An Approximate Formula for Goldbach's Problem with Applications

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1 Introduction

A significant part of the most important results on the binary (and ternary) Goldbach problems were reached by the so-called circle method, invented about a hundred years ago by Hardy, Littlewood and Ramanujan. The same applies for the analogue equation $p_1 - p_2 = m$ which is often called the Generalized Twin Prime Problem. We will use the method in the form of Vinogradov's trigonometric sums.

In the present work we will briefly discuss three approximations to the binary Goldbach problem for which the detailed proofs will be published elsewhere. We will begin with the estimate of the exceptional set in Goldbach's problem (see (1.11)) which we outline in Section 3. Another approximation, initiated by Linnik is to write every even number as the sum of two primes and a bounded number of powers of two. A joint result with I. Z. Ruzsa [PR1] about a conditional treatment (based on GRH) of the problem appeared in 2003. The unconditional treatment [PR2] is under press. This is sketched in Section 4. Finally, a third application is an improvement of a result of Brüdern, Kawada and Wooley [BKW]. This estimates the size of the set of n's for which $2\Phi(n)$ has no Goldbach representation for an arbitrary fixed polynomial $\Phi \in \mathbb{Z}[x]$ with a positive leading coefficient. This was improved in a joint work with A. Perelli which is still under preparation. The improvement is outlined in the special case of $\Phi(n) = n^2$ in Section 5.

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In the following we will present the notation and the crucial common ideas for these three approximations in Sections 1 and 2 while Sections 3–5 are devoted to some of the special ideas for the three above mentioned applications.

We note that due to the ineffective nature of the approximate formula of Section 2 the results are ineffective in their present form. However, with additional efforts one can obtain a somewhat weaker effective form of it. The author plans to return later to these problems in a future work.

If we plan to investigate primes up to a large number X we introduce the generating function

$$S(\alpha) = \sum_{X_1 < p \le X} \log p e(p^\alpha), \quad e(\alpha) = e^{2\pi i \alpha}, \quad X_1 = X^{1-\varepsilon}, \quad L = \log X \tag{1.1}$$

where p, p', p_i denote always primes, and $\varepsilon > 0$ is an arbitrary, sufficiently small fixed real number.

The Farey arcs will be introduced by the aid of the parameters P, Q satisfying

$$2 \le P \le \sqrt{X} \le Q = X/P < X. \tag{1.2}$$

According to this we define the major and minor arcs as

$$\mathfrak{M} = \bigcup_{q \le P} \bigcup_{\substack{a \\ (a,q)=1}} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right], \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] - \mathfrak{M}. \tag{1.3}$$

If we would like to consider the equations

$$p + p' = m, \ \ p, p' \in (X_1, X], \ \ m \in [X/2, X], \tag{1.4}$$

$$p - p' = m, \ \ p, p' \in (X_1, X], \ \ m \in [X/4, X/2], \tag{1.5}$$

let us denote the weighted numbers of their solutions by

$$R(m) = \sum_{\substack{p+p'=m \\ p,p' \in (X_1, X]}} \log p \cdot \log p', \quad R'(m) = \sum_{\substack{p-p'=m \\ p,p' \in (X_1, X]}} \log p \cdot \log p'. \tag{1.6}$$

These can be expressed as the sum of the contribution of the major arcs and the minor arcs as

$$R(m) = R_1(m) + R_2(m), \qquad R'(m) = R_1'(m) + R_2'(m), \tag{1.7}$$

$$(1.8) \qquad R_1(m) = \int_{\mathfrak{M}} S^2(\alpha)e(-m\alpha)d\alpha, \quad R_2(m) = \int_{\mathfrak{m}} S^2(\alpha)e(-m\alpha)d\alpha,$$

$$(1.9) \qquad R_1'(m) = \int_{\mathfrak{M}} |S^2(\alpha)|e(-m\alpha)d\alpha, \quad R_2'(m) = \int_{\mathfrak{m}} |S^2(\alpha)|e(-m\alpha)d\alpha.$$

In the traditional attack of both Hardy–Littlewood and Vinogradov one chooses the parameter P in such a way that the contribution $R_1(m)$ (or analogously that of $R_1'(m)$) could be evaluated asymptotically. This value depends on our knowledge about the distribution of primes up to X in arithmetic progressions (AP's) modulo $q \le P$. This bound was (roughly) $P = \log^2 X$ before Siegel's theorem was proved (1935) and it increased to $P = (\log X)^A$ with an arbitrary large fixed A after it (Siegel's theorem was used naturally as the corollary now called Siegel–Walfisz theorem).

In addition to this Vinogradov succeeded to develop a new estimate for $S(\alpha)$ on the minor arcs, which we quote here as [Vin].

Lemma 1. *For $\alpha \in \mathfrak{m}$ we have the estimate*

$$(1.10) \qquad S(\alpha) \ll \left(\frac{X}{\sqrt{P}} + X^{4/5} \right) L^4.$$

This was proved later in a simpler form by Vaughan [Vau1].

The above estimate and Siegel's theorem made possible in 1937 for Vinogradov to show his celebrated three-primes theorem [Vin], that every sufficiently large odd number can be written as the sum of three primes. His result was extended for all odd numbers larger than 5 in 2013 by H. A. Helfgott [Hel]. Concerning the binary problem, Vinogradov's method made it possible to show that almost all even numbers are Goldbach, that is, can be written as the sum of two primes. Namely, simultaneously and independently Čudakov [Cud], Van der Corput [VdC] and Estermann [Est] showed the same estimate for the exceptional set $E(X)$:

$$(1.11) \qquad E(X) := \#\{n \le X, 2 \mid n, n \ne p + p'\} \ll_A \frac{X}{(\log X)^A}$$

for any $A > 0$.

This was the best result for 35 years when after an initial improvement of Vaughan [Vau2] Montgomery and Vaughan [MV] reached the breakthrough result

$$(1.12) \qquad E(X) \ll X^{1-c_0}$$

with an unspecified small (but theoretically explicitly calculable) $c_0 > 0$.

One of the crucial idea of them was to choose P much larger, $P = X^{c_1}$, with a small c_1 such that one should have a valid asymptotic for the number of primes in AP's (arithmetic progressions) with the possible exception of one modulus q_1 and its multiples. This q_1 is the possible exceptional modulus of the possibly existing unique exceptional primitive character χ_1 for which the corresponding Dirichlet's L-function has a simple exceptional (real) zero $\varrho_1 = 1 - \delta_1$ (the so-called Siegel zero). Although the existence of such a zero destroys the classical uniform distribution of primes in the AP's whose difference is divisible by q_1 we can still describe the number of primes in these progressions with the aid of a secondary term depending on q_1 and δ_1.

2 The approximate formulae

The main goal of our work is to describe a more general formula (proved in [Pin1]) which enables us to work with a much higher level of P ($P \leq X^{4/9-\varepsilon}$). The cost of it is that the formula will be more complicated and it will depend on several hypothetical "generalized exceptional zeros" instead of one. However, their number can be bounded independently from X (the zeros themselves might change with X growing). This bound will just depend on the required preciseness of the formula (such a phenomenon does not occur in case of the treatment of [MV]).

Afterwards we will mention some problems where these new approximate formulae can lead to better results (usually along with other new ideas). We will give a short overall description of the advantages of this new formula, further in some cases we will give a brief indication how our formula helps in the given problem. If we summarize briefly, we obtain on a significant part of the unit interval (earlier a part of the minor arcs, which will belong after increasing P to the major arcs) instead of a mere estimate on the crucial integral a quite precise evaluation of it, which, however, depends on the unknown generalized exceptional zeros. On the other hand the formulae also show that if these zeros really exist, then their influence on the contribution of the major arcs can not be neglected (by using more clever methods, for example).

Similarly to the explicit formula for the number of primes up to a large number X in AP's our explicit formulae will depend on singularities of the logarithmic derivatives of the primitive Dirichlet L-functions mod $r \leq P$, accordingly it will depend on the zeros of the L-functions. Another similarity is that the main term will be provided by the pole of the zeta-function while secondary terms will be provided by low zeros lying near to the line $\mathrm{Re}\ s = 1$. The effect of the other zeros can be estimated. Our goal will be to obtain an estimate of type $R_1(m) > c_0\mathfrak{S}(m)m$ or $R_1'(m) > c_0\mathfrak{S}(m)m$ (with a small absolute constant $c_0 > 0$), at least apart from some exceptional values of the variable m. (The singular series $\mathfrak{S}(m)$ is defined in (2.4).)

We will choose two parameters H and U whose values will influence the preciseness of the approximate formulae obtained for $R_1(m)$ and $R_1'(m)$, respectively. It will turn out that if we allow for an error of size $\varepsilon\mathfrak{S}(m)X$ in $R_1(m)$ and $R_1'(m)$ then we can choose H and U large constants depending on ε ($H \geq H_0(\varepsilon)$, $U \geq U_0(\varepsilon)$). This means they do not need to grow with $X \to \infty$. We have to consider

(i) the generalized exceptional zeros ϱ_i ($i = 1, 2, \ldots, M$) belonging to L-functions formed with primitive characters χ_i with conductors $r_i \leq P$;

(ii) the pole $\varrho_0 = 1$ of the Riemann zeta function which we consider as a primitive Dirichlet L-function belonging to χ_0 modulo $r_0 = 1$.

We will introduce the notation

$$A(\varrho) = 1 \quad \text{if} \quad \varrho = \varrho_0 = 1, \quad A(\varrho) = -1 \quad \text{if} \quad \varrho \neq 1, \tag{2.1}$$

and the exceptional singularities ϱ_i ($i = 1, 2, \ldots, M$) of the L'/L-functions will be considered if and only if

$$\varrho_i = 1 - \delta_i + i\gamma_i, \quad \delta_i \leq \frac{H}{L}, \quad |\gamma_i| \leq U. \tag{2.2}$$

Their set will be denoted by $\mathcal{E}(H, U)$.

A crucial result is that their number (counted with multiplicity) $M + 1$ satisfies by the so-called log-free density theorems

$$M + 1 \leq Ce^{2H} \quad \text{for} \quad U \leq X \tag{2.3}$$

with an absolute constant $C > 0$. (See e.g. Jutila [Jut].) This means that the number of generalized exceptional zeros appearing in our formula will only depend on ε, not on X.

As well known, the classical asymptotics for $R_1(m)$ and $R_1'(m)$ depend on the singular series

(2.4)

$$\mathfrak{S}(m) = 2C_0 \prod_{\substack{p \mid m \\ p > 2}} \left(1 + \frac{1}{p-2}\right), \quad C_0 = \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.66016\ldots$$

introduced by Hardy and Littlewood. We have to introduce for any pairs of generalized exceptional primitive characters χ_i, χ_j mod r_i, mod r_j the generalized singular series ($[a,b] = \text{l.c.m. } [a,b]$)

$$\mathfrak{S}(\chi_i, \chi_j, m) = \sum_{\substack{q=1 \\ [r_1,r_2]|q}}^{\infty} c(\chi_1, \chi_2, q, m), \tag{2.5}$$

$$c(\chi_1, \chi_2, q, m) = \frac{1}{\varphi^2(q)} c_{\chi_1\chi_2\chi_{0,q}}(-m)\tau(\overline{\chi}_1\chi_{0,q})\tau(\overline{\chi}_2\chi_{0,q}), \tag{2.6}$$

where for a character χ mod q

$$c_\chi(m) = \sum_{h=1}^{q} \chi(h) e\left(\frac{hm}{q}\right), \quad \tau(\chi) = c_\chi(1). \tag{2.7}$$

In case of the Generalized Twin Prime Problem (cf. (1.5) and (1.9)) we have to use the similar quantities

$$\mathfrak{S}'(\chi_1, \chi_2, m) = \sum_{\substack{q=1 \\ [r_1,r_2]|q}}^{\infty} c'(\chi_1, \chi_2, q, m), \tag{2.8}$$

$$c'(\chi_1, \chi_2, q, m) = \frac{1}{\varphi^2(q)} c_{\chi_1\overline{\chi}_2\chi_{0,q}}(-m)\tau(\overline{\chi}_1\chi_{0,q})\overline{\tau(\overline{\chi}_2\chi_{0,q})}. \tag{2.9}$$

We note that in case of $\chi_1 = \chi_2 = \chi_0$ mod 1 we recover in both cases the classical singular series $\mathfrak{S}(m)$ of (2.4) which will appear in form of $\mathfrak{S}(m)m$ as the main term in our formulae. The analogue of m in the quantity for $R_1(m)$ will be played in the weighted new formulae by

$$I(\varrho_1, \varrho_2, m) := \sum_{\substack{m=k+\ell \\ X_1<k,\ell\le X}} k^{\varrho_1^{-1}} \ell^{\varrho_2^{-1}} = \frac{\Gamma(\varrho_1)\Gamma(\varrho_2)}{\Gamma(\varrho_1+\varrho_2)} m^{\varrho_1+\varrho_2-1} + O(X_1) \tag{2.10}$$

and

$$I'(\varrho_1, \varrho_2, m) := \sum_{\substack{m=k-\ell \\ x_1<k,\ell\le X}} k^{\varrho_1^{-1}} \ell^{\overline{\varrho}_2^{-1}} = \frac{\Gamma(\varrho_1)\Gamma(\overline{\varrho}_2)}{\Gamma(\varrho_1+\overline{\varrho}_2)} m^{\varrho_1+\overline{\varrho}_2-1} + O(X_1). \tag{2.11}$$

We remark that the above terms arise as natural consequences of their appearances in the explicit formulae for $\psi(x, \chi, q)$.

A crucial feature of our approximate formulae will be the fact that from the bounded number of terms appearing in our formulae only those have a non-negligible effect for which the quantities

(2.12)
$$U(\chi_1, \chi_2, m) = \max\left(\frac{r_1^2}{(r_1, r_2)^2}, \frac{r_2^2}{(r_1, r_2)^2}, \frac{r_1}{(|m|, r_1)}, \frac{r_2}{(|m|, r_2)}, \text{cond } \chi_1\chi_2\right)$$

and in case of the Generalized Twin Prime Problem

(2.13)
$$U'(\chi_1, \chi_2, m) = \max\left(\frac{r_1^2}{(r_1, r_2)^2}, \frac{r_2^2}{(r_1, r_2)^2}, \frac{r_1}{(|m|, r_1)}, \frac{r_2}{(|m|, r_2)}, \text{cond } \chi_1\overline{\chi}_2\right)$$

respectively, are bounded as a function of ε. This means that both r_1 and r_2 have to be "quasi-divisors" of m and χ_1 can differ from $\overline{\chi}_2$ (or χ_2, respectively) just by a character of bounded conductor, or equivalently

(2.14)
$$r_i \mid C(\varepsilon)m \quad (i = 1, 2), \qquad \text{cond } \chi_1\chi_2 \leq C(\varepsilon) \quad (\text{or cond } \chi_1\overline{\chi}_2 \leq C(\varepsilon), \text{ resp.}).$$

This completely new phenomenon expresses that we can specify the "suspicious" values of m for which we lose the classical asymptotics for the contribution of the major arcs and they are just the "quasi-multiples" of the conductors of the "bad" generalized exceptional characters. Even for these "bad" values of m we have an explicit form of the appearing secondary terms which will enable the estimation of their effects in many cases where the earlier known methods have much more crude estimates. Finally the new asymptotic formulae will be valid with an earlier hopelessly large choice of the parameter P,

$$P \leq X^{4/9-\varepsilon}. \tag{2.15}$$

Although the explicit forms (2.5)–(2.9) for the generalized singular series are quite complicated, their most important property is that

$$\left|\mathfrak{S}(\chi_1, \chi_2, m)\right| \leq \mathfrak{S}(m), \qquad \left|\mathfrak{S}'(\chi_1, \chi_2, m)\right| \leq \mathfrak{S}(m), \tag{2.16}$$

that is, they can be estimated by the classical singular series. (Equality is possible in non-trivial cases, too.) Similarly we have clearly

$$\left|I(\varrho_1, \varrho_2, m)\right| \leq I(m) := \sum_{\substack{m=k+\ell \\ X_1<k,\ell\leq X_1}} 1, \tag{2.17}$$

and

$$\big|I'(\varrho_1,\varrho_2,m)\big| \le I'(m) := \sum_{\substack{m=k-\ell\\ X_1<k,\ell\le X_1}} 1. \tag{2.18}$$

If we denote more generally the set of generalized exceptional singularities (2.2) for $|\gamma| \le \sqrt{X}$, and denote the resulting set by

$$\mathcal{E} = \mathcal{E}\big(H, \sqrt{X}\big) \tag{2.19}$$

we can formulate our result about the approximate formula as

Theorem 1. *Let $\varepsilon > 0$ be an arbitrary, sufficiently small fixed real number. For every $P_0 \le X^{4/9-\varepsilon}$ we can choose a $P \in [P_0X^{-\varepsilon}, P_0]$ with the following properties. We have for all $m \le X$ the explicit formulas*

$$\begin{aligned} R_1(m) = &\sum_{(\varrho_i,\chi_i)\in\mathcal{E}}\ \sum_{(\varrho_j,\chi_j)\in\mathcal{E}} A(\varrho_i)A(\varrho_j)\mathfrak{S}(\chi_i,\chi_j,m)I(\varrho_i,\varrho_j,m)\\ &+ O_\varepsilon(\mathfrak{S}(m)Xe^{-c_\varepsilon H}) + O_\varepsilon(X^{1-\varepsilon}), \end{aligned} \tag{2.20}$$

$$\begin{aligned} R_1'(m) = &\sum_{(\varrho_i,\chi_i)\in\mathcal{E}}\ \sum_{(\varrho_j,\chi_j)\in\mathcal{E}} A(\varrho_i)A(\varrho_j)\mathfrak{S}'(\chi_i,\chi_j,m)I'(\varrho_i,\varrho_j,m)\\ &+ O_\varepsilon(\mathfrak{S}(m)Xe^{-c_\varepsilon H}) + O_\varepsilon(X^{1-\varepsilon}). \end{aligned} \tag{2.21}$$

Suppose additionally $m \in [X/4, X/2]$ in case of (2.21)*. Then, replacing the summation condition in* (2.20)–(2.21) *by*

$$\sum_{\substack{(\varrho_i,\chi_i)\in\mathcal{E}\\ |\gamma_i|\le U_0\\ [r_1,r_2]\le P,}}\ \sum_{\substack{(\varrho_j,\chi_j)\in\mathcal{E}\\ |\gamma_j|\le U_0\\ U(\chi_1,\chi_2,m)\le U_0}} \tag{2.22}$$

(in case of (2.21) *$U(\chi_1,\chi_2,m)$ should be replaced by $U'(\chi_1,\chi_2,m) = U(\chi_1,\overline{\chi}_2,m)$), we obtain* (2.20)–(2.21) *with an additional error term*

$$O(\mathfrak{S}(m)X\log U_0/\sqrt{U_0}).$$

Remark. In the formulae (2.20)–(2.21) multiple zeros are listed with their multiplicity.

The proof of Theorem 1 can be found in [Pin1].

3 The exceptional set in Goldbach's problem

We now turn to applications. In most of the applications the new approximate formulae lead to a significant improvement. However, in order to obtain a stronger improvement one needs further new ideas too. In case of the estimation of the exceptional set $E(X)$ in Goldbach's problem (cf. (1.11)–(1.12)) the contribution of the minor arcs can be estimated in the same way as earlier, by the theorem (1.10) of Vinogradov, using Parseval's identity. That is, we obtain in case of $P \le X^{2/5}$ for the contribution $R_2(m)$ of the minor arcs (see (1.8)) the estimate

(3.1)

$$\sum_m R_2^2(m) = \int_{\mathfrak{m}} |S^4(\alpha)| d\alpha \le \max_{\mathfrak{m}} |S(\alpha)|^2 \int_0^1 |S(\alpha)|^2 d\alpha \ll \frac{X^2}{P} L^4 X L = \frac{X^3 L^5}{P}.$$

This means that, apart from an exceptional set $E_2(X)$, with

$$\big|E_2(X)\big| \ll_\varepsilon \frac{X^{1+3\varepsilon}}{P}, \tag{3.2}$$

we will have for $m \in [x \mid 2, x]$, $2 \mid m$

$$\big|R_2(m)\big| < \frac{X^{1-\varepsilon}}{4} < \frac{m\mathfrak{S}(m)}{2X^\varepsilon}. \tag{3.3}$$

Consequently, it is sufficient to concentrate on the contribution $R_1(m)$ of the major arcs (see (1.8)).

If we have a Siegel zero, that is, a single simple real zero belonging to a real primitive character $\bmod r_1$ with $r_1 \le P$, satisfying

$$\varrho_1 = 1 - \delta_1, \qquad \delta_1 \le \frac{c_1}{\log X} \tag{3.4}$$

(with a sufficiently small absolute constant c_1) then an elaborated argument along with the proof of an improved Deuring–Heilbronn phenomenon can yield (see [Pin1])

$$R_1(m) > X^{1-\varepsilon/2} > \frac{m\mathfrak{S}(m)}{2X^\varepsilon} \tag{3.5}$$

for all $m \in [X/2, X]$, which settles the problem by (3.2)–(3.3).

Remark. The above inequality is stronger than the implicit lower estimate $XP^{-1/3}$ in (8.3) of [MV]. The improvement is due to the fact that our treatment is ineffective since we make use of Siegel's theorem in the proof of the approximate formula in [Pin1].

If we have no Siegel zero in the sense of (3.4) then we can show with another elaborated argument (see [Pin1], [Pin2]) that

$$R_1(m) > \varepsilon m \mathfrak{S}(m) \tag{3.6}$$

holds apart from an exceptional set $E_1(X)$ of size

$$E_1(X) \leq \frac{C'(\varepsilon)X}{P}, \tag{3.7}$$

with a suitably chosen large constant $C'(\varepsilon)$, depending only on ε.

In order to give some details we mention that we will choose in the approximate formula of Theorem 1

$$H = C_1(\varepsilon), \quad U_0 = C_2(\varepsilon) \tag{3.8}$$

with a sufficiently large $C_1(\varepsilon)$ and afterwards with a sufficiently large $C_2(\varepsilon)$ depending on ε and $C_1(\varepsilon)$.

The number of the generalized exceptional singularities, consequently the number of terms in (2.20) and (2.21) will be just $O_\varepsilon(1)$ as mentioned in (2.3), where multiple zeros are counted with their multiplicity, depending just on ε and $C_1(\varepsilon)$. One such term has a non-negligible effect only for the "quasi-multiples" m of r_i and r_j. Let us denote by $\mathcal{R}$ the set of the "bad" conductors r_i of the "bad" characters χ_i and denote by $\mathcal{R}'$ an arbitrary subset of $\mathcal{R}$. Let us fix $\mathcal{R}'$ and let

$$R' = \text{l.c.m. } [r_i; r_i \in \mathcal{R}']. \tag{3.9}$$

Let us consider for a given large constant $C(\varepsilon)$ depending on ε the set

$$\mathcal{R}'(C(\varepsilon), X) = \{m \leq X;\ r_i \mid C(\varepsilon)m \Longleftrightarrow r_i \in \mathcal{R}'\}. \tag{3.10}$$

We first note that the total number of sets of type $\mathcal{R}' \subset \mathcal{R}$ is at most 2^{M+1}, a quantity which is smaller than some function of ε and $H = c_1(\varepsilon)$ (see (2.3) and (3.8)). This means that we can take an arbitrary fixed subset $\mathcal{R}' \subset \mathcal{R}$ and it is sufficient to show that the number of exceptional n's with

$n \in \mathcal{R}'(c(\varepsilon), x)$ is at most $C^*(\varepsilon)\frac{X}{P}$ with some constant $C^*(\varepsilon)$ depending only on ε.

We can distinguish two cases.

Case 1. $R' \geq P$.

Case 2. $R' < P$.

Case 1. We can discard all m values which belong to some set of type $\mathcal{R}'(C(\varepsilon), X)$ for any $\mathcal{R}'$ with $R' \geq P$ since each of them contains at most $C(\varepsilon)\frac{X}{P}$ numbers up to X.

Case 2. In this case we can discard all pairs in (2.20) or (2.21) containing at least one χ_i with $r_i \notin \mathcal{R}'(C(\varepsilon), X)$. All discarded error terms will contribute together in modulus at most ε if $C_1(\varepsilon)$ in (3.8) was first chosen large enough and later $C_2(\varepsilon)$ in (3.8) was chosen large enough depending on ε and $C_1(\varepsilon)$.

Consequently, in case of the absence of a Siegel zero in order to prove $R_1(m) > \varepsilon\mathfrak{S}(m)m$ it is sufficient to show for any $\mathcal{R}' \subset \mathcal{R}$

$$\sum_{\substack{\chi_i(r_i)\\ (\varrho_i,\chi_i)\in\mathcal{E}\\ r_i\in\mathcal{R}'\\ \varrho_i\neq 1}} \sum_{\substack{\chi_j(r_j)\\ (\varrho_j,\chi_j)\in\mathcal{E}\\ r_j\in\mathcal{R}'\\ \text{or } \varrho_j\neq 1\\ \text{cond } \chi_i\chi_j < C(\varepsilon)}} X^{-\delta_1-\delta_2} < 1 - 2\varepsilon, \qquad \delta_i = 1 - \beta_i. \tag{3.11}$$

Let us observe that all primitive generalized exceptional characters χ_i in the above double summation can be substituted by the not necessarily primitive characters $\chi_i' = \chi_i\chi_{0,R'} (\bmod R')$, where $R' < P$. Thus, extending the definition of the generalized exceptional characters in $\mathcal{E}$ to arbitrary characters induced by them and denoting this set by $\mathcal{E}'$ we need to show (3.11) for all characters mod R' where R' is an arbitrary integer with $R' \leq P$. Such a result was proved in [Pin2] for $P = X^{7/25}$ which we quote (using the earlier notation as)

Theorem 2. *For $q \leq X^{7/25}$ we have for $X > X_0(\varepsilon)$ in case of the absence of a Siegel zero*

$$\sum_{\substack{(\chi_i,q,\varrho_i)\in\mathcal{E}'\\ \varrho_i\neq 1}} \sum_{\substack{(\chi_j,q,\varrho_j)\in\mathcal{E}'\\ \text{cond } \chi_i\chi_j < C(\varepsilon)\\ \text{or } \varrho_j\neq 1}} X^{-\delta_1-\delta_2} < 1 - 2\varepsilon. \tag{3.12}$$

Remark. As we see from the above arguments, it is sufficient to show (3.12) for characters of the same modulus q. This represents a huge gain compared to previous treatments. The other huge gain is the condition cond $\chi_i\chi_j < C(\varepsilon)$.

Remark. (3.2) actually implies the same relation with the condition cond $\chi_i\overline{\chi}_j < C(\varepsilon)$ which appears in the case of the Generalized Twin Prime Problem.

Theorem 3. $E(X) \leq \widetilde{C}(\varepsilon)X^{0.72+\varepsilon}$.

Remark. In the proof of Theorem 2 we can choose $P \in \left(X^{7/25+\varepsilon}, X^{7/25+2\varepsilon}\right)$ too, which yields the stronger

Theorem 3'. *$E(X) < X^{0.72}$ for $X > X_0$, effective constant.*

Remark. Due to the separate treatment of the case of the existence of Siegel zeros in [Pin2], Theorems 3 and 3' hold unconditionally. Theorem 3 and Theorem 3' both improve the earlier best result $E(X) \ll X^{0.879}$ of Lu [Lu].

4 The Goldbach–Linnik problem

Linnik considered about 70 years ago the following approximation to the binary Goldbach problem. Is it possible to give a fixed integer K such that every sufficiently large even integer could be written as the sum of two primes and K powers of two? If the Goldbach conjecture is true then $K = 0$ (or any $K > 0$) suits for this purpose.

Linnik succeeded to answer the problem positively. In two papers he proved the assertion, first [Lin1] under the Generalized Riemann Hypothesis (GRH), later unconditionally [Lin2]. However, his constant K was very large and unspecified. After a simplification of the proof by Gallagher [Gal2] the first explicit bounds were given by Liu, Liu and Wang at the end of the last century:

$K = 54\,000$ [LLW2] (unconditionally) and
$K = 770$ [LLW1] (assuming GRH).

After several steps this was improved to
$K = 1906$ [Li1] (unconditionally) and
$K = 160$ [Wan] (assuming GRH).

Parallel to this, I obtained the results $K = 12$ (unconditionally) and $K = 10$ (assuming GRH) improving several parts of the earlier arguments, including a new treatment of the exponential sum

$$G(\alpha) = \sum_{\nu=1}^{L} e(2^\nu \alpha), \qquad L = \log_2 X - \sqrt{\log_2 X} \tag{4.1}$$

where $\log_2 X$ means the logarithm of base 2. These results were announced (without proofs) at an international meeting at Debrecen in 2000.

Somewhat later, in a joint work with I. Z. Ruzsa [PR1] and simultaneously and independently by Heath-Brown and Puchta [HP] the result $K = 7$ was obtained under the assumption of GRH. Unconditionally we announced $K = 8$ (in [PR1]) while in [HP] the bound $K = 13$ was proved. This was improved little later to $K = 12$ by C. Elsholtz (unpublished). A proof of $K = 12$ appeared independently in a work of Z. Liu and G. Liu in 2011 [LL]. The complete proof of $K = 8$ will appear in [PR2].

A new feature of the problem is that in contrast to the problem of the exceptional set we have to distinguish two parts of the minor arcs with some constant $c_1 \in (0,1)$ to be chosen later

$$\begin{aligned} &\text{(i)} \qquad \mathcal{E}^* = \big\{\alpha \in [0,1]; |G(\alpha)| \geq c_1 L\big\}, \\ &\text{(ii)} \;\; C(\mathcal{E}^*) = [0,1] \setminus \mathcal{E}. \end{aligned} \tag{4.2}$$

The method of [PR1] (which is essentially best possible in the sense that the estimates approximately reflect the truth) give for the Lebesgue measure $\mu(\mathcal{E}^*)$ of $\mathcal{E}^*$

$$\mu(\mathcal{E}^*) \ll N^{-3/5} L^{-100} \quad \text{if} \;\; c_1 = 0.789401. \tag{4.3}$$

The method could prove similar results of type

$$\mu(\mathcal{E}^*) \ll X^{-c_3} \quad \text{if} \quad c_1 = c_4 \tag{4.4}$$

with suitable pairs c_3 and c_4. The reason to choose $c_3 = 3/5$ is that if we choose $P \in [X^{0.4}, X^{0.41}]$ then by our approximate formula we can still control the integral

$$\int\limits_{\mathfrak{M}} S^2(\alpha) G^k(\alpha) e(-m\alpha) d\alpha \quad (k \leq 8) \tag{4.5}$$

on the major arcs. On the other hand that part of the minor arcs which lie within $\mathcal{E}^*$ (see (i) of (4.2)) can be neglected since by Vinogradov's estimate

(1.10) and (4.3) we obtain

$$\int_{\mathcal{E}^*} \left|S^2(\alpha)G^k(\alpha)\right| d\alpha \le \mu(\mathcal{E}^*) X^{8/5} L^{k+8} \ll X L^{-(92-k)}. \tag{4.6}$$

If the truth of the (GRH) is assumed then we can choose P quite near to $\sqrt{X}$ ($P = \sqrt{X} L^{-8}$, for example) and we can still evaluate the contribution of the major arcs. If we work unconditionally then the approximate formulae of Theorem 1 enable us to choose P up to $X^{4/9-\varepsilon}$. Since Vinogradov's estimate (1.10) does not improve if P grows over $X^{2/5}$ we are contented to take

$$P = [X^{0.4}, X^{0.41}]. \tag{4.7}$$

(Increasing it up to $X^{4/9-\varepsilon}$ would result in a small gain in the procedure but the gain is not enough to obtain $K = 7$.)

The crucial point which makes possible (in contrast to the estimate of the exceptional set in Goldbach's problem – see Section 3) an asymptotic evaluation of (4.5) is that instead of $R_1'(m)$ in (1.9) we have to evaluate an average of it for the set

$$B(m, L) = \left\{ m - 2^{\nu_1} - \cdots - 2^{\nu_k},\ 1 \le \nu_i \le L \right\}. \tag{4.8}$$

We can now utilise the useful feature of the approximate formulae that they tell us that "suspicious" values of m are those which are the "quasimultiples" of the conductor of at least one generalized exceptional character r_i (see (2.14)). However, this can not hold for a positive proportion of the elements of the set $B(m, L)$ in (4.8). We can even fix $\nu_1, \dots, \nu_{k-1}$ and it is sufficient to let run ν_k.

Our crucial result from [PR2] is the following, which helps to eliminate the effect of the bounded number of exceptional characters.

Lemma 2. *Let $m \le X$ be arbitrary, q be an odd squarefree number. Then for any $\eta > 0$*

$$\sum_{\substack{\nu \le L \\ 2^\nu < m \\ q \mid m - 2^\nu}} \mathfrak{S}(m - 2^\nu) \le \eta L \tag{4.9}$$

if $\min(q, X) > C_0(\eta)$.

The proof of Lemma 2 is non-trivial since the values of the singular series can oscillate between c_5 and $c_6 \log\log X = c_7 \log L$. However, an argument, similar but somewhat more complicated than the proof of Romanov's theorem yields Lemma 2. (See Lemma 1 of [PR2].)

The treatment of the rest of the minor arcs (see (ii) of (4.2)) follows closely the earlier ones (see [PR1] or [Gal2]) and this finally proves

Theorem 4 (J. Pintz – I. Z. Ruzsa)**.** *Every sufficiently large even integer can be written as the sum of two primes and eight powers of* 2.

5 Goldbach numbers in polynomial sequences

In this chapter we would like to investigate the problem whether the Goldbach conjecture is true for almost all elements of thin sequences, for example, of a polynomial sequence

$$2\Phi(n) = p_1 + p_2, \tag{5.1}$$

where $\Phi(x) \in Z[x]$ is a polynomial of degree k with a positive leading coefficient. In the most simple case $\Phi(x) = x$ this is exactly the binary Goldbach problem. In this case the mentioned result (1.11) of Čudakov, Estermann and Van der Corput gives a positive answer.

On the other hand, for any polynomial of degree at least 2, even the sharpest conditional result under GRH,

$$E(X) \ll_\varepsilon X^{1/2+\varepsilon}, \tag{5.2}$$

due to Hardy and Littlewood [HL] (the ε in the exponent was substituted by a power of $\log X$ by Goldston [Gol]) gives directly an estimate, weaker than the trivial one for the problem (5.1).

Consequently, new ideas were needed to show non-trivial estimates in the case $\Phi(x) = x^k$ for the size of the corresponding exceptional set ($\mathcal{P}$ denotes the set of primes)

$$\mathcal{E}_k(N) = \bigl\{n \le N;\ 2n^k \ne p_1 + p_2,\ p_i \in \mathcal{P}\bigr\}, \quad E_k(N) = \bigl|\mathcal{E}_k(N)\bigr|. \tag{5.3}$$

Perelli [Per] gave the first estimate

$$E_k(N) \ll_{k,A} N/(\log N)^A \quad \text{for any} \quad A > 0, \tag{5.4}$$

showing that almost all numbers of the form $2n^k$ are Goldbach (i.e. can be written as a sum of two primes). This represents a generalization of (1.11) It was improved in 2000 to

$$E_k(N) \ll N^{1-c/k} \tag{5.5}$$

with a small unspecified absolute constant $c > 0$, by Brüdern, Kawada and Wooley [BKW]. The stronger estimate (5.5) is a generalization of the theorem (1.12) of Montgomery and Vaughan [MV].

In case of small values of k the estimate (5.5) gives just

$$E_k(N) \ll N^{1-c_k} \tag{5.6}$$

with a small unspecified $c_k > 0$, similarly to (1.5).

Our goal is to reach an improvement of (5.5). The application of the approximate formula does not improve the dependence on k but it helps to improve the value of c. This improvement is especially useful for small values of k when the constant in (5.5)–(5.6) is very small and very hard to calculate since it depends on the prime number theorem of Gallagher [Gal1]. Let us concentrate for the case of $k = 2$, i.e., for the problem how frequently can

$$2n^2 \neq p_1 + p_2, \quad p_i \in \mathcal{P} \tag{5.7}$$

happen. We obtained actually results for all values of k in a joint work with A. Perelli, among which we state here the one for $k = 2$.

Theorem 5 (A. Perelli – J. Pintz)**.** *With the notation* (5.3) *we have*

$$E_2(N) \ll_\varepsilon N^{4/5+\varepsilon} \quad \textit{for any} \ \ \varepsilon > 0. \tag{5.8}$$

We will give a brief sketch of the proof which is somewhat similar to the proof of Theorem 3.

We will use the notations of Section 1. Further, let the Farey arcs defined by P (to be chosen later) and let

$$X = N^2, \qquad f(\alpha) = \sum_{1 \le n \le N} e(2\alpha n^2). \tag{5.9}$$

For the minor arcs we can follow the ingenious treatment of Brüdern, Kawada and Wooley but now for $k = 2$ we will use (instead of Vinogradov's mean value theorem) the well-known simple estimate (let c be a generic positive absolute constant, not necessarily the same at each occurrence)

$$\int_0^1 |f(\alpha)|^4 d\alpha \ll N^2 L^c. \tag{5.10}$$

This yields by (1.10) similarly to the estimate following (9) of [BKW], with the choice $P \in \left[X^{2/5}, X^{2/5+\varepsilon}\right]$,

(5.11)

$$\sum_{m\le N} \left|R_2(2m^2)\right| \le \Big(\sup_{\alpha\in\mathfrak{m}} |S(\alpha)|\Big)^{1/2} \Big(\int_0^1 |S(\alpha)|^2 d\alpha\Big)^{3/4} \Big(\int_0^1 |f(\alpha)|^4 d\alpha\Big)^{1/4}$$
$$\ll X^{2/5} X^{3/4} N^{1/2} L^c = X N^{4/5} L^c.$$

which means that

$$\left|R_2(2m^2)\right| \le X^{1-\varepsilon} \tag{5.12}$$

apart from an exceptional set $\mathcal{E}_{2,2}$ of size

$$\left|\mathcal{E}_{2,2}(N)\right| \ll_\varepsilon N^{4/5+2\varepsilon}. \tag{5.13}$$

The treatment of the major arcs (we can again suppose the absence of Siegel zeros) is similar to that of Section 3 with adequate changes as follows.

Define $\mathcal{R}'$ and R' as in (3.9) and (3.10) with m replaced by m^2. The new dissection of the two cases we define with an additional parameter $P^* < P$ as follows:

Case 1. $R' \ge P^*$.

Case 2. $R' < P^*$.

In Case 1 it is easy to see that the number of (exceptional) squares with

$$R' \mid C(\varepsilon)m^2, \qquad m \le N \tag{5.14}$$

is at most

$$\frac{N}{C'(\varepsilon)\sqrt{R'}} \ll_\varepsilon \frac{N}{\sqrt{P^*}}. \tag{5.15}$$

On the other hand choosing $P^* = X^{7/25}$ and applying Theorem 2 we obtain that in Case 2 we have (as we assumed the absence of Siegel zeros)

$$R_1(m^2) > \varepsilon \mathfrak{S}(m^2) X. \tag{5.16}$$

If Siegel zeros do exist then in Case 2 we can use that the work [Pin1] actually implies without a further exceptional set the inequality (3.5) for all integers m, consequently for all squares too, at least if the crucial parameter P, in our case now P^* satisfies the stronger condition $P^* < X^{16/39-\varepsilon}$ (see (11.39)–(11.40) of [Pin1]).

So we obtain for the size of the exceptional set with respect to major arcs

$$\left|\mathcal{E}_{2,1}(N)\right| \ll_\varepsilon \frac{N}{\sqrt{P^*}} = N^{0.72}. \tag{5.17}$$

From (5.13) and (5.17) we have

$$E_2(N) \le \left|\mathcal{E}_{2,1}(N)\right| + \left|\mathcal{E}_{2,2}(N)\right| \ll_\varepsilon N^{4/5+\varepsilon} \tag{5.18}$$

which proves Theorem 5.

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