Glauber dynamics [CFL09; SS07].

The commonly used underlying topology for modeling the residential areas are (toroidal) grid graphs, regular graphs, paths, cycles, and trees, see e.g. [BEL18; KKV21; OF18a]. Considering the influence of the given topology that models the

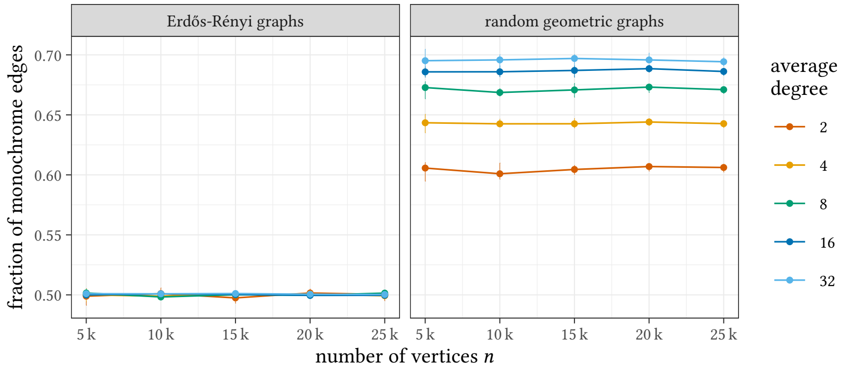


Figure 6.1: The fraction of monochrome edges after the Flip Schelling Process (FSP) in Erdős–Rényi graphs and random geometric graphs for different graph sizes (number of vertices n) and different expected average degrees. Each data point shows the average of over 1000 generated graphs with one simulation of the FSP per graph. The error bars show the interquartile range, i.e., 50 % of the measurements lie between the top and bottom end of the error bar.

residential area regarding, e.g., the existence of stable states and convergence behavior leads to phenomena like the non-existence of stable states [Aga+21] and see also our results in Chapter 3, non-convergence to stable states, see Chapter 3 and Chapter 4, and high-mixing times in corresponding Markov chains [BMR14; Ger+08]. To avoid such undesirable characteristics, we suggest investigating *random geometric graphs* [Pen03], like in [OF18b]. Random geometric graphs demonstrate, in contrast to other random graphs without geometry, such as *Erdős–Rényi graphs* [ER59; Gil59], community structures, i.e., densely connected clusters of vertices. An effect observed by simulating the FSP is that the fraction of monochrome edges is significantly higher in random geometric graphs compared to Erdős–Rényi graphs, where the fraction stays almost stable around $\frac{1}{2}$, cf. Figure 6.1.

We set out to rigorously prove this phenomenon. In particular, we prove for random geometric graphs with n vertices that if the expected average degree is $o(\sqrt{n})$, there exists a positive constant c such that, given an edge $\{u, v\}$, the probability that $\{u, v\}$ is monochrome is lower-bounded by $\frac{1}{2} + c$, cf. Theorem 6.6. In contrast, we show for Erdős–Rényi graphs that segregation is not likely to occur and that the probability that $\{u, v\}$ is monochrome is upper-bounded by $\frac{1}{2} + o(1)$, cf. Theorem 6.17.

We introduce a general framework to deepen the understanding of the influence of the underlying topology on residential segregation. To this end, we first show that a highly decisive common neighborhood supports segregation, cf. Section 6.3.1. In particular, we provide a lower bound on the probability that an edge $\{u, v\}$ is monochrome based on the probability that the difference between the majority and the minority regarding both types in the common neighborhood, i.e., the number of agents which are adjacent to u and v, is larger than their exclusive neighborhoods, i.e., the number of agents which are adjacent to either u or v. Next, we show that large sets of agents are more decisive, cf. Section 6.3.2. This implies that a large common neighborhood, compared to the exclusive neighborhood is likely to be more decisive, i.e., makes it more likely that the absolute value of the difference between the number of different types in the common neighborhood is larger than in the exclusive ones. These considerations hold for arbitrary graphs. Hence, we reduce the question concerning a lower bound for the fraction of monochrome edges in the FSP to the probability that, given $\{u, v\}$, the common neighborhood is larger than the exclusive neighborhoods of u and v, respectively.

For random geometric graphs, we prove that a large geometric region, i.e., the intersecting region that is formed by intersecting disks, leads to a large vertex set, cf. Section 6.3.3, and that random geometric graphs have enough edges that have sufficiently large intersecting regions, cf. Section 6.3.4, such that segregation is likely to occur. In contrast, for Erdős–Rényi graphs, we show that the common neighborhood between two vertices u and v is with high probability empty and therefore segregation is not likely to occur, cf. Section 6.4.

In Section 6.5, we complement our theoretical results with empirical investigations that consider multiple iterations of the FSP. We find that for random geometric graphs, the segregation strength increases with every further iteration, while Erdős–Rényi graphs become single-colored over time. However, our results also show that random geometric graphs with n vertices become single-colored with non-vanishing probability once their average degree is $\Theta(\sqrt{n})$, suggesting that our theoretical results, which hold up to average degrees of $o(\sqrt{n})$, are close to tight.

Overall, we shed light on the influence of the structure of the underlying graph and discovered the significant impact of the community structure as an important factor in the obtained segregation strength. We reveal for random geometric graphs that already after one round a provable tendency is apparent and strong segregation occurs.

6.1 Model

Remember that a random geometric graph $G \sim \mathcal{G}(n, r)$ is obtained by distributing n vertices uniformly at random in some geometric ground space and connecting vertices u and v if and only if $\operatorname{dist}(u, v) \leq r$. We use a two-dimensional toroidal Euclidean space with a total area of 1 as ground space. More formally, each vertex v is assigned to a point $(v_1, v_2) \in [0, 1]^2$ and the distance between $u = (u_1, u_2)$ and v is $\operatorname{dist}(u, v) = \sqrt{|u_1 - v_1|_\circ^2 + |u_2 - v_2|_\circ^2}$ for $|u_i - v_i|_\circ = \min\{|u_i - v_i|, 1 - |u_i - v_i|\}$. We note that using a torus instead of, e.g., a unit square, has the advantage that we do not have to consider edge cases, for vertices that are close to the boundary. A disk of radius r around any point has the same area πr^2. We consider a ground space with total area 1, $\pi r^2 = 1$. As every vertex v is connected to all vertices in the disk of radius r around it, its expected average degree is $\overline{\deg} = (n - 1)\pi r^2$.

Consider two different vertices u and v. Let $N(u \cap v) := |N(u) \cap N(v)|$ be the number of vertices in the *common neighborhood*, let $N(u \setminus v) := |N(u) \setminus N(v)|$ be the number of vertices in the *exclusive neighborhood* of u, and let $N(v \setminus u) := |N(v) \setminus N(u)|$ be the number of vertices in the exclusive neighborhood of v. Furthermore, with $N(\overline{u \cup v}) := |V \setminus (N(u) \cup N(v))|$, we denote the number of vertices that are neither adjacent to u nor v.

We consider the Flip Schelling Process with 2 types. Let G be a graph where each vertex represents an orange or blue agent. The type of each agent is chosen independently and uniformly at random. Remember that an agent is content if the fraction of agents in its neighborhood with the same type is larger than $\frac{1}{2}$. Otherwise, if the fraction is smaller than $\frac{1}{2}$, an agent is discontent and is willing to flip its type to become content. If the fraction of the same type of agents in its neighborhood is exactly $\frac{1}{2}$, an agent flips its type with probability $\frac{1}{2}$.

6.2 Preliminaries

In this section, we state several lemmas that we will use to prove our results in the next sections.

▶ **Lemma 6.1.** Let $X \sim \mathrm{Bin}(n, p)$ and $Y \sim \mathrm{Bin}(n, q)$ with $p \geq q$ be independent. Then $\Pr[X \geq Y] \geq \frac{1}{2}$. ◀

Proof. Let $Y_1, \ldots, Y_n$ be the individual Bernoulli trials for Y, i.e.,

$$Y = \sum_{i \in [n]} Y_i.$$

Define new random variables $Y'_1, \ldots, Y'_n$ such that $Y_i = 1$ implies $Y'_i = 1$ and if $Y_i = 0$, then $Y'_i = 1$ with probability $(p - q)/(1 - q)$ and $Y'_i = 0$ otherwise. Note that for each individual Y'_i, we have $Y'_i = 1$ with probability p, i.e.,

$$Y' = \sum_{i \in [n]} Y'_i \sim \mathrm{Bin}(n, p).$$

Moreover, as $Y' \geq Y$ for every outcome, we have $\Pr[X \geq Y] \geq \Pr[X \geq Y']$. It remains to show that

$$\Pr[X \geq Y'] \geq \frac{1}{2}.$$

As X and Y' are equally distributed, we have $\Pr[X \geq Y'] = \Pr[X \leq Y']$. Moreover, as one of the two inequalities, holds in any event, we get $\Pr[X \geq Y'] + \Pr[X \leq Y'] \geq 1$, and thus equivalently $2\Pr[X \geq Y'] \geq 1$, which proves the claim. ■

▶ **Lemma 6.2 ([Dar64]).** Let $n \in \mathbf{N}^+$, $p \in [0, 1)$, and let $X \sim \mathrm{Bin}(n, p)$. Then, for all $i \in [0..n]$, it holds that $\Pr[X = i] \leq \Pr[X = \lfloor p(n + 1) \rfloor]$. ◀

Proof. We interpret the distribution of X as a finite series and consider the sign of the differences $b\colon [0, n - 1] \to \mathbf{R}$ of two neighboring terms. That is, for all $d \in [0, n - 1] \cap \mathbf{N}$, it holds that

$$\begin{aligned} b(d) &= \Pr[X = d + 1] - \Pr[X = d] \\ &= \binom{n}{d+1} p^{d+1} (1 - p)^{n-d-1} - \binom{n}{d} p^d (1 - p)^{n-d}. \end{aligned}$$

We are interested in the sign of b, as a local maximum of the distribution of X is located at the position at which b switches from positive to negative. In more detail, for any $d \in [0, n - 2] \cap \mathbf{N}$, if $\mathrm{sgn}\big(b(d)\big) \geq 0$ and $\mathrm{sgn}\big(b(d + 1)\big) \leq 0$, then

$d+1$ is a local maximum. If the sign is always negative, then there is a global maximum in the distribution of X at position 0.

In order to determine the sign of b, for all $i \in [0..n-1]$, we rewrite

$$\begin{aligned} b(i) &= \frac{n!}{i!(n-i-1)!} p^i (1-p)^{n-i-1} \frac{p}{i+1} - \frac{n!}{i!(n-i-1)!} p^i (1-p)^{n-i-1} \frac{1-p}{n-i} \\ &= \frac{n!}{i!(n-i-1)!} p^i (1-p)^{n-i-1} \left(\frac{p}{i+1} - \frac{1-p}{n-i} \right). \end{aligned}$$

Since the term $n!/(i!(n-i-1)!)p^i(1-p)^{n-i-1}$ is always non-negative, the sign of $b(i)$ is determined by the sign of $p/(i+1) - (1-p)/(n-i)$. Solving for i, we get that

$$\frac{p}{i+1} - \frac{1-p}{n-i} \geq 0 \Leftrightarrow i \leq p(n+1) - 1.$$

Note that $p(n+1)-1$ is not necessarily an integer. Further note that the distribution of X is uni-modal, as the sign of b changes at most once. Thus, each local maximum is also a global maximum. As discussed above, the largest value $d \in [0, n-2] \cap \mathbf{N}$ such that $\operatorname{sgn}\big(b(d)\big) \geq 0$ and $\operatorname{sgn}\big(b(d+1)\big) \leq 0$ then results in a global maximum at position $d+1$. Since d needs to be an integer, the largest value that satisfies this constraint is $\lfloor p(n+1)-1 \rfloor$. If the sign of b is always negative ($p \leq 1/(n+1)$), then the distribution of X has a global maximum at 0, which is also satisfied by $\lfloor p(n+1)-1 \rfloor + 1$, which concludes the proof. ■

▶ **Theorem 6.3 (Stirling's Formula [Fel68, page 54]).** For all $n \in \mathbf{N}^+$, it holds that

$$\sqrt{2\pi} n^{n+1/2} \, \mathrm{e}^{-n} \cdot \mathrm{e}^{(12n+1)^{-1}} < n! < \sqrt{2\pi} n^{n+1/2} \, \mathrm{e}^{-n} \cdot \mathrm{e}^{(12n)^{-1}} .$$

◀

▶ **Corollary 6.4.** For all $n \geq 2$ with $n \in \mathbf{N}$, it holds that

$$n! > \sqrt{2\pi} n^{n+1/2} \, \mathrm{e}^{-n} \quad \text{and} \tag{6.1}$$

$$n! < n^{n+1/2} \, \mathrm{e}^{-n+1} . \tag{6.2}$$

◀

Proof. For both inequalities, we aim at using Theorem 6.3.

Equation (6.1): Note that $e^{(12n+1)^{-1}} > 1$, since $\frac{1}{12n+1} > 0$. Hence,

$$\sqrt{2\pi n}^{n+1/2}\, e^{-n} < \sqrt{2\pi n}^{n+1/2}\, e^{-n} \cdot e^{(12n+1)^{-1}} .$$

Equation (6.2): We prove this case by showing that

$$\sqrt{2\pi}\, e^{(12n)^{-1}} < e\,. \tag{6.3}$$

Note, that $e^{(12n)^{-1}}$ is strictly decreasing. Hence, we only have to check whether Equation (6.3) holds for $n = 2$.

$$\sqrt{2\pi}\, e^{(12n)^{-1}} \leq \sqrt{2\pi}\, e^{\frac{1}{24}} < 2.7 < e\,.$$ ■

▶ **Lemma 6.5.** Let A, B, and C be random variables such that $\Pr[A > C \wedge B \leq C] > 0$ and $\Pr[A > C \wedge B > C] > 0$. Then $\Pr[A > B \wedge A > C] \geq \Pr[A > B] \cdot \Pr[A > C]$. ◀

Proof. Using the definition of conditional probability, we obtain

$$\Pr[A > B \wedge A > C] = \Pr[A > B \mid A > C] \cdot \Pr[A > C].$$

Hence, we are left with bounding $\Pr[A > B \mid A > C] \geq \Pr[A > B]$. Partitioning the sample space into the two events $B > C$ and $B \leq C$ and using the law of total probability, we obtain

$$\begin{aligned}\Pr[A > B \mid A > C] = &\Pr[B > C \mid A > C] \cdot \Pr[A > B \mid A > C \wedge B > C]\\ &+ \Pr[B \leq C \mid A > C] \cdot \Pr[A > B \mid A > C \wedge B \leq C].\end{aligned}$$

Note that the condition $A > C \wedge B \leq C$ already implies $A > B$ and thus the last probability equals to 1. Moreover, using the definition of conditional probability, we obtain

$$\begin{aligned}\Pr[A > B \mid A > C] = &\Pr[B > C \mid A > C] \cdot \frac{\Pr[A > B \wedge A > C \wedge B > C]}{\Pr[A > C \wedge B > C]}\\ &+ \Pr[B \leq C \mid A > C].\end{aligned}$$

Using that $\Pr[B > C \mid A > C] \geq \Pr[A > C \wedge B > C]$, that $A > B \wedge B > C$ already implies $A > C$, that $\Pr[B \leq C \mid A > C] \geq \Pr[A > B \wedge B \leq C]$, and finally the law of total probability, we obtain

$$\begin{aligned}\Pr[A > B \mid A > C] &\geq \Pr[A > B \wedge A > C \wedge B > C] + \Pr[B \leq C \mid A > C] \\ &= \Pr[A > B \wedge B > C] + \Pr[B \leq C \mid A > C] \\ &\geq \Pr[A > B \wedge B > C] + \Pr[A > B \wedge B \leq C] \\ &= \Pr[A > B].\end{aligned}$$ ■

6.3 Monochrome Edges in Geometric Random Graphs

In this section, we prove the following main theorem.

▶ **Theorem 6.6.** Let $G \sim \mathcal{G}(n, r)$ be a random geometric graph with expected average degree $\overline{\deg} = o(\sqrt{n})$. The expected fraction of monochrome edges after the FSP is at least

$$\frac{1}{2} + \frac{9}{800} \cdot \left(\frac{1}{2} - \frac{1}{\sqrt{2\pi\lfloor\overline{\deg}/2\rfloor}}\right)^2 \cdot \left(1 - e^{-\overline{\deg}/2}\left(1 + \frac{\overline{\deg}}{2}\right)\right) \cdot (1 - o(1)).$$

◀

Note that the bound in Theorem 6.6 is bounded away from $\frac{1}{2}$ for all $\overline{\deg} \geq 2$. Moreover, the two factors depending on $\overline{\deg}$ go to $\frac{1}{2}$ and 1, respectively, for a growing $\overline{\deg}$.

Given an edge $\{u, v\}$, we prove the above lower bound on the probability that $\{u, v\}$ is monochrome in the following four steps.

(1) For a vertex set, we introduce the concept of *decisiveness* that measures how much the majority is ahead of the minority in the FSP. With this, we give a lower bound on the probability that $\{u, v\}$ is monochrome based on the probability that the common neighborhood of u and v is more decisive than their exclusive neighborhoods.

(2) We show that large neighborhoods are likely to be more decisive than small neighborhoods. To this end, we give bounds on the likelihood that two similar random walks behave differently. This step reduces the question

of whether the common neighborhood is more decisive than the exclusive neighborhoods to whether the former is larger than the latter.

(3) Turning to geometric random graphs, we show that the common neighborhood is sufficiently likely to be larger than the exclusive neighborhoods if the geometric region corresponding to the former is sufficiently large. We do this by first showing that the actual distribution of the neighborhood sizes is well approximated by independent binomial random variables. Then, we give the desired bounds for these random variables.

(4) We show that the existence of the edge $\{u, v\}$ in the geometric random graph makes it sufficiently likely that the geometric region hosting the common neighborhood of u and v is sufficiently large.

6.3.1 Monochrome Edges via Decisive Neighborhoods

Let $\{u, v\}$ be an edge of a given graph. To formally define the above-mentioned decisiveness, let $N^o(u \cap v)$ and $N^b(u \cap v)$ be the number of vertices in the common neighborhood of u and v that are occupied by orange and blue agents, respectively. Then $D(u \cap v) := |N^o(u \cap v) - N^b(u \cap v)|$ is the *decisiveness* of the common neighborhood of u and v. Analogously, we define $D(u \setminus v)$ and $D(v \setminus u)$ for the exclusive neighborhoods of u and v, respectively.

The following theorem bounds the probability for $\{u, v\}$ to be monochrome based on the probability that the common neighborhood is more decisive than each of the exclusive ones.

▶ **Theorem 6.7.** In the FSP, let $\{u, v\} \in E$ be an edge and let D be the event $\{D(u \cap v) > D(u \setminus v) \wedge D(u \cap v) > D(v \setminus u)\}$. Then $\{u, v\}$ is monochrome with probability at least $1/2 + \Pr[D]/2$. ◀

Proof. If D occurs, then the types of u and v after the FSP coincide with the predominant type before the FSP in the shared neighborhood. Thus, $\{u, v\}$ is monochrome.

Otherwise, assuming $\overline{D}$, without loss of generality, let $D(u \cap v) \leq D(u \setminus v)$ and assume further that the type of v has already been determined. If $D(u \cap v) = D(u \setminus v)$, then u chooses a type uniformly at random, which coincides with the type of v with probability $\frac{1}{2}$. Otherwise, $D(u \cap v) < D(u \setminus v)$ and thus u takes the type that is predominant in u's exclusive neighborhood, which is orange and blue with probability $\frac{1}{2}$, each. Moreover, this is independent of the type of v as v's neighborhood is disjoint to u's exclusive neighborhood.

Thus, for the event M that $\{u, v\}$ is monochrome, we get $\Pr[M \mid D] = 1$ and $\Pr\left[M \mid \overline{D}\right] = \frac{1}{2}$. Hence, we get $\Pr[M] \geq \Pr[D] + \frac{1}{2}(1 - \Pr[D]) = \frac{1}{2} + \Pr[D]/2$. ■

6.3.2 Large Neighborhoods are More Decisive

The goal of this section is to reduce the question of how decisive a neighborhood is to the question of how large it is. To be more precise, assume we have a set of vertices of size a and give each vertex the type orange and blue, respectively, each with probability $\frac{1}{2}$. Let A_i for $i \in [a]$ be the random variable that takes the value $+1$ and -1 if the i-th vertex in this set is orange and blue, respectively. Then, for $A = \sum_{i \in [a]} A_i$, the decisiveness of the vertex set is $|A|$. In the following, we study the decisiveness $|A|$ depending on the size a of the set. Note that this can be viewed as a random walk on the integer line: Starting at 0, in each step, it moves one unit either to the left or to the right with equal probabilities. We are interested in the distance from 0 after a steps.

Assume for the vertices u and v that we know that b vertices lie in the common neighborhood and a vertices lie in the exclusive neighborhood of u. Moreover, let A and B be the positions of the above random walk after a and b steps, respectively. Then the event $D(u \cap v) > D(u \setminus v)$ is equivalent to $|B| > |A|$. Motivated by this, we study the probability of $|B| > |A|$, assuming $b \geq a$. The core difficulty here comes from the fact that we require $|B|$ to be strictly larger than $|A|$. Also, note that $a + b$ corresponds to the degree of u in the graph. Thus, we have to study the random walks also for small numbers of a and b. We note that all results in this section are independent of the specific application to the FSP, and thus might be of independent interest.

Before we give a lower bound on the probability that $|B| > |A|$, we need the following technical lemma. It states that doing more steps in the random walk only makes it more likely to deviate further from the starting position.

▶ **Lemma 6.8.** For $i \in [a]$ and $j \in [b]$ with $0 \leq a \leq b$, let A_i and B_j be independent random variables that are -1 and 1 each with probability $\frac{1}{2}$. Let $A = \sum_{i \in [a]} A_i$ and $B = \sum_{j \in [b]} B_j$. Then $\Pr[|A| < |B|] \geq \Pr[|A| > |B|]$. ◀

Proof. Let Δ_k be the event that $|B| - |A| = k$. First, note that

$$\Pr[|A| < |B|] = \sum_{k \in [b]} \Pr[\Delta_k] \quad \text{and} \quad \Pr[|A| > |B|] = \sum_{k \in [a]} \Pr[\Delta_{-k}].$$

To prove the statement of the lemma, it thus suffices to prove the following claim.

▶ **Claim 6.9.** For $k \geq 0$, $\Pr[\Delta_k] \geq \Pr[\Delta_{-k}]$. ◀

We prove this claim via induction on $b - a$. For the base case $a = b$, A and B are equally distributed, and thus Claim 6.9 holds.

For the induction step, let B^+ be the random variable that takes the values $B + 1$ and $B - 1$ with probability $\frac{1}{2}$ each. Note that B^+ represents the same type of random walk as A and B but with $b + 1$ steps. Moreover B^+ is coupled with B to make the same decisions in the first b steps. Let Δ_k^+ be the event that $|B^+| - |A| = k$. It remains to show that Claim 6.9 holds for these Δ_k^+. For this, first, note that the claim trivially holds for $k = 0$. For $k \geq 1$, we can use the definition of Δ_k^+ and the induction hypothesis to obtain

$$\begin{aligned} \Pr\left[\Delta_k^+\right] &= \frac{\Pr[\Delta_{k-1}]}{2} + \frac{\Pr[\Delta_{k+1}]}{2} \\ &\geq \frac{\Pr[\Delta_{-k+1}]}{2} + \frac{\Pr[\Delta_{-k-1}]}{2} = \Pr\left[\Delta_{-k}^+\right]. \end{aligned}$$ ■

Using Lemma 6.8, we now prove the following general bound for the probability that $|A| < |B|$, depending on certain probabilities for binomially distributed variables.

▶ **Lemma 6.10.** For $i \in [a]$ and $j \in [b]$ with $0 \leq a \leq b$, let A_i and B_j be independent random variables that are -1 and 1 each with probability $\frac{1}{2}$. Let $A = \sum_{i \in [a]} A_i$ and $B = \sum_{j \in [b]} B_j$. Moreover, let $X \sim \mathrm{Bin}(a, \frac{1}{2})$, $Y \sim \mathrm{Bin}(b, \frac{1}{2})$, and $Z \sim \mathrm{Bin}(a + b, \frac{1}{2})$. Then

$$\Pr[|A| < |B|] \geq \frac{1}{2} - \Pr\left[Z = \frac{a+b}{2}\right] + \frac{\Pr\left[X = \frac{a}{2}\right] \cdot \Pr\left[Y = \frac{b}{2}\right]}{2}.$$

◀

Proof. Using that $\Pr[|A| < |B|] \geq \Pr[|A| > |B|]$ (see Lemma 6.8), we obtain

$$\begin{aligned}
& \Pr[|A| < |B|] + \Pr[|A| > |B|] + \Pr[|A| = |B|] = 1 \\
\Rightarrow \quad & 2\Pr[|A| < |B|] + \Pr[|A| = |B|] \geq 1 \\
\Leftrightarrow \quad & \Pr[|A| < |B|] \geq \frac{1}{2} - \frac{\Pr[|A| = |B|]}{2}. \qquad (6.4)
\end{aligned}$$

Thus, it remains to give an upper bound for $\Pr[|A| = |B|]$. Using the inclusion–exclusion principle and the fact that B is symmetric around 0, i.e., $\Pr[B = x] = \Pr[B = -x]$ for any x, we obtain

$$\begin{aligned}
\Pr[|A| = |B|] &= \Pr[A = B \vee A = -B] \\
&= \Pr[A = B] + \Pr[A = -B] - \Pr[A = B = 0] \\
&= 2\Pr[A = -B] - \Pr[A = B = 0]. \qquad (6.5)
\end{aligned}$$

We estimate $\Pr[A = -B]$ and $\Pr[A = B = 0]$ using bounds for binomially distributed variables. To this end, define new random variables $X_i = \frac{A_i+1}{2}$ for $i \in [a]$ and let $X = \sum_{i\in[a]} X_i$. Note that the X_i are independent and take values 0 and 1, each with probability $\frac{1}{2}$. Thus, $X \sim \mathrm{Bin}(a, \frac{1}{2})$. Moreover, $A = 2X - a$. Analogously, we define Y with $Y \sim \mathrm{Bin}(b, \frac{1}{2})$ and $B = 2Y - b$. Note that X and Y are independent and thus $Z = X + Y \sim \mathrm{Bin}(a + b, \frac{1}{2})$. With this, we get

$$\Pr[A = -B] = \Pr[2X - a = -2Y + b] = \Pr\left[Z = \frac{a+b}{2}\right], \text{ and}$$

$$\Pr[A = B = 0] = \Pr[A = 0] \cdot \Pr[B = 0] = \Pr\left[X = \frac{a}{2}\right] \cdot \Pr\left[Y = \frac{b}{2}\right].$$

This, together with Equations (6.4) and (6.5) yield the claim. ■

The bound in Lemma 6.10 becomes worse for smaller values of a and b. Considering this worst case, we obtain the following specific bound.

▶ **Theorem 6.11.** For $i \in [a]$ and $j \in [b]$ with $0 \leq a \leq b$, let A_i and B_j be independent random variables that are -1 and 1 each with probability $\frac{1}{2}$. Let $A = \sum_{i\in[a]} A_i$ and $B = \sum_{j\in[b]} B_j$. $\Pr[|A| < |B|] = 0$ if $a = b = 0$ or $a = b = 1$. Otherwise, $\Pr[|A| < |B|] \geq \frac{3}{16}$. ◀

Proof. Clearly, if $a = b = 0$, then $A = B = 0$ and thus $\Pr[|A| < |B|] = 0$. Similarly, if $a = b = 1$, then $|A| = |B| = 1$ and thus $\Pr[|A| < |B|] = 0$. For the remainder, assume that neither $a = b = 0$ nor $a = b = 1$, and let X, Y, and Z be defined as in Lemma 6.10, i.e., $X \sim \mathrm{Bin}(a, \frac{1}{2})$, $Y \sim \mathrm{Bin}(b, \frac{1}{2})$, and $Z \sim \mathrm{Bin}(a+b, \frac{1}{2})$.

If $a + b$ is odd, then $\Pr\left[Z = \frac{a+b}{2}\right] = 0$. Thus, by Lemma 6.10, we have $\Pr[|A| < |B|] \geq \frac{1}{2}$. If $a + b$ is even and $a + b \geq 6$, then

$$\Pr\left[Z = \frac{a+b}{2}\right] = \binom{a+b}{\frac{a+b}{2}}\left(\frac{1}{2}\right)^{a+b} \leq \binom{6}{3}\left(\frac{1}{2}\right)^{6} = \frac{5}{16}.$$

Hence, by Lemma 6.10, we have $\Pr[|A| < |B|] \geq \frac{1}{2} - \frac{5}{16} = \frac{3}{16}$.

If $a + b < 6$ (and $a + b$ is even), there are four cases: $a = 0, b = 2$; $a = 0, b = 4$; $a = 1, b = 3$; $a = 2, b = 2$.

If $a = 0$ and $b = 2$, then $A = 0$ with probability 1 and $|B| = 2$ with probability $\frac{1}{2}$. Thus, $\Pr[|A| < |B|] = \frac{1}{2}$.

If $a = 0$ and $b = 4$, then $|A| < |B|$ unless $B = 0$. As $\Pr[B = 0] = \binom{4}{2} \cdot (\frac{1}{2})^4 = \frac{3}{8}$, we get $\Pr[|A| < |B|] = 1 - \frac{3}{8} = \frac{5}{8}$.

If $a = 1$ and $b = 3$, then $|A| = 1$ with probability 1 and $|B| = 3$ with probability $\frac{1}{4}$ (either $B_1 = B_2 = B_3 = 1$ or $B_1 = B_2 = B_3 = -1$). Thus, $\Pr[|A| < |B|] = \frac{1}{4}$.

If $a = b = 2$, then $|A| = 0$ with probability $\frac{1}{2}$ and $|B| = 2$ with probability $\frac{1}{2}$. Thus $\Pr[|A| < |B|] = \frac{1}{4}$.

We note that the bound of $\Pr[|A| < |B|] = \frac{3}{16}$ is tight for $a = b = 3$. ■

6.3.3 Large Common Regions Yield Large Common Neighborhoods

To be able to apply Theorem 6.11 to an edge $\{u, v\}$, we need to make sure that the size of their common neighborhood (corresponding to b in the theorem) is at least the size of the exclusive neighborhoods (corresponding to a in the theorem). In the following, we give bounds for the probability that this happens. Note that this is the first time we take the graph into account. Thus, all the above considerations hold for arbitrary graphs.

Recall that we consider random geometric graphs $\mathcal{G}(n, r)$ and u and v are each connected to all vertices that lie within a disk of radius r around them. As u and v are adjacent, their disks intersect, which separates the ground space into four regions; cf. Figure 6.2. Let $R(u \cap v)$ be the intersection of the two disks. Let $R(u \setminus v)$ be the set of points that lie in the disk of u but not in the disk of v, and analogously, let $R(v \setminus u)$ be the disk of v minus the disk of u. Finally,

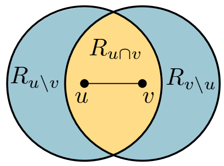


Figure 6.2: The geometric regions corresponding to the common and exclusive neighborhoods, respectively, with yellow illustrating $R(u \cap v)$ and blue illustrating $R(u \setminus v)$ and $R(v \setminus u)$. Please refer to Section 6.3.3 for details.

let $R(\overline{u \cup v})$ be the set of points outside both disks. Then, each of the $n-2$ remaining vertices ends up in exactly one of these regions with a probability equal to the corresponding measure. Let $\mu(\cdot)$ be the area of the respective region and $p = \mu(R(u \cap v))$ and $q = \mu(R(u \setminus v)) = \mu(R(v \setminus u))$ be the probabilities for a vertex to lie in the common and exclusive regions, respectively. The probability for $R(\overline{u \cup v})$ is then $1 - p - 2q$.

We are now interested in the sizes $N(u \cap v)$, $N(u \setminus v)$, and $N(v \setminus u)$ of the common and the exclusive neighborhoods, respectively. As each of the $n-2$ remaining vertices ends up in $N(u \cap v)$ with probability p, we have

$$N(u \cap v) \sim \mathrm{Bin}(n-2, p).$$

For $N(u \setminus v)$ and $N(v \setminus u)$, we already know that v is a neighbor of u and vice versa. Thus,

$$(N(u \setminus v) - 1) \sim \mathrm{Bin}(n-2, q) \text{ and } (N(v \setminus u) - 1) \sim \mathrm{Bin}(n-2, q).$$

Moreover, the three random variables are not independent, as each vertex lies in only exactly one of the four neighborhoods, i.e., $N(u \cap v)$, $(N(u \setminus v) - 1)$, $(N(v \setminus u) - 1)$, and the number of vertices in neither neighborhood together follow a multinomial distribution $\mathrm{Multi}(n-2, \boldsymbol{p})$ with $\boldsymbol{p} = (p, q, q, 1 - p - 2q)$.

The following lemma shows that these dependencies are small if p and q are sufficiently small. This lets us assume that, if the expected average degree is not too large, $N(u \cap v)$, $(N(u \setminus v) - 1)$, $(N(v \setminus u) - 1)$ are independent random variables following binomial distributions.

▶ **Lemma 6.12.** Let $\boldsymbol{p} = (p, q, q, 1 - p - 2q)$ and let $X = (X_1, X_2, X_3, X_4) \sim \mathrm{Multi}(n, \boldsymbol{p})$. Then there exist independent random variables $Y_1 \sim \mathrm{Bin}(n, p)$, $Y_2 \sim \mathrm{Bin}(n, q)$, and $Y_3 \sim \mathrm{Bin}(n, q)$ such that

$$\Pr[(X_1, X_2, X_3) = (Y_1, Y_2, Y_3)] \geq 1 - 3n \cdot \max(p, q)^2.$$

◀

Proof. Let $Y_1 \sim \mathrm{Bin}(n, p)$, and $Y_2, Y_3 \sim \mathrm{Bin}(n, q)$ be independent random variables. We define the event B to hold if each of the n individual trials increments at most one of the random variables Y_1, Y_2, or Y_3. More formally, for $i \in [3]$ and $j \in [n]$, let $Y_{i,j}$ be the individual Bernoulli trials of Y_i, i.e., $Y_i = \sum_{j \in [n]} Y_{i,j}$. For $j \in [n]$, we define the event B_j to be $Y_{1,j} + Y_{2,j} + Y_{3,j} \leq 1$, and the event

$$B = \bigcap_{j \in [n]} B_j.$$

Based on this, we now define the random variables X_1, X_2, X_3, and X_4 as follows. If B holds, we set $X_i = Y_i$ for $i \in [3]$ and $X_4 = n - X_1 - X_2 - X_3$. Otherwise, if $\overline{B}$, we draw $X = (X_1, X_2, X_3, X_4) \sim \mathrm{Multi}(n, \boldsymbol{p})$ independently from Y_1, Y_2, and Y_3 with $\boldsymbol{p} = (p, q, q, 1-p-2q)$. Note that X clearly follows $\mathrm{Multi}(n, \boldsymbol{p})$ if $\overline{B}$. Moreover, conditioned on B, each individual trial increments exactly one of the variables X_1, X_2, X_3, or X_4 with probabilities p, q, q, and $1-p-2q$, respectively, i.e., $X \sim \mathrm{Multi}(n, \boldsymbol{p})$.

Thus, we end up with $X \sim \mathrm{Multi}(n, \boldsymbol{p})$. Additionally, we have three independent random variables $Y_1 \sim \mathrm{Bin}(n, p)$, and $Y_2, Y_3 \sim \mathrm{Bin}(n, q)$ with $(X_1, X_2, X_3) = (Y_1, Y_2, Y_3)$ if B holds. Thus, to prove the lemma, it remains to show that

$$\Pr[B] \geq 1 - 3n \max(p, q)^2.$$

For $j \in [n]$, the probability that the jth trial goes wrong is

$$\begin{aligned}
\Pr\left[\overline{B}_j\right] &= 1 - \left((1-p)(1-q)^2\right) - \left(p(1-q)^2\right) - 2(q(1-p)(1-q)) \\
&= 2pq - 2pq^2 + q^2 \leq 2pq + q^2 \leq 3 \cdot \max(p, q)^2.
\end{aligned}$$

Using the union bound it follows that $\Pr\left[\overline{B}\right] \leq \sum_{j \in [n]} \Pr\left[\overline{B}_j\right] \leq 3n \cdot \max(p, q)^2$. ■

As mentioned before, we are interested in the event $N(u \cap v) \geq N(u \setminus v)$ (and likewise $N(u \cap v) \geq N(v \setminus u)$), in order to apply Theorem 6.11. Moreover, due to Lemma 6.12, we know that $N(u \cap v)$ and $(N(u \setminus v) - 1)$ almost behave like independent random variables that follow $\mathrm{Bin}(n-2, p)$ and $\mathrm{Bin}(n-2, q)$, respectively. The following lemma helps to bound the probability for $N(u \cap v) \geq N(u \setminus v)$. Note that it gives a bound for the probability of achieving strict inequality (instead of just $\geq$), which accounts for the fact that $(N(u \setminus v) - 1)$ and not $N(u \setminus v)$ itself follows a binomial distribution.

▶ **Lemma 6.13.** Let $n \in \mathbf{N}$ with $n \geq 2$, and let $p \geq q > 0$. Further, let $X \sim \mathrm{Bin}(n, p)$ and $Y \sim \mathrm{Bin}(n, q)$ be independent, let $d = \lfloor p(n+1) \rfloor$, and assume $d = \mathrm{o}(\sqrt{n})$, then

$$\Pr[X > Y] \geq \left(\frac{1}{2} - 1/\sqrt{2\pi d}\right)(1 - \mathrm{o}(1)).$$

◀

Proof. By Lemma 6.1, we get $\Pr[X \geq Y] \geq \frac{1}{2}$, and we bound

$$\Pr[X > Y] = \Pr[X \geq Y] - \Pr[X = Y] \geq \frac{1}{2} - \Pr[X = Y],$$

leaving us to bound $\Pr[X = Y]$ from above. By independence of X and Y, we get

$$\Pr[X = Y] = \sum_{i \in [n]} \Pr[X = i] \cdot \Pr[Y = i]. \tag{6.6}$$

Note that, by Lemma 6.2, for all $i \in [0..n]$, it holds that $\Pr[X = i] \leq \Pr[X = d]$. Assume that we have a bound B such that $\Pr[X = d] \leq B$. Substituting this into Equation (6.6) yields

$$\Pr[X = Y] \leq B \sum_{i \in [n]} \Pr[Y = i] = B,$$

resulting in $\Pr[X > Y] \geq \frac{1}{2} - B$. Thus, we now derive such a bound for B, noting that $\Pr[X = d]$ is increasing as long as $d - np \geq 0$, and by applying the inequality that for all $x \in \mathbf{R}$, it holds that $1 + x \leq \mathrm{e}^x$, as well as Equation (6.1). We get

$$\begin{aligned}
\Pr[X = d] = \binom{n}{d} p^d (1-p)^{n-d} &\leq \frac{n^d}{d!}\left(\frac{d}{n}\right)^d \left(1-\frac{d}{n}\right)^n \left(1-\frac{d}{n}\right)^{-d} \\
&\leq \frac{d^d}{d!} \mathrm{e}^{-d} \left(1-\frac{d}{n}\right)^{-d} \\
&\leq \frac{d^d}{\sqrt{2\pi} d^{d+1/2} \mathrm{e}^{-d}} \mathrm{e}^{-d} \left(1-\frac{d}{n}\right)^{-d} \\
&= \frac{1}{\sqrt{2\pi d}} \frac{1}{(1-d/n)^d}. \qquad (6.7)
\end{aligned}$$

By Bernoulli's inequality, we bound $(1 - d/n)^d \geq 1 - d^2/n = 1 - \mathrm{o}(1)$ by the assumption $d = \mathrm{o}(\sqrt{n})$. Substituting this back into Equation (6.7) concludes the proof. ■

Finally, in order to apply Theorem 6.11, we have to make sure not to end up in the special case where $a = b \leq 1$, i.e., we have to make sure that the common neighborhood includes at least two vertices. The probability for this to happen is given by the following lemma.

▶ **Lemma 6.14.** Let $X \sim \mathrm{Bin}(n, p)$ and let $c = np \in \mathrm{o}(n)$. Then it holds that

$$\Pr[X > 1] \geq (1 - \mathrm{e}^{-c}(1 + c))(1 - \mathrm{o}(1)).$$

◀

Proof. As $X > 1$ holds if and only if $X \neq 0$ and $X \neq 1$, we get

$$\Pr[X > 1] = 1 - \Pr[X = 0] - \Pr[X = 1] = 1 - (1-p)^n - n \cdot p \cdot (1-p)^{n-1}.$$

Using that for all $x \in \mathbf{R}$ it holds that $1 - x \leq \mathrm{e}^{-x}$, we get

$$\begin{aligned}
\Pr[X > 1] &\geq 1 - \mathrm{e}^{-pn} - n \cdot p \cdot \mathrm{e}^{-p(n-1)} \\
&= 1 - \mathrm{e}^{-c} - c \cdot \mathrm{e}^{c/n} \cdot \mathrm{e}^{-c} \\
&= 1 - \mathrm{e}^{-c}\left(1 + c \cdot \mathrm{e}^{c/n}\right).
\end{aligned}$$

As $\mathrm{e}^{c/n}$ goes to 1 for $n \to \infty$, we get the claimed bound. ■

6.3.4 Many Edges Have Large Common Regions

In Section 6.3.3, we derived a lower bound on the probability that $N(u \cap v) \geq N(u \setminus v)$ provided that the probability for a vertex to end up in the shared region $R(u \cap v)$ is sufficiently large compared to $R(u \setminus v)$. In the following, we estimate the measures of these regions depending on the distance between u and v. Then, we give a lower bound on the probability that $\mu(R(u \cap v)) \geq \mu(R(u \setminus v))$.

▶ **Lemma 6.15.** Let $G \sim \mathcal{G}(n, r)$ be a random geometric graph with expected average degree $\overline{\deg}$, let $\{u, v\} \in E$ be an edge, and let $\tau := \frac{\operatorname{dist}(u,v)}{r}$. Then,

$$\mu(R(u \cap v)) = \frac{\overline{\deg}}{(n-1)\pi}\left(2\arccos\left(\frac{\tau}{2}\right) - \sin\left(2\arccos\left(\frac{\tau}{2}\right)\right)\right) \tag{6.8}$$

and

$$\mu(R(u \setminus v)) = \mu(R(v \setminus u)) = \frac{\overline{\deg}}{n-1} - \mu(R(u \cap v)). \tag{6.9}$$

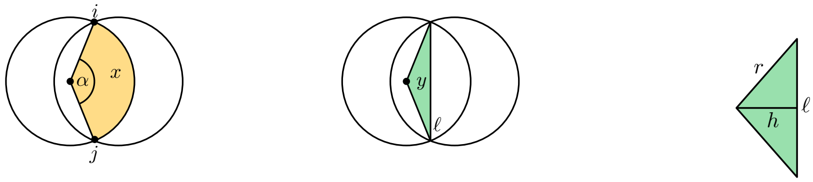
◀

Proof. We start with proving Equation (6.8). Let i and j be the two intersection points of the disks of u and v, let α be the central angle enclosed by i and j, and let x be the corresponding circular sector, cf. Figure 6.3 (a). Moreover, let the triangle y be a subarea of x determined by α and the radical axis ℓ, cf. Figure 6.3 (b). Let h denote the height of the triangle y, cf. Figure 6.3 (c). For our calculations, we restrict the length of ℓ by the intersection points i and j. Since we consider the intersection between disks and thus ℓ divides the area $\mu(R(u \cap v))$ into two subareas of equal sizes, it holds that $\mu(R(u \cap v)) = 2(\mu(x) - \mu(y))$. Considering the two areas $\mu(x)$ and $\mu(y)$, it holds that

$$\mu(x) = \frac{\alpha}{2}r^2 \quad \text{and} \quad \mu(y) = h \cdot \frac{\ell}{2} = \cos\left(\frac{\alpha}{2}\right)r \cdot \sin\left(\frac{\alpha}{2}\right)r = \frac{\sin(\alpha)}{2}r^2. \tag{6.10}$$

For the central angle α we know $\cos(\alpha/2) = h/r = \tau/2$ and therefore $\alpha = 2\arccos\left(\frac{\tau}{2}\right)$. Together with Equation (6.10), we obtain

$$\begin{aligned}\mu(R(u \cap v)) &= 2(\mu(x) - \mu(y)) \\ &= 2\left(\frac{2\arccos\left(\frac{\tau}{2}\right)}{2}r^2 - \frac{\sin\left(2\arccos\left(\frac{\tau}{2}\right)\right)}{2}r^2\right).\end{aligned} \tag{6.11}$$



(a) Let α be the central angle determined by the intersection points i and j, and let x be the corresponding circular sector (illustrated in yellow).

(b) Let y be a triangle in the intersection (illustrated in green) determined by the radical axis ℓ and the central angle α, cf. Figure 6.3 (a).

(c) The height h divides the area $\mu(y)$ (illustrated in green) of the triangle y, cf. Figure 6.3 (b), into two subareas of equal size, since adjacent and opposite legs have the same length r.

Figure 6.3: The neighborhood of two adjacent vertices u and v in a random geometric graph. See the proof of Lemma 6.15 for more details.

The area of a general circle is equal to πr^2. Since we consider a ground space with total area 1, the area of one disk in the random geometric graph equals $\frac{\overline{\deg}}{n-1}$, i.e., $r^2 = \frac{\overline{\deg}}{(n-1)\pi}$. Together with Equation (6.11), we obtain Equation (6.8).

Equation (6.9): We get the claimed equality by noting that

$$\mu(R(u \cap v)) + \mu(R(u \setminus v)) = \pi r^2. \qquad \blacksquare$$

▶ **Lemma 6.16.** Let $G \sim \mathcal{G}(n,r)$ be a random geometric graph, and let $\{u,v\} \in E$ be an edge. Then $\Pr[\mu(R(u \cap v)) \geq \mu(R(u \setminus v))] \geq \left(\frac{4}{5}\right)^2$. ◀

Proof. Let $\tau = \frac{\operatorname{dist}(u,v)}{r}$. By Lemma 6.15 with $\mu(R(u \cap v)) \geq \mu(R(v \setminus u))$, we get

$$\left(2\arccos\left(\frac{\tau}{2}\right) - \sin\left(2\arccos\left(\frac{\tau}{2}\right)\right)\right) \geq \frac{\pi}{2},$$

which is true for $\tau \geq \frac{4}{5}$. The area of a disk of radius $\frac{4}{5}r$ is $\left(\pi(\frac{4}{5}r)^2\right)/\left(\pi r^2\right) = \left(\frac{4}{5}\right)^2$ times the area of a disk of radius r. Hence, the fraction of edges with distance at most $\frac{4}{5}r$ is at least $\left(\frac{4}{5}\right)^2$, concluding the proof. $\blacksquare$

6.3.5 Proof of Theorem 6.6

By Theorem 6.7, the probability that a random edge $\{u, v\}$ is monochrome is at least $\frac{1}{2} + \Pr[D]/2$, where D is the event that the common neighborhood of u and v is more decisive than each exclusive neighborhood. It remains to bound $\Pr[D]$.

Existence of an edge yields a large shared region. Let R be the event that $\mu(R(u \cap v)) \geq \mu(R(u \setminus v))$. Note that this also implies $\mu(R(u \cap v)) \geq \mu(R(v \setminus u))$ as $\mu(R(u \setminus v)) = \mu(R(v \setminus u))$. Due to the law of total probability, we have

$$\Pr[D] \geq \Pr[R] \cdot \Pr[D \mid R].$$

Due to Lemma 6.16, we have $\Pr[R] \geq \left(\frac{4}{5}\right)^2$. Recall that the area of one disk in the random geometric graph equals $\frac{\overline{\deg}}{n-1}$, where $\overline{\deg}$ is the expected average degree. By conditioning on R in the following, since $\mu(R(u \cap v)) + \mu(R(u \setminus v)) = \frac{\overline{\deg}}{n-1}$, it holds that $\mu(R(u \cap v)) \geq \frac{\overline{\deg}}{2(n-1)} \geq \mu(R(u \setminus v)) = \mu(R(v \setminus u))$.

Neighborhood sizes are roughly binomially distributed. The next step is to go from the size of the regions to the number of vertices in these regions. Each of the remaining $n' = n - 2$ vertices is sampled independently to lie in one of the regions $R(u \cap v)$, $R(u \setminus v)$, $R(v \setminus u)$, or $R(\overline{u \cup v})$. Denote the resulting numbers of vertices with X_1, X_2, X_3, and X_4, respectively. Then (X_1, X_2, X_3, X_4) follows a multinomial distribution with parameter $\boldsymbol{p} = (p, q, q, 1 - p - 2q)$ for $p = \mu(R(u \cap v))$ and $q = \mu(R(u \setminus v)) = \mu(R(v \setminus u))$. Note that $N(u \cap v) = X_1$, $N(u \setminus v) = X_2 + 1$, and $N(v \setminus u) = X_3 + 1$ holds for the sizes of the common and exclusive neighborhoods, where the $+1$ comes from the fact that v is always a neighbor of u and vice versa.

We apply Lemma 6.12 to obtain independent binomially distributed random variables Y_1, Y_2, and Y_3 that are likely to coincide with $X_1 = N(u \cap v)$, $X_2 = N(u \setminus v) - 1$, and $X_3 = N(v \setminus u) - 1$, respectively. Let B denote the event that $(N(u \cap v), N(u \setminus v) - 1, N(v \setminus u) - 1) = (Y_1, Y_2, Y_3)$. Again, using the law of total probabilities and since R and B are independent, we get

$$\Pr[D \mid R] \geq \Pr[B \mid R] \cdot \Pr[D \mid R \cap B] = \Pr[B] \cdot \Pr[D \mid R \cap B].$$

Note that $p, q \leq \frac{\overline{\deg}}{n'}$ for the expected average degree $\overline{\deg}$. Thus, Lemma 6.12 implies that $\Pr[B] \geq \left(1 - 3\overline{\deg}^2/n'\right)$. Conditioning on B makes it correct to assume that $N(u \cap v) \sim \mathrm{Bin}(n', p)$, $(N(u \setminus v) - 1) \sim \mathrm{Bin}(n', q)$, $(N(v \setminus u) - 1) \sim \mathrm{Bin}(n', q)$ are independently distributed. Additionally conditioning on R gives us $p \geq \frac{\overline{\deg}}{2n'} \geq q$.

A large shared region yields a large shared neighborhood. In the next step, we consider an event that makes sure that the number $N(u \cap v)$ of vertices in the shared neighborhood is sufficiently large. Let N_1, N_2, and N_3 be the events that $N(u \cap v) \geq N(u \setminus v)$, $N(u \cap v) \geq N(v \setminus u)$, and $N(u \cap v) > 1$, respectively. Let N be the intersection of N_1, N_2, and N_3. We obtain

$$\begin{aligned} &\Pr[D \mid R \cap B] \\ &\geq \Pr[N \mid R \cap B] \cdot \Pr[D \mid R \cap B \cap N] \\ &\geq \Pr[N_1 \mid R \cap B] \cdot \Pr[N_2 \mid R \cap B] \cdot \Pr[N_3 \mid R \cap B] \cdot \Pr[D \mid R \cap B \cap N], \end{aligned}$$

where the last step follows from Lemma 6.5 as the inequalities in N_1, N_2, and N_3 all go in the same direction. Note that $N(u \cap v) \geq N(u \setminus v)$ is equivalent to $N(u \cap v) > N(u \setminus v) - 1$. Due to the condition on B, $N(u \cap v)$ and $N(u \setminus v) - 1$ are independent random variables following $\mathrm{Bin}(n', p)$ and $\mathrm{Bin}(n', q)$, respectively, with $p \geq q$ due to the condition on R. Thus, we can apply Lemma 6.13, to obtain

$$\Pr[N_1 \mid R \cap B] = \Pr[N_2 \mid R \cap B] \geq \left(\frac{1}{2} - \frac{1}{\sqrt{2\pi\lfloor\overline{\deg}/2\rfloor}}\right)(1 - o(1)),$$

and Lemma 6.14 gives the bound

$$\Pr[N_3 \mid R \cap B] \geq \left(1 - e^{-\overline{\deg}/2}\left(1 + \frac{\overline{\deg}}{2}\right)\right)(1 - o(1)).$$

Note that both of these probabilities are bounded away from 0 for $\overline{\deg} \geq 2$. Conditioning on N lets us assume that the shared neighborhood of u and v contains at least two vertices and that it is at least as big as each of the exclusive neighborhoods.

A large shared neighborhood yields high decisiveness. The last step is to actually bound the remaining probability $\Pr[D \mid R \cap B \cap N]$. Note that, once we know the number of vertices in the shared and exclusive neighborhoods, the decisiveness no longer depends on R or B, i.e., we can bound $\Pr[D \mid N]$ instead. For this, let D_1 and D_2 be the events that $D(u \cap v) > D(u \setminus v)$ and $D(u \cap v) > D(v \setminus u)$, respectively. Note that D is their intersection. Moreover, due to Lemma 6.5, we have $\Pr[D \mid N] \geq \Pr[D_1 \mid N] \cdot \Pr[D_2 \mid N]$. To bound $\Pr[D_1 \mid N] = \Pr[D_2 \mid N]$, we use Theorem 6.11. Note that the b and a in Theorem 6.11 correspond to $N(u \cap v)$ and $N(u \setminus v) + 1$ (the +1 coming from the fact that $N(u \setminus v)$ does not count the vertex v). Moreover, conditioning on N implies that $a \leq b$ and $b > 1$. Thus, Theorem 6.11 implies $\Pr[D_1 \mid N] \geq \frac{3}{16}$.

Conclusion. The above arguments give us that the fraction of monochrome edges is

$$\begin{aligned} &\frac{1}{2} + \frac{\Pr[D]}{2} \\ \geq\ &\frac{1}{2} + \frac{1}{2} \cdot \underbrace{\Pr[R]}_{\geq (\frac{4}{5})^2} \cdot \underbrace{\Pr[B]}_{1-o(1)} \cdot \Big(\underbrace{\Pr[N_1 \mid R \cap B]}_{\geq \frac{1}{2} - \frac{1}{\sqrt{2\pi \lfloor \overline{\deg}/2 \rfloor}}}\Big)^2 \cdot \underbrace{\Pr[N_3 \mid R \cap B]}_{\geq 1 - e^{-\overline{\deg}/2}\left(1 + \frac{\overline{\deg}}{2}\right)} \cdot \Big(\underbrace{\Pr[D_1 \mid N]}_{\geq \frac{3}{16}}\Big)^2, \end{aligned}$$

where we omitted the $o(1)$ terms for $\Pr[N_1 \mid R \cap B]$ and $\Pr[N_3 \mid R \cap B]$, as they are already covered by the $1 - o(1)$ coming from $\Pr[B]$. This yields the bound stated in Theorem 6.6:

$$\frac{1}{2} + \frac{9}{800} \cdot \left(\frac{1}{2} - \frac{1}{\sqrt{2\pi \lfloor \overline{\deg}/2 \rfloor}}\right)^2 \cdot \left(1 - e^{-\overline{\deg}/2}\left(1 + \frac{\overline{\deg}}{2}\right)\right) \cdot (1 - o(1)).$$

6.4 Monochrome Edges in Erdős–Rényi Graphs

In the following, we are interested in the probability that an edge $\{u, v\}$ is monochrome after the FSP on Erdős–Rényi graphs. In contrast to geometric random graphs, we prove an upper bound. To this end, we show that it is likely that the common neighborhood is empty, and therefore u and v choose their types to be the predominant type in their exclusive neighborhood, which is orange and blue with probability $\frac{1}{2}$, each.

▶ **Theorem 6.17.** Let $G \sim \mathcal{G}(n,p)$ be an Erdős–Rényi graph with expected average degree $\overline{\deg} = \mathrm{o}(\sqrt{n})$. The expected fraction of monochrome edges after the FSP is at most $\frac{1}{2} + \mathrm{o}(1)$. ◀

Proof. Given an edge $\{u,v\}$, let M be the event that $\{u,v\}$ is monochrome. We first split M into disjoint sets concerning the size of the common neighborhood and apply the law of total probability and get $\Pr[M] =$

$$\begin{aligned}
&\Pr[M \mid N(u \cap v) = 0] \cdot \Pr[N(u \cap v) = 0] + \\
&\Pr[M \mid N(u \cap v) > 0] \cdot \Pr[N(u \cap v) > 0] \\
\leq\ &\Pr[M \mid N(u \cap v) = 0] \cdot 1 + 1 \cdot \Pr[N(u \cap v) > 0].
\end{aligned}$$

We bound each of the summands separately. For estimating $\Pr[M \mid N(u \cap v) = 0]$, we note that the types of u and v are determined by the predominant type in disjoint vertex sets. By definition of the FSP this implies that the probability of a monochrome edge is equal to $\frac{1}{2}$.

We are left with bounding $\Pr[N(u \cap v) > 0]$. Let $n' = n - 2$ be the number of the remaining vertices. Note that $N(u \cap v) \sim \mathrm{Bin}(n', p^2)$. Thus, by Bernoulli's inequality we get

$$\Pr[N(u \cap v) > 0] = 1 - \Pr[N(u \cap v) = 0] = 1 - \left(1 - p^2\right)^{n'} \leq n'p^2.$$

Noting that $n'p^2 = \mathrm{o}(1)$ holds due to our assumption on $\overline{\deg}$, concludes the proof. ∎

6.5 Empirical Comparison for More Iterations

Our theoretical analyses in the previous sections focused on the segregation strength after the *first* iteration. In this section, we complement these results with empirical results for *multiple* iterations. That is, agents make their decision whether to change their color several times, based on the state after the previous iteration.

In Section 6.5.1, we analyze how much the fraction of monochrome edges changes in each iteration. On the one hand, for random geometric graphs, we observe that the fraction of monochrome edges converges to a value larger than $1/2$, with the first iteration contributing considerably to this change. On the other hand, for Erdős–Rényi graphs, the fraction of monochrome edges first stays close to $1/2$ before reaching 1, depending on the average degree and the number of iterations.

The behavior of Erdős–Rényi graphs reaching fully monochrome edge sets leads to the question of how evenly the two colors are distributed among the agents, which we consider in Section 6.5.2. We find that the average degree of Erdős–Rényi graphs plays an important role in whether the two colors are roughly equally distributed or whether one color takes over the entire graph. In contrast, for random geometric graphs, the two colors are equally distributed over multiple iterations. This shows that random geometric graphs evince a more stable behavior while Erdős–Rényi graphs show a more degenerated one.

Last, based on the observations of the behavior of Erdős–Rényi graphs, we investigate in Section 6.5.3 if and at which average degree the FSP on random geometric graphs results in a single color taking over all agents. We find that this is the case for some average degree in $\Theta(\sqrt{n})$, suggesting that our regime for the average degree of $o(\sqrt{n})$ in Theorem 6.6 is close to tight.

In the following, we explain our experimental setup and then go into detail about the observations mentioned above.

Experimental Setup We consider random geometric and Erdős–Rényi graphs. Recall that we use for the random geometric graphs a two-dimensional toroidal Euclidean space as the ground space. We note that we ran our experiments, in addition to what we present here, also on the (non-toroidal) unit square as ground space but could not notice any qualitative difference in our observations. For the Erdős–Rényi graphs, we used the $G(n, p)$ model. We consider graph sizes from 5 000 up to 25 000 vertices, expected average degrees between 2 and 32 as well as $0.5\sqrt{n}$ and $3.5\sqrt{n}$, respectively. Moreover, we consider up to 200 iterations and run our experiments 1 000 times to measure the fraction of monochrome edges, the fraction of vertices changing their color, and the fraction of vertices belonging to the minority. For reproducibility purposes, our code is publicly available on GitHub [Blä+21].

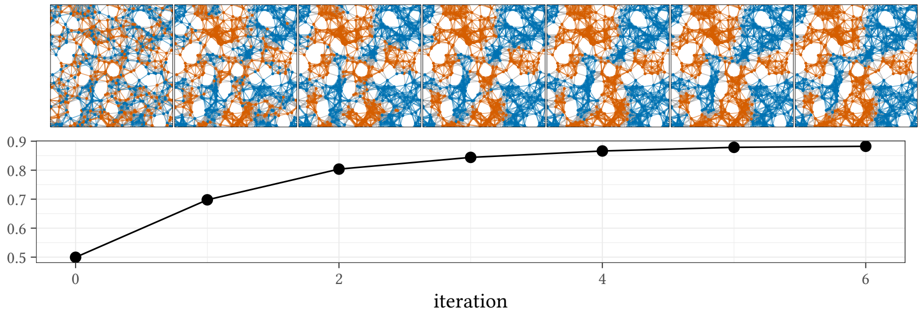


Figure 6.4: The fraction of monochrome edges for the first six iterations of the FSP on a random geometric graph with 500 vertices and average degree 16 on the torus. The top part of the figure depicts the state of the FSP after each iteration. The blue and orange edges are monochrome edges between two adjacent blue and orange agents, respectively, while a gray edge depicts an edge between an orange and blue agent. Please refer to Section 6.5.1 for more details.

6.5.1 Changes to the Colors of Agents

We are interested in how often agents change their color. To this end, we look at only the number of monochrome edges as well as the number of agents that change color.

Changes to the Fraction of Monochrome Edges

Figure 6.4 shows exemplarily the first six iterations of the FSP for a random geometric graph. As seen in Figure 6.5, we observe that in random geometric graphs, the fraction of monochrome edges increases with every iteration. However, while in the first iterations the fraction of monochrome edges is strongly rising, in particular, the strongest increase happens in the first iteration, it stabilizes quickly, and, from then on, only small changes are visible. Hence, this shows that the first iteration plays a large role since we see a clear difference in the fraction of monochrome edges which is not the case after 30 iterations, where only very small changes can be observed. Moreover, note that Figure 6.5 shows only a very low variance so the overall behavior does not depend on the specific graph.

Turning to Erdős–Rényi graphs in the first iterations the process acts expectedly: approximately half of the edges are monochrome, cf. Figure 6.5. However,

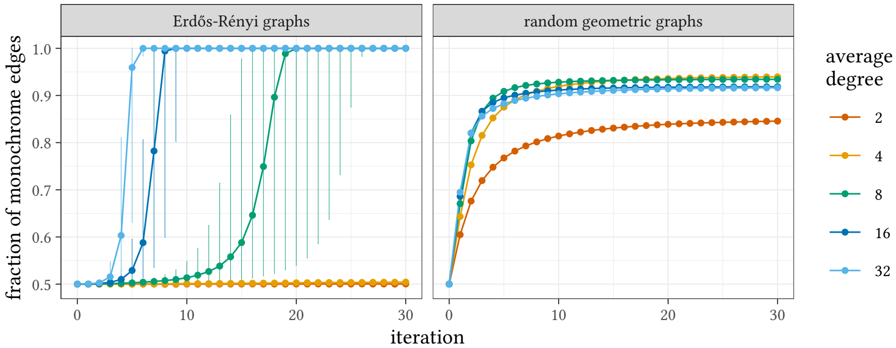


Figure 6.5: The fraction of monochrome edges over the first 30 iterations of the FSP on Erdős–Rényi graphs and random geometric graphs with 25 000 vertices for different average degrees. Each point denotes the mean of 1 000 runs. The lines around each point depict the standard deviation. In general, the segregation strength increases with the number of iterations. Please refer to Section 6.5.1 for more details.

there is a turning point from which the number of monochrome edges increases until (almost) all edges are monochrome. This is a surprising behavior since the FSP behaves differently in the subsequent iterations compared to the first ones. In Section 6.5.2, we see that this is due to one color taking over the entire graph. The turning point where the graph becomes monocolored depends on the specific graph, which leads to a high variance in the plot. Furthermore, the plot suggests that the turning point appears earlier for higher average degrees.

Number of Agents Changing Color

For random geometric graphs, Figure 6.6 shows that for small average degrees, a substantial fraction of the agents keeps on changing their color although Figure 6.5 indicates convergence in the number of monochrome edges. For higher average degrees only a very small number of agents changes their color after 30 iterations, which suggests almost stable states. Thus, while the number of monochrome edges seems to always converge, the convergence of the FSP itself concerning the colors of the agents is more dependent on the average degree. In particular, this shows that a big part of the graph is stable while there are areas in which the agents switch between strongly segregated configurations. We note that such oscillating behavior has been observed before in the literature. This

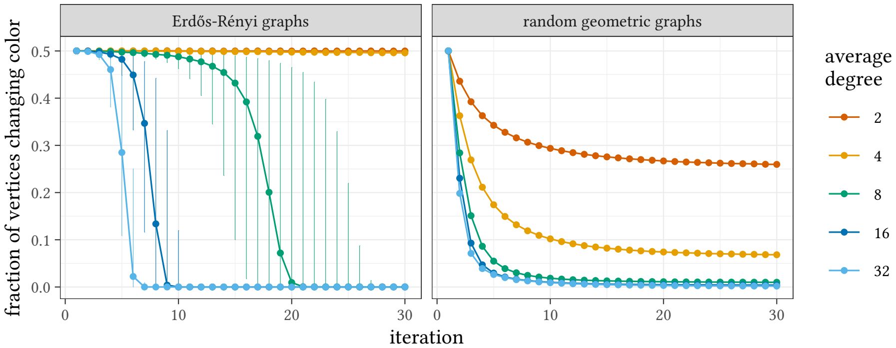


Figure 6.6: The fraction of vertices changing their color over the first 30 iterations of the FSP on Erdős–Rényi graphs and random geometric graphs with 25 000 vertices for different average degrees. Each point denotes the mean of 1 000 runs. The lines around each point depict the standard deviation. In general, except for very small average degrees, the process reaches a stable state. Please refer to Section 6.5.1 for more details.

happens heavily in regular structures commonly used for modeling residential areas, like grid graphs, regular graphs, paths, cycles, and trees. In contrast, random geometric graphs exhibit irregularities, which leads to stronger local minima concerning the number of monochrome edges and, hence, to a more stable behavior. This effect is not as strong for low expected average degrees as it is for large ones. We believe this to be an indicator of the benefit of using random geometric graphs instead of completely random structures as underlying topology.

6.5.2 The Size of the Minority

We consider the number of agents of the color that has fewer agents (the *minority*), shedding light on whether the FSP results in a graph that consists of agents of only a single color.

In Figure 6.7, we see that for random geometric graphs, the fraction of the minority is very close to $1/2$ and stays there over many iterations. Thus, both colors contribute roughly equally to the number of monochrome edges. However, for Erdős–Rényi graphs, the behavior is quite different. While the fraction of the minority stays close to $1/2$ for low average degrees (at least for the first 30 iterations), it goes to 0 for higher average degrees, and it does so more quickly the

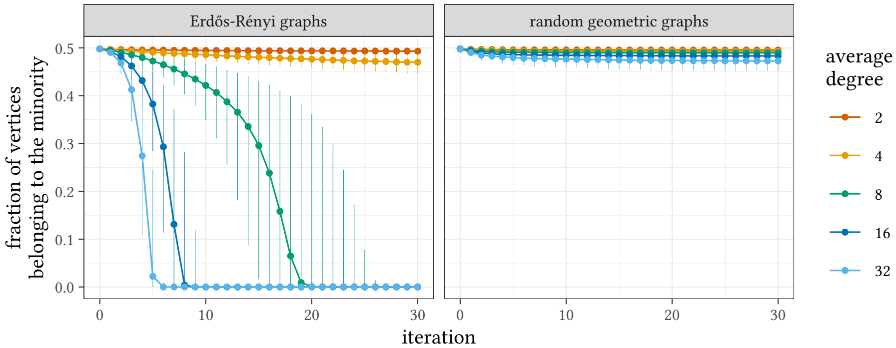


Figure 6.7: The fraction of vertices belonging to the minority over the first 30 iterations of the FSP on Erdős–Rényi graphs and random geometric graphs with 25 000 vertices for different average degrees. Each point is based on 1 000 runs. The lines around each point depict the standard deviation. In general, Erdős–Rényi graphs end up single-colored while random geometric graphs stay bi-colored. Please refer to Section 6.5.2 for more details.

higher the average degree. Note that we see in Figure 6.7 that also for low average degrees the fraction of the minority starts to move away from $1/2$ towards 0. The high variance indicates that the graph structure has some impact on when this change takes place, but all agents eventually have the same color for higher average degrees. Hence, although the probability of each color remains $1/2$ for each vertex, there are dependencies and the FSP has a reinforcing effect on an already slight imbalance. This also explains the increase of the fraction of monochrome edges, as discussed in Section 6.5.1, and the convergence of agents changing color, as discussed in Section 6.5.1.

6.5.3 Degeneracies in Random Geometric Graphs for Higher Average Degrees

The behavior of the Erdős–Rényi graphs discussed in Section 6.5.2 raises the question for random geometric graphs if and, if so, at which average degree the FSP ends in a graph where all agents have the same color.

Figure 6.8 depicts the fractions of FSPs that resulted in all agents having the same color after 200 iterations concerning the average degree, for multiple graph sizes. We see that increasing the average degree leads to a drastically

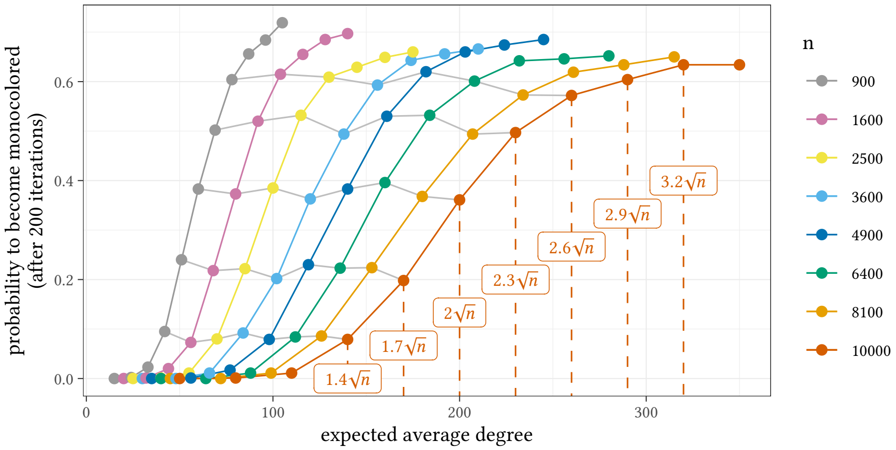


Figure 6.8: The probability that one color takes completely over after 200 iterations in the FSP on a random geometric graph depending on the average degree for different numbers of vertices n. For each value of n, the average degrees range from $0.5\sqrt{n}$ to $3.5\sqrt{n}$ in steps of $0.3\sqrt{n}$. Each point is based on 1 000 runs. In general, the higher the expected average degree the more likely the FSP ends up in a single-colored graph. Please refer to Section 6.5.3 for more details.

increased probability of the FSP converging to a single color of agents, although its probability seems to be a constant bounded away from 1. For all graph sizes considered, the transition from a probability of almost 0 to a positive probability happens for average degrees of $\Theta(\sqrt{n})$. This is in line with our main theoretical result, Theorem 6.6, which states that the FSP on random geometric graphs, after the first iteration, has a fraction of monochrome edges that is higher than $1/2$ by a constant as long as the average degree is in $o(\sqrt{n})$, suggesting that the behavior of the FSP is rather different for higher average degrees. Hence, both our theoretical result as well as our empirical studies indicate that something changes decisively for average degrees of $\Theta(\sqrt{n})$. This calls for a theoretical investigation of this threshold behavior. Moreover, we suspect that there is another threshold where the probability of becoming monochrome switches from a constant bounded away from 1 to 1.

6.6 Conclusion and Open Problems

We introduced the Flip Schelling Process (FSP), a version of Schelling's segregation model where agents choose their type based on the majority in their neighborhood. We analyzed it theoretically for a single iteration and empirically for multiple iterations. This leaves the theoretical analysis of multiple iterations open. Note that our empirical analysis shows that one should expect oscillating behavior in the FSP for low average degrees, cf. Figure 6.6. Thus, beyond studying the number of monochrome edges in an equilibrium, one additionally has to understand this oscillating behavior, e.g., by showing that there is an average degree beyond which the FSP reaches a stable state.

In this chapter, we assumed that agents choose their type based on their neighborhood, regardless of their own type. However, a natural behavior of the agents is that the type of the considered agent itself affects the agent's choice. Preliminary experiments show that the behavior of the FSP is different if we do not break ties fairly, i.e., if exactly half of the agents in the neighborhood have a different type, they choose each type with probability $\frac{1}{2}$, but agents are lazy and keep their type instead. This tie-breaking rule increases the likelihood that agents have monochrome edges since each agent influences their neighbors with their own type, which they keep, instead of choosing a random type for the next iteration. This introduces an imbalance of colors concerning an agent's own type in case of a draw in the neighborhood. Hence, we observe higher fractions of monochrome edges after the FSP in both, random geometric and Erdős–Rényi graphs. The smaller the average degree, the greater the impact of this effect seems to be, as this increases the likelihood of ties in a neighborhood.

Last, our results are based on the assumption that the type of each agent is chosen independently and uniformly at random. Hence, roughly half of the agents are orange and the other half is blue. It remains open to investigate a more general model where agents are orange with an arbitrary probability p_o and blue with probability $p_b = 1 - p_o$. Since we saw in our empirical results that the FSP has a reinforcing effect on even slight imbalances, we conjecture that for Erdős–Rényi graphs, already in the first iterations, the number of monochrome edges increases until one color takes over completely. For random geometric graphs, we conjecture that if the average degree is low enough and if p_o is constant, the fraction of orange vertices remains roughly around its initial value.

7 Conclusions & Outlook

Although Schelling introduced his model in the 1970s, and is now a landmark model to investigate residential segregation, the fact that agents strategically change their positions has not been taken into account so far. Hence, we proposed different variants of a game-theoretic version of Schelling's segregation model and conducted a thorough analysis of core game-theoretic questions.

Regarding the convergence to and existence of equilibria, we have observed that the swap version behaves radically different compared to the jump version, which converges to an equilibrium much less often. Furthermore, contrasting behavior is also found within the different variants considered. A factor not to be neglected is the underlying graph topology, modeling the residential area, where we can show, among other things, the existence and convergence for regular graphs when agents are oriented towards integration, i.e., $\tau \leq \frac{1}{2}$ and $\Lambda \leq \frac{1}{2}$, respectively. In general, it remains open whether the existence of equilibria is guaranteed if the minimum degree of the underlying graph is equal to 2. Moreover, we do not know whether there is a specific property that guarantees the existence of equilibria in general. Since our study has largely focused on the swap variant, a detailed analysis focusing on the jump variant is still open.

As a second aspect, we have examined the quality of equilibria. Here, we could also show for the swap version that the underlying graph's structure has a strong influence. In general, for $\tau = 1$ the Price of Anarchy concerning social welfare can be unbounded, but we were able to show a constant bound for grids, which are a popular topology for modeling, for instance, American cities. Also for the quality of equilibria with respect to the Degree of Integration, we proved tight bounds. In particular, we obtained tight bounds for the Price of Stability for the single-peaked version. Again, our focus was on the swap variants, so the exact influence of the topology on the stable feasible strategy profiles of the jump game remains open.

Investigating the Price of Anarchy and Price of Stability with respect to the Degree of Integration is a first step towards studying the segregation strength of equilibria. However, Weinberg and Steinmetz [WS02] propose five dimensions of segregation, which they identified as evenness, exposure, clustering, concentration, and centralization. While the Degree of Integration covers the dimension exposure, an examination of equilibria in general concerning the other dimensions or further segregation measures remains completely open.

As a third point, we studied the computational hardness, in particular, the influence of the underlying graph, and provided hardness results for computing social optimal strategy profiles as well as swap equilibria concerning the Degree of Integration as well as concerning the number of content agents. This is a first step towards the core question in this scope, the hardness of finding integrated equilibria and the least segregated equilibria, respectively. That is, to prove (or disprove) that finding integrated equilibria is NP-hard for many variants of Schelling Games. So far, not much is known about the hardness of computing equilibria. Most of the equilibria look quite segregated. However, for example, for two types of agents, the checkerboard pattern on a toroidal grid with Moore neighborhood, i.e., the 8 neighborhood, with $\tau \leq \frac{1}{2}$, is a stable feasible strategy profile. Hence, in the cases where we know that stable integrated feasible strategy profiles exist, the question of whether these states are reachable via improving response dynamics is of interest.

As the last point, we focused on segregation itself and showed that geometry has a significant influence on segregation strength. In particular, a community structure seems to be of importance. Here, it would be interesting to precisely map the border, i.e., how pronounced the cluster structure must be. Another interesting direction is to prove results on the obtained segregation strength when starting from a given initial feasible strategy profile with certain dynamics. Ideally, one wants to show that starting with a uniform random initial feasible strategy profile, the segregation strength of all reachable equilibria is (much) higher. Furthermore, our investigated utility functions model different variants where agents either actively strive for segregation, passively accept segregation, or actively strive for integration, respectively. The visual impression of the equilibria we saw in simulations suggests that the utility function of the agents naturally has an impact on the segregation strength. A detailed analysis would in any case be of great interest.

As a further conceptual contribution, we took into account that agents are locally bound to their current place of residence or value different places of residence differently. This is an interesting direction that should be expanded. Besides several open questions, for instance, whether local equilibria always exist, many other variants are also conceivable, for example, a local variant with a radius, i.e., agents can still change their strategy only locally, but they are not limited to their adjacent vertices only. Another direction is to consider agents who have an ordering over the vertices.

We had to leave some possible directions in this thesis unnoted and would like to mention a few in the following.

The first direction is to combine Schelling Games with Hedonic Diversity Games [BE20; BEI19], that is, to add a second incentive, besides similar neighborhoods, such as a diverse workplace. In contrast to impose direct diversity constraints [Ben+18], this might as well promote diversity in the neighborhood of agents without a significant welfare loss.

So far, Schelling's model has been studied mainly with uni-dimensional agents. A realistic generalization is to extend the model so that agents are multi-dimensional. In reality, people have multiple attributes such as age, profession, ethnicity, or income which all together contribute to the evaluation by other agents. Liu et al. [Liu+19] raised the question of whether multi-dimensionality can boost stability or reduce segregation in society. It would be interesting to answer these questions also from a theoretical point of view.

Another natural generalization is that all agents have a type value somewhere between 0 and 1. The utility of an agent depends on the values in its neighborhood, e.g., on the difference to the average value of the neighborhood, or the maximum difference to its own value. That is, agents have continuous attributes. This takes into account the fact that, in addition to being segregated because of race, agents are also segregated because of non-categorical attributes such as income. Economists have shown that homeowners strongly prefer living in a neighborhood with a very similar income to their own [LT12]. Kanellopoulos et al. [KKV22] already consider an ordering of the types and introduced a generalized Schelling model where agents are in principle more tolerant towards agents of types that are closer to their own according to the ordering. This can be seen as a special case.

As the last point, we would like to mention the subject of Schelling mechanisms. The goal is to study mechanism design variants of Schelling games. Agents submit preferences concerning their desired type ratio in their neighborhood and their desired favorite location in the graph to the mechanism. The mechanism then computes a feasible strategy profile with favorable properties, e.g. low segregation, that (approximately) respects the preferences.

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List of Publications

Articles in Refereed Journals

[1] Joint work with Davide Bilò, Vittorio Bilò, and Pascal Lenzner. **Topological Influence and Locality in Swap Schelling Games**. *Autonomous Agents Multi Agent Systems* 36:2 (2022), 47. DOI: 10.1007/s10458-022-09573-7.

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[3] Joint work with Maximilian Böther, Leon Schiller, Philipp Fischbeck, Martin S. Krejca, and Tobias Friedrich. **Evolutionary Minimization of Traffic Congestion**. *IEEE Transactions on Evolutionary Computation* (2023). Accepted. DOI: 10.1109/TEVC.2022.3228750.

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