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The Invariant Subspace Problem for Non-Archimedean Banach Spaces

Wiesław Śliwa

Abstract. It is proved that every infinite-dimensional non-archimedean Banach space of countable type admits a linear continuous operator without a non-trivial closed invariant subspace. This solves a problem stated by A. C. M. van Rooij and W. H. Schikhof in 1992.

1 Introduction

Let $\mathbb{K}$ be a field with a non-trivial complete non-archimedean valuation $|\cdot|\colon \mathbb{K} \to [0,\infty)$. Every infinite-dimensional (id) Banach space E of countable type over $\mathbb{K}$ is isomorphic to the Banach space $c_0(\mathbb{K})$ of all sequences in $\mathbb{K}$ converging to 0 (with the sup-norm), [8, Theorem 3.16(ii)]. Any closed subspace F of E is $(1+\epsilon)$-complemented in E for every $\varepsilon > 0$, *i.e.,* for every $\varepsilon > 0$ there exists a linear continuous projection P_ε from E onto F with $\|P_\varepsilon\| \leq 1+\varepsilon$ [8, Theorem 3.16(v)]. If the valuation of $\mathbb{K}$ is discrete, then any closed subspace of $c_0(\mathbb{K})$ is 1-complemented [8, Corollaries 2.4, 4.7].

Note that for complex Banach spaces, any closed subspace of a complex Banach space E is complemented in E if and only if E is isomorphic to a complex Hilbert space [3].

Let T be a linear operator on a linear space E. A linear subspace M of E is a non-trivial invariant subspace of T if $\{0\} \neq M \neq E$ and $T(M) \subset M$. One of the most famous problems of the operator theory is the invariant subspace problem for complex Hilbert spaces. It asks whether every linear continuous operator on an infinite-dimensional separable complex Hilbert space has a non-trivial closed invariant subspace. This problem is still open. There exists a vast literature dedicated to the invariant subspace problem for various important classes of complex Banach spaces and linear continuous operators.

P. Enflo [2] and C. J. Read [5–7] negatively solved the invariant subspace problem for complex Banach spaces. Read obtained a linear continuous operator on the complex Banach space l_1 without a non-trivial closed invariant subspace. This l_1-example was simplified by A. M. Davie and can be found in Beauzamy's book [1, Ch. XIV].

Developing this example, we shall construct a linear continuous operator T on some infinite-dimensional Banach space E of countable type over $\mathbb{K}$ with no non-trivial closed invariant subspace. Clearly, E is isomorphic to the Banach space $c_0(\mathbb{K})$. In particular, we solve the problem stated by A. C. M. van Rooij and W. H. Schikhof

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in [9]: if $\mathbb{K}$ is algebraically closed and spherically complete, does every linear continuous operator T on a Banach space E over $\mathbb{K}$ have a non-trivial closed subspace?

Note that the Banach space E of countable type is reflexive if $\mathbb{K}$ is not spherically complete [8, Corollary 4.18]. Meanwhile, for reflexive complex Banach spaces, it is not known if there exists a linear continuous operator without a non-trivial closed invariant subspace on an infinite-dimensional reflexive complex Banach space.

If $\mathbb{K}$ is algebraically closed, then it is easy to see, as in the complex case, that every linear operator T on a finite-dimensional Banach space with $\dim E > 1$ has a non-trivial invariant subspace.

If $\mathbb{K}$ is not algebraically closed (for example if $\mathbb{K} = \mathbb{Q}_p$), we can take an element $a \in (\bar{\mathbb{K}} \setminus \mathbb{K})$, where $\bar{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$. Clearly, $E = \mathbb{K}(a)$ is a finite-dimensional linear space over $\mathbb{K}$ with $\dim E > 1$ and the linear operator $T\colon E \to E, Ty = ay$, has no non-trivial invariant subspace [9].

If $\mathbb{K}$ is not spherically complete (for example if $\mathbb{K} = \mathbb{C}_p$), then (see [8, Theorem 4.49]) there exists a spherically complete valued field $\hat{\mathbb{K}}$ that is an immediate extension of $\mathbb{K}$, *i.e.*, $\mathbb{K}$ is a subfield of $\hat{\mathbb{K}}$ and no non-zero element of $\hat{\mathbb{K}}$ is orthogonal to $\mathbb{K}$. We can take an element $b \in (\hat{\mathbb{K}} \setminus \mathbb{K})$. Clearly, the closed linear span of the set $B = \{b^k : k \geq 0\}$ in $\hat{\mathbb{K}}$ is a Banach algebra E of countable type over $\mathbb{K}$ with $\dim E > 1$ (we consider $\hat{\mathbb{K}}$ as a Banach algebra over $\mathbb{K}$). In fact E is a subfield of $\hat{\mathbb{K}}$: any non-zero element $y \in E$ is not orthogonal to $\mathbb{K}$, so there exists $z \in \mathbb{K}$ with $\|y - z\| < \max\{\|y\|, \|z\|\}$. Then $\|y\| = \|z\|$ and $\|1 - yz^{-1}\| < 1$. Thus $zy^{-1} = \sum_{n=0}^{\infty}(1 - yz^{-1})^n \in E$, so $y^{-1} \in E$. The linear continuous operator $T\colon E \to E, Ty = by$ has no non-trivial closed invariant subspace. Indeed, let M be a closed invariant subspace of T. Then $V = \{y \in E : yM \subset M\}$ is a closed linear subspace of E and $B \subset E$; so $V = E$. Thus M is an ideal of E. Hence $M = \{0\}$ or $M = E$ [9].

2 Preliminaries

Let K be a field. A function $|\cdot|\colon K \to [0, \infty)$ is a *valuation* if:

(i) $\forall \alpha \in K : |\alpha| = 0 \Leftrightarrow \alpha = 0$;
(ii) $\forall \alpha, \beta \in K : |\alpha\beta| = |\alpha||\beta|$;
(iii) $\forall \alpha, \beta \in K : |\alpha + \beta| \leq |\alpha| + |\beta|$.

A valuation $|\cdot|$ on K is

- *non-trivial* if $|\alpha| > 1$ for some $\alpha \in K$;
- *complete* if metric $d\colon K \times K \to [0, \infty), d(x, y) = |x - y|$ is complete on K;
- *archimedean* if the sequence $|1|, |1 + 1|, |1 + 1 + 1|, \ldots$ is unbounded in $[0, \infty)$;
- *non-archimedean* if it is not archimedean;
- *discrete* if the set $|K| = \{|\alpha| : \alpha \in \big(K \setminus \{0\}\big)\}$ is discrete in $(0, \infty)$.

Any field with a complete archimedean valuation is topologically isomorphic to $(\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$ [8, p. 4]. A valuation $|\cdot|$ on K is non-archimedean if and only if $\forall \alpha, \beta \in K : |\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$ [8, Theorem 1.1].

A field K with a non-trivial complete non-archimedean valuation $|\cdot|$ is called non-archimedean. The field $\mathbb{Q}_p$ of p-adic numbers is non-archimedean for any prime number p.

Let $\mathbb{K}$ be a non-archimedean field. We say that $\mathbb{K}$ is *spherically complete* if any decreasing sequence of closed balls in $\mathbb{K}$ has non-empty intersection. An element $y \in \mathbb{K}$ is *orthogonal* to a subfield L if $|y - z| = \max\{|y|, |z|\}$ for any $z \in L$.

By a *norm* on a linear space E over $\mathbb{K}$ we mean a function $\|\cdot\|: E \to [0, \infty)$ such that

(i) $\forall y \in E$: $\|y\| = 0 \Leftrightarrow y = 0$;
(ii) $\forall \alpha \in \mathbb{K}, x \in E : \|\alpha x\| = |\alpha| \|x\|$;
(iii) $\forall x, y \in E : \|x + y\| \leq \max\{\|x\|, \|y\|\}$.

If $\|\cdot\|$ is a norm on a linear space E over $\mathbb{K}$ and $x, y \in E$ with $\|x\| \neq \|y\|$, then $\|x + y\| = \max\{\|x\|, \|y\|\}$.

For fundamentals of non-archimedean normed spaces we refer to [4, 8, 10].

A normed space $E = (E, \|\cdot\|)$ is of *countable type* if E contains a linearly dense countable set. We say that the normed spaces E and F over $\mathbb{K}$ are isomorphic if there is a linear bijective map $T: E \to F$ such that the maps T and T^{-1} are continuous.

A Banach space is a complete normed space. A series $\sum_{n=1}^{\infty} x_n$ in a Banach space is convergent if and only if $\lim x_n = 0$. Every n-dimensional normed space E over $\mathbb{K}$ is isomorphic to the Banach space $\mathbb{K}^n$ (with the sup-norm) and any linear operator on E is continuous.

The closed graph theorem, the open mapping theorem and the Banach–Steinhaus theorem [4, Theorems 2.49, 2.73, 3.37] hold for continuous linear operators between such Banach spaces.

Let $E = (E, \|\cdot\|)$ be a normed space. A sequence $(y_n)_{n=0}^{\infty}$ in E is a *Schauder basis* in E if each $y \in E$ can be written uniquely as $y = \sum_{n=0}^{\infty} \alpha_n y_n$ with $(\alpha_n)_{n=0}^{\infty} \subset \mathbb{K}$ and the coefficient functionals $g_n: E \to \mathbb{K}, y \to \alpha_n (n \in \mathbb{N}_0)$ are continuous; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A linearly dense sequence $(y_n)_{n=0}^{\infty}$ in E is an *orthonormal basis* in E if

$$\Big\| \sum_{i=0}^{n} c_i y_i \Big\| = \max_{0 \leq i \leq n} |c_i|$$

for all $n \in \mathbb{N}$ and $c_1, \ldots, c_n \in \mathbb{K}$.

Any orthonormal basis in E is a Schauder basis in E and every Banach space with an orthonormal basis $(y_n)_{n=0}^{\infty}$ is isometrically isomorphic to $c_0(\mathbb{K})$ [8, p. 171].

We denote by $\operatorname{lin} A$ the linear span of a subset A of a linear space E.

3 Results

Put $d_0 = 2$. Let $\alpha \in \mathbb{K}$ with $|\alpha| \geq 8$ and $(d_n) \subset \mathbb{N}$ with $d_n \geq |\alpha|^{2d_{n-1}}$ for $n \in \mathbb{N}$. It is easy to see that $d_{n-1} \geq 2n$ and $d_n \geq |\alpha|^{2n} d_{n-1} \geq 4n^2 |\alpha| d_{n-1}$ for $n \in \mathbb{N}$. Hence we have $d_n \geq 2^{3n+1}, n \in \mathbb{N}$; so $d_n \geq |\alpha|^{2d_{n-1}} \geq |\alpha|^{2n+2}, n \in \mathbb{N}$. Since the function $f(x) = x^{8/\sqrt{x}}$ is decreasing in the interval (e^2, ∞), we get $d_n^{8/\sqrt{d_n}} \leq |\alpha|^{16(n+1)/|\alpha|^{n+1}} < |\alpha|^{1/(n+1)}$ for $n \in \mathbb{N}$. Thus $d_n^{2n+2} < |\alpha|^{\sqrt{d_n}/4} < d_{n+1}$ for $n \in \mathbb{N}$.

Put $v_0 = 0, a_n = d_{2n-1}, b_n = d_{2n}$ and $v_n = (n-1)(a_n + b_n)$ for $n \in \mathbb{N}$. Then $4n < |\alpha|^{4n} \leq a_n$, $a_n^{4n} < |\alpha|^{\sqrt{a_n}/4} < b_n$, and $b_n^{4n} < |\alpha|^{\sqrt{b_n}/4} < a_{n+1}$ for every

$n \in \mathbb{N}$. Hence we get $8na_n < b_n$, $8nb_n < a_{n+1}$, $4(v_{n-1}+1) < a_n$, $|\alpha|a_n^2 < |\alpha|^{\sqrt{a_n}/4}$, $|\alpha|b_n^2 < |\alpha|^{\sqrt{b_n}/4}$, and $n^2a_n^2 < b_n$ for every $n \in \mathbb{N}$.

For $a, b \in \mathbb{Z}$ we denote the set $\{k \in \mathbb{Z} : a < k \leq b\}$ by $(a, b]$; similarly we define $[a, b)$, $[a, b]$ and (a, b). For nonempty sets $A, B \subset \mathbb{N}$ we write $A < B$ if $1 + \max A = \min B$.

For $n, r \in \mathbb{N}$ with $n > r$ we put:

$$J_{n,r} = ((r-1)a_n + v_{n-r}, ra_n), \quad I_{n,r} = [ra_n, ra_n + v_{n-r-1}],$$
$$L_{n,r} = ((n-1)a_n + (r-1)b_n, r(a_n + b_n)), \quad K_{n,r} = [r(a_n + b_n), (n-1)a_n + rb_n].$$

These sets are non-empty and $J_{n,r} < I_{n,r} < J_{n,r+1}$, $L_{n,r} < K_{n,r} < L_{n,r+1}$ for $n, r \in \mathbb{N}$ with $n > r+1$ and $J_{n,n-1} < I_{n,n-1}$, $L_{n,n-1} < K_{n,n-1}$ for $n \geq 2$. Moreover $X_n := \bigcup_{r=1}^{n-1}(J_{n,r} \cup I_{n,r}) = (v_{n-1}, (n-1)a_n]$, $Y_n := \bigcup_{r=1}^{n-1}(L_{n,r} \cup K_{n,r}) = ((n-1)a_n, v_n]$ for $n \in \mathbb{N}$ with $n \geq 2$; so $X_n < Y_n < X_{n+1}$ for $n \geq 2$ and $\bigcup_{n=2}^{\infty}(X_n \cup Y_n) = \mathbb{N}$.

Let $(\alpha_n), (\beta_n) \subset \mathbb{K}$ with $|\alpha_n| \in (a_n|\alpha|^{-1}, a_n]$ and $|\beta_n| \in (b_n|\alpha|^{-1}, b_n]$ for $n \in \mathbb{N}$. Clearly $1 < |\alpha_n| < |\beta_n| < |\alpha_{n+1}|$ for $n \in \mathbb{N}$.

Let $\langle a \rangle = k$ if $k \leq a < k+1$ and $k \in \mathbb{Z}$. It is easy to see that $\langle a \rangle - \langle b \rangle \leq \langle a - b \rangle + 1$ for $a, b \in \mathbb{R}$.

Let $F = \mathbb{K}[x]$ and $F_n = \{f \in F : \deg(f) \leq n\}$ for $n \in \mathbb{N}_0$. Then F is a linear algebra over $\mathbb{K}$ and F_n is a linear subspace of F for every $n \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Put

$$f_i = \begin{cases} x^i & \text{if } i = 0, \\ \alpha^{\langle [(2r-1)a_n - 2i]/\sqrt{4a_n} \rangle} x^i & \text{if } i \in J_{n,r} \text{ and } n, r \in \mathbb{N} \text{ with } n > r, \\ \alpha_{n-r}(x^i - x^{i-a_n}) & \text{if } i \in I_{n,r} \text{ and } n, r \in \mathbb{N} \text{ with } n > r, \\ \alpha^{\langle [(2r-1)b_n - 2i]/\sqrt{4b_n} \rangle} x^i & \text{if } i \in L_{n,r} \text{ and } n, r \in \mathbb{N} \text{ with } n > r, \\ x^i - \beta_n x^{i-b_n} & \text{if } i \in K_{n,r} \text{ and } n, r \in \mathbb{N} \text{ with } n > r. \end{cases}$$

Obviously, $\mathrm{lin}\{f_i : 0 \leq i \leq n\} = F_n$ for $n \in \mathbb{N}_0$. Thus $(f_i)_{i=0}^{\infty}$ is a linear base in F. For $f \in F$ of the form $f = \sum_{i=0}^{m} c_i f_i$, we put $\|f\| = \max_{0 \leq i \leq m} |c_i|$; $\|\cdot\|$ is a norm on F. Clearly, $(f_i)_{i=0}^{\infty}$ is an orthonormal basis in $(F, \|\cdot\|)$.

Put $A_m = |\alpha|^{\sqrt{a_m}}$ and $B_m = |\alpha|^{\sqrt{a_m}+\sqrt{b_m}}$ for $m \in \mathbb{N}$. We have the following.

Lemma 1 (i) *Let $m \in \mathbb{N}$ with $m \geq 2$. Then $\max_{0 \leq i \leq (m-1)a_m} \|x^i\| \leq A_m$ and $\max_{0 \leq i \leq v_m} \|x^i\| \leq B_m$.*

(ii) *Let $n, r \in \mathbb{N}$ with $n > r$. Then $\|x^i - x^{i-ra_n}\| \leq |\alpha_{n-r}^{-1}|$ for $i \in I_{n,r}$ and $\|x^i - \beta_n^r x^{i-rb_n}\| \leq |\beta_n|^{r-1}$ for $i \in K_{n,r}$.*

Proof For $m \in \mathbb{N}$ we have $[0, ma_{m+1}] = [0, v_m] \cup \bigcup_{r=1}^{m}(J_{m+1,r} \cup I_{m+1,r})$ and $[0, v_{m+1}] = [0, ma_{m+1}] \cup \bigcup_{r=1}^{m}(L_{m+1,r} \cup K_{m+1,r})$.

Clearly $\|x^0\| = 1$. Let $n, r \in \mathbb{N}$ with $n > r$. It is easy to check that $\|x^i\| < |\alpha|^{\sqrt{a_n}}$ for $i \in J_{n,r}$ and $\|x^i\| < |\alpha|^{\sqrt{b_n}}$ for $i \in L_{n,r}$.

Let $i \in I_{n,r}$. For $j \in [0, r)$ we have $i - ja_n \in I_{n,r-j}$, so

$$\alpha_{n-r+j}^{-1} f_{i-ja_n} = x^{i-ja_n} - x^{i-(j+1)a_n}.$$

Hence $\sum_{j=0}^{r-1} \alpha_{n-r+j}^{-1} f_{i-ja_n} = x^i - x^{i-ra_n}$, so

$$\|x^i - x^{i-ra_n}\| \leq \max_{0\leq j<r} |\alpha_{n-r+j}^{-1}| = |\alpha_{n-r}^{-1}| < 1.$$

Then $\|x^i\| \leq \max\{1, \|x^{i-ra_n}\|\}$ and $i - ra_n \in [0, v_{n-2}]$.

Let $i \in K_{n,r}$. For $j \in [0, r)$ we have $i - jb_n \in K_{n,r-j}$, so

$$x^{i-jb_n} = f_{i-jb_n} + \beta_n x^{i-(j+1)b_n}.$$

Hence $x^i = \sum_{j=0}^{r-1} \beta_n^j f_{i-jb_n} + \beta_n^r x^{i-rb_n}$, so $\|x^i - \beta_n^r x^{i-rb_n}\| \leq \max_{0\leq j<r} |\beta_n^j| = |\beta_n^{r-1}|$. Then $\|x^i\| \leq \max\{|\beta_n^{r-1}|, |\beta_n^r|\|x^{i-rb_n}\|\} \leq |\alpha|^{\sqrt{b_n}} \max\{1, \|x^{i-rb_n}\|\}$ and $i - rb_n \in [a_n, (n-1)a_n]$. Hence, by induction, we get $\max_{0\leq i\leq(m-1)a_m} \|x^i\| \leq |\alpha|^{\sqrt{a_m}}$ and $\max_{0\leq i\leq v_m} \|x^i\| \leq |\alpha|^{\sqrt{a_m}+\sqrt{b_m}}$ for $m \in \mathbb{N}$ with $m \geq 2$. ■

Denote by $E = (E, \|\cdot\|)$ the completion of the normed space $(F, \|\cdot\|)$. Then E is an infinite-dimensional Banach space of countable type and $(f_i)_{i=0}^{\infty}$ is an orthonormal basis in E; so E is linearly isometric to $c_0(\mathbb{K})$.

Lemma 2 *The linear operator* $T\colon (F, \|\cdot\|) \to (F, \|\cdot\|)$, $Tf = xf$ *is continuous and* $\|T\| \leq |\alpha|$.

Proof It is enough to show that $\|Tf_i\| \leq |\alpha|$ for any $i \in \mathbb{N}_0$. Since $1 \in J_{2,1}$ we have $f_1 = \alpha^{\langle[a_2-2]/\sqrt{4a_2}\rangle} x$. Thus $Tf_0 = x = \alpha^{-\langle[a_2-2]/\sqrt{4a_2}\rangle} f_1$; hence $\|Tf_0\| \leq 1$.

Let $i \in \mathbb{N}$. For some $n, r \in \mathbb{N}$ with $n > r$ we have $i \in J_{n,r} \cup I_{n,r} \cup L_{n,r} \cup K_{n,r}$. Consider four cases (and many subcases).

Case 1: $i \in J_{n,r}$. Then $f_i = \alpha^{\langle[(2r-1)a_n-2i]/\sqrt{4a_n}\rangle} x^i$.

1.1 $i < ra_n - 1$. Then $i+1 \in J_{n,r}$, so $f_{i+1} = \alpha^{\langle[(2r-1)a_n-2(i+1)]/\sqrt{4a_n}\rangle} x^{i+1}$. Thus $Tf_i = \alpha^{\langle[(2r-1)a_n-2i]/\sqrt{4a_n}\rangle - \langle[(2r-1)a_n-2(i+1)]/\sqrt{4a_n}\rangle} f_{i+1}$. Hence $\|Tf_i\| \leq |\alpha|^{\langle 2/\sqrt{4a_n}\rangle+1} = |\alpha|$.

1.2 $i = ra_n - 1$. Then $Tf_i = \alpha^{\langle[2-a_n]/\sqrt{4a_n}\rangle} x^{ra_n}$. By Lemma 1 we have $\|x^{ra_n} - x^0\| \leq |\alpha_{n-r}^{-1}| \leq 1$. Thus $\|x^{ra_n}\| \leq \max\{\|x^{ra_n} - x^0\|, \|x^0\|\} = 1$. Hence $\|Tf_i\| \leq |\alpha|^{-\sqrt{a_n}/4} \leq a_n^{-1} < 1$.

Case 2: $i \in I_{n,r}$. Then $f_i = \alpha_{n-r}(x^i - x^{i-a_n})$ and $Tf_i = \alpha_{n-r}(x^{i+1} - x^{i+1-a_n})$.

2.1 $i < ra_n + v_{n-r-1}$. Then $i + 1 \in I_{n,r}$, so $f_{i+1} = \alpha_{n-r}(x^{i+1} - x^{i+1-a_n})$. Thus $\|Tf_i\| = \|f_{i+1}\| = 1$.

2.2 $i = ra_n + v_{n-r-1}$. Then $4(i+1) < (4r+1)a_n < b_n$. Put $j = i - a_n + 1$.

If $r < n-1$, then $i + 1 \in J_{n,r+1}$; so $f_{i+1} = \alpha^{\langle[(2r+1)a_n-2(i+1)]/\sqrt{4a_n}\rangle} x^{i+1}$. Hence $\|x^{i+1}\| = |\alpha|^{-\langle[(2r+1)a_n-2(i+1)]/\sqrt{4a_n}\rangle} \leq |\alpha|^{-\langle\sqrt{a_n}/4\rangle}$.

If $r = n-1$, then $i + 1 \in L_{n,1}$; so $f_{i+1} = \alpha^{\langle[b_n-2(i+1)]/\sqrt{4b_n}\rangle} x^{i+1}$. Hence $\|x^{i+1}\| = |\alpha|^{-\langle[b_n-2(i+1)]/\sqrt{4b_n}\rangle} \leq |\alpha|^{-\langle\sqrt{b_n}/4\rangle}$.

If $r = 1$ and $n = 2$, then $j = 1 \in J_{2,1}$ and $\|x^j\| = |\alpha|^{-\langle[a_2-2]/\sqrt{4a_2}\rangle} \|f_j\| \leq |\alpha|^{-\langle\sqrt{a_2}/4\rangle}$.

If $r = 1$ and $n > 2$, then $j \in J_{n-1,1}$; so $f_j = \alpha^{\langle [a_{n-1}-2j]/\sqrt{4a_{n-1}}\rangle} x^j$. Hence $\|x^j\| = |\alpha|^{-\langle [a_{n-1}-2(v_{n-2}+1)]/\sqrt{4a_{n-1}}\rangle} \leq |\alpha|^{-\langle \sqrt{a_{n-1}}/4\rangle}$.

If $1 < r < n-1$, then $j \in I_{n,r-1}$ and using Lemma 1 we get $\|x^j - x^{v_{n-r-1}+1}\| \leq |\alpha_{n-r+1}^{-1}|$. Moreover we have $f_{v_{n-r-1}+1} = \alpha^{\langle [a_{n-r}-2(v_{n-r-1}+1)]/\sqrt{4a_{n-r}}\rangle} x^{v_{n-r-1}+1}$, because $v_{n-r-1} + 1 \in J_{n-r,1}$. Hence

$$\|x^{v_{n-r-1}+1}\| = |\alpha|^{-\langle [a_{n-r}-2(v_{n-r-1}+1)]/\sqrt{4a_{n-r}}\rangle} \leq |\alpha|^{-\langle \sqrt{a_{n-r}}/4\rangle}.$$

Thus $\|x^j\| \leq \max\{\|x^j - x^{v_{n-r-1}+1}\|, \|x^{v_{n-r-1}+1}\|\} \leq |\alpha|^{-\langle \sqrt{a_{n-r}}/4\rangle}$.

If $r = n-1$, then $j \in J_{n,n-1}$; so $f_j = \alpha^{\langle [(2n-3)a_n-2j]/\sqrt{4a_n}\rangle} x^j$. Hence $\|x^j\| = |\alpha|^{-\langle [a_n-2]/\sqrt{4a_n}\rangle} \leq |\alpha|^{-\langle \sqrt{a_n}/4\rangle}$.

It follows that $\|Tf_i\| \leq |\alpha_{n-r}| \max\{\|x^{i+1}\|, \|x^j\|\} \leq |\alpha_{n-r}||\alpha|^{-\langle \sqrt{a_{n-r}}/4\rangle} \leq a_{n-r}^{-1}$.

Case 3: $i \in L_{n,r}$. Then $f_i = \alpha^{\langle [(2r-1)b_n-2i]/\sqrt{4b_n}\rangle} x^i$.

3.1 $i < r(a_n+b_n)-1$. Then $i+1 \in L_{n,r}$, so $f_{i+1} = \alpha^{\langle [(2r-1)b_n-2(i+1)]/\sqrt{4b_n}\rangle} x^{i+1}$. Thus $Tf_i = \alpha^{\langle [(2r-1)b_n-2i]/\sqrt{4b_n}\rangle - \langle [(2r-1)b_n-2(i+1)]/\sqrt{4b_n}\rangle} f_{i+1}$. Hence $\|Tf_i\| \leq |\alpha|$.

3.2 $i = r(a_n+b_n)-1$. Then $Tf_i = \alpha^{\langle [-b_n-2ra_n+2]/\sqrt{4b_n}\rangle} x^{r(a_n+b_n)}$. Put $j = r(a_n+b_n)$. By Lemma 1 we have $\|x^j - \beta_n^r x^{j-rb_n}\| \leq |\beta_n|^{r-1}$. In Case 1 we have shown that $\|x^{ra_n}\| \leq 1$. Hence $\|x^j\| \leq \max\{\|x^j - \beta_n^r x^{ra_n}\|, |\beta_n^r|\|x^{ra_n}\|\} \leq b_n^{n-1}$. Thus $\|Tf_i\| \leq |\alpha|^{\langle [-b_n-2ra_n+2]/\sqrt{4b_n}\rangle} b_n^{n-1} \leq |\alpha|^{-\sqrt{b_n}/4} b_n^{n-1} \leq b_n^{-1}$.

Case 4: $i \in K_{n,r}$. Then $f_i = x^i - \beta_n x^{i-b_n}$ and $Tf_i = x^{i+1} - \beta_n x^{i+1-b_n}$.

4.1 $i < (n-1)a_n + rb_n$. Then $i+1 \in K_{n,r}$, so $f_{i+1} = x^{i+1} - \beta_n x^{i+1-b_n} = Tf_i$. Hence $\|Tf_i\| = 1$.

4.2 $i = (n-1)a_n + rb_n$. Put $j = i+1-b_n$. Then $j \in L_{n,r}$, so

$$f_j = \alpha^{\langle [(2r-1)b_n-2j]/\sqrt{4b_n}\rangle} x^j.$$

Hence $\|x^j\| = |\alpha|^{-\langle [(2r+1)b_n-2(i+1)]/\sqrt{4b_n}\rangle} \leq |\alpha|^{-\langle \sqrt{b_n}/4\rangle}$.

If $r < n-1$, then $i+1 \in L_{n,r+1}$; so $f_{i+1} = \alpha^{\langle [(2r+1)b_n-2(i+1)]/\sqrt{4b_n}\rangle} x^{i+1}$. Hence $\|x^{i+1}\| = \|x^j\| \leq |\alpha|^{-\langle \sqrt{b_n}/4\rangle}$.

If $r = n-1$, then $i+1 \in J_{n+1,1}$; so $f_{i+1} = \alpha^{\langle [a_{n+1}-2(i+1)]/\sqrt{4a_{n+1}}\rangle} x^{i+1}$. Hence $\|x^{i+1}\| = |\alpha|^{-\langle [a_{n+1}-2(i+1)]/\sqrt{4a_{n+1}}\rangle} \leq |\alpha|^{-\langle \sqrt{a_{n+1}}/4\rangle} \leq |\alpha|^{-\langle \sqrt{b_n}/4\rangle}$. Thus we have $\|Tf_i\| \leq \max\{\|x^{i+1}\|, |\beta_n|\|x^j\|\} \leq |\beta_n||\alpha|^{-\langle \sqrt{b_n}/4\rangle} \leq |\alpha|$. ■

From now on, by T we will denote the linear continuous operator on E such that $Tf = xf$ for all $f \in F$; clearly $\|T\| \leq |\alpha|$.

By the proof of Lemma 2 we get the following.

Remark 3. If $n, r \in \mathbb{N}$ with $n > r$, then $\|Tf_{ra_n-1}\| \leq a_n^{-1}$, $\|Tf_{ra_n+v_{n-r-1}}\| \leq a_{n-r}^{-1}$, and $\|Tf_{r(a_n+b_n)-1}\| \leq b_n^{-1}$.

Let $m \in \mathbb{N}$ with $m > 2$. Put $S_m = \bigcup_{n=m+1}^{\infty} I_{n,n-m}$. Let $Q_m\colon F \to F_{(m-1)a_m}$ be a linear operator such that

$$Q_m f_i = \begin{cases} f_i & \text{if } i \in [0,(m-1)a_m], \\ -\alpha_m x^{i-(n-m)a_n} & \text{if } i \in I_{n.n-m} \text{ and } n > m, \\ 0 & \text{if } i \in (\mathbb{N} \setminus S_m) \text{ with } i > (m-1)a_m. \end{cases}$$

Clearly, $\|Q_m f_i\| = 1$ for $0 \le i \le (m-1)a_m$, and $\|Q_m f_i\| = |\alpha_m| \|x^{i-(n-m)a_n}\| \le a_m B_{m-1} < \sqrt{A_m}$ for $i \in I_{n,n-m}, n > m$. Thus $\sup_{i \in \mathbb{N}_0} \|Q_m f_i\| < \sqrt{A_m}$; so the linear operator $Q_m \colon (F, \|\cdot\|) \to (F, \|\cdot\|)$ is continuous and $\|Q_m\| < \sqrt{A_m}$. From now on, by Q_m we will denote its continuous extension on $(E, \|\cdot\|)$.

We have the following lemma.

Lemma 4 *Let $m \in \mathbb{N}$ with $m > 2$ and let $s \in K_{m,1}$. Then $\|T^s - T^s Q_m\| \le |\alpha|$.*

Proof It is enough to show that $\|T^s f_i - T^s Q_m f_i\| \le |\alpha|$ for every $i \in \mathbb{N}_0$. If $i \in [0,(m-1)a_m]$, then $T^s f_i - T^s Q_m f_i = 0$. Let $i > (m-1)a_m$. For some $n, r \in \mathbb{N}$ with $n > r$ we have $i \in J_{n,r} \cup I_{n,r} \cup L_{n,r} \cup K_{n,r}$. If $i \in J_{n,r} \cup L_{n,r} \cup K_{n,r}$, then $i \notin S_m$; so $Q_m f_i = 0$ and $T^s f_i - T^s Q_m f_i = T^s f_i$. Consider four cases.

Case 1: $i \in J_{n,r}$. Then $T^s f_i = \alpha^{\langle [(2r-1)a_n - 2i]/\sqrt{4a_n} \rangle} x^{i+s}$. We have $m < n$, since $(m-1)a_m < i < (n-1)a_n$. Thus $s < 2b_m < \sqrt{a_n}$.

1.1 $i < ra_n - s$. Then $i + s \in J_{n,r}$, so $f_{i+s} = \alpha^{\langle [(2r-1)a_n - 2(i+s)]/\sqrt{4a_n} \rangle} x^{i+s}$. Hence $T^s f_i = \alpha^{\langle [(2r-1)a_n - 2i]/\sqrt{4a_n} \rangle - \langle [(2r-1)a_n - 2(i+s)]/\sqrt{4a_n} \rangle} f_{i+s}$. Thus $\|T^s f_i\| \le |\alpha|$.

1.2 $i \ge ra_n - s$. Then $T^{i+s-ra_n+1} f_{ra_n-1} = \alpha^{\langle [(2r-1)a_n - 2(ra_n-1)]/\sqrt{4a_n} \rangle} x^{i+s}$, since $ra_n - 1 \in J_{n,r}$. Thus

$$\begin{aligned} \|T^s f_i\| &= |\alpha|^{\langle [(2r-1)a_n - 2i]/\sqrt{4a_n} \rangle - \langle [-a_n+2]/\sqrt{4a_n} \rangle} \|T^{i+s-ra_n+1} f_{ra_n-1}\| \\ &\le |\alpha| \|T\|^{i+s-ra_n} \|T f_{ra_n-1}\|. \end{aligned}$$

By Remark 3 we have $\|T f_{ra_n-1}\| \le a_n^{-1}$. It follows that

$$\|T^s f_i\| \le |\alpha|^{1+i+s-ra_n} a_n^{-1} \le |\alpha|^s a_n^{-1} \le |\alpha|^{2b_m} a_{m+1}^{-1} \le 1.$$

Case 2: $i \in I_{n,r}$. Then $f_i = \alpha_{n-r}(x^i - x^{i-a_n})$, so $T^s f_i = \alpha_{n-r}(x^{i+s} - x^{i+s-a_n})$. We have $n > m$, since $(m-1)a_m < i \le ra_n + \nu_{n-r-1} \le (n-1)a_n$. Thus $4(i+s) < (4r+1)a_n$.

2.1 $r = n - m$. Then $i = ra_n + l$ for some $l \in [0, \nu_{m-1}]$ and $T^s Q_m f_i = -\alpha_m x^{l+s}$. We have $i + s \in J_{n,r+1}$, so $f_{i+s} = \alpha^{\langle [(2r+1)a_n - 2(i+s)]/\sqrt{4a_n} \rangle} x^{i+s}$. Hence $\|x^{i+s}\| = |\alpha|^{-\langle [(2r+1)a_n - 2(i+s)]/\sqrt{4a_n} \rangle} \le |\alpha|^{-\langle \sqrt{a_n}/4 \rangle} \le |\alpha_n^{-1}|$. Using Lemma 1 we get

$$\|x^{(r-1)a_n+l+s} - x^{l+s}\| \le |\alpha_{n-r+1}^{-1}| = |\alpha_{m+1}^{-1}|.$$

Thus $\|T^s f_i - T^s Q_m f_i\| = |\alpha_m| \|x^{i+s} - (x^{i-a_n+s} - x^{l+s})\| < 1$.

2.2 $r \ne n - m$. Then $i \notin S_m$, so $Q_m f_i = 0$ and $T^s f_i - T^s Q_m f_i = T^s f_i$.

2.2a $r > n - m$. Then for $j = i + s$ we have $j - a_n \in J_{n,r}$, so $f_{j-a_n} = \alpha^{\langle[(2r-1)a_n - 2(j-a_n)]/\sqrt{4a_n}\rangle} x^{j-a_n}$. Thus $\|x^{j-a_n}\| = |\alpha|^{-\langle[(2r+1)a_n-2j]/\sqrt{4a_n}\rangle} \leq |\alpha|^{-\langle\sqrt{a_n}/4\rangle}$.

If $r + 1 < n$, then $j \in J_{n,r+1}$; so $f_j = \alpha^{\langle[(2r+1)a_n-2j]/\sqrt{4a_n}\rangle} x^j$. Hence $\|x^j\| = \|x^{j-a_n}\| \leq |\alpha|^{-\langle\sqrt{a_n}/4\rangle}$.

If $r + 1 = n$, then $j \in L_{n,1}$; so $f_j = \alpha^{\langle[b_n-2j]/\sqrt{4b_n}\rangle} x^j$. Hence $\|x^j\| = |\alpha|^{-\langle[b_n-2j]/\sqrt{4b_n}\rangle} \leq |\alpha|^{-\langle\sqrt{b_n}/4\rangle}$.

It follows that

$$\|T^s f_i\| \leq |\alpha_{n-r}| \max\{\|x^j\|, \|x^{j-a_n}\|\} \leq |\alpha_n||\alpha|^{-\langle\sqrt{a_n}/4\rangle} < 1.$$

2.2b $r < n - m$. Then $i = ra_n + l$ for some $l \in [0, v_{n-r-1}]$.

If $l+s \leq v_{n-r-1}$, then $i+s \in I_{n,r}$; so $f_{i+s} = \alpha_{n-r}(x^{i+s} - x^{i+s-a_n}) = T^s f_i$. Thus $\|T^s f_i\| = 1$.

If $l + s > v_{n-r-1}$, then we have

$$T^s f_i = \alpha_{n-r}(x^{i+s} - x^{i+s-a_n}) = T^{s+l-v_{n-r-1}}[\alpha_{n-r}(x^{ra_n+v_{n-r-1}} - x^{(r-1)a_n+v_{n-r-1}})]$$
$$= T^{s+l-v_{n-r-1}-1} T f_{ra_n+v_{n-r-1}}.$$

By Remark 3 we obtain $\|T f_{ra_n+v_{n-r-1}}\| \leq a_{n-r}^{-1}$. Thus we get $\|T^s f_i\| \leq \|T\|^{s+l-v_{n-r-1}-1} a_{n-r}^{-1} \leq |\alpha|^s a_{n-r}^{-1} \leq |\alpha|^{2b_m} a_{m+1}^{-1} \leq 1$.

Case 3: $i \in L_{n,r}$. Then $T^s f_i = \alpha^{\langle[(2r-1)b_n-2i]/\sqrt{4b_n}\rangle} x^{i+s}$. Put $j = i + s$. We have $n \geq m$, since $v_{m-1} < a_m < i < r(a_n + b_n) \leq v_n$. Thus $4j < a_{n+1}$.

3.1 $n = m$ and $j > v_n$. Then $j \in J_{n+1,1}$; so $f_j = \alpha^{\langle[a_{n+1}-2j]/\sqrt{4a_{n+1}}\rangle} x^j$ and $\|x^j\| = |\alpha|^{-\langle[a_{n+1}-2j]/\sqrt{4a_{n+1}}\rangle} \leq |\alpha|^{-\langle\sqrt{a_{n+1}}/4\rangle}$. Thus $\|T^s f_i\| \leq |\alpha|^{\langle\sqrt{b_n}/2\rangle}\|x^j\| \leq |\alpha|^{\langle\sqrt{b_n}/2\rangle - \langle\sqrt{a_{n+1}}/4\rangle} \leq 1$.

3.2 $n = m$ and $j \leq v_n$. Then $j > na_n + rb_n$ and $r < n - 1$.

3.2a If $j < (r + 1)(a_n + b_n)$, then $j \in L_{n,r+1}$; so $f_j = \alpha^{\langle[(2r+1)b_n-2j]/\sqrt{4b_n}\rangle} x^j$. Thus $\|T^s f_i\| = |\alpha|^{\langle[(2r-1)b_n-2i]/\sqrt{4b_n}\rangle - \langle[(2r+1)b_n-2j]/\sqrt{4b_n}\rangle} \leq |\alpha|$, since $s - b_n < \sqrt{b_n}$.

3.2b If $(r + 1)(a_n + b_n) \leq j \leq (n - 1)a_n + (r + 1)b_n$, then using Lemma 1 we get $\|x^j - \beta_n^{r+1} x^{j-(r+1)b_n}\| \leq |\beta_n^r| \leq b_n^r$ and $\|x^{j-(r+1)b_n}\| \leq A_n < b_n^4$. Thus $\|x^j\| \leq \max\{b_n^r, b_n^{r+5}\} \leq b_n^{4n} < |\alpha|^{\sqrt{b_n}/4}$. Moreover, we have

$$\|T^s f_i\| = |\alpha|^{\langle[(2r-1)b_n-2i]/\sqrt{4b_n}\rangle}\|x^j\| \leq |\alpha|^{\langle-\sqrt{b_n}/4\rangle}\|x^j\|,$$

since $i = j - s \geq [r - (1/4)]b_n$. It follows that $\|T^s f_i\| \leq 1$.

3.2c If $j > (n - 1)a_n + (r + 1)b_n$, then $r < n - 2$ and $j \in L_{n,r+2}$; so $f_j = \alpha^{\langle[(2r+3)b_n-2j]/\sqrt{4b_n}\rangle} x^j$. Thus

$$\|T^s f_i\| = |\alpha|^{\langle[(2r-1)b_n-2i]/\sqrt{4b_n}\rangle - \langle[(2r+3)b_n-2j]/\sqrt{4b_n}\rangle} \leq 1.$$

3.3 $n > m$ and $j < r(a_n + b_n)$. Then $j \in L_{n,r}$, so $f_j = \alpha^{\langle[(2r-1)b_n-2j]/\sqrt{4b_n}\rangle} x^j$. Thus $\|T^s f_i\| = |\alpha|^{\langle[(2r-1)b_n-2i]/\sqrt{4b_n}\rangle - \langle[(2r-1)b_n-2j]/\sqrt{4b_n}\rangle} \leq |\alpha|$, since $s < 2b_m < \sqrt{b_n}$.

3.4 $n > m$ and $j \geq r(a_n + b_n)$. Put $k = r(a_n + b_n)$. Clearly $k - 1 \in L_{n,r}$, so $f_{k-1} = \alpha^{\langle[(2r-1)b_n - 2(k-1)]/\sqrt{4b_n}\rangle} x^{k-1}$. Hence $T^{j-k} T f_{k-1} = \alpha^{\langle[(2r-1)b_n-2(k-1)]/\sqrt{4b_n}\rangle} x^j$. Thus $\|x^j\| \leq |\alpha|^{-\langle[(2r-1)b_n-2(k-1)]/\sqrt{4b_n}\rangle} \|T\|^{s-1} \|T f_{k-1}\|$. Using Remark 3 we obtain $\|T f_{k-1}\| \leq b_n^{-1}$. Thus

$$\|T^s f_i\| \leq |\alpha|^{\langle[(2r-1)b_n-2i]/\sqrt{4b_n}\rangle - \langle[(2r-1)b_n-2(k-1)]/\sqrt{4b_n}\rangle} |\alpha|^{s-1} b_n^{-1}$$

$$\leq |\alpha|^s b_n^{-1} < |\alpha|^{2b_m} b_{m+1}^{-1} < 1,$$

since $k - i - 1 < s < 2b_m < \sqrt{b_n}$.

Case 4: $i \in K_{n,r}$. Then $f_i = x^i - \beta_n x^{i-b_n}$ and $T^s f_i = x^{i+s} - \beta_n x^{i+s-b_n}$. We have $n \geq m$, since $(m-1)a_m < i \leq (n-1)a_n + rb_n \leq v_n$. Put $j = i + s$.

4.1 $n = m$ and $j \leq (n-1)a_n + (r+1)b_n$. Then $j \geq (r+1)(a_n + b_n)$, so $r + 1 \leq n - 1$ and $j \in K_{n,r+1}$. Thus $f_j = x^j - \beta_n x^{j-b_n} = T^s f_i$, so $\|T^s f_i\| = 1$.

4.2 $n = m$ and $j > (n-1)a_n + (r+1)b_n$. Then $4j < (4r+5)b_n < a_{n+1}$.

If $r < n - 2$, then $j \in L_{n,r+2}$ and $f_j = \alpha^{\langle[(2r+3)b_n-2j]/\sqrt{4b_n}\rangle} x^j$; so $\|x^j\| = |\alpha|^{-\langle[(2r+3)b_n-2j]/\sqrt{4b_n}\rangle} \leq 1$.

If $r \geq n - 2$, then $j \in J_{n+1,1}$ and $f_j = \alpha^{\langle[a_{n+1}-2j]/\sqrt{4a_{n+1}}\rangle} x^j$; so $\|x^j\| = |\alpha|^{-\langle[a_{n+1}-2j]/\sqrt{4a_{n+1}}\rangle} \leq 1$.

If $r < n - 1$, then $j - b_n \in L_{n,r+1}$ and $f_{j-b_n} = \alpha^{\langle[(2r+1)b_n-2(j-b_n)]/\sqrt{4b_n}\rangle} x^{j-b_n}$; so $\|x^{j-b_n}\| = |\alpha|^{-\langle[(2r+3)b_n-2j]/\sqrt{4b_n}\rangle} \leq |\alpha|^{-\langle\sqrt{b_n}/4\rangle} \leq |\beta_n^{-1}|$.

If $r = n - 1$, then $j - b_n \in J_{n+1,1}$ and $f_{j-b_n} = \alpha^{\langle[a_{n+1}-2(j-b_n)]/\sqrt{4a_{n+1}}\rangle} x^{j-b_n}$; so $\|x^{j-b_n}\| = |\alpha|^{-\langle[a_{n+1}-2(j-b_n)]/\sqrt{4a_{n+1}}\rangle} \leq |\alpha|^{-\langle\sqrt{a_{n+1}}/4\rangle} \leq |\alpha_{n+1}^{-1}|$.

It follows that $\|T^s f_i\| \leq \max\{\|x^j\|, |\beta_n| \|x^{j-b_n}\|\} \leq 1$.

4.3 $n > m$ and $j \leq (n-1)a_n + rb_n$. Then $j \in K_{n,r}$ and $f_j = x^j - \beta_n x^{j-b_n} = T^s f_i$; so $\|T^s f_i\| = 1$.

4.4 $n > m$ and $j > (n-1)a_n + rb_n$. Then $4j < (4r+1)b_n < a_{n+1}$.

If $r < n - 1$, then $j \in L_{n,r+1}$ and $f_j = \alpha^{\langle[(2r+1)b_n-2j]/\sqrt{4b_n}\rangle} x^j$; so $\|x^j\| = |\alpha|^{-\langle[(2r+1)b_n-2j]/\sqrt{4b_n}\rangle} \leq 1$.

If $r = n - 1$, then $j \in J_{n+1,1}$ and $f_j = \alpha^{\langle[a_{n+1}-2j]/\sqrt{4a_{n+1}}\rangle} x^j$; so $\|x^j\| = |\alpha|^{-\langle[a_{n+1}-2j]/\sqrt{4a_{n+1}}\rangle} \leq 1$.

Moreover we have $j - b_n \in L_{n,r}$ and $f_{j-b_n} = \alpha^{\langle[(2r-1)b_n-2(j-b_n)]/\sqrt{4b_n}\rangle} x^{j-b_n}$; so $\|x^{j-b_n}\| = |\alpha|^{-\langle[(2r+1)b_n-2j]/\sqrt{4b_n}\rangle} \leq |\alpha|^{-\langle\sqrt{b_n}/4\rangle} \leq |\beta_n^{-1}|$. It follows that $\|T^s f_i\| \leq \max\{\|x^j\|, |\beta_n| \|x^{j-b_n}\|\} \leq 1$. ■

For $f \in F$ of the form $f = \sum_{i=0}^m c_i x^i$ we put $|f| = \max_{0 \leq i \leq m} |c_i|$. The functional $|\cdot|: F \to [0, \infty)$, $f \to |f|$ is a multiplicative norm on F [8, p. 7].

It is easy to check that for $m \in \mathbb{N}$ with $m > 2$ and $y \in F_{(m-1)a_m}$ we have $\|y\| \leq A_m |y|$ and $|y| \leq \max_{0 \leq i \leq (m-1)a_m} |f_i| \|y\| \leq \sqrt{A_m} \|y\|$.

For $n \in \mathbb{N}_0$ we denote by P_n the linear projection from F onto F_n such that $P_n(x^i) = 0$ for $i > n$. We have $x(P_n v) = P_{n+1}(xv)$ for $n \in \mathbb{N}$ and $v \in F$.

We need two more lemmas to prove our theorem.

Lemma 5 *Let $e \in E$ with $e \neq 0$ and $k \in \mathbb{N}$ with $k > 2$. Then there exists $m \in \mathbb{N}$ with $m > k$ such that $|P_{(m-k)a_m}(Q_m e)| \geq a_m^{-1}$.*

Proof Suppose by contradiction that for every $m \in \mathbb{N}$ with $m > k$ we have

$$|P_{(m-k)a_m}(Q_m e)| < a_m^{-1}. \tag{3.1}$$

For some $(e_j) \in c_0(\mathbb{K})$ we have $e = \sum_{j=0}^{\infty} e_j f_j$. Then $\|e\| = \max_{j\in\mathbb{N}_0} |e_j| > 0$. Put $c_n = (n-1)a_n$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ we have $\sum_{j=0}^{c_n} e_j f_j = \sum_{j=0}^{c_n} y_{n,j} x^j$ for some $(y_{n,j})_{j=0}^{c_n} \subset \mathbb{K}$. For $n \in \mathbb{N}$ with $n > 2$ we obtain $Q_n(\sum_{j=c_n+1}^{\infty} e_j f_j) = \sum_{i=0}^{v_{n-1}} z_{n,i} x^i$, where

$$z_{n,i} = -\alpha_n \sum_{m=n+1}^{\infty} e_{i+(m-n)a_m}. \tag{3.2}$$

So we get

$$Q_n e = \sum_{j=0}^{c_n} y_{n,j} x^j + \sum_{j=0}^{v_{n-1}} z_{n,j} x^j. \tag{3.3}$$

From (3.1) and (3.3) we obtain for $m \in \mathbb{N}$ with $m > k$

$$\max_{j\in(v_{m-1},(m-k)a_m]} |y_{m,j}| < a_m^{-1}. \tag{3.4}$$

Let $m > n > k$ and $M_{m,n} = ((m-n)a_m + v_{n-2}, (m-n)a_m + v_{n-1}]$. Clearly, $M_{m,n} \subset [a_m, c_m] = \bigcup_{s=1}^{m-1} I_{m,s} \cup \bigcup_{s=2}^{m-1} J_{m,s}$ and $f_i = \alpha_{m-s}(x^i - x^{i-a_m})$ for $i \in I_{m,s}, s \in [1, m)$ and $f_i = \alpha^{\langle[(2s-1)a_m-2i]/\sqrt{4a_m}\rangle} x^i$ for $i \in J_{m,s}, s \in [1, m)$. If $j \in M_{m,n}$, then $j \in I_{m,m-n}$; if $i \in [a_m, c_m]$ and $i - a_m \in M_{m,n}$, then $i \in J_{m,m-n+2}$. Thus $y_{m,j} = \alpha_n e_j$ for $j \in M_{m,n}$. Clearly, $M_{m,n} \subset (v_{m-1}, (m-k)a_m]$. Using (3.2) and (3.4) we obtain for $n > k$

$$\max_{v_{n-2}<j\le v_{n-1}} |z_{n,j}| \le \max_{m>n} \max_{v_{n-2}<j\le v_{n-1}} |\alpha_n||e_{j+(m-n)a_m}| = \max_{m>n} \max_{j\in M_{m,n}} |y_{m,j}| \le a_{n+1}^{-1}. \tag{3.5}$$

From (3.3) and (3.1) we get for $n > k$

$$\max_{j\in[0,v_{n-1}]} |y_{n,j} + z_{n,j}| \le |P_{a_n}(Q_n e)| < a_n^{-1}. \tag{3.6}$$

Hence, by (3.5), we have for $n > k$

$$\max_{j\in(v_{n-2},v_{n-1}]} |y_{n,j}| < a_n^{-1}. \tag{3.7}$$

Let $n > k$. Put $M_n = \sum_{j=v_{n-1}+1}^{c_n} e_j f_j - \sum_{j=v_{n-1}+1}^{c_n} y_{n,j} x^j$. Clearly, $(v_{n-1}, c_n] = \bigcup_{s=1}^{n-1}(J_{n,s} \cup I_{n,s})$. If $i \in \bigcup_{s=1}^{n-1} J_{n,s} \cup \bigcup_{s=2}^{n-1} I_{n,s}$, then $P_{v_{n-1}} f_i = 0$; if $i \in I_{n,1}$, then $P_{v_{n-1}}(f_i) \in F_{v_{n-2}}$. Thus $P_{v_{n-1}}(M_n) \in F_{v_{n-2}}$; but $M_n = \sum_{j=0}^{v_{n-1}} y_{n,j} x^j - \sum_{j=0}^{v_{n-1}} e_j f_j \in$

$F_{\nu_{n-1}}$, so $M_n \in F_{\nu_{n-2}}$. Hence $\sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} e_j f_j - \sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} y_{n,j} x^j = \sum_{j=0}^{\nu_{n-2}} t_{n,j} f_j$ for some $(t_{n,j})_{j=0}^{\nu_{n-2}} \subset \mathbb{K}$. By (3.7) and Lemma 1 we get

$$\max_{j\in(\nu_{n-2},\nu_{n-1}]} |e_j| \le \Big\| \sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} e_j f_j - \sum_{j=0}^{\nu_{n-2}} t_{n,j} f_j \Big\| \le \max_{j\in(\nu_{n-2},\nu_{n-1}]} |y_{n,j}|\|x^j\|$$
$$\le \frac{B_{n-1}}{a_n} < \frac{1}{a_n^{1/2}}.$$

Hence for $m \ge k$ we have $\max_{j\in(\nu_{m-1},c_m]} |e_j| \le \max_{j\in(\nu_{m-1},\nu_m]} |e_j| < a_{m+1}^{-1/2} < a_m^{-1}$. Using (3.2) we get for $n \ge k$:

$$\max_{j\in[0,\nu_{n-1}]} |z_{n,j}| \le |\alpha_n| \max_{m>n} \max_{j\in[0,\nu_{n-1}]} |e_{j+(m-n)a_m}| \le |\alpha_n| \max_{m>n} \max_{j\in(\nu_{m-1},\nu_m]} |e_j|$$
$$\le |\alpha_n| \max_{m>n} a_{m+1}^{-1/2} = |\alpha_n| a_{n+2}^{-1/2} \le a_{n+1}^{-1}.$$

Let $n \in \mathbb{N}$ with $n > k$. Applying (3.6) we obtain

$$\Big| P_{\nu_{n-1}}\Big(\sum_{j=0}^{c_n} e_j f_j\Big)\Big| = \Big| P_{\nu_{n-1}}\Big(\sum_{j=0}^{c_n} y_{n,j} x^j\Big)\Big| = \max_{j\in[0,\nu_{n-1}]} |y_{n,j}| < a_n^{-1}.$$

Moreover we have

$$\Big| P_{\nu_{n-1}}\Big(\sum_{j=\nu_{n-1}+1}^{c_n} e_j f_j\Big)\Big| \le \max_{j\in(\nu_{n-1},c_n]} |e_j| \max_{j\in(\nu_{n-1},c_n]} |P_{\nu_{n-1}}(f_j)| \le a_n^{-1} a_{n-1}.$$

Thus $|\sum_{j=0}^{\nu_{n-1}} e_j f_j| = |P_{\nu_{n-1}}(\sum_{j=0}^{c_n} e_j f_j) - P_{\nu_{n-1}}(\sum_{j=\nu_{n-1}+1}^{c_n} e_j f_j)| < a_n^{-1} a_{n-1}$. For some $(s_{n,j})_{j=0}^{\nu_{n-1}} \subset \mathbb{K}$ we have $\sum_{j=0}^{\nu_{n-1}} e_j f_j = \sum_{j=0}^{\nu_{n-1}} s_{n,j} x^j$. Hence

$$\max_{j\in[0,\nu_{n-1}]} |e_j| = \Big\| \sum_{j=0}^{\nu_{n-1}} s_{n,j} x^j \Big\| \le \max_{j\in[0,\nu_{n-1}]} |s_{n,j}|\|x^j\| \le \Big| \sum_{j=0}^{\nu_{n-1}} s_{n,j} x^j \Big| B_{n-1}$$
$$\le a_n^{-1} a_{n-1} B_{n-1} < a_{n-1}^{-1}$$

for every $n > k$. It follows that $\max_{j\in\mathbb{N}_0} |e_j| = 0$, so $e = 0$, which is a contradiction. ■

Lemma 6 *Let $0 < \varepsilon < 1$ and $1 < M < \varepsilon^{-1}$. Let $n \in \mathbb{N}$ and $m \in [0, n]$. Assume that $y \in F_n$ with $|y| \le M$ and $|P_m(y)| \ge M\varepsilon$. Then there exists $q \in F_n$ with $|q| \le \varepsilon^{-(n+2)!}$ such that $|P_n(qy) - x^m| < \varepsilon$.*

Proof Clearly, $y = \sum_{i=0}^{n} y_i x^i$ for some $(y_i)_{i=0}^{n} \subset \mathbb{K}$. By assumptions we have $\max_{0\le i\le n} |y_i| \le M$ and $\max_{0\le i\le m} |y_i| \ge M\varepsilon$.

If $n = 1$ and $m = 0$, then $q = y_0^{-1} x^0 - y_0^{-2} y_1 x^1$ satisfies our claim.

If $n = 1$ and $m = 1$, then we can take $q = y_1^{-1}x^0$ if $|y_0| < \varepsilon^2 M$, and $q = y_0^{-1}x^1$ if $|y_0| \geq \varepsilon^2 M$.

Suppose that our claim is true for $n = k \geq 1$. We shall prove that it is true for $n = k + 1$.

Let $m = 0$. Then we put $q_0 = y_0^{-1}$ and $q_{i+1} = -y_0^{-1}\sum_{j=1}^{i+1} y_j q_{i+1-j}$ for $0 \leq i \leq k$ and $q = \sum_{i=0}^{k+1} q_i x^i$. It is easy to see that $P_{k+1}(qy) - x^0 = 0$ and

$$|q| = \max_{0\leq i\leq k+1} |q_i| \leq \max_{0\leq i\leq k+1} \varepsilon^{-i}(M\varepsilon)^{-1} = \varepsilon^{-k-2}M^{-1} \leq \varepsilon^{-(k+3)!}.$$

Let $1 \leq m \leq k + 1$. Consider two cases.

Case 1: $|y_0| < \varepsilon^{(k+2)!+1}$. Then $\max_{1\leq i\leq k+1} |y_i| \leq M$ and $\max_{1\leq i\leq m} |y_i| \geq M\varepsilon$. By the inductive assumption for $\overline{y} = \sum_{i=1}^{k+1} y_i x^{i-1}$, there exists $q = \sum_{i=0}^{k} q_i x^i$ with $|q| \leq \varepsilon^{-(k+2)!}$ and $|P_k(q\overline{y}) - x^{m-1}| < \varepsilon$. Then we have

$$|P_{k+1}(qy) - x^m| = |P_{k+1}(q(x\overline{y} + y_0x^0)) - xx^{m-1}| = |x(P_k(q\overline{y}) - x^{m-1}) + y_0 q| < \varepsilon.$$

Case 2: $|y_0| \geq \varepsilon^{(k+2)!+1}$. Then we put $q_i = 0$ for $0 \leq i < m$, $q_m = y_0^{-1}$, and $q_{m+j} = -y_0^{-1}\sum_{i=1}^{j} y_i q_{m+j-i}$ for $1 \leq j \leq k + 1 - m$. For $q = \sum_{i=0}^{k+1} q_i x^i$ it is easy to check that $P_{k+1}(qy) - x^m = 0$ and

$$|q| = \max_{0\leq j\leq k+1-m} |q_{m+j}| \leq \max_{0\leq j\leq k+1-m} \varepsilon^{-(k+2)!-1}(M\varepsilon^{-(k+2)!-1})^j \leq \varepsilon^{-(k+3)!}.$$

■

Now we are ready to show our main result.

Theorem 7 *Assume that $d_1 \geq |\alpha|^4$ and $d_{n+1} \geq |\alpha|^{(nd_n)!}$ for every $n \in \mathbb{N}$. Then the linear continuous operator T on E has no non-trivial closed invariant subspace.*

Proof Let M be a closed subspace of E with $M \neq \{0\}$ such that $T(M) \subset M$. Then $g(T)(M) \subset M$ for every $g \in F$. Let $e \in M$ with $0 < \|e\| \leq 1$. We shall prove that for every $\delta > 0$ there exists $f \in F$ such that $\|f(T)e - x^0\| < \delta$. Let $\delta > 0$. Let $k > 2$ with $a_{k-1} > |\alpha|\delta^{-1}$. By Lemma 5 we have $|P_{(m-k)a_m}(Q_m e)| \geq a_m^{-1}$ for some $m > k$. For $R_m = (a_m A_m)^{[(m-2)a_m+2]!}$ we have $|\alpha|a_m A_m R_m < b_m$, since $a_m A_m < |\alpha|^{a_m}$ and $|\alpha|^{(ma_m)!} < b_m$. Put $y = Q_m e$. Then $|y| \leq \sqrt{A_m}\|Q_m\|\|e\| \leq A_m$. By Lemma 6 and its proof there exists $q \in F_{(m-2)a_m}$ with $|q| \leq R_m$ such that

$$|P_{(m-2)a_m}(qy) - x^{(m-k)a_m}| < (a_m A_m)^{-1}. \tag{3.8}$$

(S1) Put $f = \beta_m^{-1}x^{a_m+b_m}q$ and $S = K_{m,1}$. Then $f = \sum_{s\in S} t_s x^s$ for some $(t_s)_{s\in S} \subset \mathbb{K}$. Let $z = fy$. Using Lemma 4 we get

$$\begin{aligned}\|f(T)e - z\| &= \|\sum_{s\in S} t_s(T^s - T^s Q_m)e\| \leq \max_{s\in S} |t_s||\alpha| = |f||\alpha| \\ &= |q||\alpha||\beta_m|^{-1} < R_m|\alpha||\beta_m|^{-1} < a_m^{-1}.\end{aligned}$$

(S2) Let $a = (m-1)a_m + b_m$ and $b = 2(m-1)a_m + b_m$. Clearly $z \in F_b$; so $z = \sum_{j=0}^{b} s_j x^j$ for some $(s_j)_{j=0}^{b} \subset \mathbb{K}$. Then $\|z - P_a z\| = \|\sum_{j=a+1}^{b} s_j x^j\| \leq \max_{a<j\leq b} |s_j| \max_{a<j\leq b} \|x^j\| \leq |z|$, since

$$\|x^j\| = |\alpha|^{-\langle[3b_m - 2j]/\sqrt{4b_m}\rangle} \leq |\alpha|^{-\langle[b_m - 4(m-1)a_m]/\sqrt{4b_m}\rangle} \leq 1$$

for $j \in (a, b] \subset L_{m,2}$. We have $|z| = |f||y| = |\beta_m^{-1}||q||y| \leq |\beta_m^{-1}|R_m A_m < a_m^{-1}$. Thus $\|z - P_a z\| < a_m^{-1}$.

(S3) Let $t = x^{a_m} q y$ and $c = (m-1)a_m$. Clearly $t \in F_{2c}$; so $t = \sum_{j=a_m}^{2c} \gamma_j x^j$ for some $(\gamma_j)_{j=a_m}^{2c} \subset \mathbb{K}$. For $j \in [a_m, c]$ we have $j + b_m \in K_{m,1}$. Thus $\|\beta_m^{-1} x^{j+b_m} - x^j\| = |\beta_m^{-1}|\|f_{j+b_m}\| = |\beta_m^{-1}|$. Hence we get

$$\begin{aligned}\|P_a z - P_c t\| &= \|P_a(\beta_m^{-1} x^{b_m} t) - P_c t\| = \Big\| P_a\Big(\sum_{j=a_m}^{2c} \gamma_j \beta_m^{-1} x^{j+b_m}\Big) - P_c\Big(\sum_{j=a_m}^{2c} \gamma_j x^j\Big)\Big\| \\ &= \Big\| \sum_{j=a_m}^{c} \gamma_j \beta_m^{-1} x^{j+b_m} - \sum_{j=a_m}^{c} \gamma_j x^j \Big\| \leq \max_{a_m \leq j \leq c} |\gamma_j| \|\beta_m^{-1} x^{j+b_m} - x^j\| \\ &\leq |t||\beta_m^{-1}|.\end{aligned}$$

Thus $\|P_a z - P_c t\| \leq |\beta_m^{-1}|R_m A_m < a_m^{-1}$, since $|t| = |q||y| \leq R_m A_m$.

(S4) Using (3.8) we get

$$\begin{aligned}|P_c t - x^{(m-k+1)a_m}| &= |x^{a_m}(P_{c-a_m}(qy) - x^{(m-k)a_m})| \\ &= |P_{(m-2)a_m}(qy) - x^{(m-k)a_m}| < (a_m A_m)^{-1}.\end{aligned}$$

Hence $\|P_c t - x^{(m-k+1)a_m}\| \leq a_m^{-1}$.

(S5) By Lemma 1 we have $\|x^{(m-k+1)a_m} - x^0\| < |\alpha| a_{k-1}^{-1}$. Since

$$f(T)e - x^0 = (f(T)e - z) + (z - P_a z) + (P_a z - P_c t) + (P_c t - x^{(m-k+1)a_m}) + (x^{(m-k+1)a_m} - x^0),$$

we obtain $\|f(T)e - x^0\| \leq |\alpha| a_{k-1}^{-1} < \delta$.

We have shown that for every $\delta > 0$ there exists $f \in F$ such that $\|f(T)e - x^0\| < \delta$. It follows that $x^0 \in M$. Hence $x^n = T^n x^0 \in M$ for all $n \in \mathbb{N}$. Thus $F \subset M$, so $M = E$. ∎

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Faculty of Mathematics and Computer Science, A. Mickiewicz University, 61-614 Poznań, Poland
e-mail: sliwa@amu.edu.pl