Geometric Inequalities for Initial Data with Symmetries

by

© **Aghil Alaee**

A thesis submitted to the School of Gradate Studies in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Department of Mathematics and Statistics

Memorial University

December 2015

St. John's, Newfoundland and Labrador, Canada

Abstract

We consider a class of initial data sets $(\Sigma, \boldsymbol{h}, K)$ for the Einstein constraint equations which we define to be generalized Brill (GB) data. This class of data is simply connected, $U(1)^2$-invariant, maximal, and four-dimensional with two asymptotic ends. We study the properties of GB data and in particular the topology of Σ. The GB initial data sets have applications in geometric inequalities in general relativity. We construct a mass functional $\mathcal{M}$ for GB initial data sets and we show:(i) the mass of any GB data is greater than or equals $\mathcal{M}$, (ii) it is a non-negative functional for a broad subclass of GB data, (iii) it evaluates to the ADM mass of reduced $t - \phi^i$ symmetric data set, (iv) its critical points are stationary $U(1)^2$-invariant vacuum solutions to the Einstein equations. Then we use this mass functional and prove two geometric inequalities: (1) a positive mass theorem for subclass of GB initial data which includes Myers-Perry black holes, (2) a class of local mass-angular momenta inequalities for $U(1)^2$-invariant black holes. Finally, we construct a one-parameter family of initial data sets which we show can be seen as small deformations of the extreme Myers-Perry black hole which preserve the horizon geometry and angular momenta but have strictly greater energy.

This dissertation dedicated to my wife Mona for all her support, inspiration, and love.

Acknowledgements

I would like to express my sincere gratitude to my supervisors, Dr. Hari K. Kunduri and Dr. Ivan Booth, for their guidance, understanding, ideas, and encouragement during my PhD at the Memorial University. Most importantly for providing a great scientific and friendly environment. Besides my supervisors, I would like to thank my supervisor committee, Dr. David Pike and Dr. Ronald Haynes, for their helpful comments in improving this dissertation.

My sincere thanks also goes to my thesis examination members, Dr. Chris Radford, Dr. Sergio Dain, and Dr. Jie Xiao, for their great comments about this thesis. In particular, I would like to express my gratitude to Dr. Sergio Dain for his inspiring talk in Memorial University in July 2012 which was the main motivation for this dissertation.

Most importantly, I would like to thanks my parents and my wife. My parents receive my deepest gratitude and love for their dedication and the many years of support which provided the foundation of this work. My most heartfelt thanks goes to my wife, Mona, who is my best friend and without her love, support, and encouragement this dissertation was impossible.

Table of contents

List of tables

List of figures

Chapter 1

Introduction

General relativity (GR) is a geometrical theory of gravity which was developed by Albert Einstein in 1915. In this theory, gravity as a natural phenomena corresponds to the geometry of spacetime by the Einstein equations. Therefore, each quantity in GR has both physical interpretation and a precise geometrical definition. Moreover, one of the main results of any geometrical theory is isoperimetric inequality. A classical example is the isoperimetric inequality for closed plane curves given by

$$L^2 \geq 4\pi A\,, \tag{1.1}$$

where A is the area enclosed by a curve C of length L, and the inequality is saturated if and only if the curve is a circle (see [127] for an exposition of the topic). These types of inequalities arise in many areas of the mathematics.

Moreover, many of these inequalities arise in GR where they correspond to some physical expectations. Note that from the geometrical perspective and without any physical intuition, it would be impossible to conjecture any isoperimetric inequalities in GR. For example, physically we expect the total energy of the universe should be non-negative. This is one of the major developments in mathematical relativity, that is the Schoen-Yau [136] positive mass theorem

$$m \geq 0\,, \tag{1.2}$$

for an asymptotically flat spacetime that satisfies dominant energy condition, where m is the ADM mass and the equality happens if and only if the spacetime is Minkowski.

A black hole is one of the most mysterious objects in GR and in the universe. In four dimensions, stationary black holes have many interesting features such as uniqueness theorem [42, 143], rigidity theorem [82], topological censorship theorem [69, 72], and stability [9]. The uniqueness theorem of the stationary black holes shows that a black hole can be characterized by its mass m, angular momentum J, and charge q. Then the cross section area A of the event horizon (we denote event horizon by $N = \mathbb{R} \times H$ and H is its cross section) as a geometrical quantity can be expressed with these quantities and they satisfy some geometric inequalities [54]. Note that dynamical black holes cannot be characterized by some parameters similar to the stationary case but can we generalize the same types of geometric inequalities for dynamical black holes?

One of the important open problems in GR is the Penrose inequality (see the review article [115]). This inequality relates the ADM mass to the cross section area A of the event horizon:

$$m \geq \sqrt{\frac{A}{16\pi}}\,, \tag{1.3}$$

where the inequality is saturated if and only if the solution is the Schwarzschild black hole [128]. The Riemannian version of the Penrose inequality has been proved by Huisken and Ilmanen [99] and Bray [22]. But here we are interested in the geometric inequalities with symmetries. In general setting, angular momentum is not a conserved quantity, however, one can assume appropriate symmetry and energy condition to obtain a conserved quasi-local definition for angular momentum. This conserved quantity leads to some geometric inequalities.

First, Dain proved the following inequality

$$m \geq \sqrt{|J|}\,, \tag{1.4}$$

for complete, maximal, asymptotically flat axisymmetric vacuum initial data to the 3+1 dimensional Einstein equation. Here m is the ADM mass associated with the data and J is the conserved angular momentum associated with the $U(1)$ isometry [51, 53]. In contrast to the Penrose inequality, this inequality is saturated for the extreme Kerr black hole.

The mass-angular momentum inequality has been discussed and studied by numerous mathematicians and physicists from many different directions. They add multiple ends to the initial data [44], include conserved charges [41], and investigate non-maximal initial data [30]. We explore these developments in Chapter 3. However, to the best of the author's knowledge, the mass-angular momenta inequality has not yet obtained any attention in higher dimensions.

Recently, the investigation of general relativity in higher dimensions has attracted a great deal of interest for a number of physical reasons, such as the gauge theory-gravity correspondence and string theory (see review article [65]). Research in this field is also of intrinsic interest in mathematical physics and Riemannian geometry. There exist several important open questions that need to be answered in higher dimensions and in particular exploring geometric inequalities (mass-angular momenta inequality, mass-charge-angular momenta inequality) and the stability of black hole solutions.

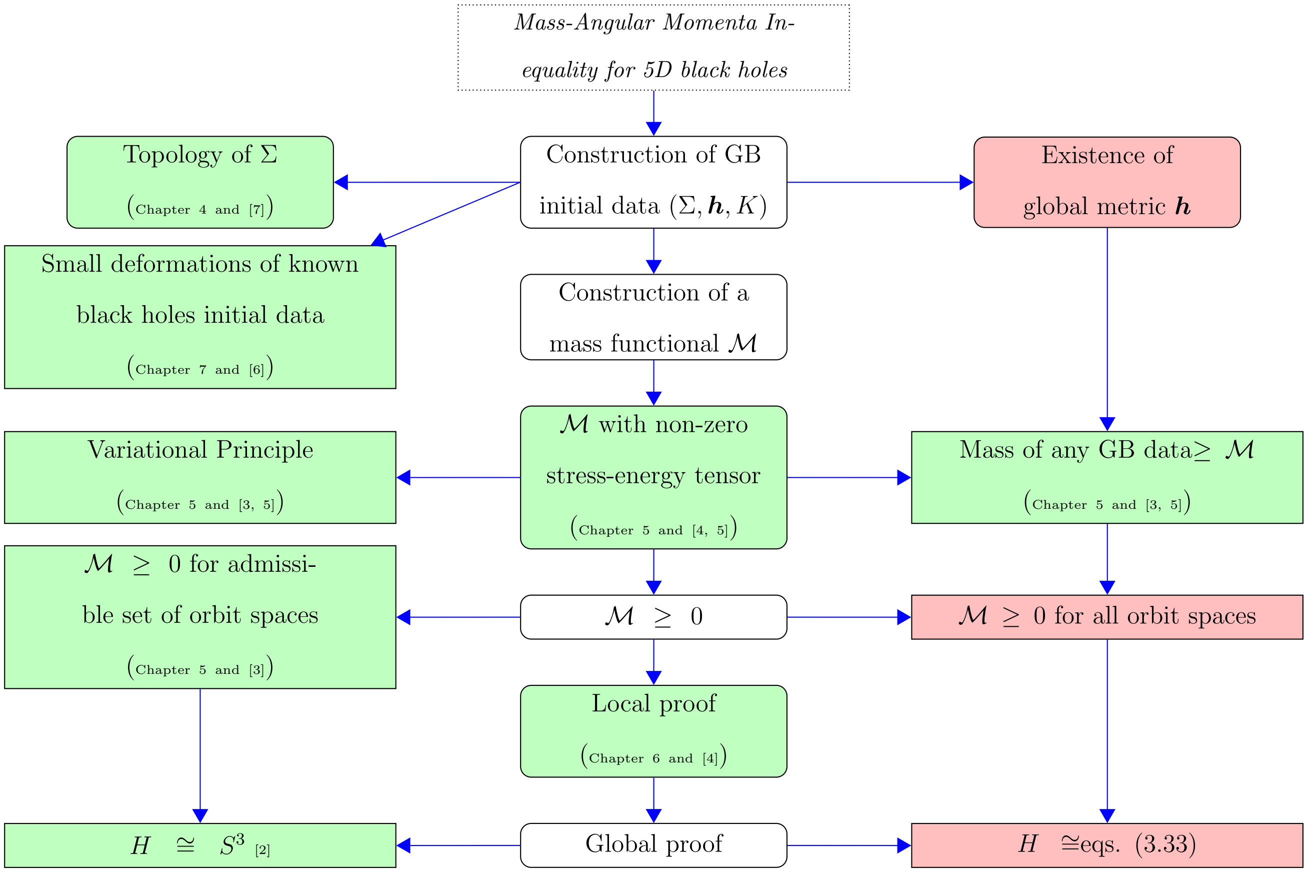
The physical motivation for these types of inequalities is the uniqueness of the stationary higher dimensional black holes with symmetry, see articles [67, 96] and review article [93]. This theorem shows a D-dimensional, analytic, stationary black hole (M, g) with $U(1)^{D-3}$ symmetry can be characterized by its angular momenta J_i, for $i = 1, \ldots, D-3$, the mass, and the *orbit space structure*, which is the boundary of the Riemannian smooth manifold $\mathcal{B} = M/\mathbb{R} \times U(1)^{D-3}$. In addition the known explicit solutions have some relations between their mass and angular momenta. This suggests existence of the geometrical inequalities between these quantities even for dynamical black holes in higher dimensions. Note that the only dimensions which can be an asymptotically flat spacetime with $U(1)^{D-3}$ symmetry are $D = 4, 5$.

In five dimensions, we have potentially two known candidates for minimizer of mass angular momenta inequality: extreme Myers-Perry black holes with $H \cong S^3$ [124], and extreme black rings with $H \cong S^1 \times S^2$ [64]. The mass of these solutions satisfy

$$M^3 = \frac{27\pi}{32}\left(|J_1| + |J_2|\right)^2 \qquad \text{(Myers-Perry)}\,, \tag{1.5}$$

$$M^3 = \frac{27\pi}{4}|J_1|(|J_2| - |J_1|) \qquad \text{(black ring)}\,, \tag{1.6}$$

where J_i are conserved angular momenta computed in terms of Komar integrals. These solutions have distinct orbit space structure. This suggests in each orbit space structure one expects a different minimizer. The central goal of this thesis is to generalize the mass-angular momenta inequalities and study the geometrical and topological aspects of five-dimensional black hole slices.



In the above flowchart, we show the steps of the procedure to obtain mass-angular momenta inequality and related chapters in this thesis. The red color means that this part remains an open problem and the green color means that we have results. As a first step towards establishing a mass-angular momenta inequality in five dimensions, we study initial data $(\Sigma, \boldsymbol{h}, K)$ of Einstein constraint equations with symmetries. First, we define n-dimensional $(n \geq 3)$ $t - \phi^i$ symmetric initial data and demonstrate properties of this class of data in Chapter 4. In fact, constant time slices of all stationary, vacuum, $U(1)^{D-3}$ spacetimes belong to this class of initial data. Then we consider a general $U(1)^2$-invariant, asymptotically flat initial data and we define as generalized Brill initial (GB) data. Then we investigate the three components of the GB initial data. In particular, we show that for appropriate energy conditions, global twist potentials exist and the norm of the extrinsic curvature K has a lower bound by a function of twist potentials Y^i and norm of the Killing vectors. Moreover, we study the possible topologies for the Riemannian manifold Σ. Note that the global existence of the slice metric $\boldsymbol{h}$ remains as an open problem.

Secondly, we investigate a generalization of Dain's mass functional $M(v, Y)$ to $D > 4$ for GB initial data. Note that most of the local analysis works equally well for $D-$dimensional spacetimes with $U(1)^{D-3}$ isometry. However, as we explain such spacetimes could only be asymptotically flat for $D = 5$. We construct the mass functional $\mathcal{M}$ which depends on five functions (v, λ', Y) in Chapter 5, where v is a function, λ' is a symmetric positive definite 2×2 matrix with $\det \lambda' = \rho^2$, and $Y = (Y^1, Y^2)$ is a column vector. We show that critical points of $\mathcal{M}$ are stationary, vacuum, $U(1)^2$-invariant, asymptotically flat spacetimes. Moreover, it is a non-negative functional for a class of orbit space which we define to be *admissible set*. By this functional we recover a positive mass theorem for GB initial data sets.

In Chapter 6, we prove the main result of this thesis and in particular we establish a class of local mass-angular momenta inequality for GB initial data sets. The argument of the proof is similar to Dain's argument [51]. However, the level of complexity increases because of more functions and different orbit spaces. We show that for different orbit spaces we have different minimizers. Moreover, in Chapter 7 we study small deformations of extreme Myers-Perry initial data set. We construct a one-parameter family of initial data with similar properties as the extreme Myers-Perry initial data. In particular this family has same angular momenta, geometries of the ends, and area of the event horizon. However, by the local mass angular inequality the mass of this family has greater energy than the extreme Myers-Perry initial data. The argument of the proof is by implicit function theorem and a classical result about the Poisson operator. Finally, except where reference is made to the work of others, all the results are original and based on the following articles.

(AA.1) Alaee, A., H. K. Kunduri, and E. M. Pedroza (2014), Notes on maximal slices of five dimensional black holes, Classical and Quantum Gravity, 31 (5), 055,004

(AA.2) Alaee, A., and H. K. Kunduri (2014), Mass functional for initial data in 4+ 1-dimensional spacetime, Physical Review D, 90 (12), 124,078

(AA.3) Alaee, A., and H. K. Kunduri (2015), Proof of the local mass-angular momenta inequality for $U(1)^2$ invariant black holes, Classical and Quantum Gravity, 32 (16), 165,020.

(AA.4) Alaee, A., and H. K. Kunduri (2015), Small deformations of extreme five dimensional myers perry black hole initial data, General Relativity and Gravitation, 47 (2), 129.

(AA.5) Alaee, A., and H. K. Kunduri (2015), Remarks on mass and angular momenta for $U(1)^2$ -invariant initial data, arXiv preprint arXiv:1508.02337, submitted to Journal of Mathematical Physics (JMP).

(AA.6) Alaee, A., Khuri, M., and H. K. Kunduri (2015), Proof of the mass-angular momenta inequality for Bi-axisymmetric black holes with spherical topology, arXiv preprint arXiv:1510.06974, submitted to Journal of Communication in Mathematical Physics (CMP).

Chapter 2

Preliminaries

In this chapter, we review some mathematical preliminaries which we will use in this thesis. We assume that the reader is familiar with basic differential geometry and then we introduce some notational conventions and collect some definitions and theorems in Section 2.1. We also review in more detail the (Bartnik's) weighted Sobolev spaces and Poisson operator in Section 2.2. We refer the interested reader to [16, 118] for an exposition of the topic.

2.1 Differential geometry

In this section, we collect some basic differential geometry concepts fixing notations and definitions.

2.1.1 Notation

We consider an n-dimensional *smooth manifold* M as a topological (i.e. Hausdorff, second countable, locally looks like $\mathbb{R}^n$) manifold with a maximal smooth atlas (smooth structure[1]) $\mathcal{A} \equiv \{(U_i, \phi_i) : M \subset \cup_i U_i$ and $\phi_i \circ \phi_j^{-1}$ is C^∞ $\}$. To fix the notation we denote a (p,q)-tensor by $\boldsymbol{T}$ as a section of $(TM)^{\otimes_p} \otimes (T^*M)^{\otimes_q}$, i.e. $\boldsymbol{T} \in \Gamma\left((TM)^{\otimes_p} \otimes (T^*M)^{\otimes_q}\right)$. A semi(pseudo)-Riemannian manifold is a pair $(M, \boldsymbol{g})$ where M is a smooth manifold and $\boldsymbol{g}$ is a non-degenerate, symmetric, $(0,2)$ tensor with signature $(\underbrace{-,\ldots,-}_{s \text{ times}}, +, \cdots, +)$ with s minus signs (i.e. $\boldsymbol{g} \in \Gamma\left[S^2(T^*M)\right]$) such that in coordinate chart (U, x) we have

$$\boldsymbol{g} = g_{ab} \mathrm{d}x^a \mathrm{d}x^b, \qquad \boldsymbol{g}^{-1} = g^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b}\,. \tag{2.1}$$

Returning to notational issues, we denote the inner product associated to $\boldsymbol{g}$ on $(TM)^{\otimes_p} \otimes (T^*M)^{\otimes_q}$ by $\langle \cdot, \cdot \rangle_{\boldsymbol{g}}$ and norm $|T|^2_{\boldsymbol{g}} = \langle T, T \rangle_{\boldsymbol{g}}$. We denote the frame on M by $\{e_a\}$ and dual frame by $\{\theta^a\}$. On a semi-Riemannian manifold $(M, \boldsymbol{g})$, there exist musical isomorphisms $\flat : TM \to T^*M$ and $\# : T^*M \to TM$ such that $\flat(X) \equiv X^\flat = g(X, \cdot)$ and $\#(\omega) \equiv \omega^\# = g^{-1}(\omega, \cdot)$ [104]. *Riemannian* and *Lorentzian manifolds* are special cases of semi(pseudo)-Riemannian manifolds with signatures $l = (l, +, \ldots, +)$ where $l = 1$ and $l = -1$, respectively. Associated to metric $\boldsymbol{g}$ there is a (torsion free and compatible) connection which is denoted by $\boldsymbol{\nabla}$. Then the *Christoffel symbols* related to the connection $\boldsymbol{\nabla}$ is Γ^a_{bc} which are defined by $\Gamma^a_{bc} = \theta^a\left(\boldsymbol{\nabla}_{e_a} e_c\right)$. In general, we denote and define the trace, divergence, and Laplacian respect to $\boldsymbol{g}$ for a (0,2)-tensor $\boldsymbol{T}$ in local frame by $\mathrm{Tr}_{\boldsymbol{g}} T_{ab} \equiv g^{ab} T_{ab}$, $\mathrm{div}_{\boldsymbol{g}} \boldsymbol{T} = g^{ab} \boldsymbol{\nabla}_a T_{bc}$, and $\Delta_{\boldsymbol{g}} \equiv \mathrm{div}_{\boldsymbol{g}} \boldsymbol{\nabla} \boldsymbol{T}$.

Coming back to arbitrary semi-Riemannian manifold $(M, \boldsymbol{g})$, the Riemannian,

[1]In general, a smooth manifold can have different smooth structures, e.g. $\mathbb{R}^4$ has an infinite number of smooth structures.

Ricci, scalar curvatures in local chart are

$$(Rm_{\boldsymbol{g}})_{abc}{}^{d} \equiv R_{abc}{}^{d} = 2\partial_{[a}\Gamma^{d}_{|b|c]} + 2\Gamma^{d}_{[a|e|}\Gamma^{e}_{b]c}\,, \tag{2.2}$$

$$(Ric_{\boldsymbol{g}})_{bc} \equiv R_{bc} = \boldsymbol{g}^{ea}\boldsymbol{g}_{ed}(Rm_{\boldsymbol{g}})_{abc}{}^{d}\,, \tag{2.3}$$

$$R_{\boldsymbol{g}} \equiv \mathrm{Tr}_{\boldsymbol{g}} Ric_{\boldsymbol{g}}\,. \tag{2.4}$$

Then in $n = 2$, $R_{\boldsymbol{g}}$ determines the full curvature tensor and in $n = 3$, $Ric_{\boldsymbol{g}}$ determines the full curvature tensor. However, for $n \geq 4$ the Riemannian curvature has another component, *Weyl tensor*

$$(\mathcal{W}_{\boldsymbol{g}})_{abcd} \equiv (Rm_{\boldsymbol{g}})_{abcd} - \frac{2}{n-2}\left(g_{a[c}\,(Ric_{\boldsymbol{g}})_{d]b} - g_{b[c]}\,(Ric_{\boldsymbol{g}})_{d]a}\right) + \frac{2}{(n-1)(n-2)} R_{\boldsymbol{g}}\, g_{a[c}g_{d]b}\,, \tag{2.5}$$

which together with Ricci tensor determines the full curvature. In all definitions, we use subindex $\boldsymbol{g}$ to indicate the curvature tensor related to connection associated to $\boldsymbol{g}$, but in general one can have a connection and compute all curvature tensors [106].

To continue fixing notations, we denote the collection of p-forms (axisymmetric $(0, p)$-tensor field) by $\Lambda^p(M)$. Then the wedge product is a map $\wedge : \Lambda^p(M) \times \Lambda^q(M) \to \Lambda^{p+q}(M)$ such that $(\alpha \wedge \beta)_{a_1 \cdots a_p b_1 \cdots b_q} \equiv \frac{(p+q)!}{p!q!}\alpha_{[a_1 \cdots a_p}\beta_{b_1 \cdots b_q]}$ and $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$ for $\alpha \in \Lambda^p(M)$ and $\beta \in \Lambda^q(M)$. Let $\delta^{a_1 \ldots a_n}_{b_1 \ldots b_n} \equiv n!\delta^{a_1}_{[b_1} \cdots \delta^{a_n}_{b_n]}$ be generalized Kronecker delta. Then the Levi-Civita tensor is $\delta^{1 \ldots n}_{a_1 \ldots a_n} = (+1, -1, 0)$ and we define Levi-Civita density tensor

$$\epsilon_{a_1 \ldots a_n} \equiv \sqrt{|\det g|}\,\delta^{1 \ldots n}_{a_1 \ldots a_n}\,, \tag{2.6}$$

such that the volume element is $\eta \equiv \frac{1}{n!}\epsilon_{a_1 \ldots a_n}\,\theta^{a_1} \wedge \cdots \wedge \theta^{a_n} = \sqrt{|\det g|}\theta^1 \wedge \cdots \wedge \theta^n$. Note that $\delta^{1 \ldots n}_{a_1 \ldots a_n}\delta^{a_1 \ldots a_n}_{1 \ldots n} = (-1)^s$ which implies $\epsilon^{a_1 \ldots a_n} = (-1)^s \frac{1}{\sqrt{|\det g|}}\delta^{1 \ldots n}_{a_1 \ldots a_n}$

In addition, the interior multiplication(derivative) and exterior derivative are $\iota : \Lambda^p(M) \to \Lambda^{p-1}(M)$ and $\mathrm{d} : \Lambda^p(M) \to \Lambda^{p+1}(M)$, respectively such that $\iota_X\alpha \equiv \alpha(X)$

and $\mathrm{d}\alpha \equiv (p+1)\nabla_{[b}\alpha_{a_1\cdots a_p]}$ for $\alpha \in \Lambda^p(M)$. For any two p-form $\alpha, \beta \in \Lambda^P$, the inner product is $\langle \alpha, \beta \rangle_{\boldsymbol{g}} = \frac{1}{p!}\alpha_{a_1\cdots a_p}\beta^{a_1\cdots a_p}$. Moreover, one can define interior multiplication by $\langle \iota_X \alpha, \beta \rangle_{\boldsymbol{g}} = \left\langle \alpha, X^\flat \wedge \beta \right\rangle_{\boldsymbol{g}}$ for $\alpha \in \Lambda^{p+1}(M)$ and $\beta \in \Lambda^p(M)$. Then the *Hodge star* is an isometry operator $\star : \Lambda^p(M) \to \Lambda^{n-p}(M)$ defined by $(\star\alpha)_{a_{p+1}\cdots a_n} \equiv \frac{1}{p!}\epsilon_{a_1\cdots a_n}\alpha^{a_1\cdots a_p}$ and

$$\star \left(\mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_p}\right) = \frac{1}{(n-p)!}\epsilon_{b_1\cdots b_{n-p}}{}^{a_1\cdots a_p}\left(\mathrm{d}x^{b_1} \wedge \cdots \wedge \mathrm{d}x^{b_{n-p}}\right) \tag{2.7}$$

with properties $\star 1 = \eta$ and $\star\eta = (-1)^s$. The Hodge dual and Hodge inverse are $\star^2\alpha = (-1)^{p(n-p)+s}\alpha$ and $\star^{-1}\alpha = (-1)^{p(n-p)+s} \star \alpha$ for $\alpha \in \Lambda^p(M)$. We have the following results

Lemma 1. *[106]*

1. $\langle \alpha, \beta \rangle_{\boldsymbol{g}}\, \eta = \alpha \wedge \star\beta = \beta \wedge \star\alpha = (-1)^s \langle \star\alpha, \star\beta \rangle_{\boldsymbol{g}}\, \eta$ *for* $\alpha, \beta \in \Lambda^p(M)$.

2. $\alpha \wedge \beta = (-1)^s \langle \star\alpha, \beta \rangle_{\boldsymbol{g}}\, \eta$ *for* $\alpha \in \Lambda^p(M)$ *and* $\beta \in \Lambda^{n-p}(M)$.

3. $\iota_X\alpha = (-1)^{n(p-1)+s} \star \left(X^\flat \wedge \star\alpha\right)$ *and* $\iota_X \star \alpha = (-1)^{np} \star \left(\alpha \wedge X^\flat\right)$ *for* $\alpha \in \Lambda^p(M)$ *and* $X \in \mathcal{X}(M)$.

4. $d\left(\alpha \wedge \star\beta\right) = d\alpha \wedge \star\beta + (-1)^{p-1}\alpha \wedge d \star \beta$ *for* $\alpha \in \Lambda^{p-1}(M)$ *and* $\beta \in \Lambda^p(M)$.

Finally, the adjoint of d is $\delta : \Lambda^p(M) \to \Lambda^{p-1}(M)$ which defined by $\delta\alpha \equiv -(-1)^{n(p+1)+s} \star \mathrm{d} \star \alpha$. More precisely, δ is the adjoint of d with respect to L^2 inner product $\langle \cdot, \cdot \rangle = \int_M \langle \cdot, \cdot \rangle_{\boldsymbol{g}}\, \eta$ where $\langle \mathrm{d}\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$. On Λ^p one can define the divergence and Hodge Laplacian by $\mathrm{div} = -\delta$ and $\Delta_H = -\left(\mathrm{d}\delta + \delta\mathrm{d}\right)$.

2.1.2 Frobenius Theorem and applications

In this thesis, we will work with Riemannian submanifolds of a semi-Riemannian manifold with signature $l = \pm 1$. Also it is important to write the relation of curvature tensors between a submanifold and the ambient manifold. Let $(M, \boldsymbol{g})$ be a semi-Riemannian manifold with signature $l = \pm 1$ and the manifold $\Sigma \subset M$ is a *hypersurface* (codimension 1 submanifold) with a unit normal vector field n, i.e. $g(n, X) = 0$ and $g(n, n) = l$ for all $X \in T_p\Sigma$. Then the *first fundamental form* (metric) and *second fundamental form* (extrinsic curvature) are

$$\boldsymbol{h} \equiv -l\mathbf{n} \otimes \mathbf{n} + \boldsymbol{g} \qquad K(X, Y) \equiv \langle \boldsymbol{\nabla}_X n, Y \rangle \,, \tag{2.8}$$

for every $X, Y \in T_pM$.

As usual, the trace of extrinsic curvature $H = \mathrm{Tr}_{\boldsymbol{h}} K$ is called *mean curvature*. Then $(\Sigma, \boldsymbol{h})$ is a *maximal (minimal) hypersurface* of an ambient manifold with signature $l = -1$ (resp. $l = 1$) if $H = 0$. Note in local frame we use Greek letters $\alpha, \beta, \ldots$ for indexes on manifold $(M, \boldsymbol{g})$ and Latin letters $a, b \ldots$ for indexes on hypersurface $(\Sigma, \boldsymbol{h})$. Moreover, the relation between the curvature tensors of submanifold and ambient manifold in a local frame on Σ are

$$(Rm_{\boldsymbol{g}})_{abcd} = (Rm_{\boldsymbol{h}})_{abcd} + 2lK_{a[c}K_{b]d} \,, \tag{2.9}$$

$$(Ric_{\boldsymbol{g}})_{ad} - l\,(Rm_{\boldsymbol{g}})_{abcd}\, n^b n^c = (Ric_{\boldsymbol{h}})_{ad} + l\,(K_{ac}K^c{}_d - HK_{ad}) \,, \tag{2.10}$$

$$R_{\boldsymbol{g}} - 2l\,(Ric_{\boldsymbol{g}})_{ac}\, n^a n^c = R_{\boldsymbol{h}} + l\left(|K|^2_{\boldsymbol{h}} - H^2\right) \,, \tag{2.11}$$

$$\nabla_b K_{ac} - \nabla_a K_{bc} = (Rm_{\boldsymbol{g}})_{abcd}\, n^d \,, \tag{2.12}$$

where (2.9) and (2.12) are *Gauss* and *Codazzi* equations, respectively.

Returning to arbitrary semi-Riemannian manifold $(M, \boldsymbol{g})$, an *m-distribution* $\mathcal{D}^m_p$ is an m-dimensional subspace of T_pM for each $p \in M$ and it is smooth distribution if $\mathcal{D}^m \equiv \sqcup_{p \in M} \mathcal{D}^m_p$ is smooth subbundle of TM [109]. A distribution $\mathcal{D}^m_p$ is *involutive* if $[X, Y] \in \mathcal{D}^m_p$ for all $X, Y \in \mathcal{D}^m_p$. An m-dimensional immersed submanifold Σ of M is *integrable* if $T_p\Sigma = \mathcal{D}_p^m$ for each $p \in \Sigma$. It is straightforward to show every integrable distribution is involutive [109]. Moreover, the converse is the following result

Theorem 2 (Frobenius theorem, [109])**.** *Let $(M, \boldsymbol{g})$ be a smooth n-dimensional semi-Riemannian manifold. An m-dimensional distribution $\mathcal{D}_p^m$ is integrable if and only if it is involutive.*

Coming back to derivatives on semi-Riemannian manifolds, we define another type of derivative which is related to isometry group of a semi-Riemannian manifold. We denote the *Lie derivative* of Killing vector X by $\mathcal{L}_X$ and it is defined for arbitrary (p,q)-tensor as

$$\mathcal{L}_X T|_p \equiv \lim_{t\to 0} \frac{\varphi_t^*(T)|_{\varphi_t(p)} - T|_p}{t}\,, \tag{2.13}$$

where φ_t denotes the flow (one-parameter family of diffeomorphism) of X and the asterisk stands for the pull-back. The general definition in local frame is

$$\mathcal{L}_X T^{a_1\cdots a_q}_{b_1\cdots b_p} \equiv X^c \partial_c T^{a_1\cdots a_q}_{b_1\cdots b_p} - \sum_{d=1}^{q} T^{a_1\cdots c\cdots a_q}_{b_1\cdots b_p} \partial_c X^{a_d} + \sum_{d=1}^{p} T^{a_1\cdots a_q}_{b_1\cdots c\cdots b_p} \partial_{b_d} X^c\,. \tag{2.14}$$

Then we have $(\mathcal{L}_X \boldsymbol{g})_{ab} = 2\nabla_{(a}X_{b)}$. Recall that a vector field X is Killing vector if flows φ_t of X are isometric maps. This means the sufficient and necessary condition for a flow φ_t to be an isometry is $\mathcal{L}_X \boldsymbol{g} = 0$. For a Killing vector field X we have the following useful equation

$$\nabla_a \nabla_b X_c = -R_{abc}{}^d X_d\,. \tag{2.15}$$

Observe that the collection of all isometries of $(M, \boldsymbol{g})$ is a group and it is called *isometry group* and denoted by $\mathrm{Iso}(M, \boldsymbol{g})$. Let $(M, \boldsymbol{g})$ be a pseudo-Riemannian manifold with arbitrary signature and Killing vector X, then for any $\alpha \in \Lambda^p(M)$ we have

$$\mathcal{L}_X \circ \mathrm{d} = \mathrm{d} \circ \mathcal{L}_X \alpha \,, \tag{2.16}$$

$$\mathcal{L}_X \star \alpha = \star \mathcal{L}_X \alpha \,, \tag{2.17}$$

$$\mathcal{L}_X \alpha = -\delta \left(\alpha \wedge X^\flat \right) + (-1)^{n+1} \left(X^\flat \wedge \delta \alpha \right) \,. \tag{2.18}$$

Now we prove a useful result and we use it in Chapter 4 to construct a traceless-transverse (TT) tensor which represents extrinsic curvature of $t - \phi^i$ symmetric initial data.

Proposition 3. *Assume that $(M, \boldsymbol{h})$ is an n-dimensional Riemannian manifold with N commuting Killing vector fields $\xi_{(i)}$, i.e. $[\xi_{(i)}, \xi_{(j)}] = 0$, $i, j = 1, \ldots, N$. Assume $n - N$ dimensional distribution $\mathcal{D}^{n-N}$ orthogonal to $\xi_{(i)}$ is integrable. Then we have the following identity*

$$\nabla_a \Phi_b = \nabla_{[a} \log \lambda \Phi_{b]} \,, \tag{2.19}$$

where $\lambda = [\lambda_{ij}] = [h(\xi_{(i)}, \xi_{(j)})]$ is Gram matrix of the Killing fields (a symmetric positive definite $N \times N$ matrix) and $\Phi = (\xi_{(1)}, \ldots, \xi_{(N)})^t$ is a column vector and t denotes transposition of a matrix.

Proof. According to Frobenius theorem $\mathcal{D}^{n-N}$ is integrable if and only if

$$\nabla_{[a} \xi_{(i)b]} = \sum_{j=1}^{N} l_{(ij)[a} \xi_{(j)b]}, \qquad i, j = 1, \ldots, N \tag{2.20}$$

where $l_{(ij)a}$ is a row vector for fixed i and $A_a = [l_{(ij)a}]$ is arbitrary matrix of one forms. Then we choose $l_{(ij)a}$ such that they are orthogonal to $\xi_{(i)}$. Since these Killing vectors are commuting, we have the following identity

$$\xi^c_{(i)}\nabla_c\xi^\flat_{(j)a} = \xi^c_{(j)}\nabla_c\xi^\flat_{(i)a} = \frac{1}{2}\nabla_a\lambda_{ij}\,. \tag{2.21}$$

Now fixing i and multiplying equation (2.20) by $\xi^b_{(k)}$ for $k = 1, \ldots, N$ and applying the equation (2.21), we obtain

$$\nabla_a\lambda_{ik} = l_{(ij)a}\lambda_{jk}\,. \tag{2.22}$$

Then we have N equations and N unknown l_{ij}. This is a solvable system and the solution is

$$A_a = \nabla_a\lambda\lambda^{-1} = \nabla_a \log\lambda\,. \tag{2.23}$$

Therefore, if we substitute (2.23) in (2.20) we have

$$\nabla_{[a}\xi_{(i)b]} = \sum_{j=1}^{N}\nabla_{[a}\log\lambda_{(ij)}\xi_{(j)b]}, \qquad i,j = 1,\ldots,N\,. \tag{2.24}$$

Thus the result (2.19) follows. □

2.1.3 Conformal geometry

In this section, we briefly review curvature relations between two conformal metrics. These relations are useful tools in general relativity and in particular for finding solutions of the Lichnerowicz equation. Thus, we have

Proposition 4. *[19, Theorem 1.159] Assume that* $(M, \boldsymbol{g})$ *is a semi-Riemannian manifold and* $u : M \longrightarrow \mathbb{R}$ *and* $\boldsymbol{g} = e^{-2u}\boldsymbol{h}$*. Let* $\boldsymbol{\nabla}$ *and* ∇ *be connections associated to* $\boldsymbol{g}$ *and* $\boldsymbol{h}$*, respectively. Then we have the following relation for Christoffel symbols,* *connection, and curvature tensors of* $\boldsymbol{g}$ *and* $\boldsymbol{h}$

$$ {}^{\boldsymbol{g}}\Gamma^i_{jk} = h^{il}\left(-(\partial_j u)h_{lk} - (\partial_k u)h_{lj} + (\partial_l u)h_{jk}\right) + \Gamma^i_{jk}, \tag{2.25} $$

$$ \boldsymbol{\nabla}_X Y = \nabla_X Y - du(X)Y - du(Y)X + h(X,Y)\nabla u, \tag{2.26} $$

$$ Rm_{\boldsymbol{g}} = e^{-2u}\left[Rm_{\boldsymbol{h}} + \left(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2_{\boldsymbol{h}}\boldsymbol{h}\right) \owedge \boldsymbol{h}\right], \tag{2.27} $$

$$ \mathcal{W}_{\boldsymbol{g}} = e^{-2u}\mathcal{W}_{\boldsymbol{h}}, \tag{2.28} $$

$$ Ric_{\boldsymbol{g}} = (n-2)\left[\nabla^2 u + \frac{\boldsymbol{h}}{(n-1)}\Delta_{\boldsymbol{h}} u + du \otimes du - \frac{1}{2}|\nabla u|^2_{\boldsymbol{h}}\boldsymbol{h}\right] + Ric_{\boldsymbol{h}}, \tag{2.29} $$

$$ R_{\boldsymbol{g}} = e^{2u}\left[2(n-1)\Delta_{\boldsymbol{h}} u - (n-1)(n-2)|\nabla u|^2_{\boldsymbol{h}} + R_{\boldsymbol{h}}\right], \tag{2.30} $$

where $\owedge$ *is Kobayashi-Nomizu product which if* A, B *are symmetric* $(0,2)$*-tensor, it is defined by*

$$ (A \owedge B)_{abcd} = 2A_{a[c}B_{d]b} + 2B_{a[c}A_{d]b}. \tag{2.31} $$

The obvious consequence of Proposition 4 is the following result.

Corollary 5. *[19, Yamabe Equation] If* $n \neq 2$, *and* $\boldsymbol{g} = \Phi^{\frac{4}{n-2}}\boldsymbol{h}$, *then*

$$ -4\frac{n-1}{n-2}\Delta_{\boldsymbol{h}}\Phi + R_{\boldsymbol{h}}\Phi = R_{\boldsymbol{g}}\Phi^{\frac{n+2}{n-2}}. \tag{2.32} $$

Proof. Let $u = -\frac{2}{n-2}\log\Phi$, using the chain rule we have

$$ \nabla u = -\frac{2}{n-2}\frac{\nabla\Phi}{\Phi}, \quad \nabla^2 u = -\frac{2}{n-2}\left(\frac{\nabla^2\Phi}{\Phi} - \frac{\nabla\Phi\otimes\nabla\Phi}{\Phi}\right). \tag{2.33} $$

Substituting these into (2.30), we get

$$ R_{\boldsymbol{g}} = \Phi^{\frac{-n+2}{n-2}}\left(-4\frac{n-1}{n-2}\Delta_{\boldsymbol{h}}\Phi + R_{\boldsymbol{h}}\Phi\right). \tag{2.34} $$

□

2.1.4 Lie Group and Group Action

A *Lie group* is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$ given by $m(g, h) = g \cdot h$ and $i^{-1}(g) = g^{-1}$ are smooth. For example, the following are Lie groups: $GL(n, \mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : A \text{ is invertible}\}$, $U(n) = \{A \in GL(n, \mathbb{R}) : A^*A = I_n\}$, $SO(n) = \{A \in GL(n, \mathbb{R}) : \det A = 1\}$, $T^n = U(1) \times \cdots \times U(1)$. But one of the important applications of a Lie group is action of the Lie group on a manifold. A left (right) action of G on M is a smooth map $\theta : G \times M \to M$, often written as $(g, p) \to g \cdot p$ (resp. $p \cdot g$) with associativity and identity properties. We call it G-action for any $p \in M$, the *orbit* of p under the G-action is the set $O_p \equiv \{g \cdot p : g \in G\}$. The set of all orbits is a manifold with the quotient topology and denoted by $\mathcal{B} = M/G$.

To classify different G-actions, a G-action is *transitive* if for any two points $p, q \in M$, there is a group element g such that $g \cdot p = q$, or equivalently if the orbit of any point is all of M. Also given $p \in M$, the *isotropy group* of p, denoted by G_p, is the set of elements $g \in G$ that fix p, i.e. $G_p \equiv \{g \in G : g \cdot p = p\}$. Then an G-action is *free* if isotropy group is identity(the action has no fixed point). A G-action on M is called *isometric* if $(M, \boldsymbol{h})$ is a Riemannian manifold and $\theta_g : M \to M$ is an isometry for all $g \in G$. Moreover, if N is a set of isometries, then $N_F = \{x \in M : \varphi(x) = x \text{ for all } \varphi \in N\}$ is also a totally geodesic submanifold in M [106].

Now we express some basic and useful results.

Theorem 6. *[106] If G is a compact Lie group acting smoothly on a smooth manifold M, then there exists a G-invariant Riemannian metric on M.*

Theorem 7. *[109, Theorem 7.10] If G is a compact Lie group and acts freely on M, then there exists a smooth structure on $\mathcal{B} = M/G$ such that $\pi : M \to \mathcal{B}$ is a principal G-bundle (and, in particular, a submersion).*

Corollary 8. *[106] Orbits of compact Lie group actions are embedded submanifolds.*

2.2 Weighted Sobolev Spaces

In this section, we collect some results in weighted Sobolev spaces [15, 116, 118]. These spaces are fundamental tools to describe asymptotic behaviour of functions on a semi-Riemannian manifold. We denote the space of smooth functions with compact support in U by $C_c^\infty(U)$ such that $\phi \in C_c^\infty(U)$ is called *test function.* Assume $n \geq 3$ and $B_R^n(0) \subset \mathbb{R}^n$ is an n-dimensional open ball centered at the origin and having radius R. Define $E_R = \mathbb{R}^n - \overline{B_R^n(0)}$ as the *exterior region* associated to $B_R^n(0)$ and we denote $E_0 = \mathbb{R}^n - \{0\}$. Let $x = (x^i)$ for $i = 1, \ldots, n$ be a fixed coordinate on $\mathbb{R}^n$ such that the weight function is $r = |x| = \sqrt{(x^i)^2}$. Then we define

Definition 9. [15] The *weighted Lebesgue space* ${L'}^p_\delta$, $1 \leq p \leq \infty$, with weight $\delta \in \mathbb{R}$, is the space of measurable functions in $L^p_{loc}(E_0)$ with standard Lebesgue measure dx, such that the norm

$$\|u\|_{{L'}^p_\delta} = \begin{cases} \left(\int_{E_0} |u|^p r^{-\delta p - n}\, dx\right)^{1/p} & p < \infty \\ \operatorname{ess\,sup}_{E_0} \left(r^{-\delta}|u|\right) & p = \infty \end{cases} \tag{2.35}$$

is finite. Then the weighted Sobolev space ${W'}^{k,p}_\delta$ is defined in the usual way

$$\|u\|_{{W'}^{k,p}_\delta} = \sum_{j=0}^{k} ||D^j u||'_{p,\delta-j}\,. \tag{2.36}$$

Relevant properties of this weighted Sobolev space are summarized in the following lemma.

Lemma 10. *[15, 55, 116] Consider the weighted Lebesgue space and the weighted*

Sobolev space, L'^{p}_{δ} and $W'^{k,p}_{\delta}$ for $1 \le p, q \le \infty$, respectively. Then

1. *If $p \le q$ and $\delta_2 < \delta_1$ then $L'^{p}_{\delta_1} \subset L'^{q}_{\delta_2}$ and the inclusion is continuous.*

2. *For $k \ge 1$ and $\delta_1 < \delta_2$ the inclusion $W'^{k,p}_{\delta_1} \subset W'^{k-1,p}_{\delta_2}$ is compact.*

3. *If $1/p < k/n$ then $W'^{k,p}_{\delta} \subset C'^{0}_{\delta}$. The inclusion is continuous. That is if $u \in W'^{k,p}_{\delta}$ then $r^{-\delta}|u| \le C \|u\|_{W'^{k,p}_{\delta}}$. Further, as proved in [55], $\lim_{r \to 0} r^{-\delta}|u| = \lim_{r \to \infty} r^{-\delta}|u| = 0$.*

2.2.1 Poisson Operator

The main goal of this section is to consider the *Poisson operator* $\mathcal{P} = \Delta_{\boldsymbol{g}} - \alpha$ on scalar functions of an asymptotically Euclidean Riemannian manifold $(M, \boldsymbol{g})$ and collect a very classical result (see [118] and [116]), that is, $\mathcal{P}$ is an isomorphism from Sobolev space $W'^{2,p}_{\delta}$ to L'^{p}_{δ}. We start by the following definition.

Definition 11. [15] Let M be a smooth, connected, complete, n-dimensional Riemannian manifold $(M, \boldsymbol{g})$, and let $\rho < 0$. We say that $(M, \boldsymbol{g})$ is *asymptotically Euclidean* of class $W'^{k,p}_{\rho}$ if

- the metric $\boldsymbol{g} \in W'^{k,p}_{\rho}(M)$, where $1/p - k/n < 0$, and $\boldsymbol{g}$ is continuous,
- there exists a finite collection $\{N_i\}_{i=1}^{m}$ of open subsets of M and diffeomorphisms $\Phi_i : E_R \to N_i$ such that $M - \cup_i N_i$ is compact, and
- for each i, $\Phi_i^* \boldsymbol{g} - \bar{\boldsymbol{g}} \in W'^{k,p}_{\rho}(E_R)$.

The maps Φ_i are called *end charts* and the corresponding coordinates are end coordinates. Now, suppose that $(M, \boldsymbol{g})$ is asymptotically Euclidean, and let $\{\Phi_i\}_{i=1}^{m}$ be its collection of end charts. Let $K = M - \cup_i \Phi_i(E_{2R})$, so K is a compact manifold. The weighted Sobolev space $W'^{k,p}_\delta(M)$ is the subset of $W'^{k,p}_{\mathrm{loc}}(M)$ such that the norm

$$\|u\|_{W'^{k,p}_\delta(M)} = \|u\|_{W'^{k,p}(K)} + \sum_i \|\Phi_i^* u\|_{W'^{k,p}_\delta(E_R)} \tag{2.37}$$

is finite. We can define similarly weighted Lebesgue space $L'^p_\delta(M)$ and C'^k_δ and $C'^\infty_\delta(M) = \cap_{k=0}^\infty C'^k_\delta(M)$.

To prove that $\mathcal{P}$ is an isomorphism we start by the following estimate [31, 32, 116]

Lemma 12. *Suppose that $(M, \boldsymbol{g})$ is asymptotically Euclidean of class $W'^{2,p}_\rho$, $p > \frac{n}{2}$, $\rho < 0$. Then if $2 - n < \delta < 0$, $\delta' \in \mathbb{R}$, and $u \in W'^{2,p}_\delta$ we have*

$$\|u\|_{W'^{2,p}_\delta} \leq \|\mathcal{L}u\|_{L'^p_{\delta-2}} + \|u\|_{L'^p_{\delta'}} . \tag{2.38}$$

Then we have following weak maximum principle (Lemma 3.2 in [116])

Lemma 13. *Suppose $(M, \boldsymbol{g})$ is asymptotically Euclidean of class $W'^{k,p}_\rho$, $k \geq 2$, $k > \frac{n}{p}$, and suppose $\alpha \in W'^{k-2,p}_{\rho-2}$ and suppose $\alpha \geq 0$. If $u \in W'^{k,p}_{loc}$ satisfies*

$$-\Delta_{\boldsymbol{g}} u + \alpha u \leq 0 \tag{2.39}$$

and if $u^+ \equiv max(u, 0)$ is $o(1)$ on each end of M, then $u \leq 0$. In particular, if $u \in W'^{k,p}_\delta$ for some $\delta < 0$ and u satisfies (2.39), then $u \leq 0$.

Proof. Fix $\epsilon > 0$, and let $v = (u - \epsilon)^+$. Since $u^+ = o(1)$ on each end, we see that v is compactly supported. Moreover, since $u \in W'^{k,p}_{\mathrm{loc}}$ we have from Sobolev embedding that $u \in W'^{1,2}_{\mathrm{loc}}$ and hence $v \in W'^{1,2}$. Now,

$$\int_M (-v\Delta_{\boldsymbol{g}} u + \alpha uv)\, \mathrm{d}x \leq 0 \quad \Longrightarrow \quad \int_M -v\Delta_{\boldsymbol{g}} u \, \mathrm{d}x \leq -\int_M \alpha uv \, \mathrm{d}x \leq 0 \tag{2.40}$$

where $\mathrm{d}x$ denotes the Lebesgue measure on $(M, \boldsymbol{g})$. Since $\alpha \geq 0$, $v \geq 0$ and u is positive wherever $v \neq 0$. Integrating by parts we have

$$\int_M |\nabla v|^2_{\boldsymbol{g}} \, dx \leq 0 \tag{2.41}$$

since $\nabla u = \nabla v$ on the support of v. So v is constant and compactly supported, so it should be zero, i.e. $\max(u - \epsilon, 0) = 0$. Then we conclude $u \leq \epsilon$. Sending ϵ to 0 we have $u \leq 0$.

Now, if $u \in W'^{k,p}_\delta$, since $W'^{k,p}_\delta \subset C'^0_\delta$, we have $u \in C'^0_\delta$. Hence if $\delta < 0$, then $u^+ = o(1)$ and we can apply the above argument to u. □

Using Lemma 13, we can prove the following interesting theorem (see similar result in[15, 118]).

Theorem 14. *Suppose that $(M, \boldsymbol{g})$ is asymptotically Euclidean of class $W'^{2,p}_\rho$, $p > \frac{n}{2}$. Then if $2 - n < \delta < 0$ and $\alpha \in L'^p_{\delta-2}$, the operator $\mathcal{P} : W'^{2,p}_\delta \to L'^p_{\delta-2}$ is Fredholm with index 0. Moreover, if $\alpha \geq 0$ then $\mathcal{P}$ is an isomorphism.*

Proof. By the estimate in Lemma 12 and [32] this operator is Fredholm. Now we show that $\mathcal{P}$ is injective. Let $\mathcal{P}u = 0$ for $u \in W'^{2,p}_\delta$. Then by weak maximum principle we have $u = 0$ on M for $2 - n < \delta < 0$ and $\mathcal{P}$ is injective. To show that $\mathcal{P}$ is surjective, it suffices to show $\mathcal{P}^*$(adjoint operator) is injective from $L'^p_{2-n-\delta} \to W'^{-2,p}_{-n-\delta}$. Now let f_1 and f_2 be smooth and compactly supported in each end of M. We have from integration by parts

$$0 = \langle f_2, \mathcal{P}^*(f_1) \rangle = \langle \mathcal{P}(f_2), f_1 \rangle = \int_M \mathcal{P}(f_2) f_1 \, \mathrm{d}x \,. \tag{2.42}$$

Thus $\int_M \mathcal{P}(f_2) f_1 \, \mathrm{d}x = 0$ for all smooth and compactly supported f_2 in each end of M, then $f_1 = 0$ and $\mathcal{P}^*$ is injective. Then $\mathcal{P}$ is surjective. Therefore, $\mathcal{P}$ is an isomorphism. □

2.3 Implicit Function Theorem

In this section we define Fréchet derivative and state the implicit function theorem. We use this theorem in Chapter 7 of thesis.

Definition 15. Let X and Z be Banach spaces and x be a point in X and let G be a mapping from neighborhood of x into Z. Then G is called *Fréchet differentiable* at the point x if there exists a linear operator $DG(x) \in L(X, Z)$ such that

$$\lim_{v \to 0} \frac{||G(x+v) - G(x) - DG(x)[v]||_Z}{||x||_X} = 0. \tag{2.43}$$

The map G is called continuously differentiable (i.e. C^1) if the derivative $DG(x)$ as an element of $L(X, Y)$ depends continuously on x. Namely, for every $\epsilon > 0$ there exist $\delta > 0$ such that

$$||x_1 - x_2||_X < \delta \quad \Longrightarrow ||DG(x_1) - DG(x_2)||_{L(X,Y)} < \epsilon. \tag{2.44}$$

Remark 16. Let $G : X_1 \times \cdots \times X_k \to Y$ be a linear map between Banach spaces $X_1, ..., X_k$, and Y. Then we define the partial derivative with respect to i^{th} argument by

$$D_i G(x_1, ..., x_k)[x] = \frac{d}{dt} G(x_1, .., x_i + tx, ..., x_k)|_{t=0} \quad \text{for} \quad i = 1, ..., k. \tag{2.45}$$

Theorem 17. *[66, Implicit Function Theorem] Suppose U is a neighborhood of x_0 in X, V is a neighborhood of y_0 in Y and $G : X \times Y \to Z$ is C^1. Suppose $G(x_0, y_0) = 0$ and $D_2G(x_0, y_0) : Y \to Z$ define a bounded operator and it is an isomorphism. Then,* *there exists a neighborhood* W *of the* x_0 *in* X *and a continuously differentiable mapping* $f : W \to Y$ *such that* $G(x, f(x)) = 0$*. Moreover, for* $||x - x_0||_X$ *and* $||y - y_0||_Y$*,* $f(x)$ *is the unique solution* y *of the equation* $G(x, y) = 0$*.*

Chapter 3

General Relativity and Black Holes

In this chapter, we provide a survey of the Einstein equations and Einstein constraint equations in general relativity. In particular, we review the causal structure of a spacetime and some basic properties of the initial data $(\Sigma, \boldsymbol{h}, K)$ of Einstein equations in Section 3.1. In Section 3.2 we give a short overview of the ADM formalism of general relativity and related formulas of mass and angular momentum. Finally, we collect major results about D-dimensional black holes with $D \geq 4$ in Section 3.3. In particular, we review the current status of the geometric inequalities in black holes theory and emphasize some of the open problems which motivate this thesis.

3.1 The Einstein Equations and the Einstein Constraint Equations

A spacetime is a Lorentzian manifold and denoted by a pair $(M, \boldsymbol{g})$. According to the least action principle [143], the Einstein equations may be obtained by a variational principle of the following action which is stable under compact perturbations of the metric:

$$A = \int_M (R_{\boldsymbol{g}} + L_m)\, \mathrm{d}V_{\boldsymbol{g}} \tag{3.1}$$

where $R_{\boldsymbol{g}}$ and $\mathrm{d}V_{\boldsymbol{g}}$ are scalar curvature and volume form with respect to the metric $\boldsymbol{g}$, respectively,and L_m is the Lagrangian associated with non-gravitational fields. Then Einstein field equations obtained from the variation of (3.1) will be

$$\mathcal{G} \equiv \mathrm{Ric}_{\boldsymbol{g}} - \frac{1}{2} R_{\boldsymbol{g}} \boldsymbol{g} = 8\pi T \tag{3.2}$$

where $\mathrm{Ric}_{\boldsymbol{g}}$ is the Ricci tensor respect to the metric $\boldsymbol{g}$, T is a symmetric 2-tensor related L_m and is called the *stress energy tensor*, and $\mathcal{G}$ is the *Einstein tensor*. When $T = 0$ we have vacuum Einstein equations

$$\mathrm{Ric}_{\boldsymbol{g}} = 0\,. \tag{3.3}$$

Coming back to spacetime $(M, \boldsymbol{g})$, the signature of metric $\boldsymbol{g}$ divides the tangent space T_pM at point $p \in M$ to three regions. Then each vector field $X \in T_pM$ is called spacelike, timelike, or nulllike if $\boldsymbol{g}(X,X) > 0$, $\boldsymbol{g}(X,X) < 0$, or $\boldsymbol{g}(X,X) = 0$, respectively. Similarly one can define a spacelike, timelike, or nulllike curve $\gamma : (a,b) \to M$ if its tangent vector has this property and it is called future (resp. past) inextendible if $\lim_{t\to b^-} \gamma(t)$ (resp. $\lim_{t\to a^+} \gamma(t)$) does not exist. The set of null vectors at p forms a double cone $\mathcal{V}_p$ in the tangent space T_pM and it is called the null cone. We say that X is *causal* (or nonspacelike) if it is timelike or null.

A Lorentzian manifold M is time-orientable if it admits a smooth timelike vector field T and the choice of this timelike vector field T fixes a time orientation on M. Then the causal vector field $X \in T_pM$ is future directed (resp. past directed) if $\boldsymbol{g}(X,T) < 0$ (resp. $\boldsymbol{g}(X,T) > 0$). Given any point p in a spacetime M, the timelike future and causal future of p are sets of all points which are related to p by timelike future-directed and causal future-directed curves and they are denoted by $I^+(p)$ and $J^+(p)$, respectively. Similarly, one can define $I^-(p)$ and $J^-(p)$.

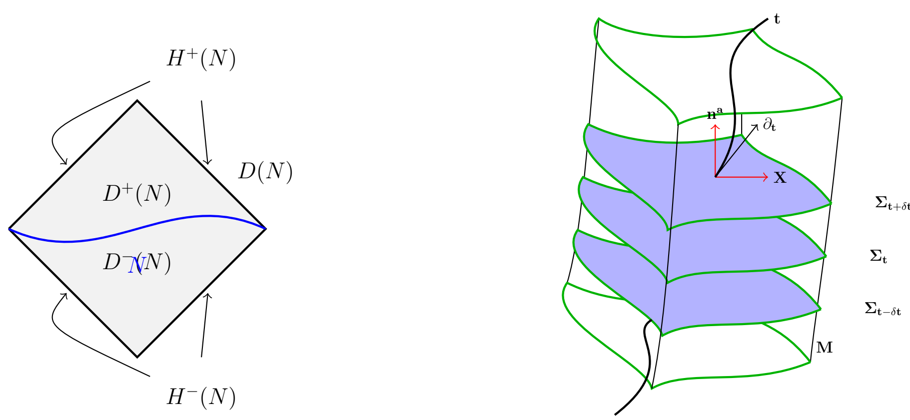
In any arbitrary spacetime, a set N is called *achronal* if no two of its points can be joined by a timelike curve. Let N be an achronal set in a spacetime M. We define the future and past domains of dependence of N, $D^+(N)$ and $D^+(N)$, by

$$D^+(N) \equiv \{p \in M : \text{every past inextendible causal curve from } p \text{ meets } N\}$$

$$D^-(N) \equiv \{p \in M : \text{every future inextendible causal curve from } p \text{ meets } N\}$$

The (total) domain of dependence of N is the union, $D(N) = D^+(N) \cup D^-(N)$. Then one can define the future and past Cauchy horizon of N by

$$H^{\pm}(N) = \overline{D^{\pm}(N)} - I^{\mp}\left[D^{\pm}(N)\right] \tag{3.4}$$



(a) Domain of dependence for set N and Cauchy horizon

(b) The foliation of M by Cauchy surfaces $\{\Sigma_a\}$

Figure 3.1: Cauchy surfaces and Cauchy horizon

By this structure the set of slices or hyperspaces $\{\Sigma_t\}$ of M is divided to three classes. We define a hypersurface Σ_t of M is timelike, spacelike, or null if its tangent space at each point has a normal vector that is spacelike, timelike, or null respectively. In initial value problem, there are two major classes of spacelike hyperspaces which are called *Cauchy* and *acausal* slices. A Cauchy (acausal) slice Σ is a spatial hypersurface of M having the property that every inextendible timelike (causal) curve in M intersects Σ not more than once. A Lorentzian manifold which admits a Cauchy surface is called *globally hyperbolic*. Now assume that M is globally hyperbolic then (Theorem 8.3.14 [143])

1. If Σ is a Cauchy surface for M then M is homeomorphic to $\mathbb{R} \times \Sigma$.

2. Any two Cauchy surfaces in M are homeomorphic.

In this setting, every globally hyperbolic Lorentzian manifold admits a continuous, globally defined, timelike vector field T which is obtained from a *time function* t, such that $T = -\nabla t$ [143]. Therefore, it is a time-oriented manifold M and $t =$ constant are leaves (slices) of this foliation, i.e. $\Sigma_t = \{t\} \times \Sigma$. Now define a coordinate chart (U, x^a) on Σ such that it corresponds to the coordinates (t, x^i) on an open neighborhood of M. Let n be the future-pointing timelike unit normal vector field on Σ. Then we have the following decomposition

$$\partial_t = Nn + X\,, \tag{3.5}$$

where N is a *lapse function* and $X = X^a \partial_a$ is a *shift vector*.

Now, suppose that $(M, \boldsymbol{g})$ is a globally hyperbolic spacetime with Cauchy surface Σ. The spacetime metric $\boldsymbol{g}$ induces two pieces of information on Σ, the first fundamental form $\boldsymbol{h}$ and the second fundamental form (or extrinsic curvature) K. Let $\mathbf{n}$

be the one-form associated with unit normal timlike vector n then

$$\boldsymbol{h} = \mathbf{n} \otimes \mathbf{n} + \boldsymbol{g} \qquad K(X,Y) = \langle \boldsymbol{\nabla}_X n, Y \rangle = \frac{1}{2}\mathcal{L}_n \boldsymbol{h}\,, \tag{3.6}$$

for every $X, Y \in T_pM$. Now by the Gauss-Codazzi equations we can find Einstein constraint equations [143] (see review article [17])

$$R_h - |K|^2_{\boldsymbol{h}} + \mathrm{Tr}_{\boldsymbol{h}} K = 2\mathcal{G}(n,n) = 16\pi\mu\,, \tag{3.7}$$

$$\mathrm{div}_{\boldsymbol{h}} \left[K - (\mathrm{Tr}_{\boldsymbol{h}} K)\, \boldsymbol{h}\right] = \mathcal{G}(n,\cdot) = -8\pi j\,, \tag{3.8}$$

$$\mathcal{C}(h,F) = 0\,, \tag{3.9}$$

where $\mathcal{C}(h,F)$ is a constraint obtained from any extra fields F, $|K|^2_{\boldsymbol{h}} = h^{ac}h^{bd}K_{ab}K_{cd}$ is full contraction of K with respect to $\boldsymbol{h}$, $\mathrm{Tr}_{\boldsymbol{h}} K$ is mean curvature, and ρ, and $j = j_a \mathrm{d}x^a$ are the energy density and the energy flux one-form, respectively. These equations are called the *Hamiltonian constraint*, *momentum constraint*, and *non-gravitational constraint*, respectively. Moreover, we have following evolution equations

$$\frac{\mathrm{d}}{\mathrm{d}t} h_{ab} = 2NK_{ab} + \mathcal{L}_X h_{ab}, \tag{3.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{ab} = \nabla_a \nabla_b N + \mathcal{L}_X K_{ab} + N\left\{2h^{cd}K_{ad}K_{bc} - (\mathrm{Tr}_{\boldsymbol{h}} K)\, K_{ab} - (Ric_{\boldsymbol{h}})_{ab}\right\}\,, \tag{3.11}$$

plus some evolution equations for the matter fields F.

Therefore, the triple $(\Sigma, \boldsymbol{h}, K)$ (or $(\Sigma, \boldsymbol{h}, K, F)$ where F is an extra field, or $(\Sigma, \boldsymbol{h}, K, \mu, j, F)$) which satisfied Einstein constraint equations is called initial data set of Einstein equations. An initial data set is called *maximal* if $\mathrm{Tr}_{\boldsymbol{h}} K = 0$. For a vacuum and maximal initial data set, the constraint equations reduce to

$$R_{\boldsymbol{h}} = |K|^2_{\boldsymbol{h}}\,, \qquad \mathrm{div}_{\boldsymbol{h}} K = 0\,. \tag{3.12}$$

In the last six decades, there has been great progress in the existence and uniqueness solutions of the Einstein constraint equations. The solutions of constraint equations is important because of the Cauchy problem in general relativity. The celebrated work of Yvonne Choquet-Bruhat shows if a set of smooth initial data which satisfies the Einstein constraint equations is given, then we have the following result

Theorem 18. *[33] Given an initial data set* (Σ, h, K) *satisfying the vacuum constraint equations there exists a unique, globally hyperbolic, maximal, spacetime* $(M, \boldsymbol{g})$ *satisfying the vacuum Einstein equations* $Ric_{\boldsymbol{g}} = 0$ *where* $\Sigma \hookrightarrow M$ *is a Cauchy surface with induced metric* h *and second fundamental form* K*. Moreover any other such solution is a subset of* $(M, \boldsymbol{g})$*.*

Thus the necessary and sufficient condition for the Cauchy problem in general relativity is the Einstein constraint equations. Many techniques have been developed to prove the existence and uniqueness of the solution of the Einstein constraint equations in different cases (constant mean curvature (CMC), near CMC, and non-CMC) such as conformal method [110], conformal sandwich method [18], barrier method [100], etc. We refer the reader to Bartnik and Isenberg's survey article [17] and an interesting paper by Maxwell [117].

One of the important questions in this subject is construction of an initial data set with desired properties. We return our attention in Chapter 7 to this question and we construct a family of initial data which has similar geometrical properties and angular momenta of the extreme Myers-Perry (EMP) black hole (see Appendix A for properties of EMP).

At this point, we will recall different energy condition on spacetimes

Definition 19. [130] Let $(M, \boldsymbol{g})$ be a spacetime. Then we have the following energy conditions

1. *Dominant energy condition*: $\mathcal{G}(u, v) \geq 0$ for all future directed and causal vectors $u, v \in T_pM$,

2. *Weak energy condition*: $\mathcal{G}(u, u) \geq 0$ for all future directed timelike vector $u \in T_pM$,

3. *Strong energy condition*: $Ric_{\boldsymbol{g}}(u, u) \geq 0$ for all future directed timelike vector $u \in T_pM$,

4. *Null energy condition*: $\mathcal{G}(u, u) \geq 0$ for all null vector $u \in T_pM$.

This definition implies geometrical restrictions on initial data set $(\Sigma, \boldsymbol{h}, K)$ with constraint equations (3.7)-(3.8). Another important class of spacetime is an *isolated system*. The geometric property of an isolated system in GR is the idea that spacetime becomes flat when we move very far from the system, and it approaches Minkowski spacetime. This motivates us to define a geometric notion independent of coordinate with conformal compactification of the spacetime and it is represented in the well-known Carter-Penrose diagram.

Definition 20. [15] An $n+1$ dimensional spacetime $(M, \boldsymbol{g})$ has *asymptotically flat end* if M contains a spacelike hypersurface Σ such that there exists a compact submanifold C, $\Sigma_{\text{ext}} = \Sigma \setminus C$ is diffeomorphic to $E_R = (\mathbb{R}^n \setminus \overline{B_R^n(0)}, \delta_n)$ for large R and in local coordinate chart $x : \Sigma_{\text{ext}} \to E_R$ for data on hypersurface $(\Sigma, \boldsymbol{h}, K)$ we have

$$h_{ab} - \delta_{ab} = o_s(r^{-p}), \quad K_{ab} = o_{s-1}(r^{-q}), \quad \partial_c h_{ab} \in L^2(\Sigma_{\text{ext}}), \tag{3.13}$$

where $r = |x|$, $p \geq \frac{n-2}{2}$, and $q > p+1$ [15].[1]

Returning to the classification of spacetimes we have the following definition.

Definition 21. [143] Let $(M, \boldsymbol{g})$ be a spacetime, it is called

- *Stationary* if there exists a complete Killing vector field k on M which is timelike in the asymptotic region of E_r.
- *Static* if it is stationary and k is hypersurface orthogonal, i.e. $k^\flat \wedge \mathrm{d}k^\flat = 0$.
- *axisymmetric* if $SO(2) = U(1)$ acts as a group of isometries on M such that the set of fixed points is a codimension-two timelike surface.

3.2 Mass and Angular Momenta in General Relativity

In this section we briefly review mass and angular momenta in general relativity (see [102, 141, 143] for comprehensive details of the topic). Energy and in particular mass in general relativity is a complicated concept. There exist various approaches to the definition of mass in general relativity, e.g. Hamiltonian approach. However, in Newtonian gravity, there is a well-defined definition of mass (locally or globally) as an integral of mass density

$$M \equiv \int_U \mu \; dV \tag{3.14}$$

where $U \subseteq \mathbb{R}^n$ is a spatial subset with Euclidean volume element dV and μ is the mass density (energy density in GR). Indeed, if we have a gravitational field $\phi = \nabla\psi$

[1]Where $f = o_s(r^\alpha)$ it means $\partial_{\beta_1} \cdots \partial_{\beta_p} f = o(r^{\alpha - p})$ for $0 \leq p \leq s$.

due to a massive object such that $\nabla\phi = -4\pi\mu$ (by Gauss Law), then

$$M = -\frac{1}{4\pi}\lim_{r\to\infty}\oint_{S_r} \nabla_a\psi\nu^a \, dS \tag{3.15}$$

where S_r is sphere of radius r centered at the origin and ν is the unit outward normal vector on S_r. Since in Newtonian gravity it is natural to suppose that $\mu \geq 0$, one can easily prove positive mass theorem locally and globally. In contrast, in general relativity there is no well-defined concept of local mass. This is a consequence of the equivalence principle, or mathematically from the invariance of the theory under diffeomorphisms [102]. This shows invalidity of (3.14) either locally or globally.

The main modern formalism for dynamic of general relativity is by Richard Arnowitt, Stanley Deser, and Charles W. Misner in 1961 [11]. They defined ADM mass (energy) and momenta by Hamiltonian approach [130, 143]. Their results motivative that a good relation which corresponds to energy density ($\mu = T_{00}$) in GR is the Hamiltonian constraint equation (3.7). The Definition 20 of asymptotically flat spacetime implies that the extrinsic curvature has strong decay conditions and it does not contribute to the mass (3.7). But the scalar curvature $R_{\boldsymbol{h}}$ contains a linear combination of second derivatives of h and quadratic terms in the Christoffel symbols. Hence after simplification we obtain

$$R_{\boldsymbol{h}} = \partial_c \underbrace{(\partial_a h_{ac} - \partial_c h_{aa})}_{o_{s-1}(r^{-p-1})} + \text{quadratic order of } \partial h \tag{3.16}$$

By integration over $U \subseteq \mathbb{R}^n$ and application of Stokes' theorem we have the following definition which is equivalent to ADM mass (energy) at each ends $(E_r^i, \boldsymbol{h})$ [11]:

Definition 22. [11] Let $(M, \boldsymbol{g})$ be an $n+1$ dimensional spacetime with some asymptotically flat ends $(E_r^i, \boldsymbol{h})$, the ADM-mass (energy) in each end $(E_r^i, \boldsymbol{h})$ is defined as

$$M_{ADM}(E_r^i, \boldsymbol{h}) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \oint_{S_r^i} (\partial_a h_{ac} - \partial_c h_{aa})\, \nu^c \, dS \tag{3.17}$$

where S_r^i is a sphere(S^{n-1}) in asymptotically flat coordinate system of $\boldsymbol{h}$ with radius r at the end E_r^i, $\omega_{n-1} = \mathrm{vol}(S^{n-1})$, and ν is unit outward normal vector on S_r^i.

By the beautiful result of Bartnik [15], the ADM-mass is a well-defined geometric quantity and remarkably, it is a geometrical invariant of the Riemannian metric on an asymptotically flat slice and independent of observer at infinity [15, 38, 123]:

Theorem 23. *[15, Theorem 4.3]. Let $\tau > 0$ be a non-exceptional constant, $k \geq 2$, and $q > n$. Suppose that a complete Riemannian manifold (Σ, h) has asymptotically flat ends $(E_r^i, h)$of type (k, q, τ) so that*

$$Ric_{\boldsymbol{h}} \in L^q_{-2-\tau} \tag{3.18}$$

If $\tau \geq \frac{n-2}{2}$, the ADM-mass exists and it is unique. Moreover, if $\tau > n-2$, the ADM-mass is zero.

One of the greatest results in general relativity is the positivity of total gravitational energy, i.e. positive mass theorem. The first proof of positive mass theorem is by Brill for time symmetric initial data (i.e. $K \equiv 0$)[23]. Then Schoen and Yau proved this problem by application of *Yamabe problem* and minimal surfaces (zero mean curvature) for 3-dimensional initial data and it has been extended to $3 \leq n \leq 7$ [136, 137]. Since there is no non-singular minimal surface in a barrier region for $n \geq 8$, this technique cannot be extended to these dimensions. Independently, Witten proved positive mass theorem by spinorial techniques for all dimensions for manifolds with a spin structure [144]. In addition, the result has been generalized for black holes, asymptotically AdS spacetimes, and some of other quasi-local definitions of mass [75, 85, 97, 138]. Here we state Schoen and Yau's positive mass theorem

Theorem 24. *[136, 137]. If (Σ, h) is an asymptotically flat Riemannian n-manifold for $3 \leq n \leq 7$ with non-negative scalar curvature, then the mass of each end is non-negative. If the manifold is geodesically complete and if the mass is zero in one end, then (Σ, h) is isometric to flat space $(\mathbb{R}^n, \delta)$.*

Because of this remarkable result one might expect to extract more interesting results (e.g. global Penrose inequality) by defining mass for a finite domain of spacetime, i.e."quasi-local mass". Note that there have been many attempts to define quasi-local mass by different authors, Penrose [129], Hawking [81], Geroch [74], Bartnik [16], York [24], etc. In spite of all these efforts, there is no generally accepted expression for quasi-local mass in general relativity (see review [141]).

Every asymptotically flat spacetime has asymptotic symmetries which preserve the asymptotic Euclidean structure of the end (3.13). This group is an infinite dimensional *Spi group* which if we impose sufficiently strong fall-off on the Weyl tensor it contains the Poincare group [12]. The translation generators of this isometry group yield to the definition of ADM momenta for each end (E^i_r, h) [141]

$$P_a(E^i_r, h) = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \oint_{S^i_r} (K_{ab} - \mathrm{Tr}_{\boldsymbol{h}} K h_{ab})\, \nu^b \, dS \,. \tag{3.19}$$

Similar to translational symmetry for ADM momenta, the ADM angular momenta are generated by rotation symmetries. In general, there are several independent rotation planes. The rotational group of an $n+1$ dimensional spacetime $(M, \boldsymbol{g})$ is $SO(n)$. This rotation group has Cartan subgroup $U(1)^N$ with $N = [\frac{n-1}{2}]$ [65]. Assume an $n+1$ dimensional spacetime $(M, \boldsymbol{g})$ has rotational isometry group $U(1)^d$ with commuting generators $\xi_{(k)}$ for $k = 1, \ldots, d \leq N$, then the ADM angular momenta are defined as

$$J_{(k)}(E^i_r, h) = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \oint_{S^i_r} \left(K_{ab} - \mathrm{Tr}_{\boldsymbol{h}} K h_{ab}\right) \xi^a_{(k)} \nu^b \, dS \,. \tag{3.20}$$

In case of spacetime with Killing vectors, one can define Komar quantities. Let $(M, \boldsymbol{g})$ be an $n+1$ dimensional stationary spacetime with timelike Killing vector n, then Komar mass is defined [141]

$$M^K \equiv -\frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \oint_{S_r} \star_{\boldsymbol{g}} \mathrm{d}n^\flat \tag{3.21}$$

where $\star$ is an Hodge star respect to spacetime metric $\boldsymbol{g}$. Note that M_K is a geometric quantity and independent of coordinate. Moreover, it equals ADM mass for vacuum stationary spacetime. Similarly, one can define Komar angular momenta if a spactime has commuting isometry group $U(1)^d$ with generators $\xi_{(k)}$ for $k = 1, \ldots, d \leq N$

$$J^K_{(k)} \equiv \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \oint_{S_r} \star_{\boldsymbol{g}} \mathrm{d}\xi^\flat_{(k)} \tag{3.22}$$

3.3 Black Holes in $D \geq 4$

A black hole is a solution of Einstein equation which informally can be defined as a region of spacetime from which no causal curve can escape to infinity. We say that a spacetime has a *black hole* if M is not contained in $I^-(\mathscr{I}^+)$, where $\mathscr{I}^+$ is null infinity. Moreover, the *black hole region* is $B = M - I^-(\mathscr{I}^+)$ and the boundary of B is a null surface and it is called the *event horizon* $N = \mathrm{bdary}\,(I^-(\mathscr{I}^+)) \cap M$ [143].

In general different kinds of black holes form through different dynamical processes. However, similar to any physical phenomena the final stage will be *equilibrium* or *stationary*. Four-dimensional black holes are known to possess a number of remarkable

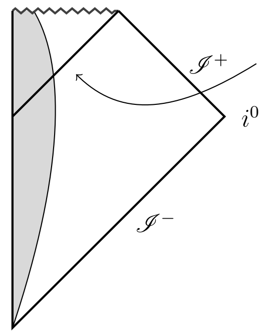


Figure 3.2: Carter-Penrose diagram of Black hole collapse

.

features, such as uniqueness, spherical topology, dynamical stability, and the laws of black hole mechanics. In the following section we will review some of these features. In contrast to 4D black holes, the higher dimensional ($D > 4$) black hole solutions have other distinctive features. But why should we study the higher dimensional black hole? The mathematical motivations to study the extension of Einstein's theory are

- The geometry of $D-1$ dimensional slices of black holes poses interesting problems in Riemannian geometry (e.g. positive mass theorem; Penrose inequalities in Riemannian geometry, discovery of inhomogeneous Einstein metrics [22, 50, 111, 136]).

- There exist interesting aspects of geometrical analysis, topology, and PDE theory of higher dimensional manifolds [7, 71, 73, 91].

- Known examples of black rings [64], black Saturn [62] and black lens [108] in higher dimension assure the existence of a rich variety of such objects whose mathematical properties are only just beginning to be uncovered.

There are many physical motivations. The string theory contains gravity and requires more than four dimensions. In fact, the first successful statistical counting of black hole entropy in string theory was performed for a five dimensional black hole [140]. In addition, the AdS/CFT correspondence relates dynamics in certain D-dimensional classical gravitational background with properties in quantum field theory in $D-1$ dimensions [114]. We refer the interested reader to the review article [65] for more physical motivations. In Section 3.3.2 we give an overview of the important results about five dimensional stationary black holes.

3.3.1 Stationary 4D Black Holes

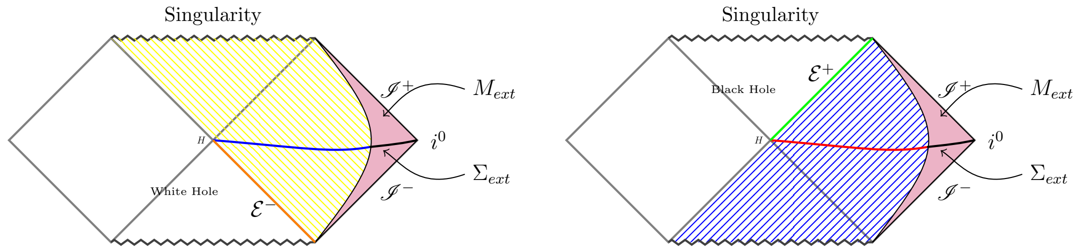
It is convenient to start with some definitions of stationary black holes (we follow the definition in [42]). Consider an asymptotically flat spacetime that has a timelike Killing vector T at Σ_{ext}. We say that a spacetime has a *black hole* (*white hole*) if M is not contained in $I^-(M_{ext})$ (resp. $I^+(M_{ext})$), where $M_{ext} = \cup\phi_t(\Sigma_{ext})$ and ϕ_t is one-parameter group of diffeomorphisms generated by of T. Moreover, the black hole region (white hole region) is $B = M - I^-(M_{ext})$ (resp. $W = M - I^-(M_{ext})$) and the boundary of B is the black hole (resp. white hole) event horizon $H^+ = \partial B$ (resp. $H^- = \partial W$) [143]. The full event horizon then is $N = H^+ \cup H^-$. Then the *domain of outer communication* or d.o.c is

$$\langle\langle M_{ext}\rangle\rangle = I^+(M_{ext}) \cap I^-(M_{ext})\,. \tag{3.23}$$

Moreover, the boundary of d.o.c contains event horizons

$$\mathcal{E}^{\pm} = \partial\,\langle\langle M_{ext}\rangle\rangle \cap I^{\pm}(M_{ext}), \qquad \mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^- \tag{3.24}$$

In the theory of black holes, there are different types of horizons such as apparent, Killing, trapping, isolated, dynamical, and slowly evolving horizons (see review articles [13, 20]). But in this section we want to review some of the fundamental results



(a) The yellow region is $I^+(M_{ext})$. (b) The blue region is $I^-(M_{ext})$

Figure 3.3: Carter-Penrose diagram of the stationary black holes.

about black holes with Killing horizons. In order to see this, let us define the Killing prehorizon(horizon).

Definition 25. Let $(M, \boldsymbol{g})$ with a Killing vector ξ. A *Killing prehorizon* of ξ is a connected, null, injectively immersed hypersurface N_ξ such that $\forall p \in N_\xi$, $\xi(p) = 0$, null and tangent. A *Killing horizon* is an embedded Killing prehorizon.

Now let N_ξ be a Killing prehorizon (horizon) associated to Killing vector field ξ with length $X = g(\xi, \xi)$. Then the *surface gravity* κ_ξ of N_ξ is defined by

$$\nabla X|_{N_\xi} = -2\kappa_\xi \xi^\flat \,. \tag{3.25}$$

The surface gravity of a horizon measures how much the parametrization of the geodesic congruence generated by ξ is not affine. A Killing prehorizon (horizon) with vanishing surface gravity is called *degenerate* or *extreme* and otherwise *non-degenerate*. An essential property of a Killing prehorizon(horizon) is it has constant surface gravity [133]. This fundamental fact implies that the surface gravity plays a similar role in the theory of equilibrium(stationary) black holes as the temperature does in ordinary thermodynamics [14, 143].

Before we move on to the features of stationary black holes, we review an important result about static black holes. The first step toward uniqueness of static black holes was by Israel. He used two integral identities from level sets of $X = g(k,k)$, where k is a complete timelike hypersurface orthogonal Killing vector, and Stokes' theorem, and he proved both static vacuum and electrovacuum black hole are spherically symmetric [101]. The technical restriction was connected horizon on the theorem has been removed by many contributors and it was completed by the beautiful gluing technique of Bunting and Masood-ul-Alam [26]. The main restriction in the theorem was analyticity which has been removed by Chrusciel and Galloway [43]. Then the statement of the theorem is

Theorem 26. *[42] Let (M, g) be an electrovacuum, four-dimensional spacetime containing a spacelike, connected, acausal hypersurface N, such that $\bar{N}$ is a topological manifold with boundary consisting of the union of a compact set and of a finite number of asymptotically-flat ends. Suppose that there exists on M a complete hypersurface-orthogonal Killing vector, that the domain of outer communication $\langle\langle M_{ext}\rangle\rangle$ is globally hyperbolic, and that $\partial N \subset M \setminus \langle\langle M_{ext}\rangle\rangle$. Then $\langle\langle M_{ext}\rangle\rangle$ is isometric to the domain of outer communications of a Reissner-Nordstrom(RN) or a Majumdar-Papapetrou(MP) spacetime(see [79, 82] for exact RN and MP solutions).*

To continue the stationary black holes, one of the main features of stationary black hole is the no-hair theorem or uniqueness theorem. But, we review some fundamental results about topological and geometrical structure of stationary black holes. One of these results is the topological censorship theorem by Friedman, Schleich, and Witt. They proved in a globally hyperbolic, asymptotically flat spacetime satisfying the null energy condition, the M_{ext} is simply connected region, i.e. $\pi_1(M_{ext}) = 0$ or every causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ is homotopic to trivial curve. Then Chrusciel and Wald showed that the domain of outer communication for stationary spacetime is simply connected. Finally, Galloway extended this result for all globally hyperbolic spacetimes that satisfy the null energy condition [71]. The precise statement of the theorem is

Theorem 27. *[45, 69, 70] Let $(M, \boldsymbol{g})$ be a stationary black hole spacetime. If the domain of outer communications is globally hyperbolic and satisfies the null energy condition then it is simply connected.*

This theorem implies another important result about stationary black holes spacetime, the Hawking topology theorem. The result has been proved by Hawking [82] and it can be recovered by topological censorship theorem.

Theorem 28. *[82] Let $(M, \boldsymbol{g})$ be an stationary four-dimensional black hole spacetime satisfying the null energy condition, then cross sections of $\mathcal{E}^+$ have spherical topology $H \cong S^2$.*

The third fundamental result about stationary black holes is the Hawking rigidity theorem.

Theorem 29. *[82] Let $(M, \boldsymbol{g})$ be an analytic spacetime with an analytic null hypersurface N such that (i) M admits a complete Killing vector ξ tangent to N,(ii) N admits a compact cross section H transverse to ξ, (iii) The average surface gravity $\langle \kappa_\xi \rangle = \frac{-1}{2|H|} \int_H < k, \nabla l > d\mu_H$ is nonzero, where l is the null generator of N satisfying $\nabla_l u = 1$ for $u : N \to \mathbb{R}$ and k is orthogonal to H, null and with $\langle l, k \rangle_N = -2$. Then there is a neighbourhood U of N and a Killing vector η on U which is null, non-zero and tangent to N. In fact, if ξ is not tangent to the generators of N then there exists a rotational commuting Killing vectors ζ with 2π period and constants Ω_N such that*

$$\eta = \xi + \Omega_N \zeta \tag{3.26}$$

One of the major breakthroughs in the mathematical study of general relativity is the uniqueness of the stationary axisymmetric black hole. The first step toward this theorem was by Carter. He used the dimensional reduction of the Einstein action with respect to the axial Killing field and obtained a linear divergence identity [27] and he showed that axisymmetry stationary black holes are unique under some restricted assumptions (see Appendix B for five dimensional version). Then Robinson and Mazur used the non-linear divergence identity and proved the uniqueness of the Kerr black hole. The Mazur identity is based on the observation that the Einstein-Maxwell equations in the presence of a Killing field describe a non-linear σ-model with coset space $G/H = SU(1,2)/S(U(1) \times U(2))$. Another approach to prove uniqueness is by Bunting [25], who applied the properties of harmonic maps in negatively curved target spaces [135].

Theorem 30. *[40, 43, 48] Let $(M, \boldsymbol{g})$ be a stationary, asymptotically-flat, electrovacuum, four dimensional analytic spacetime. If the event horizon is connected and either mean non-degenerate or rotating, then $\langle\langle M_{ext} \rangle\rangle$ is isometric to the domain of outer communications of a Kerr–Newman spacetime.*

3.3.2 Stationary $5D$ Black Holes

In higher dimensions, the Einstein theory and in particular black holes have richer features and the reason is that as the number of dimensions grows the number of degrees of freedom of the gravitational field also increases. Here we are interested in mathematical aspects of higher dimensional black holes. We refer the interested reader to the review articles [65, 93] for a comprehensive exposition of the topic. In contrast to four dimensional spacetime, in higher dimensional instead of rotations around a line (axis), the rotations are around (spatial) codimension two hypersurfaces. Thus, there are different rotational planes. This condition is related to the asymptotic property of the spacetime. The rotation group of D-dimensional asymptotically flat spacetime is $SO(D-1)$, this group has Cartan subgroup $U(1)^N$ where

$$N = \left[\frac{D-1}{2}\right] . \tag{3.27}$$

This is equivalent to the existence of N independent rotational planes $(x_1, x_2), \ldots, (x_{N-1}, x_N)$ associated to the rotational vectors $\partial_{\phi^1}, \ldots, \partial_{\phi^N}$. As pointed out in Section 3.3.1, axisymmetric stationary spacetimes have many remarkable features, the one can consider the generalization to higher dimensions. For D-dimensional stationary spacetime, we assume we have $U(1)^{D-3}$ rotational symmetry. Then the reduced manifold under $\mathbb{R} \times U(1)^{D-3}$ isometry group is a two dimensional quotient manifold, i.e. *orbit space*. Existence of this symmetry imposes an important limitation on dimension. If we demand $D-3$ rotatonal isometry in asymptotically flat spacetime, then $U(1)^{D-3} \leq U(1)^N$ where $\leq$ is subgroup notation. Therefore by equation (3.27)

$$D - 3 \leq \left[\frac{D-1}{2}\right], \qquad \Longrightarrow \qquad D = 4, 5 \tag{3.28}$$

because of this reason we only focus on five dimensional asymptotically flat black hole spacetimes in the thesis. First, we consider the topology of higher dimensional black hole spacetimes. The Yamabe invariant $Y[H]$, is the topological invariant which characterizes the topology of the cross section of event horizon H [71]

$$Y[H] = \sup_{[\gamma]} \inf_{0<\Phi\in C^\infty} \frac{\int_H R_{\tilde{\gamma}}\, \mathrm{d}V_{\tilde{\gamma}}}{\left[\int_H \mathrm{d}V_{\tilde{\gamma}}\right]^{\frac{D-4}{D-2}}} , \tag{3.29}$$

where $\tilde{\boldsymbol{\gamma}} = \Phi^2 \boldsymbol{\gamma}$ is the conformally transformed metric on H and $\mathrm{d}V_{\tilde{\gamma}}$ is the volume element with respect to $\tilde{\boldsymbol{\gamma}}$. In dimension $D = 4$, applying Gauss-Bonnet theorem , $Y[H]$ is proportional with a constant to the Euler characteristic of H. As we explain in Section 3.3.1, there are different types of horizons and here we define trapped surface or *marginally outer trapped surface (MOTS).* Consider the $(D-2)$-dimensional spacelike surface S and we define a pair of null vector fields n and l orthogonal to S and normalized as that $g(n,l) = 1$. Here, n is the future pointing null vector field which generates a congruence of affine null geodesics, i.e. a null sheet $\mathcal{N}$ and l is "outward pointing", parallel transported along $\mathcal{N}$ and it can be tangent to another congruence of null geodesics. Then n and l are completely fixed up to a rescaling, once they have been defined on each point $p \in S$. Now let θ_n and θ_l be corresponding null expansions in the directions n and l. Then the surface S is marginally outer trapped surface if [73]

$$\theta_n = 0, \qquad \mathcal{L}_l \theta_n \geq 0, \qquad \text{on S}. \tag{3.30}$$

An example of a MOTS is the event horizon cross section H of a black hole. In mathematical relativity a MOTS is very useful definition from which one can prove interesting results about black holes. Moreover, the metric of spacetime near the event horizon takes *Gaussian null form* [120] for stationary spacetime

$$\boldsymbol{g} = 2\mathrm{d}u\,(\mathrm{d}r - r\alpha \mathrm{d}u - r\beta_a \mathrm{d}x^a) + \gamma_{ab}\mathrm{d}x^a \mathrm{d}x^b\,, \tag{3.31}$$

where x^a is local coordinates on H, and α, β_a, and γ_{ab} are a scalar field, 1-form, and Riemannian metric on each of the spheres S that are parameterized by u and r. Then the Einstein equations in this form take a very simple expression [120]. Applying Einstein equations near horizon we obtain constraints on the Yamabe invariant of horizon cross sections [132]

$$Y[H] = \sup_{[\gamma]} \inf_{0<\Phi\in C^\infty} \frac{\int_H \left[4\frac{D-3}{D-4}\left|\nabla\Phi\right|^2 + R_\gamma \Phi^2\right]\,\mathrm{d}V_\gamma}{\left[\int_H \Phi^{2\frac{D-2}{D-4}}\mathrm{d}V_\gamma\right]^{\frac{D-4}{D-2}}} \geq 0\,. \tag{3.32}$$

Then we have the following generalization of Hawking topology theorem

Theorem 31. *[71, 73] Let $(M, \boldsymbol{g})$ be a D dimensional spacetime that satisfies the dominant energy condition. If H is a stably marginally outer trapped surface in M, then H is of positive Yamabe type, $Y[H] > 0$.*

Then when $D = 5$, H is a closed compact 3-manifold. Since $Y[H] > 0$, H is a connected sum

$$H \cong \#_n \left(S^3/\Gamma_n\right) \#m \left(S^1 \times S^2\right) \tag{3.33}$$

where $\Gamma_n < O(4)$ are discrete subgroups. Returning to the other features of higher dimensional black hole, the topological censorship theorem [72] and rigidity theorem for stationary spacetime [94, 121] holds. In the case of stationary five dimensional black holes, by the rigidity theorem of Hollands, Wald, and Ishibashi [94] we have at least one rotational Killing vector which generates $U(1)$ isometry group. Then the generalization of Hawking topology theorem has a refinement. For a stationary black hole in the topology of the cross sections of N is[92]

$$H \cong \begin{cases} S^3/\Gamma \\ \#m\,(S^1 \times S^2)\,\#_i L(p_i, q_i) \end{cases} \tag{3.34}$$

where $\Gamma < O(4)$ and all possible choices of Γ are given in [92], and each $L(p, q)$ is a lens space.

Recall that a uniqueness theorem is a fundamental result in mathematical relativity. As explained above, there are different choices for horizon topology, thus the uniqueness should cover all these distinctive topologies. To achieve this goal for stationary, asymptotically flat five dimensional spacetime, one needs to impose an extra $U(1)$ isometry group more than what the rigidity theorem provides. Therefore, spacetime has $G = \mathbb{R} \times U(1)^2$ isometry group. The reason for this extra symmetry is existence of non-linear sigma model for five dimensional stationary spacetime, which is a considerable simplification. By this isometry group the quotient manifold $\mathcal{B} = M/G$ is a simply connected, asymptotically flat, 2-dimensional manifold with one dimensional boundary and corners [96] (see Proposition 43). The boundary of this manifold is related to fixed points of the Killing vectors. More precisely, a linear combination of two generators ∂_{ϕ^i} of $U(1)^2$ isometry group vanish on an (finite, infinite, semi-infinite) interval I_i,

$$v_i = v^j \partial_{\phi^j} \quad \Longrightarrow \quad g(v,v) = 0 \quad \text{on } I_i \tag{3.35}$$

the coefficient of this Killing vector is called a *direction vector* and I_i is called *rod.* A corner corresponds to a point at which two direction vectors of adjacent intervals vanish simultaneously. By the Riemannian mapping theorem we can map $\mathcal{B}$ to the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ in the complex plane. By the vacuum Einstein equations one can construct a *geometrical coordinate* (ρ, z) on $\mathcal{B}$ such that the axis $\Gamma = \{\rho = 0\}$ corresponds the boundary of the manifold $\mathcal{B}$, ρ is harmonic and z is conjugate harmonic, and the metric has the following global representation[96]

$$\boldsymbol{g} = e^{2v}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + G_{ij}\mathrm{d}x^i\mathrm{d}x^j \,, \tag{3.36}$$

where $x^i = (t, \phi^1, \phi^2)$, $\boldsymbol{g}_{\mathcal{B}} = e^{2v}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right)$ is the metric on $\mathcal{B}$. Moreover, let us define the twist potential one form of the Killing vector $\xi_i = \partial_{\phi^i}$ by

$$\omega_i = \star_{\boldsymbol{g}}\left(\xi_1^\flat \wedge \xi_2^\flat \wedge \mathrm{d}\xi_i^\flat\right) \,. \tag{3.37}$$

Since the manifold is simply connected and it can be shown directly by the vacuum Einstein equations that ω_i are closed, they are exact, i.e. $\omega_i = \mathrm{d}Y^i$ and $Y = (Y^1, Y^2)$ is twist potential column vector(see Section 4.1). Then one can define the following

Carter functional

$$S = \int_{\mathcal{B}} \mathrm{Tr}\left[\left(\Phi^{-1}\mathrm{d}\Phi\right)^2\right] \mathrm{d}\mu\,, \tag{3.38}$$

where Φ is defined in equation (B.2) and $\mathrm{d}\mu = \rho \mathrm{d}\rho \mathrm{d}z$. By a variational principle, one can obtain the *Mazur divergence identity* (B.3). We refer the interested reader to the article [29] and the book [86] for a complete survey of the identity. Moreover, the vacuum Einstein equations arise as the critical points of the Carter functional above[67, 96]

$$D_a\left[\rho\Phi^{-1}D^a\Phi\right] = 0\,, \tag{3.39}$$

where D is connection with respect to the metric $\boldsymbol{g}_{\mathcal{B}}$. Hollands and Yazadjiev applied the Mazur identity (B.3), vacuum Einstein equations (3.39), and maximum principle to prove the following uniqueness theorem.

Theorem 32. *[96] Consider two stationary, asymptotically flat, vacuum black hole spacetimes of dimension 5, having two commuting axial Killing fields that commute also with the time translation Killing field. Assume that both solutions have the same interval structure, and the same values of the mass m and angular momenta J_1, J_2. Then they are isometric.*

3.3.3 Geometric Inequalities for Black Holes

In gravitational collapse (black holes) there are three classes of geometric inequalities with physical applications. These inequalities are motivated by exact stationary black hole solutions of Einstein equations. It is well known that the parameters, (m, J, Q) where J is angular momentum, and Q is electric charge, that characterize the Kerr-Newman black hole, which is a stationary, axisymmetric, electrovacuum, four dimensional, asymptotically flat spacetime [143], satisfy several important geometric inequalities:

$$m \geq \sqrt{\frac{A}{16\pi}}, \qquad \text{Penrose Inequality} \tag{3.40}$$

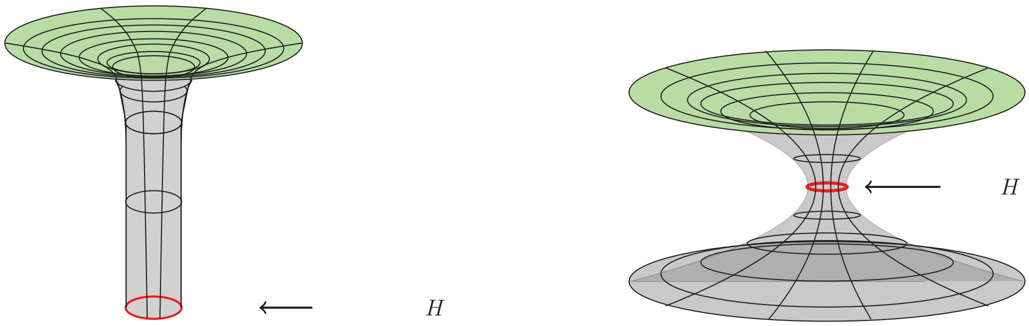
$$m^2 \geq \frac{Q^2 + \sqrt{Q^2 + 4J^2}}{2}, \qquad \text{Mass-Charge-Angular Moementum Inequality} \tag{3.41}$$

$$A \geq 4\pi\sqrt{Q^4 + 4J^2}, \qquad \text{Area-Charge-Angular Moementum Inequality} \tag{3.42}$$

These inequalities are saturated for the slice of the extreme Kerr-Newmann black holes, see Figure 3.4. By the uniqueness theorem we know Kerr-Newman black holes are the unique spactimes when we fix mass, angular momentum, and charges. Therefore, since these inequalities hold for the Kerr-Newman black hole, we expect them to be true for all stationary black holes. But the ultimate goal is to prove these inequalities for all dynamical black holes.

The first inequality is the *Penrose inequality*, which states a relationship between the area A of a cross-section of the event horizon and ADM mass m. The inequality is conjectured to hold rather generally in asymptotically flat and strongly asymptotically predictable spacetimes subject to a dominant energy condition on Ricci curvature [143], and is closely tied to the cosmic censorship conjecture (see the review article [115] and original paper [128]). The Riemannian version of this conjecture, which asserts that the area of a closed minimal surface in an asymptotically flat 3-manifold is a lower bound for the square of the mass (times 16π) whenever the scalar curvature is non-negative, has been proved by Huisken and Ilmanen, and extended by Bray [22, 99].

The second inequality is the mass-charge-angular momentum inequality. To prove this inequality we need conserved angular momentum and charges. However, in general, these quantities are not conserved. If we assume axisymmetry which is the result of rigidity theorem for stationary spacetime, then since gravitational field does not



(a) Extreme (b) Non-extreme

Figure 3.4: Extreme and non-extreme slice of black holes

carry angular momentum, angular momentum will be a conserved quantity. The first version of this inequality was proved by Dain [53] for vacuum, simply connected, axisymmetric, asymptotically flat black holes. First, he constructed a non-negative mass functional $\mathcal{M}$ which evaluates to the ADM mass for asymptotically flat, maximal, $t-\phi$ symmetric data (see Section 4.1 and [61, Hawking, Appendix C] for a survey of $t-\phi^i$ symmetric data). Then he showed that $\mathcal{M}$ is greater than or equal to the mass of any axisymmetric, maximal, asymptotically flat, complete initial data and the critical points of this functional are stationary axisymmetric black holes [52]. By these results, he proved the mass angular momentum inequality for asymptotically flat, axisymmetric vacuum, black holes [51, 53]. In global proof, he used the technique of harmonic map from $\mathbb{R}^3\setminus\Gamma \to \mathbb{H}^2$, where $\mathbb{H}^2$ is two dimensional hyperbolic space. The statement of the theorem is as follows

Theorem 33. *[53] Consider an axially symmetric, vacuum, asymptotically flat and maximal initial data set* $(\Sigma, \boldsymbol{h}, K)$ *with two asymptotic ends. Let* m *and* J *denote the total mass and angular momentum respectively at one of the ends. Then, the following inequality holds*

$$m \geq \sqrt{J}\,. \tag{3.43}$$

This result and the technique of its proof have some restrictions: (1) Σ should be a complete Riemannian manifold and not a manifold with boundary which is natural for black hole solutions,(2) Σ has only two ends and not for multiple black holes solutions, (3) the initial data set is maximal, (4) the initial data set is vacuum, (5) since the ADM mass is only defined for asymptotically flat, the technique can not be applied for other spacetime such as Kaluza-Klein ends, (6) it is for three dimensional initial data.

In the last decade, Dain and other authors like Chrusciel, Li, Costa, Weinstein, Khuri, Schoen tried to extend this result to different cases [30, 44, 49, 105, 139]. First, Chrusciel, Li, and Weinstein extended this result to complete Riemannian manifolds with N asymptoticaly flat ends and they proved that the mass of any asymptote is greater than or equals a function of angular momenta of the other asymptote, that is $m \geq f(J_1, \ldots, J_N)$[44]. Second, Chrusciel and Costa extended it for electro-vacuum case which is inequality (3.41) [49]. Then Khuri and Cha investigated the non-maximal (nonzero mean curvature) slices and they showed that the problem reduced to a system of nonlinear differential equations which are called *Jang equations* [54]. Recently, Schoen and Zhou applied a more geometrical technique and recovered the previous results. We refer readers to the review article [54] for comprehensive survey of the topic. In this thesis, we construct a mass functional which is a natural extension of Dain's mass functional for 4-dimensional initial data sets. Moreover, we prove some interesting results about characteristics of this functional. Finally we prove a class of local mass-angular momenta inequalities for four dimensional initial data sets.

The third class of inequalities is the area-angular momentum inequality for black holes or more generally, initial data containing apparent horizons [103]. The first step toward this was by Hennig, Ansorg, and Cederbaum [83, 84]. They proved an area-angular momentum inequality for axisymmetric, stationary black hole spacetimes in Einstein-Maxwell theory. Then this result was extended by Acena, Dain, and Clement

[1] to dynamical axisymmetric black holes.

Theorem 34. *[1] Consider an axisymmetric, vacuum and maximal initial data, with a non-negative cosmological constant. Assume that the initial data contains an orientable closed stable minimal axially symmetric surface S. Then*

$$A \geq 16\pi |J| \,, \tag{3.44}$$

where A is the area and J the angular momentum of S. Moreover, ir the inequality saturated then $\Lambda = 0$ and the local geometry of the surface S is an extreme Kerr throat sphere.

Moreover, Clement, Jaramillo, and Reiris add charges to these inequality [46] and cosmological constants [47]. Finally, Hollands considered spactimes with $U(1)^{D-3}$ isometry satisfy the Einstein equations with a non-negative cosmological constant. As pointed out in Section 3.3.2, in this case the spacetime is reduced to 2D orbit space with boundary which is a union of intervals. One of these intervals represents the horizon and denoted by $I_H \equiv N/U(1)^{D-3} = [a, b]$ where N is the event horizon. Suppose that $v_\pm = a^i_\pm \partial_{\phi^i}$ are Killing vectors vanish on end points of I_H. Then we have the following result.

Theorem 35. *[90] Let $(M, \boldsymbol{g})$ be a spacetime satisfying the vacuum Einstein equations with a cosmological constant $\Lambda \geq 0$ having isometry group at least $U(1)^{D-3}$. Define*

$$J_\pm = J_i a^i_\pm \tag{3.45}$$

Then:

- *The area of any stably outer marginally trapped surface (e.g. event horizon cross* *section of a black hole) satisfies*

$$A \geq 8\pi \left| J_+ J_- \right| . \tag{3.46}$$

- *Furthermore, if $\Lambda = 0$, and if $(M, \boldsymbol{g})$ is a "near horizon geometry", then the inequality is saturated. Conversely, if the inequality is saturated, then the tensor fields α, β, γ determining the induced geometry on H (see equation (3.31)) are equal to those of a near horizon geometry.*

3.4 Summary

The goal of this chapter was to give a short introduction into general relativity and black holes. In Section 3.1, we presented Einstein equations and Einstein constraint equations with matter source and vacuum. Then we reviewed causal structure of spacetimes and defined some properties of spacetimes such as energy conditions. The energy condition is equivalent to imposing geometrical restriction in different problems. In Section 3.2, we overview definitions of mass and angular momenta in general relativity. In particular, we review ADM formalism of GR and noted that ADM mass is a geometrical quantity of asymptotically flat spacetimes, i.e. isolated systems. Moreover, we stated the Schoen and Yau's positive mass theorem which is one of the major results in mathematical relativity.

Finally, in Section 3.3, we investigate black holes and express the definition of the black hole and related terminologies in dimensions $D \geq 4$. We review some of the fundamental results in the theory of stationary black holes such as the uniqueness theorem, rigidity theorem, and Hawking topology theorem. In the last subsection, we collected geometric inequalities in gravitational collapse with symmetry and stated the main results in the literature. Most of these geometrical results have been extended to higher dimensions except the mass-charge-angular momenta inequality. This motivated an open problem which we investigate in this thesis: Is there any mass-charge-angular momenta inequality in higher dimensions? We will address this question in the next chapters.

Chapter 4

Initial Data with Symmetries

In this chapter, we construct n-dimensional $t-\phi^i$ symmetric initial data sets for $n \geq 3$ and we construct a traceless-transverse symmetric $(0,2)$-tensor which is a good candidate for extrinsic curvature in Section 4.1. In Section 4.2, we provide the four dimensional generalized Brill (GB) initial data set which is a fundamental tool in the argument of mass-angular momenta inequality. Moreover, we demonstrate conservation of angular momenta and definition of the reduced $t-\phi^i$ symmetric data. Finally, in Section 4.3 we prove some topological results about the slice topology of any GB initial data and corresponding slice in the domain of outer communication of the five dimensional black holes. The results of this chapter appeared in the following journal articles: (AA.1) Classical and Quantum Gravity, 31 (5), 055,004(2014)[7], (AA.2) General Relativity and Gravitation, 47 (2), 129(2015)[6], (AA.5) arXiv:1508.02337 which was submitted to Journal of Mathematical Physics [5] in July 2015.

4.1 $t-\phi^i$ Symmetric Initial Data

In this section, we prove that the extrinsic curvature takes a particular form in the presence of a type of symmetry, $t-\phi^i$ symmetry. Gibbons in [77] and his thesis introduced a type of symmetry for initial data with axial Killing vector field (see also [61, Hawking, Appendix C]). This symmetry was generalized to n-dimensional initial data [68]. Firstly, we define limited $t-\phi^i$ symmetric data and $t-\phi^i$ symmetric data.

Definition 36. Let $(\Sigma, \boldsymbol{h}, K)$ be an n-dimensional initial data set with $U(1)^{n-2}$ isometry group with commuting generators $\xi_{(i)} = \partial_{\phi^i}$, that is $[\xi_{(i)}, \xi_{(j)}] = 0$ for $i, j = 1, \dots, n-2$, and

$$\mathcal{L}_{\xi_{(i)}} h_{ab} = 0, \qquad \mathcal{L}_{\xi_{(i)}} K_{ab} = 0\,. \tag{4.1}$$

Moreover,

(a) $\phi^i \to -\phi^i$ is a diffeomorphism which preserves $\boldsymbol{h}$

(b) $\phi^i \to -\phi^i$ is a diffeomorphism which reverses the sign K

Then the initial data set is *limited* $t-\phi^i$ *symmetric data* if the initial data set satisfies condition (a) and it is $t-\phi^i$ *symmetric data* if it satisfies conditions (a)-(b).

One of the main geometrical consequences of limited $t-\phi^i$ symmetry is the following lemma.

Lemma 37. *Let $(\Sigma, \boldsymbol{h}, K)$ be n-dimensional initial data with limited $t-\phi^i$ symmetry, then the two dimensional distribution $\mathcal{D}^2$ orthogonal to $\xi_{(i)}$ is integrable.*

Proof. The $t-\phi^i$ symmetry implies that $\boldsymbol{h}$ does not have cross terms between the Killing part of metric and two other dimensions. Thus the general form of the metric in local chart is

$$\boldsymbol{h} = q_{AB}\mathrm{d}x^A\mathrm{d}x^B + \lambda_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j \tag{4.2}$$

where $i, j, k = 1, \dots, n-2$ and $A, B, C = 1, 2$, $\xi_{(i)} = \frac{\partial}{\partial\phi^i}$ and $\lambda_{ij} = h(\xi_{(i)}, \xi_{(j)})$. Since the Christoffel symbols are

$$\Gamma^A_{ij} = -\frac{1}{2}q^{AB}\partial_B\lambda_{ij} \qquad \Gamma^j_{Ai} = \frac{1}{2}\lambda^{jk}\partial_A\lambda_{ik} \qquad \Gamma^C_{AB} = {}^2\Gamma^C_{AB} \qquad \Gamma^A_{iB} = \Gamma^i_{AB} = \Gamma^k_{ij} = 0 \tag{4.3}$$

The iA components of the Ricci tensor of $\boldsymbol{h}$ by equation (2.3) vanish, that is $R_i^A = 0$. Moreover, since $\xi_{(i)}$ are axial Killing vectors, there exists axes of rotations for $\xi_{(i)}$, i.e. there exist $p \in \Sigma$ such that $\xi_{(i)}|_p = 0$. Then we obtain the following conditions

- $\epsilon_{a_1 \dots a_n} \xi^{a_1}_{(1)} \cdots \xi^{a_{n-2}}_{(n-2)} \nabla^{a_{n-1}} \xi^{a_n}_{(i)}$ vanishes at least one point of the axis of rotations for a given $i = 1, \dots, n-2$
- $\epsilon_{a_1 \dots a_n} \xi^{a_1}_{(1)} \cdots \xi^{a_{n-2}}_{(n-2)} R^{a_{n-1}}_c \xi^c_{(i)}$ vanishes for a given $i = 1, \dots, n-2$

Hence by [63] which is a generalized version of Theorem 7.1.1 of [143], $\mathcal{D}^2$ is integrable.

□

Remark 38. The $t - \phi^i$ symmetric data set obviously implies $K_{AB} q^{AC} q^{BD} = 0$ and $K_{ab} \xi^a_{(i)} \xi^b_{(j)} = 0$ (this means only $K_{Ai} \neq 0$) and the extrinsic curvature [68] is

$$K_{ab} = 2 A^t_{(a} \Phi_{b)} \,, \tag{4.4}$$

where $\Phi = (\xi_{(1)}, \dots, \xi_{n-2})^t$ and $A = (A^1, \dots, A^{n-2})^t$ are column vectors such that $\iota_{\xi_{(j)}} A^i = 0$ for $i, j = 0, \dots, n-2$. Therefore, $t - \phi^i$-symmetry implies maximal condition on initial data, i.e. $\mathrm{Tr}_{\boldsymbol{h}} K = 0$.

Now we construct the candidate transverse traceless $(0,2)$-tensor in limited $t - \phi^i$ symmetry for extrinsic curvature.

Proposition 39. *Let $(\Sigma, \boldsymbol{h}, K)$ be n-dimensional initial data with limited $t - \phi^i$ symmetry and Σ be simply connected. Assume there exists a divergence-less one form column vector $S = (S^1, \dots, S^{n-2})^t$ such that $\iota_{\xi_{(j)}} S^i = 0$ and $\mathcal{L}_{\xi_{(j)}} S^i = 0$ for $i, j = 1, \dots, n-2$. Then there exist a traceless-traverse (TT), symmetric (0,2)-tensor H and functions Y^i such that*

$$H_{ab} \equiv 2 {S^t}_{(a} \lambda^{-1} \Phi_{b)}, \qquad S^i \equiv \frac{(-1)^n}{2 \det \lambda} \iota_{\xi_{(n-2)}} \cdots \iota_{\xi_{(1)}} \star dY^i, \qquad \mathcal{L}_{\xi_{(i)}} Y^j = 0 \tag{4.5}$$

where $\Phi = (\xi_{(1)}, \ldots, \xi_{(n-2)})^t$ *is a column vector,* $\lambda = h(\xi_{(i)}, \xi_{(j)})$ *is a* $(n-2) \times (n-2)$ *symmetric, positive definite matrix, and* $\star$ *is Hodge operator with respect to* $\boldsymbol{h}$*.*

Proof. We prove this lemma by using properties of commuting Killing vectors and Lemma 37 and Proposition 37. First, We define the following one form

$$\mathcal{K}^i \equiv 2 \star \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)}\right) \tag{4.6}$$

Then since $\xi_{(i)}$ are Killing vectors and by equation (2.17) we have

$$\begin{aligned} \mathcal{L}_{\xi_{(j)}} \mathcal{K}^i &= \star \mathcal{L}_{\xi_{(j)}} \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)}\right) \\ &= \star \left([\mathcal{L}_{\xi_{(j)}} S^i] \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)} \right) \\ &+ \star \sum_{k=1}^{n-2} \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \mathcal{L}_{\xi_{(j)}} \xi^\flat_{(k)} \wedge \cdots \wedge \xi^\flat_{(n-2)} \right) = 0 \end{aligned} \tag{4.7}$$

We take the exterior derivative d of the both sides and apply Lemma 1-3

$$\begin{aligned} \mathrm{d}\mathcal{K}^i &= 2\mathrm{d} \star \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)}\right) \\ &= 2(-1)^{n(n-2)} \mathrm{d}\iota_{\xi_{(n-2)}} \star \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-3)}\right) \\ &= 2(-1)^{n(n-2)} \left[\mathcal{L}_{\xi_{(n-2)}} - \iota_{\xi_{(n-2)}} \mathrm{d}\right] \star \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)}\right) \\ &= 2(-1)^{n(n-2)} \Big[\star \mathcal{L}_{\xi_{(n-2)}} \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-3)}\right) \\ &\quad - \iota_{\xi_{(n-2)}} \mathrm{d} \star \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-3)}\right)\Big] \\ &= -2(-1)^{n(n-2)} \iota_{\xi_{(n-2)}} \mathrm{d} \star \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-3)}\right) \\ &= -2(-1)^{n(n-2)} \iota_{\xi_{(n-2)}} \mathrm{d} \star \left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-3)}\right) . \end{aligned} \tag{4.8}$$

If we continue these steps for $\xi_{(i)}$ where $i = 2, \ldots, n-2$, we have

$$\mathrm{d}\mathcal{K}^i = 2(-1)^{n-2} \iota_{\xi_{(n-2)}} \cdots \iota_{\xi_{(1)}} \mathrm{d} \star S^i \,, \tag{4.9}$$

since div$S^i = 0$, we have d$\mathcal{K}^i = 0$. Thus $\mathcal{K}^i$ is a closed form and by simply connectedness of Σ, the $\mathcal{K}^i$ is exact. Therefore, there exists a function Y^i such that $\mathcal{K}^i = \mathrm{d}Y^i$ for each i. We multiply $\mathcal{K}^i$ and take interior multiplication

$$S^i = \frac{(-1)^n}{2\det\lambda} \star \left(\mathrm{d}Y^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)}\right) . \tag{4.10}$$

Second, we prove that H is a transverse-traceless (0,2)-tensor. Let ∇ be the covariant derivative with respect to $\boldsymbol{h}$. Then we have

$$\begin{aligned}
\mathrm{div}_{\boldsymbol{h}} H &= \nabla^a \mathrm{Tr}\left(\lambda^{-1} S_a \Phi^t_b\right) + \nabla^a \mathrm{Tr}\left(\lambda^{-1}\Phi_a S^t_b\right) \\
&= \mathrm{Tr}\left(\nabla^a \lambda^{-1} S_a \Phi^t_b\right) + \mathrm{Tr}\left(\lambda^{-1} S_a \nabla^a \Phi^t_b\right) + \mathrm{Tr}\left(\nabla^a \lambda^{-1}\Phi_a S^t_b\right) + \mathrm{Tr}\left(\lambda^{-1}\Phi_a \nabla^a S^t_b\right) \\
&= \mathrm{Tr}\left(\nabla^a \lambda^{-1} S_a \Phi^t_b\right) + 2\mathrm{Tr}\left(\lambda^{-1} S_a \nabla^a \Phi^t_b\right) \\
&= \mathrm{Tr}\left(\nabla^a \lambda^{-1} S_a \Phi^t_b\right) + \mathrm{Tr}\left(\lambda^{-1} S_a \Phi^t_b \lambda^{-1}\nabla^a \lambda\right) - \mathrm{Tr}\left(\lambda^{-1} S_a \Phi^{ta} \lambda^{-1}\nabla_b \lambda\right) \\
&= -\mathrm{Tr}\left(\lambda^{-1}\nabla^a \lambda \lambda^{-1} S_a \Phi^t_b\right) + \mathrm{Tr}\left(S_a \Phi^t_b \lambda^{-1} \nabla^a \lambda \lambda^{-1}\right) = 0 \, .
\end{aligned} \tag{4.11}$$

The first equality follows from trace property of product of matrices, i.e. $\Phi^t_a \lambda^{-1} S_b = \mathrm{Tr}\,(\lambda^{-1} S_b \Phi^t_a) = \mathrm{Tr}\,(\lambda^{-1}\Phi_a S^t_b)$. The second equality is based on Killing properties of $\xi_{(i)}$ and divergence-less property of S^i, i.e. $\nabla^a \Phi_a = 0$ and $\nabla^a S_a = 0$. The third equality follows from symmetric property of λ and $\mathcal{L}_{\xi_{(i)}} S^j = 0$, that is $\xi^a_{(i)} \nabla_a S^{jb} = S^{ja} \nabla_a \xi^b_{(i)}$. Moreover, the fact that $\xi_{(i)}$ are Killing vectors and in $t - \phi^i$ symmetry the metric is in the form of equation (4.2) we have $\Phi_a \nabla^a \lambda^{-1} = 0$. The fourth equality follows from integrability property of distribution $\mathcal{D}^2$ orthogonal to $\xi_{(i)}$ by Lemma 37 and the identity in Proposition 3. The fifth equality follows from $\iota_{\xi_{(i)}} S^i = 0$. □

Now the question is what is the relation between TT tensor H and extrinsic curvature K for $t-\phi^i$ symmetric data. By Remark 38 we have

$$K_{ab} = 2A^t_{(a}\Phi_{b)}\,. \tag{4.12}$$

Then if we multiply this by Φ^{tb} and simplify we have $A_a = K_{ab}\lambda\Phi^b$. We define $\hat{A}_a \equiv \lambda^{-1}A_a = (\hat{A}^1,\ldots,\hat{A}^{n-2})^t$, then

$$K_{ab} = 2\hat{A}^t_{(a}\lambda^{-1}\Phi_{b)}\,. \tag{4.13}$$

Then by a similar argument to the steps leading to equation (4.11), we have

$$\nabla^a\hat{A}_a = \nabla^a K_{ab}\Phi^b\,. \tag{4.14}$$

Thus $\nabla^a\hat{A}_a = 0$ if and only if $\text{div}_{\boldsymbol{h}}K = 0$ or $\iota_{\xi_{(i)}}\text{div}_{\boldsymbol{h}}K = 0$ for $i = 1,\ldots,n-2$. But in $t-\phi^i$ symmetry a straightforward computation shows $\text{div}_{\boldsymbol{h}}K = (\iota_{\xi_{(i)}}\text{div}_{\boldsymbol{h}}K)\text{d}\phi^i$ for $i = 1,\ldots,n-2$ where $\xi_{(i)} = \partial_{\phi^i}$. Therefore, $H = K$ if and only if $\text{div}_{\boldsymbol{h}}K = 0$. Now, what is the geometrical meaning of the function Y in Definition 4.5 of S? The answer is the following corollary. First, we know the twist one form is defined by [143]

$$\omega^i \equiv \star_{\boldsymbol{g}}\left(\xi^\flat_{(1)}\wedge\cdots\wedge\xi^\flat_{(n-2)}\wedge\text{d}\xi^\flat_{(i)}\right)\,, \tag{4.15}$$

where $\star_{\boldsymbol{g}}$ is Hodge star with respect to spacetime $(M,\boldsymbol{g})$ corresponding to data $(\Sigma,\boldsymbol{h},K)$ such that $\star = \iota_n\star_{\boldsymbol{g}}$, where n is a unit normal timelike vector field on Σ. Observe that $\iota_n\xi^\flat_{(i)} = 0$ for $i = 1,2$. Geometrically the twist one form ω^i measures the failure of $\xi_{(i)}$ to be hypersurface orthogonal.

Corollary 40. *Let $(\Sigma,\boldsymbol{h},K)$ be simply connected, $t-\phi^i$ symmetric data with $\text{div}_{\boldsymbol{h}}K = 0$. Then $dY^i = \omega^i$ and we call Y^i twist potential.*

Proof. Since $\mathrm{div}_{\boldsymbol{h}} K = 0$ we can define one form $S = K_{ab}\Phi^b$. Then by Proposition 39 there exits Y^i related to S^i for $i = 1, \dots, n-2$. First by definition of S^i we have

$$\begin{aligned} S^i_a &= K_{ab}\xi^b_{(i)} = h^l_a \boldsymbol{\nabla}_l n_b \xi^b_{(i)} \\ &= -h^l_a \boldsymbol{\nabla}_l \xi^b_{(i)} n_b = -\left(g^l_a + n^l n_a\right) \boldsymbol{\nabla}_l \xi^b_{(i)} n_b \\ &= -\boldsymbol{\nabla}_{\boldsymbol{a}} \xi^b_{(i)} n_b = -\frac{1}{2}(\mathrm{d}\xi_{(i)})_{ab} n^b \end{aligned} \tag{4.16}$$

where $\boldsymbol{\nabla}$ and ∇ are covariant derivatives with respect to $\boldsymbol{g}$ and $\boldsymbol{h}$. Then we multiply $\star_{\boldsymbol{g}}$ to equation (4.15) and we have

$$\star_{\boldsymbol{g}} \omega^i = (-1)^{n+1} \mathrm{d}\xi_{(i)} \wedge \xi_{(1)} \wedge \cdots \wedge \xi_{(n-2)} \,. \tag{4.17}$$

If we take interior multiplication of this equation with respect to $\xi_{(i)}$ for $i = 1, \dots, n-2$ and using Lemma 1-3, we have

$$\mathrm{d}\xi_{(i)} = \frac{(-1)^{n+1}}{\det\lambda} \star_{\boldsymbol{g}} \left(\omega^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)}\right) \,. \tag{4.18}$$

Then by equation (4.16) we have

$$\begin{aligned} S^i &= -\frac{1}{2}\iota_n \mathrm{d}\xi_{(i)} = \frac{(-1)^{n+2}}{2\det\lambda}\iota_n \star_{\boldsymbol{g}} \left(\omega^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)}\right) \\ &= \frac{(-1)^n}{2\det\lambda} \star \left(\omega^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)}\right) = \frac{(-1)^n}{2\det\lambda}\iota_{\xi_{(n-2)}} \cdots \iota_{\xi_{(1)}} \star \omega^i \,. \end{aligned} \tag{4.19}$$

Hence $\mathrm{d}Y^i = \omega^i$. □

Note that in the construction of the mass functional in the next chapter, we need the full contraction of extrinsic curvature. In $t - \phi^i$ symmetric initial data it has the following simple expression

$$
\begin{aligned}
|K|^2_{\boldsymbol{h}} &= \left(S^t_a\lambda^{-1}\Phi_b + S^t_b\lambda^{-1}\Phi_a\right)\left(\Phi^{ta}\lambda^{-1}S^b + \Phi^{tb}\lambda^{-1}S^a\right) \\
&= 2\mathrm{Tr}\left(\lambda^{-1}S^aS^t_a\right) \\
&= \frac{1}{2\det\lambda}\mathrm{Tr}\left(\lambda^{-1}\nabla^aY\nabla_aY^t\right) = \frac{1}{2\det\lambda}q^{AB}D_AY^t\lambda^{-1}D_BY\,, \qquad (4.20)
\end{aligned}
$$

where D and ∇ are covariant derivative with respect to the two dimensional metric $\boldsymbol{q}$ orthogonal to $\xi_{(i)}$ in equation (4.2) and $\boldsymbol{h}$, respectively. The first equality is based on the trace property of matrices and symmetric property of λ, i.e. $S^t_a\lambda^{-1}\Phi_b = \Phi^t_b\lambda^{-1}S_a = \mathrm{Tr}(\lambda^{-1}S_a\Phi^t_b) = \mathrm{Tr}(\Phi_bS^t_a\lambda^{-1})$. The second equality follows from $\iota_{\xi_{(j)}}S^i = 0$ and $\Phi_a\Phi^{ta} = \lambda$ and definition of trace. The third equality follows from

$$
\begin{aligned}
S^i_aS^{ja} &= \left\langle S^i, S^j\right\rangle_{\boldsymbol{h}} = \frac{1}{4(\det\lambda)^2}\left\langle \star\left(\mathrm{d}Y^i\wedge\alpha\right), \star\left(\mathrm{d}Y^j\wedge\alpha\right)\right\rangle_{\boldsymbol{h}} \\
&= \frac{1}{4(\det\lambda)^2}\epsilon_{aca_1\dots a_{n-2}}\epsilon^{bcb_1\dots b_{n-2}}\nabla^aY^i\nabla_bY^j\alpha^{a_1\dots a_{n-2}}\alpha_{b_1\dots b_{n-2}} \\
&= \frac{1}{4(\det\lambda)^2}\delta^{bb_1\dots b_{n-2}}_{aa_1\dots a_{n-2}}\nabla^aY^i\nabla_bY^j\alpha^{a_1\dots a_{n-2}}\alpha_{b_1\dots b_{n-2}} \\
&= \frac{1}{4(\det\lambda)^2}\nabla^aY^i\nabla_aY^j\delta^{b_1\dots b_{n-2}}_{a_1\dots a_{n-2}}\alpha^{a_1\dots a_{n-2}}\alpha_{b_1\dots b_{n-2}} \\
&= \frac{1}{4\det\lambda}\nabla^aY^i\nabla_aY^j = \frac{1}{4\det\lambda}q^{AB}D_AY^iD_BY^j\,, \qquad (4.21)
\end{aligned}
$$

where we define $\alpha \equiv \xi^\flat_{(1)}\wedge\cdots\wedge\xi^\flat_{(n-2)}$ and $\alpha^{a_1\dots a_{n-2}} \equiv \xi^{a_1}_{(1)}\cdots\xi^{a_n}_{(n-2)}$ and third equality of (4.21) follows from $\mathcal{L}_{\xi_j}Y^i = 0$. Moreover, since $\lambda_{ij} = h(\xi_{(i)}, \xi_{(j)})$, we have $\det\lambda = \delta^{b_1\dots b_{n-2}}_{a_1\dots a_{n-2}}\alpha^{a_1\dots a_{n-2}}\alpha_{b_1\dots b_{n-2}}$ and yields final equality. Now we prove existence of twist potential vector Y for general $U(1)^{n-2}$ invariant initial data

Corollary 41. *Assume an n-dimensional simply connected, initial data set $(\Sigma, \boldsymbol{h}, K)$ with isometry group $U(1)^{n-2}$ which its generators commute ($[\xi_{(i)}, \xi_{(j)}] = 0$) and $\iota_{\xi_i}div_{\boldsymbol{h}}K = 0$. Then there exist global twist potentials $Y = (Y^1, \dots, Y^{n-2})^t$.*

Proof. We define the following $U(1)^{n-2}$ invariant vector of one form

$$S_a \equiv K_{ab}\Phi^b - K_{cd}\Phi_a\lambda^{-1}\Phi^c\Phi^{td} \tag{4.22}$$

Since $\xi_{(i)}$ are Killing vectors, we have $\iota_{\xi_{(i)}}S^i = 0$ for $i = 1, \ldots, n-2$. Moreover, since $\iota_{\xi_i}\mathrm{div}_{\boldsymbol{h}}K = 0$ and $\iota_{\xi_{(i)}}S^i = 0$, we have

$$\mathrm{div}S = \nabla^a K_{ab}\Phi^b + K_{ab}\nabla^a\Phi^b - \nabla^a\Phi_a K_{cd}\lambda^{-1}\Phi^c\Phi^{td} - \Phi^a\nabla_a\left(K_{cd}\lambda^{-1}\Phi^c\Phi^{td}\right) = 0\,. \tag{4.23}$$

Then by the argument of Proposition 39 and Corollary 40 the function vector Y exists and is defined by equation (4.6), that is

$$\mathrm{d}Y^i \equiv 2\star\left(S^i \wedge \xi^\flat_{(1)} \wedge \cdots \wedge \xi^\flat_{(n-2)}\right) \tag{4.24}$$

for $i = 1, \ldots, n-2$. □

4.2 Generalized Brill (GB) Initial Data

In this section, we introduce *generalized Brill (GB) data set* $(\Sigma, \boldsymbol{h}, K)$ with some assumptions and follow the argument in [3, 39, 96]. Recall that Brill data set is a three dimensional initial data with vanishing extrinsic curvature, *time-symmetric*, and $U(1)$-action is orthogonal transitive. The GB initial data set has three main characteristics

1. $(\Sigma, \boldsymbol{h})$ is complete four dimensional Riemannian manifold with two ends (at least one asymptotically flat end).

2. Σ is a simply connected manifold.

3. The data set is $U(1)^2$-invariant.

In general, the GB data set has two asymptotic ends. We consider one asymptotically flat end and another end can be asymptotically flat or cylindrical end. By Definition 20 and [15] asymptotically flatness for ends of GB data is equivalent to the asymptotic behaviour in local chart (U, x) at each end

$$h_{ab} - \delta_{ab} = o_2(r^{-1}), \quad K_{ab} = o_1(r^{-2}), \quad \partial_c h_{ab} \in L^2(\Sigma_{\text{ext}})\,. \tag{4.25}$$

Moreover, the initial data has an asymptotically cylindrical end if Σ_{ext} is diffeomorphic to $(C_R \equiv \mathbb{R}^+ \times N, h^c)$, where N is a closed 3 dimensional manifold and h^c is fixed Riemannian metric on N [36], such that for some positive constants c_1, c_2 in local chart $x : \Sigma_{ext} \to C_R$ we have

$$c_1\left(\mathrm{d}y^2 + h^c\right) \leq h \leq c_2\left(\mathrm{d}y^2 + h^c\right) \tag{4.26}$$

i.e. the metric is equipped to a cylindrical geometry, and

$$h = \mathrm{d}y^2 + h^c + O(e^{-\nu y}) \tag{4.27}$$

where $\nu > 0$. Before we continue to construct this class of data, we have the following definition of symmetry.

Definition 42. A four dimensional initial data set $(\Sigma, \boldsymbol{h}, K)$ is called $U(1)^2$-invariant if there exist two commuting rotational Killing vector fields $\xi_{(i)}$ which generate the compact Lie group $U(1) \times U(1)$ that acts smoothly with no discrete isotropy groups on Riemannian manifold $(\Sigma, \boldsymbol{h})$ and

$$\mathcal{L}_{\xi_{(i)}} \boldsymbol{h} = \mathcal{L}_{\xi_{(i)}} K = 0\,. \tag{4.28}$$

Thus the GB data has isometry group $G = U(1) \times U(1)$ with elements $k = (e^{i\tau_1}, e^{i\tau_2})$ for $0 \leq \tau_1, \tau_2 \leq 2\pi$ and generators $\xi_{(i)}$. Since G-action does not have discrete isotropy groups, the only isotropy groups are $G_1 \equiv \{e\}$, $G_2 \equiv U(1)$, and $G_3 \equiv G$. If there exist any other two commuting Killing vectors $\tilde{\xi}_{(i)}$ which generate G, then they are related to $\xi_{(i)}$ by the matrix

$$\tilde{\xi}_{(i)} = \sum_{j=1}^{2} N_i^j \xi_{(j)}, \qquad N_i^j \in GL(2, \mathbb{Z}) \tag{4.29}$$

with Gram matrix $\lambda_{ij} = h(\xi_{(i)}, \xi_{(j)})$. Then the orbit space $\mathcal{B} = \Sigma/G$ is a 2 dimensional manifold with three different region types corresponding to three isotropy groups. We have the following result [96].

Proposition 43. *[96, Proposition 1]The orbit space $\mathcal{B} = \Sigma/G$ is a 2-dimensional simply connected smooth manifold with boundaries and corners, i.e. a manifold locally modelled over $\mathbb{R}\times\mathbb{R}$ (interior points), $\mathbb{R}^+\times\mathbb{R}$ (1- dimensional boundary segments) and $\mathbb{R}^+ \times \mathbb{R}^+$ (corners). Furthermore, for each of the 1-dimensional boundary segments, the rank of the Gram matrix λ is precisely 1, and there is a vector $w = (w_1, w_2)$ with integer entries such that $\lambda_{ij} w^j = 0$ for each point of the segment. If w_i and w_{i+1} are the vectors associated with two adjacent boundary segments meeting in a corner, then we must have*

$$\begin{pmatrix} w_i^1 & w_{i+1}^1 \\ w_i^2 & w_{i+1}^2 \end{pmatrix} \in GL(2, \mathbb{Z}) \quad \Longleftrightarrow \quad \det(w_i, w_{i+1}) = \pm 1. \tag{4.30}$$

On the corners, the Gram matrix has rank 0, and in the interior it has rank 2.

The proof of this proposition is based on proving that tangent space of the orbits at each point has 0, 1, and 2 dimensions. We refer the interested reader to the Proposition 1 of [96] for the proof (see the Figure 4.1). Now, let $\pi : \Sigma \to \mathcal{B}$ a be canonical

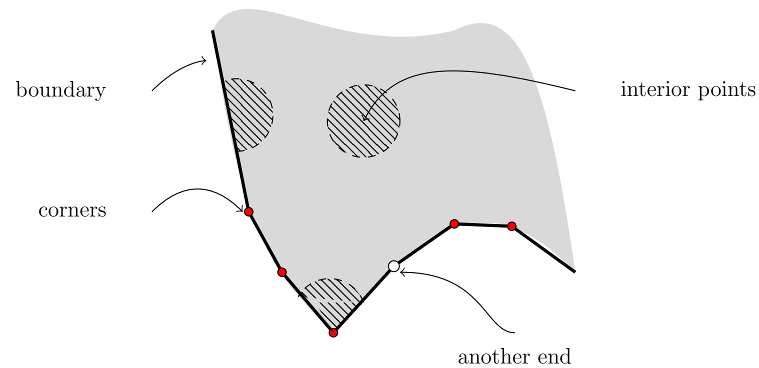


Figure 4.1: Orbit space with boundaries and corners. The empty circle which is a point removed from boundary represents another end.

projection of Σ onto orbit space $\mathcal{B}$. For any point $p \in \mathcal{B}$ which lifts to an orbit of *principal type* (interior points of $\mathcal{B}$ have orbits with principal(minimal) type), there exists a metric q defined by $X, Y \in T_p\mathcal{B}$. Let $\hat{p} \in \Sigma$ such that $\det \lambda|_{\hat{p}} \neq 0$ and $\pi(\hat{p}) = p$. Then the push forward map $\pi_* : \mathcal{X}_{\xi_{(i)}}(\Sigma) \to \mathcal{X}(\mathcal{B})$ such that $\pi_*(\hat{X})|_p = \pi_*|_{\hat{p}}(\hat{X}|_{\hat{p}})$, where $\mathcal{X}_{\xi_{(i)}}(\Sigma) = \{X \in \mathcal{X}(\Sigma) : h(\xi_{(i)}, X) = 0, \quad \mathcal{L}_{\xi_{(i)}} X = 0\}$, is an isomorphism. Then there exist unique vectors $\hat{X}, \hat{Y} \in T_{\hat{p}}\Sigma$ orthogonal to $\xi_{(i)}$ such that $\pi_*(\hat{Y}) = Y$ and $\pi_*(\hat{X}) = X$ and we have

$$q(X, Y) \equiv h(\hat{X}, \hat{Y})\,. \tag{4.31}$$

Then by the principal orbit theorem (Theorem 1.13 of [60]) there exists an open dense set of $\mathcal{B}$ which can at least locally be modeled on smooth sub-manifold (perhaps with boundary and corners) say S of Σ which meets orbits of $\xi_{(i)}$ precisely once and it is called *cross-section* of the G-action. Let

$$p \in \bar{S} \equiv S \backslash \{\det \lambda_{ij} = 0\}\,, \tag{4.32}$$

where $\lambda_{ij} = h(\xi_{(i)}, \xi_{(j)})$ and for any $X, Y \in T_p\bar{S} \subset T_p\mathcal{B}$ we have

$$\begin{aligned} q(X,Y) &= h(X,Y) - \frac{1}{\det h_{ij}}\Big[h(\xi_{(2)},\xi_{(2)})h(\xi_{(1)},X)h(\xi_{(1)},Y) \\ &+ h(\xi_{(1)},\xi_{(1)})h(\xi_{(2)},X)h(\xi_{(2)},Y) - h(\xi_{(1)},\xi_{(2)})h(\xi_{(1)},X)h(\xi_{(2)},Y) \\ &- h(\xi_{(1)},\xi_{(2)})h(\xi_{(2)},X)h(\xi_{(1)},Y)\Big]\,. \end{aligned} \tag{4.33}$$

This definition coincides with (4.31) when X, Y are tangent to orbit space and in indices we have the projection map

$$q^a_b = \delta^a_b - h^{ij}\xi^a_{(i)}\xi^\flat_{(j)b}\,. \tag{4.34}$$

Thus all information about h is contained in q and in the one-form

$$\xi^\flat_{(i)} = h(\xi_{(i)}, \cdot)\,. \tag{4.35}$$

This means that there exists a projection $P_{\xi_{(i)}} : T\Sigma \to T\bar{S}$ such that $X \to P_{\xi_{(i)}}(X)$. Thus $X = Y + \alpha^i\xi_{(i)}$ where $Y \in T\bar{S}$ and $P_{\xi_{(i)}}(X) \equiv Y$. Assume that $x = (x^A)$, for $A = 1, 2$, is a local coordinate on S and propagate these coordinates of S such that $\mathcal{L}_{\xi_{(i)}}x^A = 0$. Moreover, let ϕ^i be coordinates that vanish on S and $\mathcal{L}_{\xi_{(i)}}\phi^i = 1$. Then $\xi_{(i)} = \partial_{\phi^i}$ and $P_{\xi_{(i)}}(X^A\partial_A + X^i\partial_{\phi^i}) = X^A\partial_A$ and the metric will be

$$\boldsymbol{h} = q_{AB}\mathrm{d}x^A\mathrm{d}x^B + \lambda_{ij}(\mathrm{d}\phi^i + A^i_B\mathrm{d}x^B)(\mathrm{d}\phi^j + A^j_B\mathrm{d}x^B)\,, \tag{4.36}$$

where $\partial_{\phi^i}q_{ab} = \partial_{\phi^i}A^i_B = \partial_{\phi^i}h_{ij} = 0$. Thus this is a smooth metric on Σ. Since h is an asymptotically flat metric with asymptotic condition (4.25), it suggests the following assumption[1]

[1]see Remark 47.

Assumption 1 We assume that q is a conformally flat smooth 2-dimensional metric on $\mathcal{B}$ (interior and axis) with *Weyl coordinates* (ρ, z) as a part of four-dimensional Riemannian metric h such that ρ is harmonic with $\mathrm{d}\rho \neq 0$ nonzero on interior of $\mathcal{B}$. Moreover, the metric has global representation

$$q_{AB}\mathrm{d}x^A\mathrm{d}X^B = e^{2V+2v}\frac{(\mathrm{d}\rho^2 + \mathrm{d}z^2)}{2\sqrt{\rho^2 + z^2}}\,, \tag{4.37}$$

and $\lambda_{ij} = \lambda'_{ij}e^{2v}$.

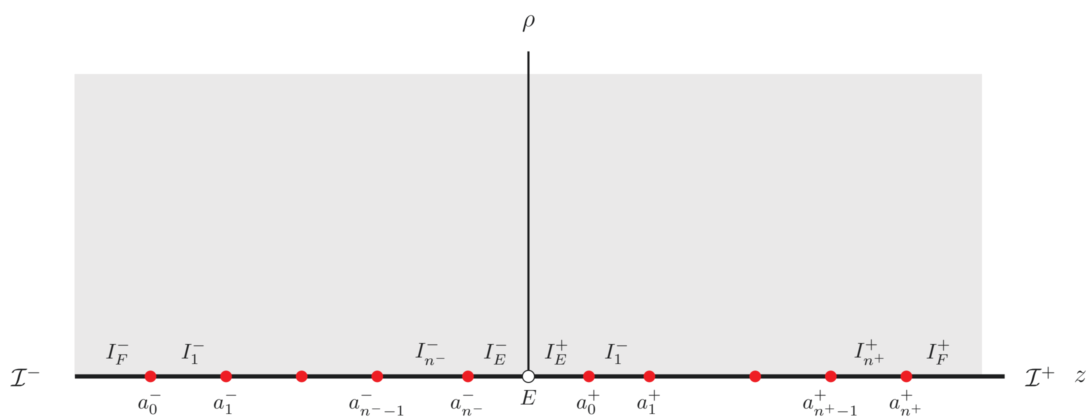


Figure 4.2: The orbit space as half plane with two ends.

Moreover, since $\mathcal{B}$ is an (orientable) simply connected 2-dimensional analytic manifold with boundaries and corners, we may map it analytically to the upper complex half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$ by the generalized version of Riemann mapping theorem, Osgood-Caratheodory theorem, such that one asymptote represents the infinity of upper half plane and another represented by origin space (see Figure 4.2)[3, 67, 96]. The boundary of the orbit space $\partial\mathcal{B}$ lies on the z-axis and it is denoted by $\Gamma \equiv \{\rho = 0\}$. By proposition 43, we can define $\Gamma = \mathcal{I}^+ \cup \mathcal{I}^-$ where

$$\mathcal{I}^{\pm} = \left(\bigcup_{i=1}^{n^{\pm}} I_i\right) \bigcup \left(\bigcup_{i=1}^{m^{\pm}} a_{i-1}^{\pm}\right) \bigcup \left(I_F^{\pm} \bigcup I_E^{\pm}\right) \tag{4.38}$$

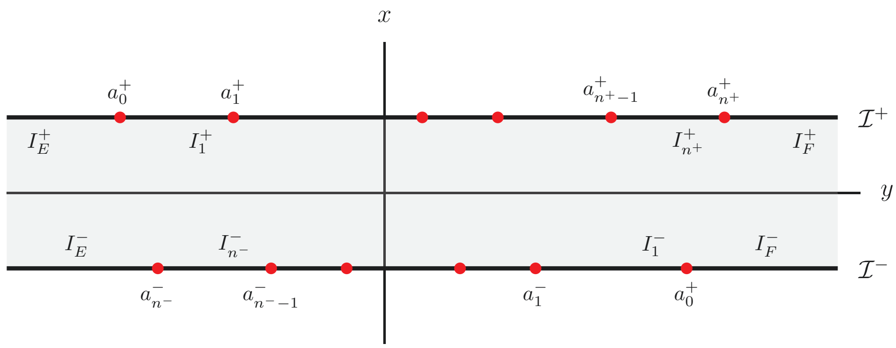


Figure 4.3: Orbit space as infinite strip. The map from the $z + i\rho$ complex plane to the $y + ix$ complex plane where $y = \log r$.

where $n^\pm \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and if $n^\pm \neq 0$ then $m^\pm = n^\pm + 1$ and if $n^\pm = 0$ then $m^\pm = 1$ or 0. These points $a_i^\pm$ represent the corners and $I_i^\pm = (a_{i-1}^\pm, a_i^\pm)$ represent finite intervals. On each interval a particular integer linear combination of the $\xi_{(i)}$ vanishes. The semi-infinite rods $I_F^\pm$ and $I_E^\pm$ at the ends correspond to the axes of rotation of the fixed asymptotically flat $\mathbb{R}^4$ region and the axes of rotation of the asymptotic E region, respectively. It is possible that these two coincide (e.g. Myers-Perry initial data) and then $\mathcal{I}^\pm = I_F^\pm = I_E^\pm$. Without loss of generality, we can choose $\xi_{(1)}, \xi_{(2)}$ to vanish on I_F^+ and I_F^- respectively. The finite-length intervals (rods), on the other hand, correspond to 2-cycles in Σ.

In order to understand the topology of the two asymptotic ends it is useful to use *the quasi-isotropic coordinate* (r, x) with transformation

$$\rho = \frac{r^2}{2}\sqrt{1 - x^2}, \qquad z = \frac{r^2}{2}x \tag{4.39}$$

such that it maps conformally the half plane $\{(\rho, z) : \rho \in \mathbb{R}^+ \cup \{0\}, \quad z \in \mathbb{R}\}$ to infinite strip $\{(r, x) : r \in \mathbb{R}^+ \cup \{0\}, \quad -1 \leq x \leq 1\}$ (see Figure 4.3). In characteristics of GB data, we consider asymptotically flat and cylindrical ends. But what are the possible topologies of each end by just considering orbit space $\mathcal{B}$? Clearly each end is a three dimensional closed manifold that we denote by N. In orbit space picture

N is a closed interval, i.e. $N/G = [a, b]$ where end points a and b correspond to isotropy group $G_2 = \{U(1)\}$ which one Killing vector $v = v^i \xi_{(i)}$ vanishes. Note that it is impossible to have isotropy group $G_3 (= U(1) \times U(1))$ at a point $p \in N$, which means both Killing vector fields $\xi_{(i)}$ vanish at p. This fact is due to the following argument. We know $\xi_{(i)}$ are commuting Killing vector fields and then the derivative of $[\xi_{(i)}, \xi_{(2)}] = 0$ at p is

$$\nabla_a \xi^c_{(1)} \nabla_c \xi^b_{(2)} - \nabla_a \xi^c_{(2)} \nabla_c \xi^b_{(1)} = 0 \qquad \text{at } p\,. \tag{4.40}$$

Moreover, since $\nabla_a \xi^c_{(i)} : T_p N \to T_p N$ are linear transformations (skew-symmetric 3×3 matrix) on $T_p N$ and linearly independent, they can be viewed as elements of the Lie-algebra $so(3)$ which commute. But the Cartan subalgebra (abelian subalgebra) of $so(3)$ has rank 1 [98]. Thus it is impossible to have isotropy group G_3. Then we have the following proposition which the proof is exactly similar to proposition 2 in [96].

Proposition 44. *Let N be a 3 dimensional closed manifold which represents ends of simply connected, complete, Riemannian manifold Σ with G-action ($G = U(1) \times U(1)$). Then N is topologically either a ring $S^1 \times S^2$, a sphere S^3, or Lens space $L(p, q)$, with $p, q \in \mathbb{Z}$.*

Remark 45. The Lens spaces $L(p, q)$ are the spaces obtained by factoring the unit sphere S^3 in $\mathbb{C}^2$ by the group action $(z_1, z_2) \to (e^{2i\pi/p} z_1, e^{2i\pi/q} z_2)$ and fundamental group of the Lens space is $\pi_1(L(p, q)) = \mathbb{Z}_p$, and q is determined only up to integer multiples of p and homology groups are $\mathcal{H}_k(L(p, q)) = \mathbb{Z}$ for $k = 0, 3$ and $\mathcal{H}_1(L(p, q)) = \mathbb{Z}_p$ [80, Example 2.43].

Note that $N \cong S^3, S^1 \times S^2, L(p, q)$ are all spaces with positive Yamabe types, that is they admit a metric with positive scalar curvature [19, 71]. Then based on

Proposition 44, Assumption 1 and equation (4.36) and asymptotic fall off equations (4.25) and (4.27) we have the following assumption

Assumption 2 The coordinate system (ρ, z, ϕ^i) forms a global coordinate[2] system on Σ where $\rho \in \mathbb{R}^+ \cup \{0\}$, $z \in \mathbb{R}$, and ϕ^i have period 2π. The functions v, V, A^i_B, and λ'_{ij} satisfy

(i) as $r \to \infty$

$$v = o_1(r^{-1}), \quad A^i_\rho = \rho o_1(r^{-5}), \quad A^i_z = o_1(r^{-3}), \quad V = o_1(r^{-1}), \qquad (4.41)$$

$$\lambda'_{ii} = \left(1 + (-1)^{i-1} f_{11} r^{-1-\epsilon} + o_1(r^{-2})\right) \sigma_{ii}, \quad \lambda'_{12} = \rho^2 o_1(r^{-5}) \qquad (4.42)$$

where $0 < \epsilon \leq 1$, $\sigma_{ij} = \frac{r^2}{2} \mathrm{diag}\,(1-x, 1+x)$.

(ii) As $r \to 0$ which represents asymptotically flat end we have

$$v = -2\log(r) + O_1(1), \quad V = o_1(r) \quad A^i_\rho = \rho o_1(r), \quad A^i_z = o_1(r^3) \quad (4.43)$$

$$\lambda'_{ii} = \left(1 + (-1)^{i-1} f_{22} r^{1+\epsilon} + o_1(r^2)\right) \sigma_{ii}, \quad \lambda'_{12} = \rho^2 o_1(r^{-1}). \qquad (4.44)$$

(iii) As $r \to 0$ which represents cylindrical end with topology $\mathbb{R}^+ \times N$ where $N \cong S^3, S^1 \times S^2, L(p,q)$ we have

$$v = -\log(r) + O_1(1), \quad A^i_\rho = \rho o_1(r), \quad A^i_z = o_1(r^3)$$
$$\lambda'_{ij} - r^2 \bar{\sigma}_{ij} = o_1(r), \quad V = O_1(1) \qquad (4.45)$$

where $h^c = e^{2V} \frac{dx^2}{4(1-x^2)} + \bar{\sigma}_{ij} \mathrm{d}\phi^i \mathrm{d}\phi^j$ is a metric on N.

[2]see Remark 47

(iv) as $\rho \to 0$ and $w = w^i \frac{\partial}{\partial \phi_i}$ is the Killing vector vanishes on the rod I_i

$$V = O_1(1), \qquad v = O_1(1), \quad A^i_\rho = O_1(\rho), \quad A^i_z = O_1(1), \tag{4.46}$$

$$\lambda'_{ij} w^j = O(\rho^2), \qquad \text{and otherwise } \lambda'_{ij} = O(1)\,, \tag{4.47}$$

and to avoid conical singularities on the axis Γ we have

$$V(z) = \frac{1}{2} \lim_{\rho \to 0} \log \left(\frac{2\sqrt{\rho^2 + z^2}\lambda'_{ij} w^i w^j}{\rho^2} \right) \equiv \frac{1}{2} \log V_i, \tag{4.48}$$

for $z \in I_i = (a_i, a_{i+1})$, $w^i \in \mathbb{Z}$, and $\log V_i \in L^1(\mathbb{R})$.

Definition 46. A generalized Brill (GB) initial data set $(\Sigma, \boldsymbol{h}, K)$ is a $U(1)^2$-invariant, simply connected, complete, maximal, initial data set and h on Σ admits a global representation of the form

$$\boldsymbol{h} = e^{2U+2v} \left(\mathrm{d}\rho^2 + \mathrm{d}z^2 \right) + e^{2v} \lambda'_{ij} \left(\mathrm{d}\phi^i + A^i_B \mathrm{d}x^B \right) \left(\mathrm{d}\phi^j + A^j_B \mathrm{d}x^B \right) \tag{4.49}$$

where $(x^1, x^2) = (\rho, z)$, $\det \lambda' = \rho^2$, $\rho \in [0, \infty)$, $z \in \mathbb{R}$, and $\phi^i \in [0, 2\pi]$ and $U = V - \frac{1}{2} \log \left(2\sqrt{\rho^2 + z^2} \right)$ and the functions v, V, A^i_B and λ'_{ij} satisfy in Assumption 2 and extrinsic curvature satisfy

$$|K|_{\boldsymbol{h}} = o_1(r^{-2}), \quad \text{as } r \to \infty \qquad |K|_{\boldsymbol{h}} = o_1(r^2) \quad \text{as } r \to 0 \tag{4.50}$$

$$|K|_{\boldsymbol{h}} = O_1(1) \quad \text{as } \rho \to 0 \tag{4.51}$$

Remark 47. Note that based on argument in [39] for 3 dimensional data with $U(1)$-action, it suggests assumptions 1 and 2 are unnecessary and we will investigate this problem in future and in this thesis we continue by the assumptions.

Remark 48. Since two known extreme vacuum, $U(1)^2$-invariant stationary black hole solutions have the following property

$$V = \bar{V}(x)r^{-2} + o_1(r^{-2}), \qquad \int_{-1}^{1} \bar{V}(x)\mathrm{d}x = 0, \qquad r \to \infty \tag{4.52}$$

We assume all extreme vacuum, $U(1)^2$-invariant stationary black hole solutions have this decay..

We have the following interesting result about lower bound of $|K|_{\boldsymbol{h}}$ for any GB initial data.

Lemma 49. *Let $(\Sigma, \boldsymbol{h}, K)$ be a GB initial data with $\iota_{\xi_{(i)}} div_{\boldsymbol{h}} K = 0$. Then*

$$|K|^2_{\boldsymbol{h}} \geq e^{-2U} \frac{\nabla Y^t \lambda^{-1} \nabla Y}{2 \det \lambda} \tag{4.53}$$

where ∇ is the covariant derivative with respect to flat metric $\delta_3 = d\rho^2 + dz^2 + \rho^2 d\varphi^2$ on $\mathbb{R}^3$.

Proof. The metric for Brill data is

$$\boldsymbol{h} = e^{2U+2v}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + e^{2v}\lambda'_{ij}\left(\mathrm{d}\phi^i + A^i_B \mathrm{d}x^B\right)\left(\mathrm{d}\phi^j + A^j_B \mathrm{d}x^B\right) . \tag{4.54}$$

Now introduce the co-frame of one forms $\{\theta^a\}$

$$\theta^B = e^{v+U}\mathrm{d}x^B, \qquad \theta^{i+2} = e^v\left(\mathrm{d}\phi^i + A^i_B \mathrm{d}x^B\right), \tag{4.55}$$

so that the metric can be expressed as

$$h = (\delta_2)_{BC}\,\theta^B\theta^C + \lambda'_{ij}\theta^{i+2}\theta^{j+2} \tag{4.56}$$

where $\delta_2 = \mathrm{d}\rho^2 + \mathrm{d}z^2$ is flat 2 dimensional Riemannian metric and the associated dual frame of basis vectors

$$e_B = e^{-(v+U)}\left(\partial_B - A^i_B\partial_{\phi^i}\right) \qquad e_{i+2} = e^{-v}\partial_{\phi^i} \tag{4.57}$$

Since $\iota_{\xi_{(i)}}\text{div}K = 0$ and Σ is simply connected, then by Remark 41 we have

$$\text{d}Y^i = 2\star\left(S^i \wedge \xi^\flat_{(1)} \wedge \xi^\flat_{(2)}\right) \tag{4.58}$$

where $S_a \equiv K_{ab}\Phi^b - K_{cd}\Phi_a\lambda^{-1}\Phi^c\Phi^{td}$. Then we have

$$\begin{aligned}
\frac{\text{d}Y^i}{2} &= \epsilon_{abcd}K^b_e\xi^e_{(i)}\xi^c_{(1)}\xi^d_{(2)}\text{d}x^\alpha \\
&= \epsilon(\partial_B, \partial_C, \partial_{\phi_1}, \partial_{\phi_2})K(\text{d}x^C, \partial_{\phi_i})\text{d}x^B \\
&= e^{3v}\epsilon(e_B, e_C, e_3, e_4)K(\theta^C, \partial_{\phi^i})\theta^B \\
&= e^{3v}\rho\epsilon_{BC}K(\theta^C, e_{i+2})\theta^B
\end{aligned} \tag{4.59}$$

where ϵ_{BC} is the volume form on the flat two-dimensional metric. Noting $K_{C(i+2)} = K(e_C, e_{i+2}) = K(\theta^C, e_{i+2})$ we read off

$$K_{2(i+2)} = \frac{e^{-(4v+U)}}{2\rho}\partial_\rho Y^i\ , \qquad K_{1(i+2)} = -\frac{e^{-(4v+U)}}{2\rho}\partial_z Y^i. \tag{4.60}$$

Noting that in this basis,

$$\begin{aligned}
|K|^2_h &= K^2_{11} + K^2_{22} + 2K^2_{12} + 2\lambda'^{ij}K_{1(i+2)}K_{1(j+2)} + 2\lambda'^{ij}K_{2(i+2)}K_{2(j+2)} \\
&+ \lambda'^{ij}\lambda'^{kl}K_{(i+2)(k+2)}K_{(j+2)(l+2)} \geq 2\lambda'^{ij}K_{1(i+2)}K_{1(j+2)} + 2\lambda'^{ij}K_{2(i+2)}K_{2(j+2)} \\
&= \frac{e^{-2(4v+U)}}{2\rho^2}(\delta_3)^{AB}\text{Tr}\left[\lambda'^{-1}\nabla_A Y^t\nabla_B Y\right] \equiv \frac{e^{-2(4v+U)}}{2\rho^2}\left[\nabla Y^t\lambda'^{-1}\nabla Y\right]
\end{aligned} \tag{4.61}$$

since the functions Y^i are independent of auxiliary angle φ in δ_3 we can assume capital

Latin indexes are $A, B = 1, 2, 3$. Now we define $\lambda = e^{2v}\lambda'$ which yields $\det\lambda = e^{4v}\rho^2$ and the result. □

Remark 50. Consider GB initial data $(\Sigma, \boldsymbol{h}, K)$. Then by definition of twist potential $Y = (Y^1, Y^2)$ in equations (4.24) and (4.60) we have

$$\partial_A Y^i = (-1)^{A+1}\rho e^{2U} K_{A(i+2)} + (-1)^A \rho e^{4U} A^i_A K_{(i+2)(j+2)} \tag{4.62}$$

with norm

$$|\nabla Y^i| = \left(|\partial_\rho Y^i|^2 + |\partial_z Y^i|^2\right)^{\frac{1}{2}} \leq C r^{-1} \rho e^{4U+\alpha} \left(|K_{1(i+2)}| + |K_{2(i+2)}|\right). \tag{4.63}$$

Furthermore, asymptotics for $K_{A(i+2)}$, $A = 1, 2$ may be obtained from the asymptotics of $|K|_h$ and λ' through the inequality

$$2\sum_{A=1,2} \lambda'^{ij} K_{A(i+2)} K_{A(i+2)} \leq |K|_h^2. \tag{4.64}$$

Therefore, we have the following asymptotes for twist potentials Y^i at each ends and the axis.

(a) as $r \to \infty$

$$Y^1 = \text{const} + \rho^2 \sqrt[4]{\frac{1-x}{2}} o_1(r^{-2}), \qquad \left|\nabla Y^1\right| = \rho \sqrt[4]{\frac{1-x}{2}}\, o_1(r^{-2}), \tag{4.65}$$

$$Y^2 = \text{const} + \rho^2 \sqrt[4]{\frac{1+x}{2}} o_1(r^{-2}), \qquad \left|\nabla Y^2\right| = \rho \sqrt[4]{\frac{1+x}{2}}\, o_1(r^{-2}). \tag{4.66}$$

(b) as $r \to 0$ and asymptotically flat end

$$\begin{aligned} Y^1 &= \text{const} + \rho^2 \sqrt[4]{\frac{1-x}{2}} o_1(r^{-6}), \qquad \left|\nabla Y^1\right| = \rho \sqrt[4]{\frac{1-x}{2}}\, o_1(r^{-6}), \quad (4.67) \\ Y^2 &= \text{const} + \rho^2 \sqrt[4]{\frac{1+x}{2}} o_1(r^{-6}), \qquad \left|\nabla Y^2\right| = \rho \sqrt[4]{\frac{1+x}{2}}\, o_1(r^{-6})\,. \quad (4.68) \end{aligned}$$

(c) as $r \to 0$ and $k > 2$ for asymptotically cylindrical end

$$\begin{aligned} Y^1 &= \text{const} + \rho^2 \sqrt[4]{\frac{1-x}{2}} o_1(r^{-2}), \qquad \left|\nabla Y^1\right| = \rho \sqrt[4]{\frac{1-x}{2}}\, o_1(r^{-2}), \quad (4.69) \\ Y^2 &= \text{const} + \rho^2 \sqrt[4]{\frac{1+x}{2}} o_1(r^{-2}), \qquad \left|\nabla Y^2\right| = \rho \sqrt[4]{\frac{1+x}{2}}\, o_1(r^{-2})\,. \quad (4.70) \end{aligned}$$

(d) as $\rho \to 0$ and $z > 0$

$$\begin{aligned} Y^1 &= \text{const} + \sqrt[4]{\frac{1-x}{2}} O(\rho^3), \qquad \left|\nabla Y^1\right| = \rho \sqrt[4]{\frac{1-x}{2}}\, O(\rho), \quad (4.71) \\ Y^2 &= \text{const} + \sqrt[4]{\frac{1+x}{2}} O(\rho^2), \qquad \left|\nabla Y^2\right| = \rho \sqrt[4]{\frac{1+x}{2}}\, O(1)\,. \quad (4.72) \end{aligned}$$

(e) as $\rho \to 0$ and $z < 0$

$$\begin{aligned} Y^1 &= \text{const} + \sqrt[4]{\frac{1-x}{2}} O(\rho^2), \qquad \left|\nabla Y^1\right| = \rho \sqrt[4]{\frac{1-x}{2}}\, O(1), \quad (4.73) \\ Y^2 &= \text{const} + \sqrt[4]{\frac{1+x}{2}} O(\rho^3), \qquad \left|\nabla Y^2\right| = \rho \sqrt[4]{\frac{1+x}{2}}\, O(\rho)\,. \quad (4.74) \end{aligned}$$

Now we have the following interesting result about the angular momenta.

Proposition 51. *Let* $(\Sigma, \boldsymbol{h}, K)$ *be a GB initial data set with* $\iota_{\xi_i} div_{\boldsymbol{h}} K = 0$ *. Then ADM angular momenta* $J_{(i)}$ *are conserved quantities and*

$$J_i = \frac{\pi}{4}\left[Y^i(x=1) - Y^i(x=-1)\right]\,. \quad (4.75)$$

Proof. By ADM angular momenta formula (3.20), we know angular momenta of a 3 dimensional closed surface $\mathcal{S}$ are

$$J_i(\mathcal{S}) = \frac{1}{8\pi}\oint_{\mathcal{S}} K_{ab}\nu^a\xi^b_{(i)}\,\mathrm{d}S\,. \tag{4.76}$$

Let $\mathcal{S}\subset\Sigma$ with unit normal vector ν and $\mathcal{S}_1$, $\mathcal{S}_2$ be two 3 dimensional surface with isometry $U(1)^2$ such that $\partial\mathcal{S} = \mathcal{S}_1\cup\mathcal{S}_2$. Then if we consider $\iota_{\xi_{(i)}}\mathrm{div}_{\boldsymbol{h}}K = 0$ we have

$$0 = \frac{1}{8\pi}\int_{\mathcal{S}}\iota_{\xi_{(i)}}\mathrm{div}_{\boldsymbol{h}}K\,\mathrm{d}\mu_0 = \frac{1}{8\pi}\oint_{\mathcal{S}_1\cup\mathcal{S}_2} K_{ab}\nu^a\xi^b_{(i)}\,\mathrm{d}S = J_i(\mathcal{S}_2) - J_i(\mathcal{S}_1)\,. \tag{4.77}$$

Thus angular momenta are conserved quantities. Since $\iota_{\xi_i}\mathrm{div}_{\boldsymbol{h}}K = 0$ and by Corollary 41 the twist potentials globally exist and it is

$$\mathrm{d}Y^i = 2\star\left(S^i\wedge\xi^\flat_{(1)}\wedge\xi^\flat_{(2)}\right)\,, \tag{4.78}$$

where $(S^1,S^2)^t = K_{ab}\Phi^b - K_{cd}\Phi_a\lambda^{-1}\Phi^c\Phi^{td}$. On GB initial data $(\Sigma,\boldsymbol{h},K)$ we can construct an orthonormal frame $(\zeta,\nu,\xi_{(1)},\xi_{(2)})$ where

$$\zeta = (\det\lambda)^{-1/2}\epsilon_{abcd}\nu^a\xi^c_{(1)}\xi^b_{(2)}\,, \tag{4.79}$$

where ζ is in the direction of x. Then the ADM angular momenta is

$$\begin{aligned} J_i &= \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r} K_{ab}\nu^a\xi^b_{(i)}\,\mathrm{d}S = \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r} S^i_a\nu^a\,\mathrm{d}S \\ &= \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r}\frac{1}{2\det\lambda}\epsilon_{abcd}\xi^c_{(1)}\xi^b_{(2)}\nabla^d Y^{(i)}\nu^a\,\mathrm{d}S \\ &= \frac{1}{16\pi}\lim_{r\to\infty}\oint_{S_r}\frac{1}{\sqrt{\det\lambda}}\zeta_a\nabla^a Y^i\,\mathrm{d}S \\ &= \frac{\pi}{4}\int_{-1}^{1}\partial_x Y^i\,\mathrm{d}x = \frac{\pi}{4}\left[Y^i(x=1) - Y^i(x=-1)\right]\,, \end{aligned} \tag{4.80}$$

where $\mathrm{d}S = \sqrt{h_{xx}\det\lambda}\,\mathrm{d}x\mathrm{d}\phi^1\mathrm{d}\phi^2$ and $\zeta_a = \sqrt{h_{xx}}\,\mathrm{d}x$. □

By Section 4.1 for the vacuum ($\mu = j = 0$) $t-\phi^i$-symmetric data, the metric takes the form (4.49) with $A^i_a = 0$ and the extrinsic curvature is determined fully from the twist potentials Y^i. Thus this suggests that the data is characterized by five scalar functions, or equivalently, the triple $u = (v, \lambda', Y)$, where v is a function, λ' is a positive definite symmetric 2×2 matrix, and Y is a column vector. Explicitly, for vacuum $t-\phi^i$ symmetric initial data set, we can express the extrinsic curvature as

$$K_{ab} = 2e^{-2v} S^t_{(b}\lambda'^{-1}\Phi_{a)}\,. \tag{4.81}$$

where $\Phi^a = (\xi^a_{(1)}, \xi^a_{(2)})^t$ is a column vector and $S = (S^1, S^2)^t$ is a column vector with components S^i defined by (4.78). This motivates the following definition.

Definition 52. Let (Σ, h, K) be a GB initial data set with $\mu \geq 0$ and $\iota_{\xi_i}\mathrm{div}_{\boldsymbol{h}} K = 0$. We define the associated *reduced data set* to be the vacuum $t-\phi^i$-symmetric data set characterized by the triple $u = (v, \lambda', Y)$ where (v, λ') is extracted from the original data set and Y is defined in (4.78) and we denote this initial data by $(\mathcal{B}, u)$.

Then we have the following result about this class of data.

Lemma 53. *Let $(\mathcal{B}, u)$ be the associated reduced data set of a GB data set. Then the associated reduced data set can be characterized by a triple $u = (v, \lambda', Y)$ and orbit space.*

Proof. A vacuum $t-\phi^i$-symmetric data set obtained from GB initial data set has the following metric and extrinsic curvature

$$\boldsymbol{h} = e^{2v}\tilde{\boldsymbol{h}}, \qquad \tilde{\boldsymbol{h}} = e^{2v}\left[e^{2U}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \lambda'_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j\right], \qquad K_{ab} = 2S^t_{(a}\lambda^{-1}\Phi_{b)} \tag{4.82}$$

where $U = V - \frac{1}{2}\log\left(2\sqrt{\rho^2+z^2}\right)$ and

$$S^i = \frac{1}{2\det\lambda}\iota_{\xi_{(2)}}\iota_{\xi_{(1)}}\star\,\mathrm{d}Y^i\,. \tag{4.83}$$

Therefore, the vacuum $t{-}\phi^i$-symmetric data set characterized by six functions (U, v, λ', Y) with boundary conditions (4.41)-(4.45). These functions are coupled by Hamiltonian constraint (3.12). If we assume $K_{ab} = e^{-2v}\tilde{K}_{ab}$ and apply Corollary 5, the Hamiltonian constraint convert to the Lichnerowiscz equation

$$\Delta_{\tilde{h}}\Phi - \frac{1}{6}R_{\tilde{h}}\Phi + \frac{1}{6}\tilde{K}_{ab}\tilde{K}^{ab}\Phi^{-5} = 0\,, \tag{4.84}$$

where $\Phi = e^v$. Now we substitute scalar curvature which is equation (5.48) (set $H^{ij} = 0$), extrinsic curvature (4.21), and $\Delta_{\tilde{h}} = e^{2U}\Delta_3$ to get

$$\Delta_3 v + \frac{1}{3}\Delta_2 U - \frac{\det\nabla\lambda'}{12\rho^2} = e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{12\rho^2} \tag{4.85}$$

where Δ_3 is three dimensional Laplace operator with respect to metric δ_3. Now for given (v, λ'_{ij}, Y), and definition of U we have a linear two dimensional Poisson equation for V with Dirichlet boundary conditions:

$$\begin{cases} \Delta_2 V = F(\rho, z) & \text{in } \mathcal{B} \equiv \mathbb{R}^2_+ \\ V = g(z) = -\frac{1}{2}\sum_{rods}\log V_i(z)\chi_{I_i} & \text{on } \Gamma \equiv \partial\mathbb{R}^2_+ \end{cases} \tag{4.86}$$

where $V = o(r^{-1})$ and $V = o(1), o(r^1)$ as $r \to \infty$ and $r \to 0$, respectively. Moreover, $\chi_{I_i}(x) = 0$ if $x \notin I_i$ and $\chi_{I_i}(x) = 1$ if $x \in I_i$. Now we define a set

$$\mathcal{A}_1 \equiv \{u = (v, \lambda', Y) : V \text{ is a solution of } (4.86)\}\,, \tag{4.87}$$

to be a class data set u such that the solution of equation (4.86) exists. □

4.3 Global Topology of Slice

In this section we discuss global topology of the GB data $(\Sigma, \boldsymbol{h}, K)$ (argument of [3, 5, 91]). Consider the GB data set, then Σ is complete, oriented, simply connected, and it has two asymptotic ends, each of which is asymptotically flat or asymptotically cylindrical. There is always at least one end of the former type. As a simple example, the t =constant hypersurface is a maximal initial data slice of the Schwarzschild-Tangherlini spacetime with the topology $\mathbb{R} \times S^3$ [142], which has two asymptotically flat ends. Asymptotically cylindrical ends (the geometry approaches a product metric on $\mathbb{R} \times N$ where N is a closed 3-manifold of positive Yamabe type by Proposition 44) arise in the context of initial data set for extreme black holes.

In general, assume $\mathscr{N}$ is a four dimensional, closed, simply connected oriented smooth manifold admitting an effective torus $U(1)^2$ action, Orlik and Raymond classified these manifolds [126]. They proved since $\pi_1(M) = 0$ (M is simply connected), the only finite isotropy group is the identity. Moreover, they show that such manifolds must have the topology of connected sums of copies of $S^2 \times S^2$, $\mathbb{CP}^2$, and $\overline{\mathbb{CP}}^2$ (note that taking the connected sum with S^4 is the identity operation), i.e.

$$\mathscr{N} \cong n_1 \left(S^2 \times S^2\right) \# n_2 \left(\mathbb{CP}^2\right) \# n_3 \left(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\right) \tag{4.88}$$

where # denotes the connected sum of manifolds [80]. Note that if $\mathscr{N}$ has spin structure, then $n_2 = n_3 = 0$. One may obtain asymptotically flat ends by removing points, or equivalently, taking the connected sum with $\mathbb{R}^4$. For example, the topology of the maximal slice of Schwarzschild discussed above can be obtained simply by removing two points from S^4. But what about other types of end? The slice with cylindrical topology arises in the context of an extreme black hole. Since the horizon of an extreme black hole has infinite distance from a point in the slice, the end of slice has horizon topology. Therefore, topology of a slice of an extreme black hole is equivalent to the slice topology in the domain of outer communication and we denote it by Σ_0. In [91] Holland, Hollands, and Ishibashi prove the following result about topology of slice of black holes with $\mathbb{R} \times U(1)$ isometry.

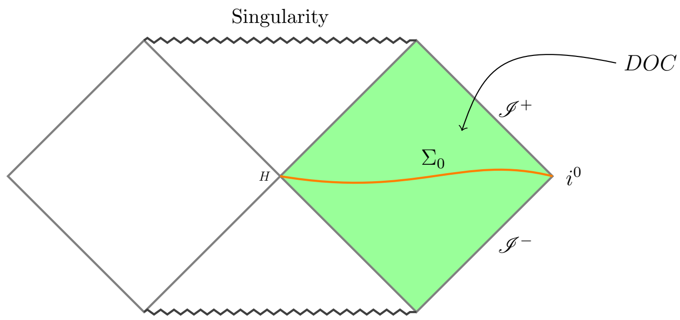


Figure 4.4: The domain of outer communication is the green region and it has topology $\mathbb{R} \times \Sigma_0$.

Theorem 54. *[91, Result 2] Consider an analytic, stationary, rotating vacuum black hole spacetime with isometry group $\mathbb{R} \times U(1)$. Then the domain of outer communication has topology $M \cong \mathbb{R} \times \Sigma_0$ where*

$$\Sigma_0 \cong \left[\mathbb{R}^4 \# n_1 \left(S^2 \times S^2\right) \# n_2 \left(\mathbb{CP}^2\right) \# n_3 \left(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\right)\right] - B \tag{4.89}$$

where B is a compact manifold without boundary such that $\partial \bar{B} \cong H$ and $n_i \in \mathbb{Z}$.

Since the slice of GB data has all assumptions of Theorem 54 with extra $U(1)$ symmetry, we can obtain similar result for GB data with cylindrical topology. Therefore we can summarize with the following corollary.

Corollary 55. *Let $(\Sigma, \boldsymbol{h}, K)$ be a GB data set, then the topology of Σ is one of the following* *(a)* Σ *has two asymptotically flat ends*

$$\Sigma \cong \mathbb{R}^4 \# n_1 \left(S^2 \times S^2\right) \# n_2 \left(\mathbb{CP}^2\right) \# n_3 \left(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\right) \# \mathbb{R}^4 \tag{4.90}$$

where $n_1, n_2, n_3 \in \mathbb{Z}$. *Note that if* Σ *has spin structure then* $n_2 = n_3 = 0$.

(b) Σ *has one asymptotically flat end and one cylindrical end with topology* $R \times N$ *such that* $N \cong S^3, S^1 \times S^2, L(p,q)$

$$\Sigma \cong \left[\mathbb{R}^4 \# m_1 \left(S^2 \times S^2\right) \# m_2 \left(\mathbb{CP}^2\right) \# m_3 \left(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\right)\right] - \bar{B} \tag{4.91}$$

where $m_1, m_2, m_3 \in \mathbb{Z}$ *and* $\bar{B}$ *is a compact manifold with boundary* $\partial \bar{B} = N$. *Note that if* Σ *has spin structure then* $m_2 = m_3 = 0$.

For explicit example of five dimensional stationary black holes with $U(1)^2$ symmetry, the argument is different. The t constant slice Σ_0 of these solutions is a 4 dimensional manifold with horizon boundary. If the black hole is extreme, H is infinitely far away and if we remove H we obtain Σ. In non-extreme case one can compactify Σ_0 by gluing in a closed 4-ball D^4_∞ and get $\mathscr{Q} = \Sigma_0 \cup D^4_\infty$, and then apply the doubling procedure to obtain a closed manifold. Note that the double of $\mathscr{Q}$ is the quotient space of $\mathscr{Q} \sqcup \mathscr{Q}$ obtained by identifying each point in the boundary $\partial \mathscr{Q}$ of the first copy of $\mathscr{Q}$ with the corresponding point in the boundary of the second copy [109] and denoted by $\mathscr{Q}_D$. Then we remove two points from $\mathscr{Q}_D$ and the slice topology is $\Sigma \cong \mathbb{R}^4 \# \mathscr{Q}_D \# \mathbb{R}^4$ [3].

For the case of the Myers-Perry black hole the topology of Σ_0^{MP} is $\Sigma_0^{MP} \cong [0,1) \times S^3$ [7]. Note that Σ_0^{MP} is asymptotically flat, simply connected, and has an inner boundary $\partial \Sigma_0^{MP} = -S^3$. Then the topology of extreme Myers-Perry is $\Sigma^{EMP} = (0,1) \times S^3$. For the non-extreme case, we compactify this manifold and we get $\mathscr{Q}^{MP} \cong$

$[0,1]\times S^3 \cong D^4$ and the doubling is $\mathscr{Q}_D^{MP} \cong S^4$. Then topology of non-extreme Myers-Perry slice is $\Sigma^{MP} \cong \mathbb{R}^4 \# S^4 \# \mathbb{R}^4 \cong (0,1) \times S^3$. The spatial slice Σ_0^{BR} of the doubly spinning black ring spacetime has topology

$$\Sigma_{BR} \cong \mathbb{R}^4 - \mathsf{Int}(R)$$

where R is a regular neighborhood of an embedded S^1 in $\mathbb{R}^4$. Note that $\mathsf{Int}(R)$ is the standard choice and it is possible that we choose another 4-dimensional compact manifold with boundary $S^1 \times S^2$. We call $\mathsf{Int}(R)$ the *standard black hole region.* The choice of the embedding of S^1 into $\mathbb{R}^4$ is not relevant due to the fact that any pair of 'locally flat' embeddings of S^1 in $\mathbb{R}^4$ (or S^4) differ by a homeomorphism of $\mathbb{R}^4$ (respectively S^4), for this topological result see [21, 78].

Consider the 4-dimensional sphere S^4. Let B be a closed 4-dimensional ball in S^4, let R be a regular closed neighborhood of a locally flat embedded S^1 in S^4, and assume that $B \cap R = \emptyset$. Since $S^4 - B \cong \mathbb{R}^4$, it is immediate that

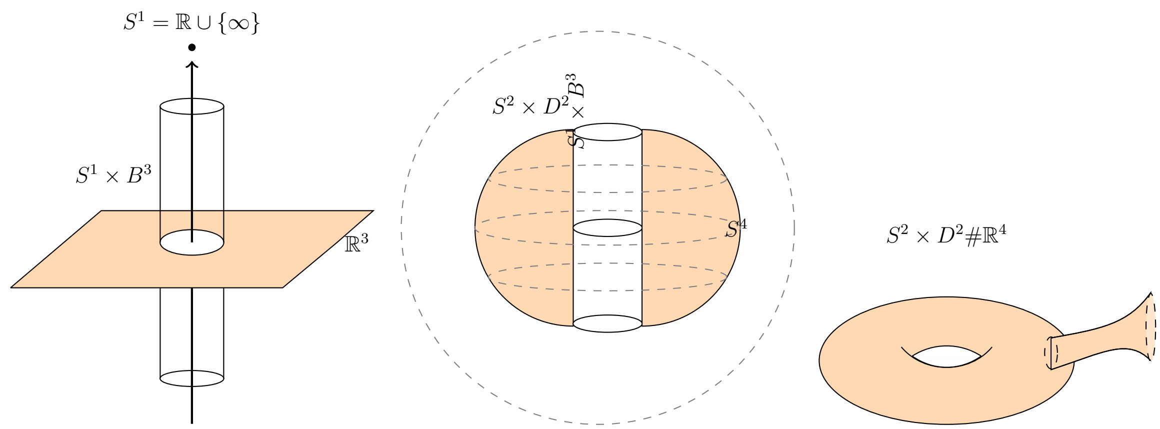
$$\Sigma_0^{BR} \cong S^4 - [B \sqcup \mathsf{Int}(R)]\,. \tag{4.92}$$

Regard S^4 as the one-point compactification $\mathbb{R}^4 \cup \{\infty\}$ of $\mathbb{R}^4$, and without loss of generality assume that R is a regular neighborhood of the S^1 formed by the w-axis of $\mathbb{R}^4$ together with $\{\infty\}$. Then one verifies that

$$S^4 - \mathsf{Int}(R) \cong S^2 \times D^2, \tag{4.93}$$

where D^2 denotes the 2-dimensional closed disk; we refer the reader to Figure 4.5. It follows that

$$\Sigma_0^{BR} \cong (S^2 \times D^2) - B. \tag{4.94}$$



(a) We delete $S^1 \times B^3$ from S^4 (b) $S^2 \times D^2$ is space around $S^1 \times B^3$ (c) $S^2 \times D^2 \# \mathbb{R}^4$

Figure 4.5: The black ring slice as $(S^2 \times D^2)\#\mathbb{R}^4$. (a) shows a regular neighborhood $R \cong S^1 \times B^3$ of $S^1 = \{\text{w-axis}\} \cup \{\infty\}$ is deleted from $S^4 \cong \mathbb{R}^4 \cup \{\infty\}$. (b) the space obtained is homeomorphic to $S^2 \times D^2$ (c) The black ring slice topology, $S^2 \times D^2 \# \mathbb{R}^4$.

Since removing a closed ball from a 4-manifold is equivalent to a connected sum with $\mathbb{R}^4$, we also have that

$$\Sigma_0^{BR} \cong (S^2 \times D^2)\#\mathbb{R}^4,$$

The homology of Σ_0^{BR} is computed as follows. Since $\mathbb{R}^4 = \Sigma_0^{BR} \cup R$ and $R \cap \Sigma_0^{BR} = \partial R \cong S^1 \times S^2$ and R is homotopic to S^1; the Mayer-Vietoris sequence for $\mathbb{R}^4 = \Sigma_0^{BR} \cup R$

$$0 \to \mathcal{H}_i(S^1 \times S^2) \to \mathcal{H}_i(\Sigma_0^{BR}) \oplus \mathcal{H}_i(S^1) \to 0, \quad i \geq 1\,,$$

determines all homology groups of Σ_0^{BR} and its Euler characteristic; namely[7]

$$\mathcal{H}_n(\Sigma_0^{BR}) = \begin{cases} \mathbb{Z} & n = 0, 2, 3 \\ 0 & \text{others} \end{cases} \qquad \chi(\Sigma_0^{BR}) = \sum_{n=0}^{4} (-1)^n \dim \mathcal{H}_n(M) = 1.$$

This computation is a particular case of our Theorem 56.

Since $\Sigma_0^{BR} \cong \mathbb{R}^4 - \mathsf{Int}(R)$ where R is the regular neighborhood of an embedded S^1, a standard dimension argument shows that Σ_0^{BR} is simply-connected. Since $\mathcal{H}_2(\Sigma_0^{BR}) \cong \mathbb{Z}$, the Hurewicz theorem shows that $\pi_2(\Sigma_0^{BR}) \cong \mathbb{Z}$. More generally,

$$\pi_1(\Sigma_0^{BR}) = 0, \qquad \pi_2(\Sigma_0^{BR}) = \mathbb{Z}, \qquad \pi_3(\Sigma_0^{BR}) = \mathbb{Z} \qquad \pi_4(\Sigma_0^{BR}) \neq 0, \tag{4.95}$$

where the claims about $\pi_3(\Sigma_0^{BR})$ and $\pi_4(\Sigma_0^{BR})$ are prove in Appendix C.1.

Note that Σ_0^{BR} is asymptotically flat, simply connected, and has an inner boundary $\partial\Sigma_0^{BR} = -S^1 \times S^2$. Then the topology of extreme black ring is $\Sigma^{EBR} = S^2 \times B^2 \# \mathbb{R}^4$ where B^2 is a 2 dimensional open ball. For the non-extreme case, we compactify this manifold and we obtain $\mathscr{Q}_0^{BR} \cong S^2 \times D^2$ and the doubling is $\mathscr{Q}_D^{BR} \cong S^2 \times S^2$. Then topology of non-extreme black ring slice is $\Sigma^{BR} \cong \mathbb{R}^4 \# S^2 \times S^2 \# \mathbb{R}^4$.

4.3.1 Topology of Σ_0

In this section we study topology of Σ_0 for multiple black holes[7]. We do not consider Lens spaces topology and we assume Σ_0 has spin structure with $n_1 = 0$ in Theorem 54. We define the standard region of a black hole with $H \cong \# m\,(S^1 \times S^2)$ as $\mathcal{N}_\epsilon\left(\vee_{i=1}^{l} S_i^1\right)$ where $\mathcal{N}_\epsilon$ represents a regular neighbourhood. This definition requires us to show that there are no knotted embeddings of $\mathcal{N}_\epsilon\left(\vee_{i=1}^{l} S_i^1\right)$ into $\mathbb{R}^4$. We compute the Euler number of a slice of a five-dimensional spacetime containing m black holes (the existence of which is consistent with all known constraints). Observe that this means we are considering spacetimes which contain a disjoint union of horizons, each of which is consistent with the horizon classification of [91, 94]. Although there are no explicit solutions for such geometries, we can still discuss aspects of their topology.

Theorem 56. *Consider an asymptotically flat stationary spacetime* (M, g) *containing* $m = n_1 + n_2 + n_3$ *black holes and horizon*

$$H \cong \left(\coprod_{i=1}^{n_1} S^3\right) \coprod \left(\coprod_{i=1}^{n_2} \left(S^1 \times S^2\right)\right) \coprod \left(\coprod_{i=1}^{n_3} \#l \left(S^1 \times S^2\right)\right). \tag{4.96}$$

Assume that the domain of outer communication has the form $\mathbb{R} \times \Sigma_0$ *where* $\Sigma_0 \cong \mathbb{R}^4 - B$ *and* B *is the standard black hole region for* H*. Then the Euler number of* Σ_0 *is* $\chi = 1 - n_1 + n_3(l-1)$ *and the homology of* Σ_0 *is given by expression* (4.104)*.*

Proof. In this case the black hole region is

$$B \cong \left(\coprod_{i=1}^{n_1} \mathsf{Int}(B^4)\right) \coprod \left(\coprod_{i=1}^{n_2} \left(S^1 \times \mathsf{Int}(B^3)\right)\right) \coprod \left(\coprod_{i=1}^{n_3} \mathcal{N}_\epsilon \left(\vee_{i=1}^{l} S_i^1\right)\right). \tag{4.97}$$

The homology of the black hole region is

$$\mathcal{H}_n(B) = \bigoplus_{i=1}^{n_1} \mathcal{H}_n\left(\mathsf{Int}(B^4)\right) \bigoplus_{i=1}^{n_2} \mathcal{H}_n\left(S^1 \times \mathsf{Int}(B^3)\right) \bigoplus_{i=1}^{n_3} \mathcal{H}_n\left(\mathcal{N}_\epsilon\left(\vee_{i=1}^{l} S_i^1\right)\right). \tag{4.98}$$

Since $\mathcal{H}_n\left(\mathcal{N}_\epsilon\left(\vee_{i=1}^{l} S_i^1\right)\right) = \bigoplus_{i=1}^{l} \mathcal{H}_n(S_i^1)$ we have

$$\mathcal{H}_n(B) = \begin{cases} \mathbb{Z}^m & n = 0 \\ \mathbb{Z}^{n_2 + n_3 l} & n = 1 \\ 0 & \text{others} \end{cases} \tag{4.99}$$

Also the homology of the horizon is

$$\mathcal{H}_n(H) = \bigoplus_{i=1}^{n_1} \mathcal{H}_n\left(S^3\right) \bigoplus_{i=1}^{n_2} \mathcal{H}_n\left(S^1 \times S^2\right) \bigoplus_{i=1}^{n_3} \mathcal{H}_n\left(\#l\ \left(S^1 \times S^2\right)\right) \tag{4.100}$$

By the long exact sequence and the excision theorem, a calculation shows

$$\mathcal{H}_n\left(\#l\left(S^1\times S^2\right)\right)=\begin{cases}\mathbb{Z} & n=0,3\\ \mathbb{Z}^l & n=1,2\\ 0 & n\geq 4\end{cases}\tag{4.101}$$

then

$$\mathcal{H}_n(H)=\begin{cases}\mathbb{Z}^m & n=0,3\\ \mathbb{Z}^{n_2+n_3l} & n=1,2\\ 0 & n\geq 4\end{cases}\tag{4.102}$$

Since $\mathbb{R}^4\cong\Sigma_0\cup B$ with $B\cap\Sigma_0\cong H$, from the long exact sequence

$$\mathcal{H}_{n+1}(\mathbb{R}^4)\to\mathcal{H}_n(H)\to\mathcal{H}_n(B)\oplus\mathcal{H}_n(\Sigma_0)\to\mathcal{H}_n(\mathbb{R}^4)\tag{4.103}$$

we deduce the following:

$\dim\mathcal{H}_n$	$\mathbb{R}^4$	H	B	Σ_0
$n=0$	1	m	m	1
$n=1$	0	n_2+n_3l	n_2+n_3l	0
$n=2$	0	n_2+n_3l	0	n_2+n_3l
$n=3$	0	m	0	m
$n\geq 4$	0	0	0	0

Table 4.1: Homology groups of Σ_0

The homology of a slice of spacetime containing m stationary black holes is

$$\mathcal{H}_n(\Sigma_0)=\begin{cases}\mathbb{Z} & n=0\\ \mathbb{Z}^{n_2+n_3l} & n=2\\ \mathbb{Z}^m & n=3\\ 0 & n=1\,\text{and}\,n\geq 4\end{cases}\tag{4.104}$$

Then the Euler number is

$$\chi(\Sigma_0) = \sum_{n=0}^{4} (-1)^n \mathrm{dim}\left(\mathcal{H}_n\left(\Sigma\right)\right) = 1 - n_1 + n_3\left(l-1\right). \tag{4.105}$$

□

We remark that under the assumptions of Theorem 56, Hurewics' theorem implies $\pi_2(\Sigma_0) = \mathbb{Z}^{n_2+n_3 l}$. Further, suppose that $(M, \boldsymbol{g})$ is an asymptotically flat stationary spacetime containing a black hole with horizon H such that $\mathbb{R}^4 \cong \Sigma_0 \cup B$, $H \cong \Sigma_0 \cap B$. We are unaware whether the following statement is true: topologically, the only possibilty for the black hole region B with horizon $\#m\left(S^1 \times S^2\right)$ is a regular neighbourhood of $\vee_{i=1}^m S_i^1$. We can, however, show the following:

Theorem 57. *Suppose that (M, g) is an asymptotically flat stationary spacetime containing a black hole with horizon H such that $\mathbb{R}^4 \cong \Sigma_0 \cup B$ and $H \cong \Sigma_0 \cap B$ where $H \cong \#m\left(S^1 \times S^2\right)$. Then the homology of B is the same as the homology of the standard black hole region for H. Moreover, the homology of Σ_0 is the same as the homology of the Σ_0 obtained by removing the standard black hole region from $\mathbb{R}^4$.*

Proof. Consider the Mayer-Vietoris sequence

$$...\mathcal{H}_{n+1}(\mathbb{R}^4) \to \mathcal{H}_n(\#m\left(S^1 \times S^2\right)) \to \mathcal{H}_n(B) \oplus \mathcal{H}_n(\Sigma_0) \to \mathcal{H}_n(\mathbb{R}^4)... \tag{4.106}$$

Observe that

- $\mathcal{H}_0(B) = \mathcal{H}_0(\Sigma_0) = \mathbb{Z}$ since B and Σ_0 are connected.
- $\mathcal{H}_1(B) = \mathbb{Z}^m$. This follows since by topological censorship [69, 70], asymptotic flatness implies Σ_0 is simply connected so $\mathcal{H}_1(\Sigma) = 0$.

- $\mathcal{H}_2(B) = 0$ and $\mathcal{H}_2(\Sigma_0) = \mathbb{Z}^m$. Indeed, by Alexander duality [80, Theorem 3.44], $\mathcal{H}_2(\Sigma_0 \cup \{\infty\}) = \mathcal{H}_2(S^4 - B) = \mathcal{H}^1(B) = \mathbb{Z}^m$. Then, observe that, $\mathcal{H}_2(\Sigma_0) = \mathcal{H}_2(\Sigma_0 \cup \{\infty\})$ since removing an interior point of a 4-manifold does not change the second homology group. This last statement is well known and is proved as follows: consider the sequence for the pair $(M^4, M^4 - \{p\})$ and use that $\mathcal{H}_2(M^4, M^4 - \{p\}) = \mathcal{H}_2(\mathbb{R}^4, \mathbb{R}^4 - \{p\}) = \mathcal{H}_2(S^3)$ which holds by excision.

- $\mathcal{H}_3(B) = 0$ and $\mathcal{H}_3(\Sigma) = \mathbb{Z}$. Analogously, by Alexander duality [80, Theorem 3.44], $\mathcal{H}_3(\Sigma_0 \cup \{\infty\}) = \mathcal{H}_3(S^4 - B) = \widetilde{\mathcal{H}}^0(B) = \mathbb{Z}$. Then $\mathcal{H}_3(\Sigma_0) = \mathcal{H}_3(\Sigma \cup \{\infty\}) = \mathbb{Z}$.

- $\mathcal{H}_n(B) = \mathcal{H}_n(\Sigma_0) = 0$ for $n \geq 4$ since B and Σ_0 are 4-manifolds with boundary.

Observe this agrees with the homology of the standard case for $H \cong \#m\,(S^1 \times S^2)$. □

Recall that we previously defined the *standard* black hole region B for a black hole with horizon a connected sum $\#m\,(S^1 \times S^2)$ of m copies of $S^1 \times S^2$ to be a smooth regular neighbourhood of a *subspace* homemorphic to $\vee_{i=1}^m S_i^1$ of S^4. We want to make sure that generically different ways to consider B are equivalent. Specifically, if B_1 and B_2 are two different possible standard regions, then there is a diffeomorphism $h\colon S^4 \to S^4$ such that $h(B_1) = B_2$. In fact the stronger statement that h is differentiable and isotopic to the identity map follows from standard results in differential topology as informally described below.

First, the notion of subspace is restricted to being a subcomplex in a triangulation of S^4, and smooth regular neighborhood is defined as done by Hirsch [89]. Then any pair of smooth regular neighbourhoods of a subcomplex $\vee_{i=1}^m S_i^1$ of S^4 are differentiable isotopic. By reasons of dimension, any pair of subcomplexes of S^4 homemorphic to $\vee_{i=1}^m S_i^1$ of S^4 are isotopic, and this isotopy extends to an ambient differentiable isotopy by the Isotopy Extension Theorem [88].

4.4 Summary

This chapter is the first step toward mass-charge-angular momenta inequality. In Section 4.1, we construct a class of n dimensional initial data with $U(1)^{n-2}$ isometry plus extra restrictions in Definition 36 and it is called $t-\phi^i$ symmetric data. This class of initial data set includes initial data set of the all vacuum stationary, $U(1)^{n-2}$-invariant solutions of Einstein equations. Moreover, we construct a traceless-trancevese symmetric (0,2)-tensor which represents the extrinsic curvature in vacuum. In Section 4.2, we consider a more general class of four dimensional data which is a generalization of the three dimensional Brill's data(see [39, 53]).

In our analysis we consider some assumptions regarding existence of the global representation of $\boldsymbol{h}$ and we leave this as an open problem. Moreover, we study the orbit space geometry of the GB initial data. In Section 4.3, we presented the topology Σ for known black hole solutions and multiple black holes. If the topology of horizon is not spherical or Lens space, we showed topology of Σ in the extreme and non-extreme cases are different.

Chapter 5

A Mass Functional $\mathcal{M}$ and Positive Mass Theorem

In this chapter we construct a mass functional $\mathcal{M}$ for the class of generalized Brill (GB) initial data sets which we defined in Chapter 4. We prove that mass of any GB initial data set is greater than or equal to the mass of vacuum $t - \phi^i$ symmetric initial data sets, i.e. associated reduced data. Moreover, the critical points of the mass functional are stationary, $U(1)^2$-invariant black solutions of the Einstein equation. Finally, we prove a positive mass theorem for restricted class of GB data sets. The results of this chapter appeared in the following journal article: (AA.2) Physical Review D, 90 (12), 124,078(2014)[3], and (AA.5) arXiv:1508.02337 which was submitted to Journal of Mathematical Physics [5] in July 2015.

5.1 Construction of $\mathcal{M}$

An important step in the proof of the mass angular momenta inequality was the construction of a well-defined mass functional $\mathcal{M}$, which is a lower bound for ADM mass of any GB initial data set. In 3+1 dimensional case, Dain constructed a mass functional for reduced vacuum $t-\phi$ symmetric initial data. In that case, $\mathcal{M}=\mathcal{M}(v,Y)$ depends on two scalar functions v and Y which can be shown to fully specify the initial data set. The proof shows that $m=\mathcal{M}(v,Y)$ for $t-\phi$ symmetric maximal initial data, and that $m\geq\mathcal{M}$ for arbitrary axisymmetric maximal data [53]. $\mathcal{M}(v,Y)$ can be shown to be positive-definite and the unique minimizer is extreme Kerr, completing the elegant argument [52].

It is natural to expect an analogous inequality would hold in $D=5$ dimensions, under suitable restrictions on the initial data. The situation is particularly interesting as there are potentially two known candidates for minimizers: extreme Myers-Perry black holes with $H\cong S^3$ [124], and extreme black rings with $H\cong S^1\times S^2$ [64]. The masses of these solutions satisfy

$$M^3 = \frac{27\pi}{32}\left(|J_1|+|J_2|\right)^2 \qquad \text{(Myers-Perry)} \tag{5.1}$$

$$M^3 = \frac{27\pi}{4}|J_1|(|J_2|-|J_1|) \qquad \text{(black ring)} \tag{5.2}$$

where J_i are conserved angular momenta computed in terms of Komar integrals. Of course it is not manifestly clear how an expression which is derived from the ADM mass (i.e. evaluated at spatial infinity) would capture information on the topology of the horizon - indeed, at the level of the initial data, the horizon is a minimal surface in the interior. It is worth noting that another, related class of geometric inequalities relating the *area* of marginally outer trapped surfaces to the angular momenta (and charge) have also been established in three spatial dimensions[59, 103]. Once again the geometries which uniquely saturate the bound were the horizon geometries corresponding to the extreme Kerr geometry. As pointed out in Chapter 3, recently, Hollands has derived an area-angular momenta inequality in general dimension D, for spaces admitting a $U(1)^{D-3}$ action as isometries[90]. In this case, the inequality depends on the topology of the marginally outermost trapped surface.

In this chapter, we construct a positive-definite functional $\mathcal{M}$ which evaluates to the mass for GB initial data set. We prove a variational principle for this mass functional. In particular, we show critical points of $\mathcal{M}$ are stationary, $U(1)^2$-invariant, vacuum black holes. In this sense our mass functional is an extension of Dain's functional $\mathcal{M}(v, Y)$, which also has this property. However, there are a number of important differences. As we will elaborate, our functional contains boundary terms which encode the 'rod structure' of the initial data.

Now , let $(\Sigma, \boldsymbol{h}, K)$ be a GB initial data set with metric

$$\boldsymbol{h} = e^{2U+2v}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + e^{2v}\lambda'_{ij}\left(\mathrm{d}\phi^i + A^i_B \mathrm{d}x^B\right)\left(\mathrm{d}\phi^j + A^j_B \mathrm{d}x^B\right) \tag{5.3}$$

with asymptotic behaviors in Definition 46. Then we parametrized the metric and extrinsic curvature with the following conformal rescaling

$$h_{ab} = \Phi^2 \tilde{h}_{ab}, \qquad K_{ab} = \Phi^{-2}\tilde{K}_{ab} \tag{5.4}$$

where $\Phi = e^v$. Then by the asymptotic behavior of GB initial data at the asymptotically flat end in the chart (U, x), i.e. $r \to \infty$ we have

$$\Phi - 1 = o(r^{-2}), \qquad \tilde{h}_{ab} = \delta_{ab} + o(r^{-2})\,. \tag{5.5}$$

Then the integrand of ADM mass equation (3.17) is

$$\begin{aligned}\partial_a h_{ac} - \partial_c h_{aa} &= 2\Phi\left(\partial_a \Phi\right)\tilde{h}_{ac} + \Phi^2\left(\partial_a \tilde{h}_{ac}\right) - 2\Phi\left(\partial_c \Phi\right)\tilde{h}_{aa} - \Phi^2\left(\partial_c \tilde{h}_{aa}\right) \\ &= -6\partial_c \Phi + \partial_a \tilde{h}_{ac} - \partial_c \tilde{h}_{aa} + o(r^{-3}).\end{aligned} \tag{5.6}$$

Therefore we find

$$
\begin{aligned}
M_{ADM} &= \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r} -6\partial_c\Phi\, n^c\, dS + m_{\tilde{h}} \\
&= \frac{1}{64\pi}\lim_{r\to\infty}\int_{S_r} -6v_{,r}\, r^3 \mathrm{d}x\mathrm{d}\phi^1\mathrm{d}\phi^2 + M_{ADM}^{\tilde{h}} \qquad (5.7)\\
&= -\frac{3\pi}{8}\int_{\mathcal{I}_F} v_{,r}\, dx + M_{ADM}^{\tilde{h}} \qquad (5.8)
\end{aligned}
$$

where we that used $\Phi = e^v = 1 + o_1(r^{-1})$ as $r \to \infty$ in first equality. The second equality follows from $U(1)^2$-invariant symmetry of v. Recall the boundary of the orbit space consists of the asymptotic regions

$$
\begin{aligned}
\mathcal{B}_F &\equiv \{z, \rho \to \infty, z(\rho^2 + z^2)^{-1/2} \text{ finite}\} = \{r \to \infty, -1 \le x \le 1\}, \qquad (5.9)\\
\mathcal{B}_E &\equiv \{z, \rho \to 0\} = \{r \to 0, -1 \le x \le 1\}. \qquad (5.10)
\end{aligned}
$$

and the axis Γ. Now we find the ADM mass of the conformal metric $\tilde{h}$.

Lemma 58. *Consider a GB data* (Σ, h, K, μ, j) *with the rescaling* (5.4). *Then*

$$
M_{ADM}^{\tilde{h}} = -\frac{\pi}{4}\int_{\mathcal{B}_F}\left(\frac{r^3}{2}V_{,r} - r^2 V\right) dx. \qquad (5.11)
$$

Proof. Let us consider the flat metric in coordinate (y_i)

$$
\delta_4 = \mathrm{d}y_1^2 + \mathrm{d}y_2^2 + \mathrm{d}y_3^2 + \mathrm{d}y_4^2\,, \qquad (5.12)
$$

with the following transformation to (r, x, ϕ^1, ϕ^2)(or $(\rho = \frac{r^2}{2}\sqrt{1-x^2}, z = \frac{r^2}{2}x, \phi^1, \phi^2)$)

$$
\begin{aligned}
y_1 &= r\sqrt{\frac{1+x}{2}}\cos\phi^1 \quad y_2 = r\sqrt{\frac{1+x}{2}}\sin\phi^1, \\
y_3 &= r\sqrt{\frac{1-x}{2}}\cos\phi^2, \quad y_4 = r\sqrt{\frac{1-x}{2}}\sin\phi^2,
\end{aligned}
$$

$$\begin{aligned} r &= \sqrt{\sum_i y_i^2}, \qquad x = \frac{(y_1^2+y_2^2)-(y_3^2+y_4^2)}{r^2}, \\ \phi^1 &= \arctan\left(\frac{y_2}{y_1}\right), \quad \phi^2 = \arctan\left(\frac{y_4}{y_3}\right). \end{aligned}$$

First we write the conformal metric in (r, x, ϕ^1, ϕ^2) coordinate.

$$\begin{aligned} \tilde{\boldsymbol{h}} &= \delta_4 + \underbrace{\left(e^{2V}-1\right)\left(\mathrm{d}r^2 + \frac{r^2}{4(1-x^2)}\mathrm{d}x^2\right)}_{B_I} + \underbrace{\left(\lambda'_{ij}-\sigma_{ij}\right)\mathrm{d}\phi^i\mathrm{d}\phi^i}_{B_{II}} + 2\underbrace{\lambda'_{ij}A^i_a\mathrm{d}x^a\mathrm{d}\phi^j}_{B_{III}} \\ &+ \text{terms quadratic in } A^i_a. \end{aligned} \tag{5.13}$$

The mass of δ_4 is zero. By Assumption 2 in Definition 46, the last quadratic terms in (5.13) will not give any contribution to the mass integral. Now we compute the mass of each term B_I, B_{II}, and B_{III}. First, by the asymptotic behavior of functions (equations (4.41) and (4.42)) we have

$$\begin{aligned} B_I + B_{II} &= \left(e^{2V}-1\right)\delta_4 + \frac{1}{2}\left[\frac{f_{11}}{r^2} - \left(e^{2V}-1\right)\right] r^2(1+x)\left(\mathrm{d}\phi^1\right)^2 \\ &+ \frac{1}{2}\left[-\frac{f_{11}}{r^2} - \left(e^{2V}-1\right)\right] r^2(1-x)\left(\mathrm{d}\phi^2\right)^2 \\ &= \underbrace{\left(e^{2V}-1\right)}_{C_I}\delta_4 + \underbrace{\left[f_{11}r^{-1-\epsilon} - \left(e^{2V}-1\right)\right]\frac{(y_1\mathrm{d}y_2 - y_2\mathrm{d}y_1)^2}{y_1^2+y_2^2}}_{C_{II}} \\ &+ \underbrace{\left[-f_{11}r^{-1-\epsilon} - \left(e^{2V}-1\right)\right]\frac{(y_4\mathrm{d}y_3 - y_3\mathrm{d}y_4)^2}{y_3^2+y_4^2}}_{C_{III}}, \\ &+ \underbrace{\lambda'_{12}\frac{(y_4\mathrm{d}y_3 - y_3\mathrm{d}y_4)(y_2\mathrm{d}y_1 - y_1\mathrm{d}y_2)}{(y_1^2+y_2^2)(y_3^2+y_4^2)}}_{C_{IV}}. \end{aligned} \tag{5.14}$$

Now we compute ADM mass of each one of these terms :

- C_I : The C_I is a conformal flat metric so the mass by argument similar to the

(5.7) will be

$$M_{ADM}^{C_I} = \frac{1}{16\pi} \lim_{r\to\infty} \oint_{S_r} -3\partial_r \left(e^{2V} - 1\right) \mathrm{d}S. \tag{5.15}$$

- C_{II} : we consider C_{II} as a metric $(C_{II})_{ab}$ such that only nonzero components are

$$(C_{II})_{ab} = \frac{\left[f_{11} r^{-1-\epsilon} - \left(e^{2V} - 1\right)\right] (y_2^2 \mathrm{d}y_1^2 + y_1^2 \mathrm{d}y_2^2 - 2y_1 y_2 \mathrm{d}y_1 \mathrm{d}y_2)}{y_1^2 + y_2^2}. \tag{5.16}$$

Then the mass is

$$\begin{aligned} M_{ADM}^{C_{II}} &= \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \left[\partial_{y_1} (C_{II})_{y_1 y_2} \frac{y_2}{r} + \partial_{y_2} (C_{II})_{y_1 y_2} \frac{y_1}{r} - \partial_{y_i} (C_{II})_{y_2 y_2} \frac{y_i}{r} \right. \\ &- \left. \partial_{y_i} (C_{II})_{y_1 y_1} \frac{y_i}{r} + \partial_{y_1} (C_{II})_{y_1 y_1} \frac{y_1}{r} + \partial_{y_2} (C_{II})_{y_2 y_2} \frac{y_2}{r} \right] \mathrm{d}s \\ &= -\frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \left(\partial_r + \frac{1}{r}\right) \left[f_{11} r^{-1-\epsilon} - \left(e^{2V} - 1\right)\right] \mathrm{d}s. \end{aligned}$$

- C_{III} : This is similar to C_{II} and we have

$$M_{ADM}^{C_{III}} = -\frac{1}{16\pi} \lim_{r\to\infty} \oint_{S_r} \left(\partial_r + \frac{1}{r}\right) \left[-f_{11} r^{-1-\epsilon} - \left(e^{2V} - 1\right)\right] \mathrm{d}s.$$

- C_{IV}: This is similar to C_{II}. Let the metric be

$$(C_{IV})_{ab} = \frac{\lambda'_{12} \left(y_1 y_4 \mathrm{d}y_2 \mathrm{d}y_3 - y_1 y_3 \mathrm{d}y_2 \mathrm{d}y_4 + y_2 y_3 \mathrm{d}y_1 \mathrm{d}y_4 - y_2 y_4 \mathrm{d}y_1 \mathrm{d}y_3\right)}{(y_1^2 + y_2^2)(y_3^2 + y_4^2)}. \tag{5.17}$$

Then

$$\begin{aligned} M_{ADM}^{C_{IV}} &= \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \left[\left(\partial_{y_1} (C_{IV})_{y_1 y_3} + \partial_{y_2} (C_{IV})_{y_2 y_3}\right) \frac{y_3}{r} \right. \\ &- \left(\partial_{y_3} (C_{IV})_{y_1 y_3} + \partial_{y_4} (C_{IV})_{y_1 y_4}\right) \frac{y_1}{r} + \left(\partial_{y_3} (C_{IV})_{y_2 y_3} + \partial_{y_4} (C_{IV})_{y_2 y_4}\right) \frac{y_2}{r} \\ &+ \left. \left(\partial_{y_1} (C_{IV})_{y_1 y_4} + \partial_{y_2} (C_{IV})_{y_2 y_4}\right) \frac{y_4}{r} \right] \mathrm{d}s = 0 \end{aligned} \tag{5.18}$$

Then the ADM mass will be

$$\begin{aligned} M_{ADM}^{B_I+B_{II}} &= \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r}\left\{-\partial_r\left(e^{2V}-1\right)+\frac{2}{r}\left(e^{2V}-1\right)\right\}\mathrm{d}S \quad (5.19)\\ &= \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r}\left\{-\partial_r\left(e^{2V}-1\right)+\frac{2}{r}\left(e^{2V}-1\right)\right\}\frac{r^3}{4}\mathrm{d}x\mathrm{d}\phi^1\mathrm{d}\phi^2 \quad (5.20)\\ &= \frac{\pi}{4}\lim_{r\to\infty}\int_{S_r}\left\{-2V_{,r}+\frac{4V}{r}\right\}\frac{r^3}{4}\mathrm{d}x = -\frac{\pi}{4}\int_{\mathcal{B}_F}\left(\frac{r^3}{2}V_{,r}-r^2V\right)\mathrm{d}x. \end{aligned}$$

where in the third line we have used equation (4.42). Now if we consider the term B_{III}

$$B_{III} = \underbrace{\frac{1}{2}r^2(1+x)\mathrm{d}\phi^1\left(A_\rho^1\mathrm{d}\rho + A_z^1\mathrm{d}z\right)}_{D_I+D_{II}} + \underbrace{\frac{1}{2}r^2(1-x)\mathrm{d}\phi^2\left(A_\rho^2\mathrm{d}\rho + A_z^2\mathrm{d}z\right)}_{D_{III}+D_{IV}} + o(r^{-3}). \quad (5.21)$$

We prove that ADM mass of D_I and D_{II} parts are zero and others are exactly similar. Consider D_I as a metric

$$\begin{aligned} (D_I)_{ab} &= \frac{1}{2}r^2(1+x)\mathrm{d}\phi^1 A_\rho^1\mathrm{d}\rho = (y_1\mathrm{d}y_2 - y_2\mathrm{d}y_1)\,A_\rho^1\mathrm{d}\sqrt{(y_1^2+y_2^2)\,(y_3^2+y_4^2)}\\ &= \frac{A_\rho^1}{\rho}(y_3^2+y_4^2)(y_1\mathrm{d}y_2 - y_2\mathrm{d}y_1)(y_1\mathrm{d}y_1 + y_2\mathrm{d}y_2)\\ &+ \frac{A_\rho^1}{\rho}(y_1^2+y_1^2)(y_1\mathrm{d}y_2 - y_2\mathrm{d}y_1)(y_3\mathrm{d}y_3 + y_3\mathrm{d}y_3). \quad (5.22) \end{aligned}$$

Then the integrand of ADM mass is

$$\left(\partial_a(D_I)_{ac} - \partial_c(D_I)_{aa}\right)n^c = \rho\,\underbrace{(y_1\partial_{y_2} - y_2\partial_{y_1})A_\rho^1}_{=0} = 0. \quad (5.23)$$

Note that $\xi_{(1)} = \partial_{\phi^1} = y_1\partial_{y_2} - y_2\partial_{y_1}$, which is a generator of the isometry group. Now consider D_{II} as a metric

$$\begin{aligned}(D_{II})_{ab} &= \frac{1}{2}r^2(1+x)\mathrm{d}\phi^1 A_z^1 \mathrm{d}z = \frac{1}{2}\left(y_1\mathrm{d}y_2 - y_2\mathrm{d}y_1\right) A_z^1 \mathrm{d}\left[\left(y_1^2+y_2^2\right) - \left(y_3^2+y_4^2\right)\right] \\ &= \frac{A_z^1}{z}(y_1\mathrm{d}y_2 - y_2\mathrm{d}y_1)(y_1\mathrm{d}y_1 + y_2\mathrm{d}y_2) \\ &- \frac{A_z^1}{z}(y_1\mathrm{d}y_2 - y_2\mathrm{d}y_1)(y_3\mathrm{d}y_3 + y_3\mathrm{d}y_3). \end{aligned} \tag{5.24}$$

Then the ADM mass is

$$\left(\partial_a D_{II\,ac} - \partial_c (D_{II})_{aa}\right) n^c = \frac{z}{2}\underbrace{(y_1\partial_{y_2} - y_2\partial_{y_1})A_z^1}_{=0} = 0. \tag{5.25}$$

There the ADM mass of conformal metric is zero, that is $M_{ADM}^{\tilde{h}} = 0$. □

Returning to the mass of GB data we have

$$M_{ADM} = \frac{\pi}{4}\int_{\mathcal{B}_F}\left[-\frac{3}{2}r^3 v_{,r} - \left(\frac{r^3}{2}V_{,r} - r^2 V\right)\right]\mathrm{d}x\,, \tag{5.26}$$

Then we define three one-form ω, α and χ such that

$$\omega \equiv 2\alpha + 6\chi \tag{5.27}$$

where

$$\alpha \equiv (\rho V_{,\rho} - V)\mathrm{d}z - \rho V_{,z}\mathrm{d}\rho \tag{5.28}$$

$$= \left(-r(1-x^2)V_{,x} - rxV\right)\mathrm{d}r + \left(\frac{r^3}{4}V_{,r} - \frac{r^2}{2}V\right)\mathrm{d}x \tag{5.29}$$

$$\chi \equiv \rho\left(v_{,\rho}\mathrm{d}z - v_{,z}\mathrm{d}\rho\right) = -r(1-x^2)v_{,x}\mathrm{d}r + \frac{r^3}{4}v_{,r}\mathrm{d}x. \tag{5.30}$$

Then

$$d\alpha = \Delta_2 V \rho d\rho dz, \quad d\chi = \Delta_3 v \rho d\rho dz, \quad d\omega = (2\Delta_2 V + 6\Delta_3 v)\, \rho d\rho dz, \tag{5.31}$$

where Δ_3 is Laplace operator with respect to the metric

$$\delta_3 = d\rho^2 + dz^2 + \rho^2 d\varphi^2 \tag{5.32}$$

on $\mathbb{R}^3$ and φ is an auxiliary 2π angle. Note that with transformation (4.39) in the chart (r, x, ϕ) the metric is

$$\delta_3 = r^2 \left[dr^2 + \frac{r^2}{4}\left((1-x^2)^{-1} dx^2 + (1-x^2) d\varphi^2 \right) \right], \tag{5.33}$$

and $\Delta_2 = \partial_\rho^2 + \partial_z^2$. Now by asymptotes of GB data set, we list the behaviour of χ_1 and χ_2 at boundary of the orbit space $\partial\mathcal{B} = \Gamma \cup \mathcal{B}_F \cup \mathcal{B}_E$.

$$\alpha = \left(\frac{r^3}{4} V_{,r} - \frac{r^2}{2} V \right) dx, \quad \chi = \frac{r^3}{4} v_{,r} dx, \qquad \text{on } \mathcal{B}_F \tag{5.34}$$

$$\alpha = -rxV dr, \quad \chi = 0, \qquad \text{on } \Gamma \tag{5.35}$$

$$\alpha = \chi = 0, \qquad \text{on } \mathcal{B}_E, \tag{5.36}$$

Now if we integrate equation(5.31) with coefficient $\frac{\pi}{4}$ over the orbit space $\mathcal{B}$ we have

$$\begin{aligned} \frac{\pi}{4}\int_{\mathcal{B}} d\omega &= \frac{\pi}{4}\int_{\partial\mathcal{B}} \omega \\ &= -\frac{\pi}{2}\int_\Gamma rxV dr + \frac{\pi}{4}\int_{\mathcal{B}_F} \left[\left(\frac{r^3}{2} V_{,r} - r^2 V \right) + \frac{3r^3}{2} v_{,r} \right] dx \\ &= \frac{\pi}{2}\int_0^\infty r\,[V(x=1) + V(x=-1)]\, dr - M_{ADM} \\ &= \frac{\pi}{4}\sum_{\text{rods}} \int_{I_i} \log V_i \, dz - M_{ADM}. \end{aligned} \tag{5.37}$$

The first equality follows from Stokes theorem and the last equality follows from equation (5.26) and orientation of (r, x) chart. We next compute the scalar curvature of $\tilde{h}_{ab}$. Then by equation (2.30) we have

$$-R_{\boldsymbol{h}}e^{2v} = -R_{\tilde{\boldsymbol{h}}} + 6e^{-2U}\left[\Delta_3 v + (\nabla v)^2\right] \tag{5.38}$$

and ∇ is the derivative with respect to δ_3. Now we compute the Ricci scalar $R_{\boldsymbol{h}}$ by the following remark.

Remark 59. Consider a metric of the form

$$\boldsymbol{g} = q_{AB}\mathrm{d}x^A\mathrm{d}x^B + \lambda'_{ij}(\mathrm{d}\phi^i + A^i_B\mathrm{d}x^B)(\mathrm{d}\phi^j + A^j_B\mathrm{d}x^B). \tag{5.39}$$

The vector field $\partial/\partial\phi^i$ are Killing fields and so all functions appearing in the metric are independent of x^A. Explicitly the metric components are

$$g_{BC} = q_{BC} + \lambda'_{ij}A^i_BA^j_C, \qquad g_{ij} = \lambda'_{ij}, \qquad g_{Bi} = \lambda'_{ij}A^j_B\,, \tag{5.40}$$

and the inverse is

$$g^{BC} \;\;=\;\; q^{AB}, \qquad g^{ij} = \lambda'^{ij} + q^{BC}A^i_BA^j_C \qquad g^{Bi} = -q^{BC}A^j_C\,. \tag{5.41}$$

A tedious but straightforward computation shows that in this coordinate basis the

Ricci tensor is

$$
\begin{aligned}
(R_{\boldsymbol{g}})_{ij} &= -\frac{1}{4}D_A \log(\det\lambda')q^{AB}D_B\lambda'_{ij} + \frac{1}{4}q^{AE}q^{BC}\lambda'_{ik}\lambda'_{jl}F^k_{EB}F^l_{AC} \\
&+ \frac{1}{2}\lambda'^{kl}q^{AB}D_A\lambda'_{jl}D_B\lambda'_{ik} + \frac{1}{2}\bar{\Gamma}^B_{AC}q^{AC}D_B\lambda'_{ij} - \frac{1}{2}q^{AB}D_AD_B\lambda'_{ij} \\
(R_{\boldsymbol{g}})_{BE} &= -\frac{1}{2}D_BD_E(\log\det\lambda') + (R_{\boldsymbol{q}})_{BE} - \frac{1}{4}\mathrm{Tr}(\lambda'^{-1}D_E\lambda'\lambda'^{-1}D_B\lambda') \\
&- (R_{\boldsymbol{g}})_{ij}A^i_BA^j_E + (R_{\boldsymbol{g}})_{iB}A^i_E + (R_{\boldsymbol{g}})_{iE}A^i_B - \frac{1}{2}q^{AC}\lambda'_{ij}F^i_{EC}F^j_{AB}b \qquad (5.42) \\
(R_{\boldsymbol{g}})_{iB} &= \frac{1}{2}D_C(q^{CE}\lambda'_{ij}F^j_{BE}) + \frac{1}{2}q^{CE}\lambda'_{ij}F^j_{BE}\bar{\Gamma}^A_{AC} + \frac{1}{4}q^{CE}\lambda'_{ij}F^j_{BC}\partial_E(\log(\det\lambda')) \\
&- \frac{1}{2}q^{CE}\lambda'_{ij}F^j_{AE}\bar{\Gamma}^A_{BC} + (R_{\boldsymbol{g}})_{ij}A^j_B \\
&= (R_{\boldsymbol{g}})_{ij}A^j_B + \frac{1}{2\sqrt{\det\lambda'}\sqrt{\det q}}q_{AB}D_C\left(\sqrt{\det\lambda'}\sqrt{\det q}\lambda'_{ij}q^{AE}q^{CN}F^j_{EN}\right)
\end{aligned}
$$

where $F^i_{AB} \equiv 2\partial_{[A}A^i_{B]}$, and D and $\bar{\Gamma}^B_{AC}$ are the covariant derivative and Christoffel symbols of q, respectively.

Now by Remark 59 and fixing $\boldsymbol{q} = e^{2U}\delta_2$ where

$$
\delta_2 \equiv \mathrm{d}\rho^2 + \mathrm{d}z^2 = r^2\left[\mathrm{d}r^2 + \frac{r^2}{4(1-x^2)}\mathrm{d}x^2\right], \qquad (5.43)
$$

and the Ricci curvature for $\tilde{h}$ is

$$
\begin{aligned}
(R_{\tilde{\boldsymbol{h}}})_{ij} &= -\frac{1}{2}\nabla_A\nabla^A\lambda'_{ij} - \frac{1}{2}\nabla_A(\log\rho)\nabla^A\lambda'_{ij} + \frac{1}{2}\nabla^A\lambda'_{ik}\lambda'^{kl}\nabla_A\lambda'_{lj} \\
&+ \frac{1}{4}e^{-4U}\lambda'_{ik}\lambda'_{jl}H^{kl} \qquad (5.44) \\
(R_{\tilde{\boldsymbol{h}}})_{iA} &= (R_{\tilde{\boldsymbol{h}}})_{ij}A^j_A + \frac{1}{2\rho}(\delta_2)_{AB}\nabla_C\left(\rho e^{-2U}\lambda'_{ij}\delta_2^{BN}\delta_2^{CE}F^j_{NE}\right) \qquad (5.45) \\
(R_{\tilde{\boldsymbol{h}}})_{AB} &= -(R_{\tilde{\boldsymbol{h}}})_{ij}A^i_AA^j_B + (R_{\tilde{\boldsymbol{h}}})_{iA}A^i_B + (R_{\tilde{\boldsymbol{h}}})_{iB}A^i_A - \frac{1}{2}e^{-2U}\delta_2^{CE}\lambda'_{ij}F^i_{AC}F^j_{BE} \\
&- D_AD_B\log\rho - \frac{1}{4}\mathrm{Tr}\left[\lambda'^{-1}\nabla_A\lambda'\lambda'^{-1}\nabla_B\lambda'\right] + (R_{\boldsymbol{q}})_{AB} \qquad (5.46)
\end{aligned}
$$

where $F^i_{AB} \equiv 2\nabla_{[A}A^i_{B]}$, and

$$H^{ij} \equiv \delta_2^{AC}\delta_2^{BE}F^i_{AB}F^j_{CE} = (A^i_{\rho,z} - A^i_{z,\rho})(A^j_{\rho,z} - A^j_{z,\rho}) \,. \tag{5.47}$$

Here ∇_A is the covariant derivative with respect to flat 3 dimensional metric δ_3 equation (5.32). Since $R_{\boldsymbol{q}} = -2e^{-2U}\Delta_2 U$ where Δ_2 is the Laplace operator respect to δ_2, the scalar curvature is

$$R_{\tilde{\boldsymbol{h}}}e^{2U} = -\frac{1}{4}e^{-2U}\lambda'_{ij}H^{ij} - 2\Delta_2 U + \frac{1}{\rho^2} - \frac{1}{4}\mathrm{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right] \,. \tag{5.48}$$

By equations (5.38) and (5.48) we have

$$-R_{\boldsymbol{h}}e^{2v+2U} = \frac{1}{4}e^{-2U}\lambda'_{ij}H^{ij} + 2\Delta_2 U - \frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right] + 6\Delta_3 v + 6\left|\nabla v\right|^2 \tag{5.49}$$

where $\left|\nabla v\right|^2 = (\nabla_\rho v)^2 + (\nabla_z v)^2$. Now we integrate equation (5.49) over $\mathcal{B}$ and use (5.37)

$$\begin{aligned} M_{ADM} &= \frac{\pi}{4}\int_{\mathcal{B}}\left[R_h e^{2v+2U} + \frac{1}{4}e^{-2U}\lambda'_{ij}H^{ij} - \frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right] + 6\left|\nabla v\right|^2\right] \mathrm{d}\mu \\ &+ \frac{\pi}{4}\sum_{\text{rods}}\int_{I_i}\log V_i \,\mathrm{d}z \end{aligned} \tag{5.50}$$

Then we have the following theorem about properties of the mass functional.

Theorem 60. *Assume* $(\Sigma, \boldsymbol{h}, K)$ *is a GB initial data set of Einstein constraint equations* (3.7) *and* (3.8) *with unit normal timelike vector* n*. Assume* $\iota_{\xi_{(i)}}\mathcal{G}(n,\cdot) = 0$ *for*

$i = 1, 2$ and $\mathcal{G}(n,n) \geq 0$, where $\mathcal{G}$ is Einstein tensor. Then the mass functional is

$$\begin{aligned} \mathcal{M}\left(v, \lambda', Y\right) \;\; &\equiv \;\; \frac{\pi}{4}\int_{\mathcal{B}}\left(-\frac{\det \nabla\lambda'}{2\rho^2} + e^{-6v}\frac{\nabla Y^t \lambda'^{-1}\nabla Y}{2\rho^2} + 6\left|\nabla v\right|^2\right)\; d\mu \\ &+ \;\; \frac{\pi}{4}\sum_{rods}\int_{I_i} \log V_i \, dz \end{aligned} \tag{5.51}$$

where V_i is defined by

$$V_i(z) = \lim_{\rho \to 0} \frac{2\sqrt{\rho^2+z^2}\lambda'_{ij}w^i w^j}{\rho^2}, \qquad z \in I_i = (a_i, a_{i+1}), \quad w^i \in \mathbb{Z}\,, \tag{5.52}$$

to avoid the conical singularity. Then we have

(a) Mass of any GB initial data is greater than or equal to $\mathcal{M}$.

(b) $\mathcal{M}$ evaluates ADM mass for $t - \phi^i$ symmetric, vacuum data $(\mathcal{B}, u)$ and it is finite.

(c) $\mathbb{R} \times U(1)^2$-invariant, vacuum solutions of Einstein equations are critical points of $\mathcal{M}$.

(d) $\mathcal{M} \geq 0$ for admissible set Ξ of orbit spaces which is defined in Definition 63 and extreme $\mathbb{R} \times U(1)^2$ invariant black holes.

Proof. We prove parts (a) and (b) here and parts (c) and (d) are Section 5.2 and Section 5.3, respectively.

(a) The ADM mass of any GB initial data is equation (5.50). Then

$$\begin{aligned} M_{ADM} &= \frac{\pi}{4}\int_{\mathcal{B}}\left[R_h e^{2v+2U} + \frac{1}{4}e^{-2U}\lambda'_{ij}H^{ij} - \frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right]\right. \\ &\quad \left. +6\left|\nabla v\right|^2\right]\mathrm{d}\mu + \frac{\pi}{4}\sum_{\mathrm{rods}}\int_{I_i}\log V_i\,\mathrm{d}z \\ &\geq \frac{\pi}{4}\int_{\mathcal{B}}\left[\left|K\right|_{\boldsymbol{h}} e^{2v+2U} - \frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right] + 6\left|\nabla v\right|^2\right]\mathrm{d}\mu \\ &+ \frac{\pi}{4}\sum_{\mathrm{rods}}\int_{I_i}\log V_i\,\mathrm{d}z \\ &\geq \frac{\pi}{4}\int_{\mathcal{B}}\left[\frac{e^{-6v}}{2\rho^2}\left[\nabla Y^t\lambda'^{-1}\nabla Y\right] + -\frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right]\right. \\ &\quad \left. +6\left|\nabla v\right|^2\right]\mathrm{d}\mu + \frac{\pi}{4}\sum_{\mathrm{rods}}\int_{I_i}\log V_i\,\mathrm{d}z \end{aligned} \tag{5.53}$$

The first inequality follows from Hamiltonian constraint equation (3.7) and eliminating the positive term $\frac{1}{4}e^{-2U}\lambda'_{ij}H^{ij}$. The second inequality follows from Lemma 49. Now, consider the term

$$-\frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}\left[\left(\lambda'^{-1}\mathrm{d}\lambda'\right)^2\right]\,. \tag{5.54}$$

By matrix identities $\mathrm{Tr}(A\mathrm{d}A) = \frac{\nabla \det A}{\det A}$ and $[\mathrm{Tr}(A)]^2 = \mathrm{Tr}(A^2) + 2\det A$ for matrix A and the fact that $\det\lambda' = \rho^2$ we have

$$\begin{aligned} -\frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right] &= -\frac{1}{4}\left(\mathrm{Tr}\left(\lambda'^{-1}\nabla\lambda'\right)\right)^2 + \frac{1}{4}\mathrm{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right] \\ &= -\frac{1}{2}\frac{\det\nabla\lambda'}{\det\lambda'} \qquad \text{for } 2\times 2 \text{ matrices} \end{aligned} \tag{5.55}$$

where we are using the notation $\det\nabla\lambda' = \frac{1}{2}\epsilon^{ik}\epsilon^{jl}\nabla\lambda'_{ij}\cdot\nabla\lambda'_{kl}$. Then we have

$$M_{ADM} \geq \mathcal{M}(v, \lambda', Y) \tag{5.56}$$

where mass functional $\mathcal{M}$ is defined in (5.51).

(b) We know in $t-\phi^i$ symmetry in vacuum $A^i_B=0$, $\mathcal{G}(n,n)=0$, and $|K|_{\boldsymbol{h}}=\frac{e^{-6v}}{2\rho^2}\left[\nabla Y^t\lambda'^{-1}\nabla Y\right]$. Thus all inequalities in equation (5.53) are equalities. Hence we have $M_{ADM}=\mathcal{M}(v,\lambda',Y)$. Now we use the asymptotic conditions of GB data in Definition 46 and show $\mathcal{M}$ has finite energy. Let $r\to\infty$ then

$$\frac{\det\nabla\lambda'}{2\rho^2}=o(r^{-6}),\qquad e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{2\rho^2}=o(r^{-2k-4}),\qquad |\nabla v|^2=o(r^{-6})\,,\tag{5.57}$$

and as $r\to 0$ and asymptotically flat

$$\frac{\det\nabla\lambda'}{2\rho^2}=o(r^{-2}),\qquad e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{2\rho^2}=o(r^{2k-8}),\qquad |\nabla v|^2=o(r^{-4})\,,\tag{5.58}$$

and as $r\to 0$ and asymptotically cylindrical

$$\frac{\det\nabla\lambda'}{2\rho^2}=o(r^{-2}),\qquad e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{2\rho^2}=o(r^{2k-2}),\qquad |\nabla v|^2=o(r^{-4})\,.\tag{5.59}$$

since the volume element is $O(r^5)$, the functional $\mathcal{M}$ is finite.

□

Remark 61. One can write the mass functional in terms of the flat metric δ_3 on $\mathbb{R}^3$ in cylindrical coordinates

$$\begin{aligned}\mathcal{M}\left(v,\lambda',Y\right) &= \frac{1}{8}\int_{\mathbb{R}^3}\left(-\frac{\det\nabla\lambda'}{2\rho^2}+e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{2\rho^2}+6\left|\nabla v\right|^2\right)\,\mathrm{d}\mu_0\\ &+\frac{\pi}{4}\sum_{\text{rods}}\int_{I_i}\log V_i\,\mathrm{d}z\,,\end{aligned}\tag{5.60}$$

where $\mathrm{d}\mu_0=\rho\,\mathrm{d}\rho\mathrm{d}z\mathrm{d}\varphi=\frac{r^5}{4}\mathrm{d}r\mathrm{d}x\mathrm{d}\varphi$. Moreover, we have two ends, one of which is at the origin $E=\{\rho=z=0\}$ of $\mathbb{R}^3$ and it is not part of integration domain, because the integrand of $\mathcal{M}$ by asymptotic conditions (4.44) and (4.45) on E is finite. Hence the domain of integration is auxiliary $\mathbb{R}^3$. Note that if we consider a region $\Omega \subset \mathbb{R}^3 \backslash \Gamma$, then the mass functional is

$$\mathcal{M}_\Omega = \frac{1}{8} \int_\Omega \left(-\frac{\det \nabla \lambda'}{2\rho^2} + e^{-6v} \frac{\nabla Y^t \lambda'^{-1} \nabla Y}{2\rho^2} + 6 \left| \nabla v \right|^2 \right) \, \mathrm{d}\mu_0 + \frac{1}{4} \oint_{\partial\Omega} \alpha \wedge \mathrm{d}\varphi \, , \tag{5.61}$$

where α is defined in (5.28).

5.2 Critical Points of $\mathcal{M}$

In this section we use two different methods to show the critical points or Euler-Lagrange equations of the mass functional $\mathcal{M}$ are vacuum stationary, $U(1)^2$-invariant spacetime (Theorem 60-c). Let we have vacuum solutions with $\mathbb{R} \times U(1)^2$ isometry group. The metric takes the canonical form [67, 96]

$$g = -H \mathrm{d}t^2 + \frac{\lambda'_{ij}}{H^{1/2}} (\mathrm{d}\phi^i - w^i \mathrm{d}t)(\mathrm{d}\phi^j - w^j \mathrm{d}t) + e^{2\nu} (\mathrm{d}\rho^2 + \mathrm{d}z^2) \, , \tag{5.62}$$

where $\rho^2 = \det \lambda'$ is harmonic on the orbit space. Remarkably, the vacuum field (Euler-Lagrange) equations for this spacetime can be derived from the critical points of the following Dirichlet energy (Carter functional) E which is defined for maps $(\lambda, Y) : \mathbb{R}^3 \to SL(3, \mathbb{R})/SO(3)$[93, 113, 119], as first discussed by Carter for $D = 4$ in [28](see [96] for general dimension):

$$E(\lambda, Y) = \frac{1}{32} \int_{\mathbb{R}^3} \left\{ \left(\frac{\nabla \det \lambda}{\det \lambda} \right)^2 + \mathrm{Tr} \left[\left(\lambda^{-1} \nabla \lambda \right)^2 \right] + 2 \frac{\nabla Y^t \lambda^{-1} \nabla Y}{\det \lambda} \right\} \mathrm{d}\mu_0 \, , \tag{5.63}$$

where ∇ is with respect to δ_3 and the integration domain, $\mathbb{R}^3$, is the auxiliary space which is obtained from the orbit space of *spacetime* $\tilde{\mathcal{B}}$ in the same approach as Remark

61 for mass functional $\mathcal{M}$ and $\lambda_{ij} = \frac{\lambda'_{ij}}{H^{1/2}}$. It follows that $H = \rho^2(\det\lambda)^{-1}$. Here E is just a particular harmonic map energy for mapping $(\lambda, Y) : \mathbb{R}^3 \to SL(3,\mathbb{R})/SO(3)$ with the field equations of E [67]

$$G_\lambda : \quad \operatorname{div}\left(\lambda^{-1}\nabla\lambda\right) \;=\; -\frac{\lambda^{-1}}{\det\lambda}\nabla Y \cdot \nabla Y^t \,, \tag{5.64}$$

$$G_Y : \quad \operatorname{div}\left(\frac{\lambda^{-1}}{\det\lambda}\nabla Y\right) \;=\; 0 \,, \tag{5.65}$$

where div is respect to δ_3 and $\cdot$ is inner product respect to δ_3 . Now in the following sections we prove Euler-Lagrange equations of $\mathcal{M}$ are same as G_λ and G_Y.

Remark 62. Note that Dirichlet energy E has the field equations G_λ and G_Y which are stationary, $U(1)^2$-invariant vacuum solutions written in spacetime Weyl coordinates with orbit space $\tilde{\mathcal{B}}$. We show in the next two sections that critical points of mass functional $\mathcal{M}$ are same as E in spacetime Weyl coordinate. However, the mass functional is defined over spatial slice orbit space $\mathcal{B}$ and associated Weyl (quasi-isotropic) coordinate (ρ, z). For non-extreme black holes, spacetime Weyl coordinates only covers the exterior region of the black hole spacetime and the manifold has an interior boundary. In particular in these coordinates the mass functional is singular on the inner boundary. One can always find quasi-isotropic coordinates on the initial data slice Σ to complete the slice manifold Σ and compute the mass, but then the resulting geometry is *not* a critical point of $\mathcal{M}$. But for extreme black holes, the usual spacetime Weyl coordinates and quasi-isotropic coordinates coincide, and the mass functional is well defined on these critical points.

5.2.1 $\mathcal{M}$ = Reduced Energy

In this section we show the mass functional $\mathcal{M}$ can be thought of a reduced energy of a Dirichlet energy [3]. This means over a region $\Omega \subset \mathbb{R}^3\backslash\Gamma$ it equals Dirichlet energy

E plus a boundary term. In fact, $\mathcal{M}$ is a regularization of E in this special case since we are removing the infinite boundary term. Consider a constant-time spatial slice of the stationary, axisymmetric metric (5.62). The metric can be placed in the general form of GB initial data metric with $A^i_a = 0$. Then we have

$$\lambda = e^{2v}\lambda', \qquad e^{2v} = \frac{\sqrt{\det\lambda}}{\rho}\,. \tag{5.66}$$

We wish to express the terms in $\mathcal{M}$ in terms of λ_{ij}. First we have

$$\nabla\lambda = \frac{1}{2}\left(\frac{\det\lambda}{\rho^2}\right)^{-\frac{1}{2}}\left(\frac{\nabla\det\lambda}{\rho^2} - 2\frac{\det\lambda\nabla\rho}{\rho^3}\right)\lambda' + \left(\frac{\det\lambda}{\rho^2}\right)^{\frac{1}{2}}\nabla\lambda'\,, \tag{5.67}$$

and

$$\lambda^{-1}\nabla\lambda \;=\; \frac{1}{2}\left(\frac{\nabla\det\lambda}{\det\lambda} - 2\frac{\nabla\rho}{\rho}\right)\mathbb{I} + \lambda'^{-1}\nabla\lambda'\,, \tag{5.68}$$

where $\mathbb{I}$ is the identity matrix. Hence

$$\begin{aligned}
&\operatorname{Tr}\left[\left(\lambda^{-1}\nabla\lambda\right)^2\right] \\
&= \frac{1}{2}\left(\frac{\nabla\det\lambda}{\det\lambda} - 2\frac{\nabla\rho}{\rho}\right)^2 + \operatorname{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right] + \left(\frac{\nabla\det\lambda}{\det\lambda} - 2\frac{\nabla\rho}{\rho}\right)\operatorname{Tr}\left[\lambda'^{-1}\nabla\lambda'\right] \\
&= \frac{1}{2}\left(\frac{\nabla\det\lambda}{\det\lambda}\right)^2 - 2\left(\frac{\nabla\rho\cdot\nabla\rho}{\rho^2}\right) + \operatorname{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right]\,.
\end{aligned} \tag{5.69}$$

Note $\nabla\rho\cdot\nabla\rho = 1$. Moreover, by taking determinant of equation (5.66) we have

$$v = \frac{1}{4}\log(\det\lambda) - \frac{\log\rho}{2}\,. \tag{5.70}$$

We then deduce

$$|\nabla v|^2 = \left(\frac{\nabla \det\lambda}{4\det\lambda} - \frac{\nabla\rho}{2\rho}\right)^2 = \frac{1}{16}\left(\frac{\nabla\det\lambda}{\det\lambda}\right)^2 - \frac{1}{4}\nabla\left(\log\rho\right)\cdot\nabla\log\left(\frac{\rho}{\det\lambda}\right)\ . \quad (5.71)$$

Then by equations (5.69) and (7.15), $\mathcal{M}$ over a bounded region $\Omega\subset\mathbb{R}^3\backslash\Gamma$ is

$$\begin{aligned}
\mathcal{M}_\Omega &= \frac{1}{8}\int_\Omega\Bigg(-\frac{1}{\rho^2} - \frac{1}{8}\left(\frac{\nabla\det\lambda}{\det\lambda}\right)^2 + \frac{1}{2}\left(\frac{\nabla\rho\cdot\nabla\rho}{\rho^2}\right) + \frac{1}{4}\mathrm{Tr}\left[\left(\lambda^{-1}\nabla\lambda\right)^2\right] \\
&+ \left(\frac{\det\lambda}{\det\lambda'}\right)^{-3/2}\left(\frac{\det\lambda}{\det\lambda'}\right)^{1/2}\frac{\mathrm{Tr}\left(\lambda^{-1}\nabla Y\mathrm{d}Y^t\right)}{2\det\lambda'} + \frac{6}{16}\left(\frac{\nabla\det\lambda}{\det\lambda}\right)^2 \\
&+ \frac{6}{4}\left(\frac{\nabla\rho\cdot\nabla\rho}{\rho^2}\right) - \frac{6}{4}\frac{\nabla\rho\cdot\nabla\det\lambda}{\rho\det\lambda}\Bigg)\,\mathrm{d}\mu_0 + \frac{1}{4}\oint_{\partial\Omega}\alpha\wedge\mathrm{d}\varphi \\
&= E_\Omega + \frac{1}{8}\int_\Omega\left\{\left(\frac{\nabla\rho\cdot\nabla\rho}{\rho^2}\right) - \frac{3}{2}\frac{\nabla\rho\cdot\nabla\det\lambda}{\rho\det\lambda}\right\}\,\mathrm{d}\mu_0 + \frac{1}{4}\oint_{\partial\Omega}\alpha\wedge\mathrm{d}\varphi \\
&= E_\Omega + \frac{1}{8}\int_\Omega\nabla\ln\left(\rho\right)\cdot\nabla\ln\left(\frac{\rho}{(\det\lambda)^{3/2}}\right)\,\mathrm{d}\mu_0 + \frac{1}{4}\oint_{\partial\Omega}\alpha\wedge\mathrm{d}\varphi \\
&= E_\Omega - \frac{1}{8}\int_\Omega\Delta_3\ln\left(\rho\right)\ln\left(\frac{\rho}{(\det\lambda)^{3/2}}\right)\,\mathrm{d}\mu_0 + \frac{1}{4}\oint_{\partial\Omega}\alpha\wedge\mathrm{d}\varphi \\
&+ \frac{1}{8}\oint_{\partial\Omega}\rho^{-1}\ln\left(\frac{\rho}{(\det\lambda)^{3/2}}\right)\nabla\rho\cdot\nu\,\mathrm{d}S\,, \qquad (5.72)
\end{aligned}$$

where ν is a normal unit vector on $\partial\Omega$ and $\mathrm{d}S = \frac{r^4}{2}\mathrm{d}x\mathrm{d}\varphi$. Moreover,

$$\Delta_3\ln\rho = 0, \qquad \text{for } \Omega\subset\mathbb{R}^3\backslash\Gamma. \quad (5.73)$$

Hence, if we define

$$g = 2\log\rho. \quad (5.74)$$

we have

$$\mathcal{M}_\Omega = E_\Omega - \frac{1}{16}\oint_{\partial\Omega}\left(g + 6v\right)\nabla g\cdot\nu\,\mathrm{d}S + \frac{1}{4}\oint_{\partial\Omega}\alpha\wedge\mathrm{d}\varphi\,. \quad (5.75)$$

This shows that the mass functional and harmonic energy are the same up to the boundary terms. Therefore, they have same Euler-Lagrange equations.

5.2.2 First Variation of $\mathcal{M}$

Consider the mass functional and perturb it as following. Assume $C_{\rho_0} \equiv \{\rho \geq \epsilon\}$ is the cylinder centered on the z axis Γ of radius ρ_0 and we define $\Omega_{\rho_0} \equiv \mathbb{R}^3 \backslash C_{\rho_0}$. We set

$$\boldsymbol{v} = v + t\bar{v}, \qquad \boldsymbol{\lambda}' = \lambda' + t\bar{\lambda}', \qquad \boldsymbol{Y} = Y + t\bar{Y} \tag{5.76}$$

where $\bar{v} \in C_c^\infty(\mathbb{R}^3)$, and $\bar{\lambda}', \bar{Y} \in C_c^\infty(\Omega_{\rho_0})$. Then we have a one parameter family of functional $\mathcal{E}(t) = \mathcal{M}(\boldsymbol{v}, \boldsymbol{\lambda}', \boldsymbol{Y})$. Now we compute

$$\mathcal{E}'(0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{E}(t) \ . \tag{5.77}$$

Clearly the boundary term is independent of perturb terms and its derivative with respect to t vanishes. We take variation of the first term in $\mathcal{E}(t)$:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[-\frac{\det \nabla \boldsymbol{\lambda}'}{2\rho^2}\right] = -\frac{1}{2\rho^2} \mathrm{Tr}\left[\mathrm{adj} \nabla \bar{\lambda}' \cdot \nabla \lambda'\right] \ . \tag{5.78}$$

We take variation of the second term

$$\begin{aligned} &\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[e^{-6\boldsymbol{v}} \frac{\nabla \boldsymbol{Y}^t \boldsymbol{\lambda}'^{-1} \nabla \boldsymbol{Y}}{2\rho^2}\right] \\ &= e^{-6v} \left[\frac{\nabla Y^t \mathrm{adj} \bar{\lambda}' \nabla Y}{2\rho^4} + \frac{\nabla Y^t \lambda'^{-1} \nabla \bar{Y}}{\rho^2} - 6\bar{v} \frac{\nabla Y^t \lambda'^{-1} \nabla Y}{2\rho^2}\right] \ . \end{aligned} \tag{5.79}$$

The last term is

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} |\nabla \boldsymbol{v}|^2 = 12\nabla v \cdot \nabla \bar{v} \ , \tag{5.80}$$

Thus the first variation is

$$\begin{aligned} \mathcal{E}'(0) &= \frac{1}{8}\int_{\mathbb{R}^3} \Bigg\{ e^{-6v}\left[\frac{\nabla Y^t \mathrm{adj}\bar{\lambda}'\nabla Y}{2\rho^4} + \frac{\nabla Y^t \lambda'^{-1}\nabla \bar{Y}}{\rho^2} - 6\bar{v}\frac{\nabla Y^t \lambda'^{-1}\nabla Y}{2\rho^2}\right] \\ &- \frac{1}{2\rho^2}\mathrm{Tr}\left[\mathrm{adj}\nabla\bar{\lambda}' \cdot \nabla\lambda'\right] + 12\nabla v \cdot \nabla \bar{v} \Bigg\} \, d\mu_0 \ . \end{aligned} \tag{5.81}$$

To find critical point of $\mathcal{E}(\epsilon)$, we set

$$\mathcal{E}'(0) = 0 \ . \tag{5.82}$$

Let $\Omega \subset \mathbb{R}^3 \backslash \Gamma$ and consider $\mathcal{E}'(0)$ over domain Ω. First term by integration of (5.79) by parts and Stokes' theorem is

$$\begin{aligned} &\int_\Omega \left(12\nabla v \cdot \nabla \bar{v} - 6\bar{v}e^{-6v}\frac{\nabla Y^t \lambda'^{-1}\nabla Y}{2\rho^2}\right) \mathrm{d}\mu_0 \\ = &-\int_\Omega \left(12\Delta_3 v\bar{v} + 6\bar{v}e^{-6v}\frac{\nabla Y^t \lambda'^{-1}\nabla Y}{2\rho^2}\right) \mathrm{d}\mu_0 + \oint_{\partial\Omega} 12\bar{v}\nu \cdot \nabla v \, \mathrm{d}S \\ = &-12\int_\Omega \bar{v}\left(2\Delta_3 v + e^{-6v}\frac{\nabla Y^t \lambda'^{-1}\nabla Y}{2\rho^2}\right) \mathrm{d}\mu_0 \ , \end{aligned} \tag{5.83}$$

where ν is outward normal vector to $\partial\Omega$. The second term is

$$\begin{aligned} &\int_\Omega e^{-6v}\frac{\nabla \bar{Y}^t \lambda'^{-1}\nabla Y}{\rho^2} \, \mathrm{d}\mu_0 = \\ &-\int_\Omega \bar{Y}^t \mathrm{div}\left(e^{-6v}\frac{\lambda'^{-1}\nabla Y}{\rho^2}\right) \mathrm{d}\mu_0 + \oint_{\partial\Omega} e^{-6v}\frac{\bar{Y}^t \lambda'^{-1}\nabla Y \cdot \nu}{\rho^2} \, \mathrm{d}S \\ &-\int_\Omega \bar{Y}^t \mathrm{div}\left(e^{-6v}\frac{\lambda'^{-1}\nabla Y}{\rho^2}\right) \mathrm{d}\mu_0 \ , \end{aligned} \tag{5.84}$$

The third term again by Stokes' theorem is

$$\begin{aligned}
&\int_\Omega -\frac{1}{2\rho^2}\mathrm{Tr}\left[\mathrm{adj}\nabla\bar{\lambda}'\cdot\nabla\lambda'\right]\,\mathrm{d}\mu_0 \\
&= \int_\Omega \mathrm{Tr}\left[\mathrm{adj}\bar{\lambda}'\mathrm{div}\left(\frac{\nabla\lambda'}{2\rho^2}\right)\right]\,\mathrm{d}\mu_0 + \oint_{\partial\Omega}\mathrm{Tr}\left[\mathrm{adj}\bar{\lambda}'\nabla\lambda'\cdot\nu\right]\,\mathrm{d}S \\
&= \int_\Omega \mathrm{Tr}\left[\mathrm{adj}\bar{\lambda}'\mathrm{div}\left(\frac{\nabla\lambda'}{2\rho^2}\right)\right]\,\mathrm{d}\mu_0\ , \qquad (5.85)
\end{aligned}$$

and last term is

$$\int_\Omega e^{-6v}\left[\frac{\nabla Y^t\mathrm{adj}\bar{\lambda}'\nabla Y}{2\rho^4}\right]\,\mathrm{d}\mu_0\ . \qquad (5.86)$$

Then the Euler-Lagrange equations are

$$\begin{aligned}
G_X: &\qquad 4\Delta_3 v + e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{\rho^2} &=&\ 0\ , \qquad (5.87)\\
G_{\lambda'}: &\qquad \mathrm{div}\left(\frac{\nabla\lambda'}{\rho^2}\right)+\frac{e^{-6v}}{\rho^4}\nabla Y\cdot\nabla Y^t &=&\ 0\ , \qquad (5.88)\\
G_Y: &\qquad \mathrm{div}\left(\frac{e^{-6v}}{\rho^2}\lambda'^{-1}\nabla Y\right) &=&\ 0\ . \qquad (5.89)
\end{aligned}$$

We prove directly these equations are same as G_λ and G_Y. By form of vacuum $\mathbb{R}\times U(1)^2$ metric we set

$$\lambda = e^{2v}\lambda', \qquad \det\lambda = \rho^2 e^{4v}\ . \qquad (5.90)$$

Then we substitute equation (5.90) in G_Y and G_λ and we have

$$\begin{aligned}
G_\lambda: &\qquad \mathrm{div}\left(e^{-2v}\lambda'^{-1}\nabla\left(e^{2v}\lambda'\right)\right) &=&\ -\frac{e^{-6v}\lambda'^{-1}}{\rho^2}\nabla Y\cdot\nabla Y^t\ , \qquad (5.91)\\
G_Y: &\qquad \mathrm{div}\left(\frac{e^{-6v}}{\rho^2}\lambda'^{-1}\nabla Y\right) &=&\ 0\ . \qquad (5.92)
\end{aligned}$$

The equation (5.92) is exactly equation (5.87). We show G_λ equals equations (5.88) and (5.89). First, we take trace of G_λ

$$\begin{aligned}
0 &= \mathrm{Tr}\left\{\mathrm{div}\left(e^{-2v}\lambda'^{-1}\nabla\left(e^{2v}\lambda'\right)\right)+\frac{e^{-6v}\lambda'^{-1}}{\rho^2}\nabla Y\cdot\nabla Y^t\right\}\\
&= \mathrm{div}\left(\mathrm{Tr}\left\{e^{-2v}\lambda'^{-1}\nabla\left(e^{2v}\lambda'\right)\right\}\right)+e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{\rho^2}\\
&= \mathrm{div}\left(\frac{\nabla\left(e^{4v}\rho^2\right)}{e^{4v}\rho^2}\right)+e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{\rho^2}\\
&= 4\Delta_3 v+2\Delta_3\ln\rho+e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{\rho^2}=4\Delta_3 v+e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{\rho^2}\ ,
\end{aligned}\tag{5.93}$$

and this equals equation (5.87). Since the trace operator and derivative commute we have the second equality. The third equality follows from identity $\mathrm{Tr}\left(A^{-1}\nabla A\right)=\frac{\nabla\det A}{\det A}$. The final equality follows from equation (5.73). Finally, we need to show traceless part of G_λ equals equation (5.88). Let us simplify G_λ:

$$\begin{aligned}
0 &= \mathrm{div}\left(e^{-2v}\lambda'^{-1}\nabla\left(e^{2v}\lambda'\right)\right)+\frac{e^{-6v}\lambda'^{-1}}{\rho^2}\nabla Y\cdot\nabla Y^t\\
&= 2\Delta v\mathbb{I}+\mathrm{div}\left(\lambda'^{-1}\nabla\lambda'\right)+\frac{e^{-6v}\lambda'^{-1}}{\rho^2}\nabla Y\cdot\nabla Y^t\\
&= 2\Delta v\mathbb{I}+\mathrm{div}\left(\frac{\mathrm{adj}\left(\lambda'\right)}{\rho^2}\nabla\lambda'\right)+\frac{e^{-6v}\mathrm{adj}\left(\lambda'\right)}{\rho^4}\nabla Y\cdot\nabla Y^t\\
&= 2\Delta v\mathbb{I}+\frac{1}{\rho^2}\nabla\left(\mathrm{adj}\left(\lambda'\right)\right)\cdot\nabla\lambda'+\mathrm{adj}\left(\lambda'\right)\mathrm{div}\left(\frac{\nabla\lambda'}{\rho^2}\right)+\frac{e^{-6v}\mathrm{adj}\left(\lambda'\right)}{\rho^4}\nabla Y\cdot\nabla Y^t\\
&= 2\Delta v\mathbb{I}+\frac{\det\nabla\lambda'}{\rho^2}\mathbb{I}+\mathrm{adj}\left(\lambda'\right)\left\{\mathrm{div}\left(\frac{\nabla\lambda'}{\rho^2}\right)+\frac{e^{-6v}}{\rho^4}\nabla Y\cdot\nabla Y^t\right\}\\
&= 2\Delta v\mathbb{I}-\frac{1}{2}\mathrm{Tr}\left\{\mathrm{adj}\left(\lambda'\right)\mathrm{div}\left(\frac{\nabla\lambda'}{\rho^2}\right)\right\}\mathbb{I}+\mathrm{adj}\left(\lambda'\right)\left\{\mathrm{div}\left(\frac{\nabla\lambda'}{\rho^2}\right)+\frac{e^{-6v}}{\rho^4}\nabla Y\cdot\nabla Y^t\right\}\\
&= \left(2\Delta v+e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{2\rho^2}\right)\mathbb{I}+\mathrm{adj}\left(\lambda'\right)\left\{\mathrm{div}\left(\frac{\nabla\lambda'}{\rho^2}\right)+\frac{e^{-6v}}{\rho^4}\nabla Y\cdot\nabla Y^t\right\}\\
&- e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{2\rho^2}\mathbb{I}-\frac{1}{2}\mathrm{Tr}\left\{\mathrm{adj}\left(\lambda'\right)\mathrm{div}\left(\frac{\nabla\lambda'}{\rho^2}\right)\right\}\mathbb{I}\\
&= \mathrm{Tracefree}\left[\mathrm{adj}\left(\lambda'\right)\left\{\mathrm{div}\left(\frac{\nabla\lambda'}{\rho^2}\right)+\frac{e^{-6v}}{\rho^4}\nabla Y\cdot\nabla Y^t\right\}\right]\ ,
\end{aligned}\tag{5.94}$$

where $\mathbb{I}$ is the identity 2×2 matrix. The first to fourth equalities are straightforward. The fifth equality follows from identity $(\text{adj}A)A = \det A$. The sixth equality follows from the identity

$$0 = \Delta_3 \ln \rho = \frac{\det \nabla \lambda'}{\rho^2} + \frac{1}{2}\text{Tr}\left\{ \text{adj}\,(\lambda')\,\text{div}\left(\frac{\nabla \lambda'}{\rho^2}\right)\right\} \;, \tag{5.95}$$

for $\Omega \subset \mathbb{R}^3 \backslash \Gamma$. The last two equalities follow from decomposition of a matrix to trace and trace-free part and equation (5.87). Therefore, any component of the left hand side matrix in equation (5.94) is a linear combination of equation (5.88). Hence the critical points of $\mathcal{M}$ are vacuum, stationary $U(1)^2$ invariant solutions to the Einstein equations.

5.3 Positive Mass Theorem for GB Initial Data

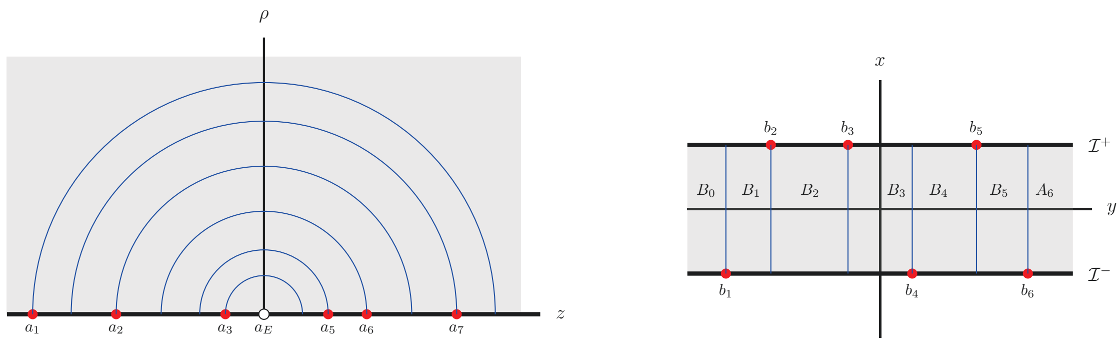
In this section we investigate the positivity of $\mathcal{M}$ for a particular class of orbit spaces Ξ which is defined in Definition 63 and extreme vacuum, $\mathbb{R} \times U(1)^2$-invariant black holes with arbitrary orbit space (Theorem 60-d). In addition, we extend Brill's positive mass theorem for GB initial data with $\mathcal{B} \in \Xi$. Positivity is a desirable property as it plays a key role in applications to geometric inequalities for three-dimensional initial data [51, 53] and investigating the linear stability of extreme black holes [56]. We will show that for a particular set of initial data, $\mathcal{M}$ can be expressed in a non-negative form. A proof of positivity for arbitrary rod data remains to be found. In the following, we will consider asymptotically flat data with a single additional asymptotic end with $N \cong S^3$.

As we discuss in Section 4.2, it is better to work with coordinates (r, x) for orbit space. This is equivalent to introducing a map from $\mathcal{B} \cong \mathbb{R} \times \mathbb{R}^+ \backslash \{a_E\}$ to the infinite strip $\mathcal{B} \cong \mathbb{R} \times [-1, 1]$ [3, 67]. We are given $m^\pm$ rod points $a_i^\pm$ in $\mathcal{I}^\pm$ (see Figure 4.3).

Subdivide the infinite strip into $n \leq m^+ + m^-$ rectangular columns B_s with

$$B_s = \{-1 \leq x \leq 1, b_s < r < b_{s+1}\} \ , \quad s = 0, \ldots, n \tag{5.96}$$

where b_i correspond to the location of the rod points a_i after ordering along the



(a) Orbit space as half plane (b) Orbit space as infinite strip

Figure 5.1: The orbit space can be subdivided into subregions B_s which are half-annuli in the (ρ, z) plane and rectangles in the (y, x) plane. In this case $n = 7$.

$y = \log r$ axis (see Figure 5.1). For convenience, we have chosen $b_1 < b_2 < \cdots < b_{n-1}$ We take $b_0 = 0$ to correspond to the asymptotic end $\mathcal{B}_E$ and b_{n+1} to correspond to the asymptotically flat end $\mathcal{B}_F$. Fix a region A_s. Then one of the following two possibilities must occur: (a) distinct Killing fields $v_{(s)}$ and $w_{(s)}$ vanish on $A_s \cap \mathcal{I}^+$ and $B_s \cap \mathcal{I}^-$ respectively (in this case A_i is topologically $S^3 \times \mathbb{R}$), or (b) the same Killing field $v_{(s)} = v^i_{(s)}\xi_i$ vanishes on both of the disjoint sub-intervals $B_s \cap \mathcal{I}^\pm$ (in this case B_s is topologically $S^2 \times D$ where D is a non-contractible disc).

Definition 63. The *admissible set* Ξ of orbit spaces is a collection of $\mathcal{B}$ such that different Killing vectors vanish on $\Gamma \cap B_s$.

Consider orbit space $\mathcal{B} \in \Xi$. The mass functional is

$$\begin{aligned}\mathcal{M} &= \frac{\pi}{16}\int_{\mathcal{B}}\left(-\frac{\det\nabla\lambda'}{2\det\lambda'}+e^{-6v}\frac{\nabla Y^t\lambda'^{-1}\nabla Y}{2\det\lambda'}+6\left|\nabla v\right|^2\right)r^3\mathrm{d}r\mathrm{d}x\\ &+ \frac{\pi}{2}\int_{\Gamma\cup\mathcal{B}_E\cup\mathcal{B}_F}\alpha. \end{aligned} \tag{5.97}$$

The boundary is $\Gamma\cup\mathcal{B}_F\cup\mathcal{B}_E = \mathcal{I}^+\cup\mathcal{I}^-\cup\mathcal{B}_F\cup\mathcal{B}_E$ where $\mathcal{B}_E$ and $\mathcal{B}_E$ are defined in (5.10) and (5.9), respectively. Then by equation (5.37) we have

$$\int_{\Gamma\cup\mathcal{B}_F\cup\mathcal{B}_E}\alpha = \int_0^\infty r\left(V|_{x=1}+V|_{x=-1}\right)\mathrm{d}r = \sum_{i=0}^{n}\int_{b_i}^{b_{i+1}} r\left(V|_{x=1}+V|_{x=-1}\right)\mathrm{d}r. \tag{5.98}$$

Consider the integral (5.97). We then express (5.97) as

$$\mathcal{M} = \sum_{s=0}^{n}\int_{B_s}\mathcal{M}_s \tag{5.99}$$

where $\mathcal{M}_s$ is the restriction of $\mathcal{M}$ to B_s. Now fix B_s and without loss of generality we can select the following parametrization of the 3 independent functions contained in λ'_{ij} and v:

$$\begin{aligned}\lambda'_{11} &= \frac{r^2(1-x)}{2\sqrt{1-W^2}}e^{V_1-V_2} & \lambda'_{22} &= \frac{r^2(1+x)}{2\sqrt{1-W^2}}e^{V_2-V_1}\\ \lambda'_{12} &= \frac{r^2\sqrt{1-x^2}W}{2\sqrt{1-W^2}} & v &= \frac{V_1+V_2+\log\sqrt{1-W^2}}{2}\end{aligned} \tag{5.100}$$

where $v_{(s)} = \partial_{\bar{\phi}_s^1}$ and $w_{(s)} = \partial_{\bar{\phi}_s^2}$ vanish on $\mathcal{I}^+\cap B_s$ and $\mathcal{I}^-\cap B_s$, respectively such that

$$\frac{\partial}{\partial_{\bar{\phi}_s^k}} = \alpha_{sk}^j\frac{\partial}{\partial_{\phi^j}}, \qquad k,j=1,2, \quad s=1,\dots,n, \qquad \alpha_{sk}^j\in\mathbb{Z}\,, \tag{5.101}$$

where for fixed s we have $\det(\alpha^j_{sk}) = \det \begin{pmatrix} \alpha^1_{s1} & \alpha^2_{s1} \\ \alpha^1_{s2} & \alpha^2_{s2} \end{pmatrix} = \pm 1$ [96]. Let V_1, V_2 and W be C^1 functions whose boundary conditions on the axis are induced from those of λ'_{ij} and v (4.42) and (4.41). In particular, we have $\det \lambda' = \rho^2$ and to remove conical singularities on $\mathcal{I}^{\pm}$ (4.48) we require:

$$2V - V_1 + V_2 = 0 \quad \text{on } \mathcal{I}^+, \quad 2V - V_2 + V_1 = 0 \quad \text{on } \mathcal{I}^-, \quad W = 0 \quad \text{on } \mathcal{I}^{\pm}. \tag{5.102}$$

Note that since λ'_{ij} and v are continuous across the boundary of A_i, this will impose boundary conditions on the parameterization functions in adjacent subregions. We take covariant derivative of fuctions λ'_{ij} and v

$$\begin{aligned}
\nabla \lambda'_{11} &= \lambda'_{11} \left((\nabla V_1 - \nabla V_2) + \frac{W \nabla W}{1 - W^2} + \frac{2}{r}\mathrm{d}r - \frac{1}{1-x}\mathrm{d}x \right) \\
\nabla \lambda'_{22} &= \lambda'_{22} \left((\nabla V_2 - \nabla V_1) + \frac{W \nabla W}{1 - W^2} + \frac{2}{r}\mathrm{d}r + \frac{1}{1+x}\mathrm{d}x \right) \\
\nabla \lambda'_{12} &= \lambda'_{12} \left(\frac{\nabla W}{W} + \frac{W \nabla W}{1 - W^2} + \frac{2}{r}\mathrm{d}r - \frac{x}{1-x^2}\mathrm{d}x \right) \\
\nabla v &= \frac{1}{2}(\nabla V_1 + \nabla V_2) - \frac{W \nabla W}{2(1 - W^2)}.
\end{aligned} \tag{5.103}$$

Then we rewrite the second and fourth terms of $\mathcal{M}$ as functions of V_1, V_2, and W, yielding:

$$\begin{aligned}
\frac{\det \nabla \lambda'}{\det \lambda'} &= \frac{-1}{(1 - W^2)} \Bigg[|\nabla V_1 - \nabla V_2|^2 \underbrace{- \frac{8}{r^2} \partial_x (V_1 - V_2)}_{\equiv \mathscr{X}} + (\nabla W)^2 + \frac{W^2 |\nabla W|^2}{1 - W^2} \\
&+ \frac{4W^2}{r^2(1 - x^2)} \Bigg]
\end{aligned} \tag{5.104}$$

and

$$6\,|\nabla v|^2 = \frac{3}{2}|\nabla V_1 + \nabla V_2|^2 + \frac{3}{2}\frac{W^2|\nabla W|^2}{(1-W^2)^2} - \frac{3W}{1-W^2}(\nabla V_1 \cdot \nabla W + \nabla V_2 \cdot \nabla W) \tag{5.105}$$

Now we substitute equations (5.104) and (5.105) in $\mathcal{M}_i$

$$\begin{aligned}
\mathcal{M}_s &= \frac{\pi}{16}\int_{A_s}\Bigg(e^{-6v}\frac{\nabla Y^t \lambda'^{-1}\nabla Y}{2\det\lambda'} + |\nabla V_1 + \nabla V_2|^2 + |\nabla V_1|^2 + |\nabla V_2|^2 + \frac{|\nabla W|^2}{2(1-W^2)} \\
&+ \frac{W^2}{2(1-W^2)}\left[|\nabla V_1 - \nabla V_2|^2 - \frac{6}{W}(\nabla V_1 \cdot \nabla W + \nabla V_2 \cdot \mathrm{d}W)\right] + \frac{2W^2|\nabla W|^2}{(1-W^2)^2} \\
&+ \frac{W^2}{r^2(1-W^2)}\left[\underbrace{4\partial_x V_2 - 4\partial_x V_1}_{\equiv \mathscr{X}_1} + \frac{2}{(1-x^2)}\right]\Bigg) r^3 \mathrm{d}x\mathrm{d}r \\
&+ \frac{\pi}{4}\int_{b_s}^{b_{s+1}} \underbrace{r(V_1 - V_2)\Big|_{x=1}^{x=-1}}_{\equiv \mathscr{X}_2} \mathrm{d}r + \frac{\pi}{2}\int_{b_s}^{b_{s+1}} r\left(V|_{x=1} + V|_{x=-1}\right)\mathrm{d}r \\
&= \frac{\pi}{16}\int_{A_s}\Bigg(e^{-6v}\frac{\nabla Y^t \lambda'^{-1}\nabla Y}{2\det\lambda'} + |\nabla V_1 + \nabla V_2|^2 + |\nabla V_1|^2 + |\nabla V_2|^2 \\
&+ \frac{W^2}{2(1-W^2)}\left[|\nabla V_1 - \nabla V_2|^2 - \frac{6}{W}(\nabla V_1 \cdot \nabla W + \nabla V_2 \cdot \nabla W)\right] + \frac{|\nabla W|^2}{2(1-W^2)} \\
&+ \frac{W^2}{r^2(1-W^2)}\left[4\partial_x V_2 - 4\partial_x V_1 + \frac{2}{(1-x^2)}\right] + \frac{2W^2|\nabla W|^2}{(1-W^2)^2}\Bigg) r^3 \mathrm{d}x\mathrm{d}r
\end{aligned} \tag{5.106}$$

Consider the first equality. The bulk terms follow from (5.104) and (5.105) while second boundary term follows from (5.98) and the first boundary follows from $\mathscr{X}$ in

(5.104)

$$\begin{aligned}\int_{A_s} \frac{\pi \mathscr{X} r^3}{32(1-W^2)} \mathrm{d}x\mathrm{d}r &= \int_{A_s} \frac{-\pi}{4r^2}\left(1-\frac{W^2}{(1-W^2)}\right)\partial_x(V_1-V_2)r^3\mathrm{d}x\mathrm{d}r \\ &= \frac{\pi}{4}\int_{b_s}^{b_{s+1}} r(V_1-V_2)\Big|_{x=1}^{x=-1}\mathrm{d}r \\ &+ \frac{\pi}{4}\int_{A_i} \frac{W^2(\partial_x V_2-\partial_x V_1)}{r^2(1-W^2)} r^3\mathrm{d}x\mathrm{d}r \end{aligned} \tag{5.107}$$

where terms in the right hand side of above equation are $\mathscr{X}_1$ and $\mathscr{X}_2$ in (5.106), respectively. The second equality is obtained by noting the boundary contributions cancel by regularity on the axis (5.102). The remaining terms can be shown to be positive. Now let us write all terms with partial derivative with respect to r:

$$2(V_{1,r})^2+2(V_{2,r})^2+2V_{1,r}V_{2,r}+\frac{2W^2(W_r)^2}{(1-W^2)^2}-\frac{3W}{(1-W^2)}\left(V_{1,r}W_r+V_{2,r}W_r\right)$$
$$+\frac{W^2}{2(1-W^2)}(V_{1,r}-V_{2,r})^2$$

The last term is clearly positive. For others, if we define

$$a=V_{1,r} \qquad b=V_{2,r} \qquad c=\frac{WW_r}{(1-W^2)}, \tag{5.108}$$

then we have

$$a^2+b^2+c^2+ab-\frac{3}{2}bc-\frac{3}{2}ac=\frac{1}{4}\left((a-b)^2+c^2\right)+\frac{3}{4}(a+b-c)^2\geq 0. \tag{5.109}$$

Now write all terms with partial derivative respect to x. First we define

$$A=V_{1,x} \qquad B=V_{2,x} \qquad C=\frac{WW_x}{(1-W^2)} \tag{5.110}$$

and then we have

$$
\begin{aligned}
& \beta\Bigg\{2(V_{1,x})^2+2(V_{2,x})^2+2V_{1,x}V_{2,x}+\frac{2W^2(W_x)^2}{(1-W^2)^2}-\frac{3W}{(1-W^2)}\left(V_{1,x}W_x+V_{2,x}W_x\right) \\
+ & \frac{W^2}{2(1-W^2)}(V_{1,x}-V_{2,x})^2+\frac{W^2}{(1-x^2)(1-W^2)}\left[\partial_x V_2-\partial_x V_1+\frac{1}{2(1-x^2)}\right]\Bigg\} \\
= & \beta\Bigg\{2(V_{1,x})^2+2(V_{2,x})^2+\frac{2W^2(W_x)^2}{(1-W^2)^2}-\frac{3W}{(1-W^2)}\left(V_{1,x}W_x+V_{2,x}W_x\right) \\
+ & \frac{W^2}{2(1-W^2)}\left[\partial_x V_2-\partial_x V_1+\frac{1}{(1-x^2)}\right]^2+2V_{1,x}V_{2,x}\Bigg\} \\
= & \beta\Bigg\{2A^2+2B^2+2AB+2C^2-3AC-3BC \\
+ & \frac{W^2}{2(1-W^2)}\left[\partial_x V_2-\partial_x V_1+\frac{1}{(1-x^2)}\right]^2\Bigg\} \\
= & 2\beta\Bigg\{\frac{3}{4}\left(A+B-C\right)^2+\frac{1}{4}(A-B)^2+\frac{1}{4}C^2 \\
+ & \frac{W^2}{4(1-W^2)}\left[\partial_x V_2-\partial_x V_1+\frac{1}{(1-x^2)}\right]^2\Bigg\}\geq 0 \qquad (5.111)
\end{aligned}
$$

where $\beta=\frac{4(1-x^2)}{r^2}$. Therefore, $\mathcal{M}_s\geq 0$ and $\mathcal{M}\geq 0$ over $\mathcal{B}\in\Xi$. In particular, the orbit space of Myers-Perry initial data belongs to the admissible set. One might expect a similar argument to hold for class (b). This case of course includes initial data for black rings (the same Killing vector field vanishes on either side of the asymptotic end). By choosing a general parametrization for the various functions in this region, one finds that the boundary term has an indefinite sign. However our strategy is merely sufficient to demonstrate positivity, and we expect positivity will hold for general rod structure.

Consider the class of extreme, stationary, $U(1)^2$-invariant vacuum solutions of Einstein equations. These solutions are critical points of mass functional (by Theorem

60-c) and their initial data sets (time-constant slices) belong to vacuum $t-\phi^i$ symmetric class. Then, the mass functional evaluates the ADM mass of these initial data (by Theorem 60-b). By Remark 62, for extreme solutions the spacetime Weyl coordinate coincide with the quasi-isotropic coordinate of the associated initial data sets. Hence, the initial data set in the quasi-isotropic coordinate gives mass and it satisfies in the field equation (5.87)

$$-\Delta_3 v = e^{-6v}\frac{\nabla Y^t \lambda'^{-1} \nabla Y}{4\rho^2} \tag{5.112}$$

By Remark 48 and ADM mass formula (5.37) and equation (5.26), we have

$$\mathcal{M}_{cp} = -\frac{3\pi}{2}\int_{\mathcal{B}} \Delta_3 v \, \mathrm{d}\mu = \frac{3\pi}{8}\int_{\mathcal{B}} \frac{\nabla Y^t \lambda'^{-1} \nabla Y}{\rho^2} \, \mathrm{d}\mu \geq 0\,. \tag{5.113}$$

Therefore, $\mathcal{M}$ is non-negative for extreme $\mathbb{R}\times U(1)^2$ invariant black holes with arbitrary orbit space.

In [23] Brill proved a positive energy theorem for a certain class of maximal, axisymmetric initial data sets on $\mathbb{R}^3$. Brill's theorem has been extended by Dain [53], Gibbons and Holzegel [76] for a larger class of 3 dimensional initial data. Subsequently, Chruściel [39] generalized the result to the maximal initial data set on a simply connected manifold (with multiple asymptotically flat ends) admitting a $U(1)$ action by isometries. Moreover, in [76] a positive energy theorem was proved for a restricted class of maximal, $U(1)^2$-invariant, four-dimensional initial data sets on $\mathbb{R}^4$. The purpose of this section is to generalize these results to a larger class of 4+1 initial data. In particular, our result extends the work of [76] in four main directions:

1. We consider the general form of a $U(1)^2$-invariant metric (i.e. we do not assume the initial data has an orthogonally transitive $U(1)^2$ isometry group) on asymptotically flat, simply connected, four-dimensional manifolds Σ admitting a torus action.

2. The orbit space $\mathcal{B} \cong \Sigma/U(1)^2$ of Σ belongs to a larger class Ξ which is defined below in Definition 63.

3. The boundary conditions on axes and fall-off conditions at spatial infinity are weaker than those considered in [76]. In particular they include the data corresponding to maximal spatial slices of the Myers-Perry black hole.

4. The manifold Σ possesses an additional end (either asymptotically flat or asymptotically cylindrical of the form $\mathbb{R} \times S^3$).

First, we have the following definition

Definition 64. Let $(\Sigma, \boldsymbol{h}, K)$ be a GB initial data set. Then we define the following subclass of GB data by

$$\mathscr{A} \equiv \left\{(\Sigma, \boldsymbol{h}, K) : R_{\boldsymbol{h}}, \rho^{-2} \det \nabla \lambda' \in L^1(\mathcal{B}), rV \in L^1(\mathbb{R}^+), \quad (A^i_{\rho,z} - A^i_{z,\rho}), v \in L^2(\mathcal{B})\right\} , \tag{5.114}$$

where

$$\left\|A^i_{\rho,z} - A^i_{z,\rho}\right\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathcal{B}} (A^i_{\rho,z} - A^i_{z,\rho})(A^j_{\rho,z} - A^j_{z,\rho})\, \mathrm{d}\mu_0 \,. \tag{5.115}$$

Then we have the following result.

Theorem 65 (Positive mass theorem)**.** *Consider a GB initial data set* $(\Sigma, \boldsymbol{h}, K)$*. If* $R_{\boldsymbol{h}} \geq 0$ *and* $\mathcal{B} \in \Xi$ *where* Ξ *is defined in Definition 63, then*

$$0 \leq M_{ADM} \leq \infty \,. \tag{5.116}$$

Moreover, we have $M_{ADM} < \infty$ *if and only if the initial data set belongs to the set* $\mathscr{A}$*. Finally, provided* Σ *has a single asymptotic end,* $M_{ADM} = 0$ *if and only if* $\boldsymbol{h}$ *is Euclidean metric on* $\Sigma = \mathbb{R}^4$.

Proof. The ADM mass of any GB data is equation (5.50). Then if $R_{\boldsymbol{h}} \geq 0$ we have

$$\begin{aligned}\infty &\geq M_{ADM} \\ &\geq \frac{\pi}{4}\int_{\mathcal{B}}\left[-\frac{\det\nabla\lambda'}{2\rho^2}+6\,|\nabla v|^2\right]\mathrm{d}\mu\,+\frac{\pi}{2}\int_0^\infty r\,[V(x=1)+V(x=-1)]\,\mathrm{d}r.\end{aligned}$$

By the argument of the first part of this section, the right hand side is non-negative if $\mathcal{B}\in\Xi$. Thus we have (5.116). If $M_{ADM}<\infty$, then all terms in equation (5.50) are bounded and belong to $\mathscr{A}$ in Definition 64. Conversely, if the initial data set belongs to $\mathscr{A}$, then by equation (5.50) $M_{ADM}<\infty$. Now if we assume $\boldsymbol{h}$ is Euclidean metric on $\Sigma=\mathbb{R}^4$, clearly $M_{ADM}=0$. Conversely, If $M_{ADM}=0$, then by (5.50) we have

$$R_{\boldsymbol{h}} = A^i_{\rho,z} - A^i_{z,\rho} = 0. \tag{5.117}$$

Now we need to show $V=0$ and $\lambda'_{ij}=\sigma_{ij}=\frac{r^2}{2}\mathrm{diag}(1+x,1-x)$. We prove it by the technique we used to prove positivity of $\mathcal{M}$ in each B_s. Fix B_s and parametrization (5.100). By equation (5.106) we have

$$\begin{aligned}0 &= \frac{\pi}{4}\int_{B_s}\left[-\frac{\det\nabla\lambda'}{2\rho^2}+6\,|\nabla v|^2\right]\mathrm{d}\mu+\frac{\pi}{2}\int_0^\infty r\,[V(x=1)+V(x=-1)]\,\mathrm{d}r \\ &= \frac{\pi}{16}\int_{B_s}\Bigg(|\nabla V_1+\nabla V_2|^2+|\nabla V_1|^2+|\nabla V_2|^2 \\ &+ \frac{W^2}{2(1-W^2)}\left[|\nabla V_1-\nabla V_2|^2-\frac{6}{W}(\nabla V_1\cdot\nabla W+\nabla V_2\cdot\nabla W)\right] \\ &+ \frac{W^2}{r^2(1-W^2)}\left[4\partial_x V_2-4\partial_x V_1+\frac{2}{(1-x^2)}\right]+\frac{|\nabla W|^2}{2(1-W^2)}+\frac{2W^2|\nabla W|^2}{(1-W^2)^2}\Bigg)\,r^3\mathrm{d}x\mathrm{d}r \\ &\geq 0.\end{aligned} \tag{5.118}$$

Then by equations (5.111) and (5.109), we have

$$\nabla V_1 = \nabla V_2 = \nabla (W)^2 = 0\,. \tag{5.119}$$

Since $W = 0$ on $\mathcal{I}^{\pm}$, we have $W \equiv 0$. Also by equations (5.100) and (5.119), we have $\nabla v = 0$ and by Definition 46, v vanishes at infinity. This implies $v \equiv 0$. Note that in particular this implies there could not be another asymptotic end as $r \to 0$, since $v \propto -\log r$ in that case. Moreover, by definition of v in the parametrization (5.100) and $v = 0$, we have $V_1 = -V_2 =$constant. This means for each B_s we have

$$\lambda'_{kk} = \frac{r^2(1-x)}{2} e^{2V_1^s} \qquad \lambda'_{jj} = \frac{r^2(1+x)}{2} e^{-2V_1^s}\,, \qquad \lambda'_{12} = 0 \qquad v = 0\,. \tag{5.120}$$

where $k \neq j$ and $k, j = 1, 2$. If we consider the last annulus B_n which extends to spatial infinity, i.e. $\mathcal{B}_F$, then by the asymptotic conditions of λ'_{ij} in Definition 46 and $\nabla V_1^n = 0$, we obtain $V_1^n = V_2^n \equiv 0$. Moreover, if we consider the common boundary of B_{n-1} and B_n, by the continuity of V_1^s through boundary of B_s and (5.101), we have

$$4V_1^{n-1} = \pm \log\left(\frac{\alpha^k_{(n-1)1}\sigma_{kl}\alpha^{tl}_{(n-1)1}}{\alpha^k_{(n-1)2}\sigma_{kl}\alpha^{tl}_{(n-1)2}}\right) + \log\left(\frac{1+x}{1-x}\right)\,, \quad 0 = \alpha^k_{(n-1)1}\sigma_{kl}\alpha^{tl}_{(n-1)2} \tag{5.121}$$

where for fixed k, $\alpha^l_{(n-1)k} = (\alpha^1_{(n-1)k}, \alpha^2_{(n-1)k})$ and $\alpha^{tl}_{(n-1)k} = (\alpha^1_{(n-1)k}, \alpha^2_{(n-1)k})^t$. These conditions arise by expressing λ'_{ij} in B_{n-1} in the fixed basis $\xi_{(i)}$ using the transformation (5.101). Since $V_1^{n-1} =$ constant in the above equation and the right hand side is a function of x for some $\alpha^l_{(n-1)k}$, then we reach to a contradiction and this implies $n = 1$. This is equivalent to Σ having the trivial orbit space, i.e. $\mathcal{B}_\Sigma = \mathcal{B}_{\mathbb{R}^4}$. Moreover, we obtain $\lambda'_{ij} = \sigma_{ij} = \frac{r^2}{2}\text{diag}(1+x, 1-x)$ and by straightforward computation it implies

$$-\frac{1}{\rho^2} + \frac{1}{4}\text{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)^2\right] = 0. \tag{5.122}$$

Hence equation (5.49) yields to

$$\Delta_2 V = 0, \qquad V \text{ vanishes at axis and infinity}\,. \tag{5.123}$$

By maximum principle on open set $O_{R,\epsilon} = \{(\rho, z): \quad \epsilon < \rho < R\}$, we have $V \equiv 0$ after passing to the limits $R \to \infty$ and $\epsilon \to 0$. By equation (5.117) the one form $\beta^i = A^i_\rho \mathrm{d}\rho + A^i_z \mathrm{d}z$ is closed differential one-form and simply connectedness of Σ implies that there exists function ψ^i such that $\beta^i = \mathrm{d}\psi^i$, i.e. β^i is exact. Then the metric is

$$\boldsymbol{h} = \frac{\mathrm{d}\rho^2 + \mathrm{d}z^2}{2\sqrt{\rho^2 + z^2}} + \sigma_{ij}\mathrm{d}\left(\phi^i + \psi^i\right)\mathrm{d}\left(\phi^j + \psi^j\right) = \frac{\mathrm{d}\rho^2 + \mathrm{d}z^2}{2\sqrt{\rho^2 + z^2}} + \sigma_{ij}\mathrm{d}\gamma^i\mathrm{d}\gamma^j\,, \tag{5.124}$$

where γ^i are new rotational angles. Hence, $(\Sigma, \boldsymbol{h})$ is isometric to the Euclidean space $(\mathbb{R}^4, \delta_4)$. □

5.4 Summary

This chapter contains two main results of this thesis. The first result is the construction of a mass functional $\mathcal{M}$ for non-zero stress-energy tensor and study its properties. We showed that $\mathcal{M}$ is the lower bound of the ADM mass of any GB data and it evaluates to the ADM mass of the associated reduced data. Moreover, its critical points are stationary $U(1)^2$- invariant, vacuum black holes. However, our analysis for positivity only works for a particular admissible set of orbit spaces and there is an open problem for the general orbit space. The second result is a generalization of Brill's positive mass theorem for GB initial data. We established this result by following the argument of 3+1 dimensional case and we proved the rigidity by a simple contradiction argument. By this functional in hand, we will prove a local version of mass-angular momenta inequalities for any GB initial data in the next chapter.

Chapter 6

Mass-Angular Momenta Inequalities

This chapter is dedicated to a central theorem of this thesis. Roughly speaking, we prove a class of local geometric inequalities for generalized Brill initial data sets with positive energy density and vanishing energy flux in the direction of $U(1)^2$-isometries. In particular, we prove for any GB initial data $(\Sigma, \boldsymbol{h}, K)$ with mass m and angular momenta J_1 and J_2 with corresponding vacuum $t - \phi^i$ symmetric part $(\mathcal{B}, u)$, where $u = (v, \lambda', Y)$, if we have an extreme initial data set which is sufficiently close to the associated reduced data $(\mathcal{B}, u)$ with same angular momenta and same orbit, we obtain

$$m \geq f(J_1, J_2) \tag{6.1}$$

where f depends on the orbit space. The results of this chapter appeared in the following journal article: (AA.3) Classical and Quantum Gravity, 32 (16), 165,020.(2015)[4], and (AA.5) arXiv:1508.02337 which was submitted to Journal of Mathematical Physics [5] in July 2015.

6.1 Statement of the Problem and Main Result

Dain has proven the inequality $m \geq |J|$ for complete, maximal, asymptotically flat axisymmetric vacuum initial data to the 3+1 dimensional Einstein equation. Here m is the ADM mass associated with the data and J is the conserved angular momenta associated with the $U(1)$ isometry [51, 53]. A thorough account of this program with references to further generalizations can be found in the review [54]. A natural problem is to investigate whether these results can be generalized to higher dimensions. The area-angular momenta inequalities (see [54] for a survey) have been shown to admit such a generalization in all dimensions D for black holes with $U(1)^{D-3}$ rotational isometries [90]. Here we will focus on extending mass-angular momenta inequalities in $D = 5$, as this is the only other possibility that admits asymptotically flat spacetimes with these isometries.

Initial data sets with cylindrical ends arise within the context of stationary, extreme black holes. Extreme black holes with degenerate Killing horizons have vanishing surface gravity $\kappa = 0$, and in the limit as one approaches the horizon, Einstein's equations decouple in a precise manner into a set of equations defined only on the horizon [107]. This gives rise to the notion of a *near-horizon geometry*, which often thought of as an infinite 'throat' region in the spacetime (indeed the proper length to a spatial section of the horizon is infinite).

Extreme black holes have attracted a great deal of interest in recent years. Due to the decoupling described above, classifying near-horizon geometries is tractable and yields important information on the full space of extreme solutions (e.g. allowed geometries and topologies of spatial cross sections). Furthermore, extreme black hole geometries saturate a number of geometric inequalities which must hold for initial data sets and for marginally outer trapped surfaces in four dimensions [41, 51, 53] (see also [4, 90] for work on the latter problem in $D > 4$). Finally, extreme black holes have the simplest microscopic description within string theory, and so are an important testing ground for various calculations in quantum gravity, the most well-known of which is black hole entropy counting. Recently, due to the work of Aretakis and others [9, 10, 112, 122], extreme black holes have been shown to be unstable to a certain horizon instability. An alternative approach to studying the non-linear instability of the extreme Kerr-Newman family using perturbations of the initial data of extreme Reisnner-Nordstsrom also has recently appeared [134].

Moreover, the slice Σ of such a near-horizon geometry has the form of the geometry of a cylindrical end, where $N \cong H$ (see Figure 3.4a). In Theorem 60 of Chapter 5 we constructed a mass functional $\mathcal{M}$ as a lower bound for any GB initial data. The mass functional evaluates to the ADM mass for vacuum $t - \phi^i$ symmetric data and by Lemma 50 it is characterized by the triple $u = (v, \lambda', Y)$. We also showed that the critical points of this mass functional amongst this class of data are precisely the $\mathbb{R} \times U(1)^2$-invariant, vacuum solutions of the five-dimensional Einstein equation. The goal of this chapter is to show the extreme $\mathbb{R} \times U(1)^2$-invariant vacuum solutions have minimum mass among all initial data sets with same orbit spaces and angular momenta. To summarize we have the following remark.

Remark 66. Let $(\Sigma, \boldsymbol{h}, K)$ be a GB initial data set with positive energy density μ and vanishing energy flux in direction of $U(1)^2$-isometries, i.e. $\iota_{\xi_{(i)}} j = 0$. Then the associated reduced data $(\mathcal{B}, u)$ has less or equal mass with respect to the GB initial data.

The uniqueness results of Figueras and Lucietti [67] imply that, for fixed angular momenta J_1, J_2 and interval structure, there is *at most* one asymptotically flat extreme black hole. We will consider the case where an extreme solution exists. Then for a fixed structure we can write the mass of the extreme black hole as $m_{ext} = f(J_1, J_2)$

for some function f which depends on the interval structure. We have shown (under suitable conditions) that for small variations with fixed angular momenta about the extreme black hole initial data, the mass m_{ext} is a minimum; that is

$$m \geq f(J_1, J_2). \tag{6.2}$$

Note that m could be the mass of a dynamic black hole. This is shown by demonstrating that the extreme black holes are local minima of the mass functional. Of course, *within* the two explicitly known families of stationary black holes, the extreme Myers-Perry [125] and extreme doubly-spinning black ring [131] for fixed angular momenta, the extreme member of the family has the minimum mass, as is the case for Kerr. However, as we showed in the Section 4.3 the orbit space and slice topology of the extreme black ring and the non-extreme black ring are different. Therefore, this result does not provide any information about the relation of these two types of black ring. Moreover, for more general interval structure, there is no reason to expect this to occur, or indeed that a non-extreme family of solutions with a given interval structure contains an extreme limit.

By Remark 66 from now on we restrict attention to the mass functional, as it is a lower bound for the mass of our original initial data. We set $\varphi = (\bar{v}, \bar{\lambda}', \bar{Y})$ where $\bar{\lambda}'$ is a symmetric 2×2 matrix such that $\det \bar{\lambda}' = 0$. As will be explained in the following sections, φ will represent a perturbation about some fixed initial data u_0 defined in Definition 67 . This should consist of five free degrees of freedom, and the apparent restriction $\det \bar{\lambda}' = 0$ is simply a gauge choice. Let Ω be a (unbounded) domain and we introduce the following weighted spaces of C^1 functions with norm

$$\|f\|_{C^1_\beta(\Omega)} = \sup_{x \in \Omega} \{\sigma^{-\beta} |f| + \sigma^{-\beta+1} |\nabla f|\} \tag{6.3}$$

and $\beta < -1$ and $\sigma = \sqrt{r^2+1}$ and for a column vector and a matrix we define respectively

$$|\bar{Y}| \equiv \left(\bar{Y}^t \lambda_0'^{-1} \bar{Y}\right)^{1/2} \ , \quad |\bar{\lambda}'| \equiv \left(\mathrm{Tr}\left[\bar{\lambda}'^t \bar{\lambda}'\right]\right)^{1/2} . \tag{6.4}$$

Let $\rho_0 > 0$ be a constant and K_{ρ_0} be the cylinder $\rho \leq \rho_0$ in $\mathbb{R}^3$. We define the domain $\Omega_{\rho_0} = \mathbb{R}^3 \backslash K_{\rho_0}$. The perturbations $\bar{Y}$ and $\bar{\lambda}$ are assumed to vanish in K_{ρ_0}. This is consistent with the physical requirement that the perturbations keep fixed the angular momenta J_i and fixed orbit space. The Banach space B is defined by

$$\|\varphi\|_B = \|\bar{v}\|_{C^1_\beta(\mathbb{R}^3)} + \left\|\bar{\lambda}'\right\|_{C^1_\beta(\Omega_{\rho_0})} + \left\|\bar{Y}\right\|_{C^1_\beta(\Omega_{\rho_0})} . \tag{6.5}$$

Now we define the class of extreme data. Note that we will denote non-negative constants which depend on parameters of data such as mass and angular momenta by C, C_i, and C'.

Definition 67. The set of *extreme class* E is the collection of data arising from vacuum extreme, asymptotically flat, $\mathbb{R} \times U(1)^2$ invariant black holes which consist of triples $u_0 = (v_0, \lambda_0', Y_0)$ where v_0 is a scalar, $\lambda_0' = [\lambda_{ij}]$ is a positive definite 2×2 symmetric matrix, and Y_0 is a column vector with the following bounds for $\rho \leq r^2$:

1. $\frac{\nabla Y_0^t \lambda_0^{-1} \nabla Y_0}{X_0} \leq C r^{-4}$ and $e^{-2v_0} \frac{\nabla Y_0^t \lambda_0^{-1} \nabla Y_0}{X_0} \leq C r^{-2}$ in $\mathbb{R}^3$ where $\lambda_0 = e^{2v_0} \lambda_0'$,

2. $C_1 \rho I_{2\times 2} \leq \lambda_0 \leq C_2 \rho I_{2\times 2}$ and $C_3 \rho^{-1} I_{2\times 2} \leq \lambda_0^{-1} \leq C_4 \rho^{-1} I_{2\times 2}$ in Ω_{ρ_0},

3. $\rho^2 \leq X_0$ in $\mathbb{R}^3$ where $X_0 = \det \lambda_0$ and $X_0^2 \leq C' \rho^4$ in Ω_{ρ_0} where $\lim_{\rho_0 \to 0} C' = \infty$,

4. $|\nabla v_0|^2 \leq C r^{-4}$, $|\nabla \ln X_0|^2 \leq C \rho^{-2}$ in $\mathbb{R}^3$ and $\left|\nabla \lambda_0 \lambda_0^{-1}\right|^2 \leq C \rho^{-2}$ in Ω_{ρ_0} .

The choice of these bounds are consistent with the two known extreme black hole initial data sets, extreme Myers-Perry and extreme doubly spinning black ring. It is difficult to prove directly because the expressions in terms of the (ρ, z) coordinates are unwieldy. However, we have checked numerically that these bounds hold for a wide range of parameters for these two cases. It is possible that there exists an extreme data set which has slightly different bounds (i.e. this would correspond to another extreme black hole with different orbit space). In that case we expect that the arguments used in the proof of theorem 68 can be extended to take into account these different estimates.

Note that by what has been proved in [3], $\mathcal{M}$ evaluated on the extreme class is non-negative and given by (5.113) Now denote an extreme data set of this class by $u_0 = (v_0, \lambda'_0, Y_0) \in E$. Then we have the following result

Theorem 68 (Mass angular momenta inequality)**.**

(a) Let $\varphi = (\bar{v}, \bar{\lambda}', \bar{Y}) \in B$ where B is the Banach space defined above and $u_0 = (v_0, \lambda'_0, Y_0) \in E$ is extreme data with fixed $\mathcal{B}$. Then the functional $\mathcal{M} : B \to \mathbb{R}$ has a strict local minimum at u_0. That is, there exists $\epsilon > 0$ such that

$$\mathcal{M}(u_0 + \varphi) > \mathcal{M}(u_0) \tag{6.6}$$

for all $\varphi \in B$ with $\|\varphi\|_B < \epsilon$ and $\varphi \neq 0$.

(b) Let (Σ, h_{ab}, K_{ab}) be a GB initial data set with mass m and fixed angular momenta J_1 and J_2 and fixed orbit space $\mathcal{B}$ satisfies in Einstein constraint equations (3.7) *and* (3.8)*. Assume $\iota_{\xi_{(i)}} j = 0$ for $i = 1, 2$ and $\mu \geq 0$. Let $u = (v, \lambda', Y)$ describe the associated $t - \phi^i$ vacuum symmetric data as in Remark 66 and write $u = u_0 + \varphi$ where u_0 is extreme data with the same J_1, J_2 and orbit space $\mathcal{B}$. If φ is sufficiently small (as in (a)) then*

$$m \geq f(J_1, J_2) = \mathcal{M}(u_0) \tag{6.7}$$

for some f which depends on the orbit space $\mathcal{B}$. Moreover, $m = f(J_1, J_2)$ for data

(Σ, h, K) *in a neighbourhood if and only if the data are extreme data.*

For the sake of illustration we mention two special cases of the theorem.

1. In dimension 5, a possible horizon topology is $H \cong S^3$. Consider fixed angular momenta J_1 and J_2 and fixed orbit space $\tilde{\mathcal{B}}$ consisting of a finite timelike interval (the event horizon) and two semi-infinite spacelike intervals extending to asymptotic infinity (representing rotation axes). Then the orbit space of the slice will be $\mathcal{B} \cong \tilde{\mathcal{B}} \backslash \{\text{horizon interval}\}$ which corresponds to slice topology $\Sigma \cong \mathbb{R} \times S^3$ [3, 7]. By the uniqueness Theorem [67] extreme Myers-Perry solution is the unique solution with this orbit space and fixed angular momenta. Thus there exists $f(x, y) = 3\left[\frac{\pi}{32}(|x| + |y|)^2\right]^{1/3}$ such that mass of extreme Myers-Perry is equal to $f(J_1, J_2)$. Then by theorem 68 mass of any GB initial data sufficiently close (in the sense made precise above) with the same interval structure and angular momenta is greater than $f(J_1, J_2)$.

2. Now consider the horizon topology $H \cong S^2 \times S^1$. Consider fixed angular momenta J_1 and J_2 and fixed orbit space $\tilde{\mathcal{B}}$ consisting of a point, a finite spatial interval, and two semi-infinite intervals extending to asymptotic infinity. Then the orbit space of the slice will be $\mathcal{B} \cong \tilde{\mathcal{B}}$ which corresponds to slice topology $\Sigma \cong S^2 \times B^2 \# \mathbb{R}^4$ [3, 7]. By the uniqueness theorem [67] the extreme doubly spinning black ring is the unique solution with orbit space $\tilde{\mathcal{B}}$ and fixed angular momenta. Thus there exists $f(x, y) = 3\left[\frac{\pi}{4}|x|(|y| - |x|)\right]^{1/3}$ such that mass of extreme doubly spinning black rings is equal to $f(J_1, J_2)$. Then by Theorem 68 the mass of any GB initial data with the same orbit structure and fixed angular momenta is greater than $f(J_1, J_2)$.

Theorem 68 is a local inequality which should be satisfied for a wide class of (possibly dynamical) black holes with a fixed interval structure with a geometry sufficiently near an extreme black hole. One may expect to prove a global result showing that this inequality holds all data with fixed J_1, J_2 and $\mathcal{B}$. Such a global inequality has been proved in the electrovacuum in 3+1 dimensions [53]. A major obstacle to extending this result to the present case is showing positivity of $\mathcal{M}$ for arbitrary interval structures consistent with asymptotic flatness. However, for a class of interval structures (including Myers-Perry black hole initial data) one can show $\mathcal{M} \geq 0$ (Theorem 60 and in [3]). We are currently investigating whether a global inequality can be demonstrated in this particular setting. In this context, it is worth noting that $\mathbb{R} \times U(1)^2$-invariant vacuum spacetimes can be cast as harmonic maps from the orbit space to $SL(3,\mathbb{R})/SO(3)$ [93]. The target space metric is easily checked to be Einstein with negative curvature (it is not conformally flat). This can be contrasted with the four-dimensional case where the $\mathbb{R} \times U(1)$-invariant vacuum solutions are harmonic maps to $SL(2,\mathbb{R})/SO(2) \cong \mathbb{H}^2$ equipped with its standard Einstein metric.

6.2 Properties of The Second Variation of $\mathcal{M}$

In this section we will study the properties of second variation of mass functional $\mathcal{M}$. Let $\varphi \in B$ and consider the real-valued function

$$\mathcal{E}_\varphi(t) \equiv \mathcal{M}(u_0 + t\varphi) \tag{6.8}$$

and we assume

$$(v, \lambda', Y) \equiv (v(t), \lambda'(t), Y(t)) = (v_0 + t\bar{v}, \lambda'_0 + t\bar{\lambda}', Y_0 + t\bar{Y}) \tag{6.9}$$

where $\det\lambda' = \rho^2$. This choice for determinant of λ' requires that $\det\bar{\lambda} = 0$, that is

$$0 \;=\; \det\left(\lambda'_0 + t\bar{\lambda}'\right) - \rho^2 = t\rho^2 \mathrm{Tr}\left(\lambda_0'^{-1}\bar{\lambda}'\right) + t^2\det\bar{\lambda}' = t^2\det\bar{\lambda}'.$$

Moreover we have

$$\lambda \equiv \lambda(t) = e^{2v}\lambda'(t) \qquad X \equiv X(t) = \det\lambda' = e^{4v}\rho^2 \tag{6.10}$$

and $X_0 = X(0)$. Then by Section 5.2.2 the first variation is

$$\begin{aligned}\mathcal{E}'_\varphi(t) \;&=\; \frac{1}{8}\int_{\mathbb{R}^3}\left[\frac{e^{-6v}}{2\rho^4}\left(\nabla Y^t \mathrm{adj}(\bar{\lambda}')\nabla Y + 2\nabla Y^t \mathrm{adj}(\lambda')\nabla\bar{Y} - 6\bar{v}\nabla Y^t\mathrm{adj}(\lambda')\nabla Y\right)\right.\\ &- \left.\frac{1}{2\rho^2}\mathrm{Tr}\left(\mathrm{adj}(\nabla\bar{\lambda}')\nabla\lambda'\right) + 12\nabla v.\nabla\bar{v} + \right] d\mu_0.\end{aligned} \tag{6.11}$$

Now we compute the second variation by taking variation of terms in $\mathcal{E}'_\varphi(t)$. The first term is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(12\nabla v.\nabla\bar{v}\right) = 12(\nabla\bar{v})^2. \tag{6.12}$$

The second term is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{e^{-6v}}{2\rho^4}\left[\nabla Y^t\mathrm{adj}(\bar{\lambda}')\nabla Y + \nabla Y^t\mathrm{adj}(\lambda')\nabla\bar{Y} - 6\bar{v}\nabla Y^t\mathrm{adj}(\lambda')\nabla Y\right]\\ =\;& \frac{e^{-6v}}{\rho^4}\left[2\nabla Y^t\mathrm{adj}(\bar{\lambda}')\nabla\bar{Y} + \nabla\bar{Y}^t\mathrm{adj}(\lambda')\nabla\bar{Y}\right.\\ -\;& \left. 6\bar{v}\nabla Y^t\mathrm{adj}(\bar{\lambda}')\nabla Y - 12\bar{v}\nabla Y^t\mathrm{adj}(\lambda')\nabla\bar{Y} + 18\bar{v}^2\nabla Y^t\mathrm{adj}(\lambda')\nabla Y\right].\end{aligned} \tag{6.13}$$

The last term is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2\rho^2}\mathrm{Tr}\left(\mathrm{adj}(\nabla\bar{\lambda}')\nabla\lambda'\right) = \frac{\det\nabla\bar{\lambda}'}{\rho^2}. \tag{6.14}$$

Then the second variation is

$$\begin{aligned}\mathcal{E}''_{\varphi}(t) &= \frac{1}{8}\int_{\mathbb{R}^3}\Bigg(12\,|\nabla\bar{v}|^2 - \frac{\det\nabla\bar{\lambda}'}{\rho^2} + \frac{e^{-6v}}{\rho^4}\Bigg[2\nabla Y^t\mathrm{adj}(\bar{\lambda}')\nabla\bar{Y} + \nabla\bar{Y}^t\mathrm{adj}(\lambda')\nabla\bar{Y} \\ &- 6\bar{v}\nabla Y^t\mathrm{adj}(\bar{\lambda}')\nabla Y - 12\bar{v}\nabla Y^t\mathrm{adj}(\lambda')\nabla\bar{Y} + 18\bar{v}^2\nabla Y^t\mathrm{adj}(\lambda')\nabla Y\Bigg]\Bigg)\, d\mu_0. \quad (6.15)\end{aligned}$$

Note that the integrand of the functional $\mathcal{M}$ is singular at $\rho = 0$. However, we have defined the Banach space B only for functions $\bar{Y}$ and $\bar{\lambda}'$ with support in Ω_{ρ_0}. Therefore, the domain of integration of the terms in which $\nabla\bar{Y}$ and $\nabla\bar{\lambda}'$ appear are in fact Ω_{ρ_0} and hence the integrand is regular for those terms.

We now introduce axillary Hilbert spaces $\mathcal{H}_i$, which are defined in terms of the weighted Sobolev spaces

$$\|\bar{v}\|^2_{\mathcal{H}_1} = \int_{\mathbb{R}^3} |\nabla\bar{v}|^2\, r^{-2}\mathrm{d}\mu_0 + \int_{\mathbb{R}^3} |\bar{v}|^2\, r^{-4}\mathrm{d}\mu_0 \quad (6.16)$$

$$\left\|\bar{\lambda}'\right\|^2_{\mathcal{H}_2} = \int_{\Omega_{\rho_0}} \left|\nabla\bar{\lambda}'\right|^2 \rho^{-2}\mathrm{d}\mu_0 + \int_{\Omega_{\rho_0}} \left|\bar{\lambda}'\right|^2 \rho^{-4}\mathrm{d}\mu_0 \quad (6.17)$$

$$\left\|\bar{Y}\right\|^2_{\mathcal{H}_3} = \int_{\Omega_{\rho_0}} \left|\nabla\bar{Y}\right|^2 \rho^{-2}\mathrm{d}\mu_0 + \int_{\Omega_{\rho_0}} \left|\bar{Y}\right|^2 \rho^{-4}\mathrm{d}\mu_0 \quad (6.18)$$

and their corresponding inner products. The following auxiliary Hilbert space for ϕ with norm defined by

$$\|\varphi\|^2_{\mathcal{H}} = \|\bar{v}\|^2_{\mathcal{H}_1} + \left\|\bar{\lambda}\right\|^2_{\mathcal{H}_2} + \left\|\bar{Y}\right\|^2_{\mathcal{H}_3}, \quad (6.19)$$

with its corresponding inner product. We have $B \subset \mathcal{H}$ and the following Poincaré inequalities

Lemma 69. *Let $\varphi \in \mathcal{H}$ and $\delta \neq 0$ is a real number. Then*

(a) $|\delta|^{-2}\int_{\mathbb{R}^3} |\nabla\bar{v}|^2\, r^{-2\delta-1} d\mu_0 \geq \int_{\mathbb{R}^3} |\bar{v}|^2\, r^{-2\delta-3} d\mu_0.$

(b) $|\delta|^{-2}\int_{\Omega_{\rho_0}}\left|\nabla\bar{\lambda}'\right|^2\rho^{-2\delta}d\mu_0 \geq \int_{\Omega_{\rho_0}}\left|\bar{\lambda}'\right|^2\rho^{-2\delta-2}d\mu_0.$

(c) $2\,|\delta|^{-2}\int_{\Omega_{\rho_0}}\nabla\bar{Y}^t\nabla\bar{Y}\rho^{-3\delta}d\mu_0 \geq 3\int_{\Omega_{\rho_0}}\bar{Y}^t\bar{Y}\rho^{-3\delta-2}d\mu_0.$

Proof. (a) The proof of this part is similar to Theorem 1.3 of [15].

(b) The proof of part (b) is as following. We know for any symmetric matrices $\bar{\lambda}$ we have

$$\left|\bar{\lambda}'\right|^2 = \bar{\lambda}'^2_{11} + \bar{\lambda}'^2_{22} + 2\bar{\lambda}'^2_{12}. \tag{6.20}$$

Let Δ_3 be Laplace operator respect to δ_3 on $\mathbb{R}^3$.

$$\Delta_3(\ln\rho) = 0, \qquad \text{on } \Omega_{\rho_0}. \tag{6.21}$$

Then for each one of these functions $\bar{\lambda}'_{ij}$ we have

$$\begin{aligned} 0 &= -\int_{\Omega_{\rho_0}}\left(\rho^{-2\delta}\bar{\lambda}'^2_{ij}\right)\Delta_3(\ln\rho)\,\mathrm{d}\mu_0 \\ &= -\oint_{\partial\Omega_{\rho_0}}\left(\rho^{-2\delta}\bar{\lambda}'^2_{ij}\right)\nabla(\ln\rho)\cdot n\,\mathrm{d}S + \int_{\Omega_{\rho_0}}\nabla\left(\rho^{-2\delta}\bar{\lambda}'^2_{ij}\right)\nabla(\ln\rho)\,\mathrm{d}\mu_0 \\ &= \int_{\Omega_{\rho_0}}\nabla\left(\rho^{-2\delta}\bar{\lambda}'^2_{ij}\right)\nabla(\ln\rho)\,\mathrm{d}\mu_0 \\ &= \int_{\Omega_{\rho_0}}\left(-2\delta\rho^{-2\delta-2}\bar{\lambda}'^2_{ij}\nabla\rho + 2\bar{\lambda}'_{ij}\rho^{-2\delta-1}\nabla\bar{\lambda}'_{ij}\right)\mathrm{d}\mu_0 \end{aligned} \tag{6.22}$$

where n is unit normal vector on Ω_{ρ_0} and $\mathrm{d}S = \rho\,\mathrm{d}z\mathrm{d}\varphi$. The second equality follows by Stokes' theorem. The third equality follows from compact supportness of $\bar{\lambda}'_{ij}$ on Ω_{ρ_0}. Now if we expand the derivatives in the integrand and use the Hölder inequality we have

$$|\delta|^{-2}\int_{\Omega_{\rho_0}}\left|\nabla\bar{\lambda}'_{ij}\right|^2\rho^{-2\delta}\mathrm{d}\mu_0 \geq \int_{\Omega_{\rho_0}}\left|\bar{\lambda}'_{ij}\right|^2\rho^{-2\delta-2}\mathrm{d}\mu_0. \tag{6.23}$$

Then we have the following inequality

$$|\delta|^{-2}\int_{\Omega_{\rho_0}}\left|\nabla\bar{\lambda}'\right|^2\rho^{-2\delta}\mathrm{d}\mu_0 \geq \int_{\Omega_{\rho_0}}\left|\bar{\lambda}'\right|^2\rho^{-2\delta-2}\mathrm{d}\mu_0. \tag{6.24}$$

(c) The proof is similar to part (b). We have

$$\begin{aligned}
0 &= -\int_{\Omega_{\rho_0}}\left(\rho^{-3\delta}\bar{Y}^t\bar{Y}\right)\Delta_3(\ln\rho)\,\mathrm{d}\mu_0 \\
&= -\oint_{\partial\Omega_{\rho_0}}\left(\rho^{-3\delta}\bar{Y}^t\bar{Y}\right)\nabla(\ln\rho)\cdot n\,\mathrm{d}S + \int_{\Omega_{\rho_0}}\nabla\left(\rho^{-3\delta}\bar{Y}^t\bar{Y}\right)\nabla(\ln\rho)\,\mathrm{d}\mu_0 \\
&= \int_{\Omega_{\rho_0}}\nabla\left(\rho^{-3\delta}\bar{Y}^t\bar{Y}\right)\nabla(\ln\rho)\,\mathrm{d}\mu_0 \\
&= \int_{\Omega_{\rho_0}}\left[-3\delta\rho^{-3\delta-2}\bar{Y}^t\bar{Y}\nabla\rho + 2\rho^{-3\delta-1}\left(\nabla\bar{Y}^t\right)\bar{Y}\right]\mathrm{d}\mu_0
\end{aligned} \tag{6.25}$$

where n is unit normal vector on Ω_{ρ_0} and $\mathrm{d}S = \rho\,\mathrm{d}z\mathrm{d}\varphi$. Now if we expand the derivatives in the integrand and use the Cauchy-Schwarz inequality ($u^t w \leq (u^t u)^{1/2}(w^t w)^{1/2}$ for vectors u and w) we have

$$2\,|\delta|^{-2}\int_{\Omega_{\rho_0}}\nabla\bar{Y}^t\nabla\bar{Y}\rho^{-3\delta}\mathrm{d}\mu_0 \geq 3\int_{\Omega_{\rho_0}}\bar{Y}^t\bar{Y}\rho^{-3\delta-2}\mathrm{d}\mu_0. \tag{6.26}$$

□

Lemma 70. *If $\varphi \in B$ and $0 < t < 1$, then*

(a) The function $\mathcal{E}_\varphi(t)$ is C^2 in the t variable.

(b) For every $\epsilon > 0$ there exist $\eta(\epsilon)$ such that for $\|\varphi\|_B < \eta(\epsilon)$ we have

$$\left|\mathcal{E}''_\varphi(t) - \mathcal{E}''_\varphi(0)\right| \leq \epsilon\,\|\varphi\|^2_{\mathcal{H}}. \tag{6.27}$$

Proof. (a) To show $\mathcal{E}_\varphi(t)$ is C^2 it is enough to to show the third derivatives exist for all $0 < t < 1$. First we have

$$\mathcal{E}_\varphi'''(t) = \frac{1}{8}\int_{\mathbb{R}^3}\frac{e^{-6v}}{\rho^4}\Bigg(3\nabla\bar{Y}^t\mathrm{adj}(\bar{\lambda}')\nabla Y - 42\bar{v}\nabla\bar{Y}^t\mathrm{adj}(\bar{\lambda}')\nabla\bar{Y} - 12\bar{v}\nabla\bar{Y}^t\mathrm{adj}(\lambda')\nabla\bar{Y}$$
$$+ 108\bar{v}^2\nabla Y^t\mathrm{adj}(\bar{\lambda}')\nabla Y + 144\bar{v}^2\nabla Y^t\mathrm{adj}(\lambda')\nabla\bar{Y} - 216\bar{v}^3\nabla Y^t\mathrm{adj}(\lambda')\nabla Y\Bigg)\,\mathrm{d}\mu_0.$$

Note $\nabla\bar{Y}$ and $\bar{\lambda}'$ have compact support in Ω_{ρ_0}. Therefore, by parts (1) and (2) of Definition 67 and relation $\mathrm{adj}\bar{\lambda}' = -\frac{1}{\rho^2}\mathrm{adj}\lambda_0'\bar{\lambda}'\mathrm{adj}\lambda_0'$ and $\det\bar{\lambda}' = 0$ it is straightforward but tedious to show that all terms are bounded by the norm B. The only term with different domain is

$$-\frac{216\bar{v}^3}{X_0}\nabla Y_0^t\lambda_0^{-1}\nabla Y_0 \tag{6.28}$$

which is bounded on $\mathbb{R}^3$ by part 1 of Definition 67. Then $\mathcal{E}_\varphi(t)$ is C^2.

(b) First by integrand of $\mathcal{E}_\varphi''(t)$ we have

$$\mathcal{E}_\varphi''(t) - \mathcal{E}_\varphi''(0) = \int_{\mathbb{R}^3}\left(A_1|_0^t + ... + A_6|_0^t\right)\,\mathrm{d}\mu_0 \tag{6.29}$$

where

$$A_1 = 18\frac{e^{-6v}\bar{v}^2}{\rho^4}\nabla Y_0^t\mathrm{adj}\lambda_0'\nabla Y_0 \qquad A_2 = \frac{e^{-6v}}{\rho^4}(18\bar{v}^2t - 6\bar{v})\nabla Y_0^t\mathrm{adj}\bar{\lambda}'\nabla Y_0$$
$$A_3 = \frac{e^{-6v}}{\rho^4}(36\bar{v}^2t - 12\bar{v})\nabla\bar{Y}^t\mathrm{adj}\lambda_0'\nabla Y_0$$
$$A_4 = \frac{e^{-6v}}{\rho^4}(18\bar{v}^2t^2 - 12\bar{v}t + 1)\nabla\bar{Y}^t\mathrm{adj}\lambda_0'\nabla\bar{Y}$$
$$A_5 = \frac{e^{-6v}}{\rho^4}(36\bar{v}^2t^2 - 24\bar{v}t^2 + 2)\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla Y_0$$
$$A_6 = \frac{e^{-6v}}{\rho^4}(18\bar{v}^2t^3 - 18\bar{v}t^2 + 3t)\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla\bar{Y}.$$

All of these terms satisfy (6.101) by similar steps as in [51]. First we have

$$|\bar{v}| \leq \sigma^{\beta} \left\|\bar{v}\right\|_{C^1_\beta(\mathbb{R}^3)} \leq \left\|\bar{v}\right\|_{C^1_\beta(\mathbb{R}^3)} \leq \left\|\varphi\right\|_B \leq \eta\,. \tag{6.30}$$

By part (1) of Definition 67 we have

$$\begin{aligned}\int_{\mathbb{R}^3} A_1|_0^t \mathrm{d}\Sigma_0 &= \int_{\mathbb{R}^3} 18\bar{v}^2 \frac{\nabla Y_0^t \lambda_0^{-1} \nabla Y_0}{X_0}\left[e^{-6t\bar{v}} - 1\right] \mathrm{d}\mu_0 \\ &\leq 18C\left[e^{6\eta} - 1\right] \int_{\mathbb{R}^3} \bar{v}^2 r^{-4} \mathrm{d}\mu_0 \\ &\leq 18C\left[e^{6\eta} - 1\right] \left\|\bar{v}\right\|^2_{\mathcal{H}_1} \leq 18C\left[e^{6\eta} - 1\right] \left\|\varphi\right\|^2_{\mathcal{H}}\,. \end{aligned}\tag{6.31}$$

Now we write $A_2 = B_1 + B_2$ where

$$B_1 = \frac{e^{-6v}}{\rho^4} 18\bar{v}^2 t \nabla Y_0^t \mathrm{adj}\bar{\lambda}' \nabla Y_0 \tag{6.32}$$

$$B_2 = -6\frac{e^{-6v_0}}{\rho^4} \bar{v} \nabla Y_0^t \mathrm{adj}\bar{\lambda}' \nabla Y_0 \left[e^{-6t\bar{v}} - 1\right]. \tag{6.33}$$

We will prove it for B_1 and B_2 is similar. We have

$$\begin{aligned}\int_{\mathbb{R}^3} B_1 \mathrm{d}\Sigma &= -\int_{\mathbb{R}^3} \frac{e^{-6v}}{\rho^6} 18\bar{v}^2 t \nabla Y_0^t \mathrm{adj}\lambda_0' \bar{\lambda}' \mathrm{adj}\lambda_0' \nabla Y_0 \,\mathrm{d}\mu_0 \\ &\leq 18 e^{6\eta}\eta \int_{\mathbb{R}^3} \frac{e^{-6v_0}}{\rho^6} \left|\bar{\lambda}'\right| \bar{v} \nabla Y_0^t (\mathrm{adj}\lambda_0')^2 \nabla Y_0 \,\mathrm{d}\mu_0 \\ &\leq 18C\eta e^{6\eta} \int_{\mathbb{R}^3} \left|\bar{\lambda}'\right| \bar{v} \rho^{-1} r^{-6} \,\mathrm{d}\mu_0 \\ &\leq 18C\eta e^{6\eta} \left\|\bar{v}\right\|_{\mathcal{H}_1} \left\|\bar{\lambda}'\right\|_{\mathcal{H}_2} \leq 18C\eta e^{6\eta} \left\|\varphi\right\|^2_{\mathcal{H}}\,. \end{aligned}\tag{6.34}$$

We used the identity $\mathrm{adj}\bar{\lambda}' = -\frac{1}{\rho^2}\mathrm{adj}\lambda_0'\bar{\lambda}'\mathrm{adj}\lambda_0'$ in the first line. The first inequality arise from (6.30) and the matrix inequality $u^t A u \leq |A|\, u^t u$ for any 2×2 matrix A. The second inequality is a consequence of parts (1) and (2) of Definition 67. Finally, the third inequality follows from Hölder's inequality, $\rho < r^2$, and part (1) of Lemma 10.

The term A_3 can be expressed as $A_3 = B_3 + B_4$ where

$$B_3 = 36\frac{e^{-6v}}{\rho^4}\bar{v}^2 t\nabla\bar{Y}^t \mathrm{adj}\lambda_0' \nabla Y_0 \tag{6.35}$$

$$B_4 = -12\frac{e^{-6v_0}}{\rho^4}\bar{v}\nabla\bar{Y}^t \mathrm{adj}\lambda_0' \nabla Y_0 \left[e^{-6t\bar{v}} - 1\right]. \tag{6.36}$$

Then the bound of B_3 is

$$\begin{aligned}
\int_{\mathbb{R}^3} B_3 \mathrm{d}\Sigma &\leq 36\eta e^{6\eta}\int_{\Omega_{\rho_0}} \frac{1}{X_0}\bar{v}\nabla\bar{Y}^t\lambda_0'^{-1}\nabla Y_0 \, \mathrm{d}\mu_0 \\
&\leq 36\eta e^{6\eta}\int_{\Omega_{\rho_0}} \frac{\bar{v}}{X_0}\left(\nabla\bar{Y}^t\lambda_0'^{-1}\nabla\bar{Y}\right)^{1/2}\left(\nabla Y_0^t\lambda_0^{-1}\nabla Y_0\right)^{1/2} \mathrm{d}\mu_0 \\
&\leq 36C\eta e^{6\eta}\left(\int_{\Omega_{\rho_0}} \rho^{-2}\nabla\bar{Y}^t\lambda_0'^{-1}\nabla\bar{Y}\, \mathrm{d}\mu_0\right)^{1/2}\left(\int_{\Omega_{\rho_0}} \bar{v}^2 r^{-4}\, \mathrm{d}\mu_0\right)^{1/2} \\
&\leq 36C\eta e^{6\eta}\left\|\bar{v}\right\|_{\mathcal{H}_1}\left\|\bar{Y}\right\|_{\mathcal{H}_3} \leq 36C\eta e^{6\eta}\left\|\varphi\right\|_{\mathcal{H}}^2 .
\end{aligned} \tag{6.37}$$

The first inequality uses (6.30). We know λ_0^{-1} is a positive definite symmetric matrix. Thus it has a square root matrix $\lambda_0^{-1/2}$, that is $\lambda_0^{-1} = \left(\lambda_0^{-1/2}\right)^2$. Then the integrand in the first line is equal to $X_0^{-1}\bar{v}u^t w$ where $u^t = \nabla\bar{Y}^t\lambda_0^{-1/2}$ and $w = \lambda_0^{-1/2}\nabla Y_0$. Since $u^t w \leq (u^t u)^{1/2}(w^t w)^{1/2}$ we have the second inequality. The third inequality follows from Hölder's inequality and parts (1) and (2) of Definition 1. The fourth inequality is by the definition of norm. B_4 is exactly similar to B_3. We have $A_4 = B_5 + B_6$ where

$$B_5 = \frac{e^{-6v}}{\rho^4}(18\bar{v}^2 t^2 - 12\bar{v}t)\nabla\bar{Y}^t \mathrm{adj}\lambda_0' \nabla\bar{Y}, \tag{6.38}$$

$$B_6 = \frac{e^{-6v_0}}{\rho^4}\nabla\bar{Y}^t \mathrm{adj}\lambda_0' \nabla\bar{Y}\left[e^{-6t\bar{v}} - 1\right] \tag{6.39}$$

Then the bound of B_5 is

$$\begin{aligned}\int_{\mathbb{R}^3} B_5 \,\mathrm{d}\mu_0 &\leq (18\eta^2+12\eta)e^{6\eta}\int_{\Omega_{\rho_0}} X_0^{-1}\nabla\bar{Y}^t\lambda_0'^{-1}\nabla\bar{Y}\,\mathrm{d}\mu_0 \\ &\leq (18\eta^2+12\eta)e^{6\eta}\int_{\Omega_{\rho_0}} \rho^{-2}\nabla\bar{Y}^t\lambda_0'^{-1}\nabla\bar{Y}\,\mathrm{d}\mu_0 \\ &\leq (18\eta^2+12\eta)e^{6\eta}\left\|\varphi\right\|_{\mathcal{H}}^2. \end{aligned} \tag{6.40}$$

The first equality follows by (6.30) and definition of X. The second equality is by part (3) of Definition 67. Proof of B_6 is exactly similar to B_5. Next, we have $A_5 = B_7 + B_8$ where

$$B_7 = \frac{e^{-6v}}{\rho^4}(36\bar{v}^2t^2 - 24\bar{v}t^2)\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla Y_0, \tag{6.41}$$

$$B_8 = 2\frac{e^{-6v_0}}{\rho^4}\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla Y_0\left[e^{-6t\bar{v}}-1\right] \tag{6.42}$$

Then the bound of B_7 is

$$\begin{aligned}&\int_{\mathbb{R}^3} B_7\,\mathrm{d}\mu_0 \\ \leq\; & (36\eta^2+24\eta)e^{6\eta}\int_{\Omega_{\rho_0}}\left|\bar{\lambda}'\right| X_0^{-1}e^{2v_0}\nabla\bar{Y}^t(\lambda_0^{-1})^2\nabla Y_0\,\mathrm{d}\mu_0 \\ \leq\; & C(36\eta^2+24\eta)e^{6\eta}\int_{\Omega_{\rho_0}}\left|\bar{\lambda}'\right|\rho^{-1}X_0^{-1}\nabla\bar{Y}^t\lambda_0^{-1}\nabla Y_0\,\mathrm{d}\mu_0 \\ \leq\; & C(36\eta^2+24\eta)e^{6\eta}\int_{\Omega_{\rho_0}}\left|\bar{\lambda}'\right|\rho^{-1}X_0^{-1}\left(\nabla\bar{Y}^t\lambda_0^{-1}\nabla\bar{Y}\right)^{1/2}\left(\nabla Y_0^t\lambda_0^{-1}\nabla Y_0\right)^{1/2}\mathrm{d}\mu_0 \\ \leq\; & C(36\eta^2+24\eta)e^{6\eta}\left(\int_{\Omega_{\rho_0}}\rho^{-2}\nabla\bar{Y}^t\lambda_0'^{-1}\nabla\bar{Y}\,\mathrm{d}\mu_0\right)^{1/2}\left(\int_{\Omega_{\rho_0}}\left|\bar{\lambda}'\right|^2\rho^{-4}\,\mathrm{d}\mu_0\right)^{1/2} \\ \leq\; & C(36\eta^2+24\eta)e^{6\eta}\left\|\bar{Y}\right\|_{\mathcal{H}_3}\left\|\bar{\lambda}'\right\|_{\mathcal{H}_2} \leq C(36\eta^2+24\eta)e^{6\eta}\left\|\varphi\right\|_{\mathcal{H}}^2. \end{aligned} \tag{6.43}$$

The first inequality uses the identity $\mathrm{adj}\bar{\lambda}' = -\frac{1}{\rho^2}\mathrm{adj}\lambda_0'\bar{\lambda}'\mathrm{adj}\lambda_0'$, inequality (6.30), and the matrix inequality $u^tAu \leq |A|\,u^tu$ for any 2×2 matrix A. The second inequality is a consequence of parts (1) and (2) of Definition 67. The third inequality follows from a similar argument for B_3 by Hölder and Cauchy-Schwartz inequalities for vectors and the fact that $e^{2v_0} \leq 1$ on Ω_{ρ_0}. The fourth inequality is a consequence of parts (1) and (3) of Definition 67. The fifth inequality follows from part (1) of Definition 67 and $\rho \leq r^2$. Similarly the argument holds for B_8. Finally we have,

$$\begin{aligned}
\int_{\mathbb{R}^3} A_6 \mathrm{d}\mu_0 &\leq C(18\eta^2 + 18\eta + 3)e^{6\eta} \int_{\Omega_{\rho_0}} \left|\bar{\lambda}'\right| \frac{e^{-6v_0}}{\rho^6} \nabla \bar{Y}^t (\mathrm{adj}\lambda_0')^2 \nabla \bar{Y} \mathrm{d}\mu_0 \\
&\leq C(18\eta^2 + 18\eta + 3)e^{6\eta} \int_{\Omega_{\rho_0}} \left|\bar{\lambda}'\right| \rho^{-3} \nabla \bar{Y}^t \lambda_0'^{-1} \nabla \bar{Y} \mathrm{d}\mu_0 \\
&\leq C(18\eta^2 + 18\eta + 3)e^{6\eta}\eta \int_{\Omega_{\rho_0}} \rho^{-2} \nabla \bar{Y}^t \lambda_0'^{-1} \nabla \bar{Y} \mathrm{d}\mu_0 \\
&\leq C(18\eta^2 + 18\eta + 3)e^{6\eta}\eta \left\|\bar{Y}\right\|^2_{\mathcal{H}_3} \\
&\leq C(24\eta + 18\eta^2 + 3)e^{8\eta}\eta \left\|\varphi\right\|^2_{\mathcal{H}}. \qquad (6.44)
\end{aligned}$$

The first inequality uses the identity $\mathrm{adj}\bar{\lambda}' = -\frac{1}{\rho^2}\mathrm{adj}\lambda_0'\bar{\lambda}'\mathrm{adj}\lambda_0'$, inequality (6.30), and the matrix inequality $u^t A u \leq |A|\, u^t u$ for any 2×2 matrix A. The second inequality is a consequence of part (2) of Definition 67. The third inequality arises from the inequality $\rho^{-1}\left|\bar{\lambda}'\right| \leq \sigma^\beta \left\|\bar{\lambda}'\right\|_{C^1_\beta(\Omega_{\rho_0})} \leq \left\|\bar{\lambda}'\right\|_{C^1_\beta(\Omega_{\rho_0})} \leq \|\varphi\|_B \leq \eta$.Therefore, $\mathcal{E}''_\varphi(t)$ is uniformly continuous.

□

It is important to show that the second variation is non-negative. We will achieve this by using Carter identity in Appendix B. We consider our parametrization of data with relations (6.9) and (6.10) we have

$$\dot{X} = 4\bar{v}X, \qquad \dot{\lambda} = \bar{\lambda} = 2\bar{v}\lambda + \lambda\lambda'^{-1}\bar{\lambda}', \qquad \dot{Y} = \bar{Y}\,. \qquad (6.45)$$

Thus

$$\lambda^{-1}\dot{\lambda} = 2\bar{v}I + \lambda'^{-1}\bar{\lambda}' \tag{6.46}$$

since $\mathrm{Tr}\left(\lambda'^{-1}\bar{\lambda}'\right) = \delta \det\lambda'/\det\lambda' = 0$ we have $\mathrm{Tr}\left(\lambda^{-1}\bar{\lambda}\right) = 4\bar{v}$. Then by Carter identity (6.47), the following identity holds for *arbitrary* v, $\bar{v}$, Y, $\bar{Y}$, λ, and $\bar{\lambda}$.

$$\begin{aligned} & X^{-1}\bar{Y}^t\lambda^{-1}\bar{Y}G_X - 4\bar{v}\dot{G}_X - 8\bar{v}G_Y^t\bar{Y} - 2\bar{Y}^t\lambda^{-1}\bar{\lambda}G_Y \\ + \; & X^{-1}\bar{Y}^t\lambda^{-1}G_\lambda^t\bar{Y} - \mathrm{Tr}\left(\lambda^{-1}\bar{\lambda}\dot{G}_\lambda^t\right) - 2\dot{G}_Y^t\bar{Y} \\ = \; & F(t) - \Delta\left(X^{-1}\bar{Y}^t\lambda^{-1}\bar{Y} + 16\bar{v}^2\right) \end{aligned} \tag{6.47}$$

where G_X, G_Y, and G_λ are defined in (B.6), (5.87),(5.64)-(5.65) and

$$\begin{aligned} F(t) &= \left(4\nabla\bar{v} + X^{-1}\bar{Y}^t\lambda^{-1}\nabla Y\right)^2 + X\left(\dot{U}_2^t\lambda\dot{U}_2 + \nabla U_1^t\lambda\nabla U_1\right) \\ &+ \mathrm{Tr}\left[\left(\nabla\left(\bar{\lambda}\lambda^{-1}\right) + X^{-1}\nabla Y\bar{Y}^t\lambda^{-1}\right)^2\right] \\ \dot{G}_X &= 4\Delta_3\bar{v} + \frac{e^{-6v}}{\rho^4}\left\{2\nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla Y + \nabla Y^t\mathrm{adj}\bar{\lambda}'\nabla Y - 6\bar{v}\nabla Y^t\mathrm{adj}\lambda'\nabla Y\right\} \\ \dot{G}_\lambda &= 2\Delta_3\bar{v}I + \mathrm{div}\delta\left(\lambda'^{-1}\nabla\lambda'\right) \\ &+ \frac{e^{-6v}}{\rho^4}\left\{2\mathrm{adj}\lambda'\nabla Y\cdot\nabla\bar{Y}^t + \mathrm{adj}\bar{\lambda}'\nabla Y\cdot\nabla Y^t - 6\bar{v}\mathrm{adj}\lambda'\nabla Y\cdot\nabla Y^t\right\} \\ \dot{G}_Y &= \mathrm{div}\left(\frac{e^{-6v}}{\rho^4}\left\{\mathrm{adj}\lambda'\nabla\bar{Y} + \mathrm{adj}\bar{\lambda}'\nabla Y - 6\bar{v}\mathrm{adj}\lambda'\nabla Y\right\}\right). \end{aligned} \tag{6.48}$$

The identity (6.47) can be verified directly. Now we show relation of the identity (6.47) and second variation $\mathcal{E}''_\varphi(t)$. First we have

$$\begin{aligned} I &\equiv -4\bar{v}\dot{G}_X - \mathrm{Tr}\left(\lambda^{-1}\bar{\lambda}\dot{G}_\lambda\right) - 2\dot{G}_Y^t\bar{Y} \\ &= -4\bar{v}\dot{G}_X - \mathrm{Tr}\left(\left[2\bar{v}I + \lambda'^{-1}\bar{\lambda}'\right]\dot{G}_\lambda\right) - 2\dot{G}_Y^t\bar{Y} \\ &= -6\bar{v}\dot{G}_X - \mathrm{Tr}\left(\lambda'^{-1}\bar{\lambda}'\dot{G}_\lambda\right) - 2\dot{G}_Y^t\bar{Y} \end{aligned}$$

$$
\begin{aligned}
&= -6\bar{v}\left[4\Delta_3\bar{v} + \frac{e^{-6v}}{\rho^4}\left\{2\nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla Y + \nabla Y^t\mathrm{adj}\bar{\lambda}'\nabla Y - 6\bar{v}\nabla Y^t\mathrm{adj}\lambda'\nabla Y\right\}\right] \\
&- \mathrm{Tr}\Bigg(2\lambda'^{-1}\bar{\lambda}'\Delta_3\bar{v}I + \lambda'^{-1}\bar{\lambda}'\mathrm{div}\delta\left(\lambda'^{-1}\nabla\lambda'\right) \\
&+ \frac{e^{-6v}}{\rho^4}\lambda'^{-1}\bar{\lambda}'\left\{2\mathrm{adj}\lambda'\nabla Y\cdot\nabla\bar{Y}^t + \mathrm{adj}\bar{\lambda}'\nabla Y\cdot\nabla Y^t - 6\bar{v}\mathrm{adj}\lambda'\nabla Y\cdot\nabla Y^t\right\}\Bigg) \\
&- 2\mathrm{div}\left(\frac{e^{-6v}}{\rho^4}\left\{\mathrm{adj}\lambda'\nabla\bar{Y} + \mathrm{adj}\bar{\lambda}'\nabla Y - 6\bar{v}\mathrm{adj}\lambda'\nabla Y\right\}\right)^t\bar{Y}. \qquad (6.49)
\end{aligned}
$$

We integrate by parts for the first term of the first line and all terms in the last line and we use the following identities

$$
\det\bar{\lambda}' = 0, \qquad \mathrm{adj}\bar{\lambda}'\bar{\lambda}' = \det\bar{\lambda}'I = 0, \qquad -\lambda'^{-1}\bar{\lambda}'\lambda'^{-1} = \frac{\mathrm{adj}\bar{\lambda}'}{\rho^2}. \qquad (6.50)
$$

Then we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} I\,\mathrm{d}\mu_0 \\
&= \int_{\mathbb{R}^3}\left(-4\bar{v}\dot{G}_X - \mathrm{Tr}\left(\lambda^{-1}\bar{\lambda}\dot{G}_\lambda\right) - 2\dot{G}_Y^t\bar{Y}\right)\mathrm{d}\mu_0 \qquad (6.51) \\
&= \int_{\mathbb{R}^3}\Bigg\{24\left|\nabla\bar{v}\right|^2 + \frac{e^{-6v}}{\rho^4}\Bigg[4\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla Y + 2\nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla\bar{Y} - 12\bar{v}\nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla Y \\
&- 24\bar{v}\nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla Y + 36\bar{v}^2\nabla Y^t\mathrm{adj}\lambda'\nabla Y\Bigg] + \underbrace{\mathrm{Tr}\left[\nabla\left(\lambda'^{-1}\bar{\lambda}'\right)\delta\left(\lambda'^{-1}\nabla\lambda'\right)\right]}_{\mathscr{Z}}\Bigg\}.\mathrm{d}\mu_0
\end{aligned}
$$

In equation (6.51), all terms are in second variation except $\mathscr{Z}$. First we integrate $\mathscr{Z}$ over $\mathbb{R}^3$ and use Stokes' theorem and the property of compact supportness of $\bar{\lambda}'$ on

$\Omega_{\rho_0} \subset \mathbb{R}^3$ and we obtain

$$\begin{aligned}
\mathscr{Z} &= \mathrm{Tr}\left\{\lambda'^{-1}\bar{\lambda}'\delta\mathrm{div}\left(\lambda'^{-1}\nabla\lambda'\right)\right\} \\
&= \delta\mathrm{Tr}\left\{\lambda'^{-1}\bar{\lambda}'\mathrm{div}\left(\lambda'^{-1}\nabla\lambda'\right)\right\} - \mathrm{Tr}\left\{\delta\left(\frac{\mathrm{adj}\lambda'\bar{\lambda}'}{\rho^2}\right)\mathrm{div}\left(\lambda'^{-1}\nabla\lambda'\right)\right\} \\
&= \delta\mathrm{Tr}\left\{\lambda'^{-1}\bar{\lambda}'\mathrm{div}\left(\lambda'^{-1}\nabla\lambda'\right)\right\} - \mathrm{Tr}\left\{\frac{\mathrm{adj}\bar{\lambda}'\bar{\lambda}'}{\rho^2}\mathrm{div}\left(\lambda'^{-1}\nabla\lambda'\right)\right\} \\
&= \delta\left[\mathrm{Trdiv}\left\{\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} - \mathrm{Tr}\left\{\nabla\left(\lambda'^{-1}\bar{\lambda}'\right)\left(\lambda'^{-1}\nabla\lambda'\right)\right\}\right] \\
&= \delta\left[\mathrm{Trdiv}\left\{\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} + \mathrm{Tr}\left\{\lambda'^{-1}\nabla\lambda'\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} - \mathrm{Tr}\left\{\lambda'^{-1}\nabla\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\}\right] \\
&= \delta\left[\mathrm{Trdiv}\left\{\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} - \frac{1}{\rho^2}\mathrm{Tr}\left\{\lambda'^{-1}\nabla\lambda'\mathrm{adj}\bar{\lambda}'\nabla\lambda'\right\} - \frac{1}{\rho^2}\mathrm{Tr}\left\{\lambda'^{-1}\nabla\bar{\lambda}'\mathrm{adj}\lambda'\nabla\lambda'\right\}\right] \\
&= \delta\left[\mathrm{Trdiv}\left\{\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} - \frac{1}{\rho^2}\mathrm{Tr}\left\{\lambda'^{-1}\left(\nabla\lambda'\mathrm{adj}\bar{\lambda}' + \nabla\bar{\lambda}'\mathrm{adj}\lambda'\right)\nabla\lambda'\right\}\right] \\
&= \delta\left[\mathrm{Trdiv}\left\{\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} - \frac{1}{\rho^2}\mathrm{Tr}\left\{\lambda'^{-1}\delta\left(\nabla\lambda'\mathrm{adj}\lambda\right)\nabla\lambda'\right\}\right] \\
&= \delta\left[\mathrm{Trdiv}\left\{\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} - \frac{1}{\rho^4}\mathrm{Tr}\left\{\delta\left(\nabla\lambda'\mathrm{adj}\lambda\right)\nabla\lambda'\mathrm{adj}\lambda'\right\}\right] \\
&= \delta\left[\mathrm{Trdiv}\left\{\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} - \frac{1}{2\rho^4}\delta\mathrm{Tr}\left\{\left(\nabla\lambda'\mathrm{adj}\lambda\right)^2\right\}\right] \\
&= \delta\left[\mathrm{Trdiv}\left\{\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} - \frac{1}{2}\delta\mathrm{Tr}\left\{\left(\lambda'^{-1}\nabla\lambda'\right)^2\right\}\right] \\
&= \mathrm{divTr}\delta\left\{\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right\} - \frac{1}{2}\delta^2\mathrm{Tr}\left\{\left(\lambda'^{-1}\nabla\lambda'\right)^2\right\} \\
&= \mathrm{divTr}\delta\left\{-\frac{\mathrm{adj}\bar{\lambda}'}{\rho^2}\nabla\lambda'\right\} - 2\delta^2\left(-\frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}\left\{\left(\lambda'^{-1}\nabla\lambda'\right)^2\right\}\right) \\
&= -\mathrm{divTr}\left\{\frac{\mathrm{adj}\bar{\lambda}'}{\rho^2}\nabla\bar{\lambda}'\right\} + 2\frac{\det\nabla\bar{\lambda}'}{\rho^2}. \qquad (6.52)
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_{\mathbb{R}^3}\mathscr{Z}\,\mathrm{d}\mu_0 &= \int_{\mathbb{R}^3}\left\{\mathrm{divTr}\left\{\frac{\mathrm{adj}\bar{\lambda}'}{\rho^2}\nabla\bar{\lambda}'\right\} - 2\frac{\det\nabla\bar{\lambda}'}{\rho^2}\right\}\mathrm{d}\mu_0 \\
&= \int_{\partial\Omega_{\rho_0}} -\mathrm{Tr}\left\{\frac{\mathrm{adj}\bar{\lambda}'}{\rho^2}\nabla\bar{\lambda}'\right\}\mathrm{d}S - \int_{\mathbb{R}^3} 2\frac{\det\nabla\bar{\lambda}'}{\rho^2}\mathrm{d}\mu_0 \\
&= -\int_{\mathbb{R}^3} 2\frac{\det\nabla\bar{\lambda}'}{\rho^2}\mathrm{d}\mu_0. \qquad (6.53)
\end{aligned}$$

The last equality follows from compact support of $\bar{\lambda}'$ on Ω_{ρ_0}. For $\varphi \in B$ we have the following remarkable relation

$$\int_{\mathbb{R}^3} \left(-4\bar{v}\dot{G}_X - \mathrm{Tr}\left(\lambda^{-1}\bar{\lambda}\dot{G}_\lambda\right) - 2\dot{G}_Y^t\bar{Y}\right) \mathrm{d}\mu_0 = 16\mathcal{E}''_\varphi(t). \tag{6.54}$$

Thus if $t = 0$, the field equations $G_X(0) = G_\lambda(0) = G_Y(0) = 0$ hold and we have from (6.47) (the integral over the divergence term vanishes by our boundary conditions)

$$\mathcal{E}''_\varphi(0) = \frac{1}{16}\int_{\mathbb{R}^3} F(0)\mathrm{d}\Sigma \geq 0 \tag{6.55}$$

where

$$\begin{aligned} F(0) &= \left(4\nabla\bar{v} + X_0^{-1}\bar{Y}^t\lambda_0^{-1}\nabla Y_0\right)^2 + X_0\left(\dot{U}_2^t\lambda\dot{U}_2 + \nabla U_1^t\lambda\nabla U_1\right) \\ &+ \mathrm{Tr}\left[\left(\nabla\left(\bar{\lambda}\lambda_0^{-1}\right) + X_0^{-1}\nabla Y_0\bar{Y}^t\lambda_0^{-1}\right)^2\right] \geq X_0\nabla U_1^t\lambda_0\nabla U_1. \end{aligned} \tag{6.56}$$

Now if $\mathcal{E}''_\varphi(0) = 0$, then $F(0) = 0$. Therefore, by inequality (6.56) we have $\nabla U_1 = 0$. Also, since $\varphi \in B$, we have $\bar{Y} = 0$. Therefore, by $F = 0$ and $\bar{Y} = 0$ we have $\bar{v} = 0$ and $\bar{\lambda} = 0$. This is, however, not sufficient to prove that the extreme data set u_0 is a *strict* local minimum. For this one needs a stronger positivity result on $\mathcal{E}''_\varphi(0)$ (see for example, Theorem 40.B of [145]) which we now demonstrate. Now, we show the following observation

Remark 71. Assume $\varphi \in B$, then we have the following identity

$$\int_{\Omega_{\rho_0}} 2\rho^{-2}\mathrm{Tr}\left(\lambda'^{-1}\nabla\lambda'\mathrm{adj}\bar{\lambda}'\nabla\bar{\lambda}'\right) \mathrm{d}\mu_0 = -\int_{\Omega_{\rho_0}} \left(\mathrm{Tr}\left[\bar{\lambda}'\nabla\left(\lambda'^{-1}\right)\right]\right)^2 \mathrm{d}\mu_0. \tag{6.57}$$

Proof of Remark 71. To prove this we start by $\mathscr{L}$. We have the following relations

$$\nabla\left(\lambda'^{-1}\bar{\lambda}'\right) = \nabla\left(\frac{\mathrm{adj}\lambda'}{\rho^2}\right)\bar{\lambda}' + \lambda'^{-1}\nabla\bar{\lambda}', \quad \delta\left(\lambda'^{-1}\nabla\lambda'\right) = -\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda' + \lambda'^{-1}\nabla\bar{\lambda}' \tag{6.58}$$

Then

$$\begin{aligned}
\mathscr{L} &= \mathrm{Tr}\left[\nabla\left(\lambda'^{-1}\bar{\lambda}'\right)\delta\left(\lambda'^{-1}\nabla\lambda'\right)\right] \\
&= \mathrm{Tr}\left(\left[\delta\left(\lambda'^{-1}\nabla\lambda'\right)\right]^2\right) + \mathrm{Tr}\left\{\left[\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda' + \nabla\left(\frac{\mathrm{adj}\lambda'}{\rho^2}\right)\bar{\lambda}'\right]\delta\left(\lambda'^{-1}\nabla\lambda'\right)\right\} \\
&= \mathrm{Tr}\left(\left[\delta\left(\lambda'^{-1}\nabla\lambda'\right)\right]^2\right) - \mathrm{Tr}\left(\left[\frac{\mathrm{adj}\bar{\lambda}'}{\rho^2}\nabla\lambda'\right]^2\right).
\end{aligned} \tag{6.59}$$

Also we have

$$\begin{aligned}
-\frac{\det\nabla\bar{\lambda}'}{\rho^2} &= \delta^2\left(-\frac{\det\nabla\lambda'}{2\rho^2}\right) \\
&= \frac{1}{2}\delta\left(\mathrm{Tr}\left[\left(\lambda'^{-1}\nabla\lambda'\right)\delta\left(\lambda'^{-1}\nabla\lambda'\right)\right]\right) \\
&= \frac{1}{2}\mathrm{Tr}\left(\left[\delta\left(\lambda'^{-1}\nabla\lambda'\right)\right]^2\right) + \frac{1}{2}\mathrm{Tr}\left(\left(\lambda'^{-1}\nabla\lambda'\right)\delta^2\left(\lambda'^{-1}\nabla\lambda'\right)\right) \\
&= \frac{1}{2}\mathrm{Tr}\left(\left[\delta\left(\lambda'^{-1}\nabla\lambda'\right)\right]^2\right) + \frac{1}{\rho^2}\mathrm{Tr}\left(\lambda'^{-1}\nabla\lambda'\mathrm{adj}\bar{\lambda}'\nabla\bar{\lambda}'\right).
\end{aligned} \tag{6.60}$$

Then by equations (6.53), (6.60), and (6.59) we have the following identity

$$\begin{aligned}
\int_{\Omega_{\rho_0}}\frac{2}{\rho^2}\mathrm{Tr}\left(\lambda'^{-1}\nabla\lambda'\mathrm{adj}\bar{\lambda}'\nabla\bar{\lambda}'\right)\,\mathrm{d}\mu_0 &= -\int_{\Omega_{\rho_0}}\mathrm{Tr}\left(\left[\frac{\mathrm{adj}\bar{\lambda}'}{\rho^2}\nabla\lambda'\right]^2\right)\mathrm{d}\mu_0 \\
&= -\int_{\Omega_{\rho_0}}\left(\mathrm{Tr}\left[-\lambda'^{-1}\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\right]\right)^2\,\mathrm{d}\mu_0 \\
&= -\int_{\Omega_{\rho_0}}\left[\mathrm{Tr}\left[-\bar{\lambda}'\lambda'^{-1}\nabla\lambda'\lambda'^{-1}\right]\right]^2\,\mathrm{d}\mu_0 \\
&= -\int_{\Omega_{\rho_0}}\left[\mathrm{Tr}\left[\bar{\lambda}'\nabla\left(\lambda'^{-1}\right)\right]\right]^2\,\mathrm{d}\mu_0.
\end{aligned} \tag{6.61}$$

The second equality follows from $\det\bar{\lambda}' = 0$ and the identity $\mathrm{Tr}(A^2) = (\mathrm{Tr}A)^2 - 2\det A$

for 2×2 matrix A. The third equality follows from property of trace. The final equality arises from the derivative of inverse matrix. $\square$

Then we prove a coercive condition required for u_0 to be a local minimum.

Lemma 72. *There exists $\mu > 0$ such that for all $\varphi \in B$ we have*

$$\mathcal{E}''_{\varphi}(0) \geq \mu \left\|\varphi\right\|^2_{\mathcal{H}}. \tag{6.62}$$

Proof. Let $\varphi \in B$. Note that $\mathcal{E}''_{\varphi}(0)$ defines a bilinear form

$$a(\varphi, \varphi) \equiv \mathcal{E}''_{\varphi}(0) = \int_{\mathbb{R}^3} F(0)\mathrm{d}\mu_0 \tag{6.63}$$

as function of φ. The inequality (6.62) is equivalent to the following variational problem

$$\mu = \inf_{\varphi \in B, \|\varphi\|^2_{\mathcal{H}}=1} a(\varphi, \varphi). \tag{6.64}$$

Since $a(\varphi, \varphi)$ is positive definite, we have $\mu \geq 0$. Now we prove $\mu > 0$. Assume $\mu = 0$, then there exists a sequence $\{\varphi_n\}$ such that

$$\|\varphi_n\|^2_{\mathcal{H}} = 1 \qquad \text{for all } n \tag{6.65}$$

and

$$\lim_{n\to\infty} a(\varphi_n, \varphi_n) = 0. \tag{6.66}$$

Then we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} a(\varphi_n,\varphi_n) = \lim_{n\to\infty}\int_{\mathbb{R}^3} F(0)\mathrm{d}\mu_0 \\ &\geq \lim_{n\to\infty}\int_{\Omega_{\rho_0}} X_0\nabla U_1^t\lambda_0\nabla U_1\mathrm{d}\mu_0 \geq C_1\lim_{n\to\infty}\int_{\Omega_{\rho_0}}\rho^3\nabla U_1^t\nabla U_1\mathrm{d}\mu_0 \\ &\geq \frac{3C_1}{2}\lim_{n\to\infty}\int_{\Omega_{\rho_0}}\rho U_1^tU_1\mathrm{d}\mu_0 \geq \frac{3C_1C_3}{2C'}\lim_{n\to\infty}\int_{\Omega_{\rho_0}}\rho^{-4}\bar{Y}_n^t\lambda_0^{-1}\bar{Y}_n\mathrm{d}\mu_0. \end{aligned} \tag{6.67}$$

In the first inequality we used (6.56). The second follows from part 2 and 3 of Definition 67. The third inequality follows from Lemma 69-(c). The fourth inequality follows from part 3 of Definition 67. Therefore,

$$\lim_{n\to\infty}\int_{\Omega_{\rho_0}}\rho^{-4}\bar{Y}_n^t\lambda_0^{-1}\bar{Y}_n\mathrm{d}\mu_0 = 0. \tag{6.68}$$

Next we establish some inequalities. First rewrite $F(0)$ in the form

$$\begin{aligned} F(0) &= \left(4\nabla\bar{v}_n + \frac{\bar{Y}_n^t\lambda_0^{-1}\nabla Y_0}{X_0}\right)^2 + 2A_1^t\lambda_0A_1 + 2A_2^t\lambda_0A_2 \\ &+ \mathrm{Tr}\left[\left(\nabla\left(\bar{\lambda}_n\lambda_0^{-1}\right) + \frac{\nabla Y_0\bar{Y}_n^t\lambda_0^{-1}}{X_0}\right)^2\right] \end{aligned} \tag{6.69}$$

where

$$A_1 = \frac{\sqrt{X_0}}{2}\left[B_I + B_{II} + B_{III}\right], \qquad A_2 = \frac{\sqrt{X_0}}{2}\left[B_{II} - B_I\right] \tag{6.70}$$

and

$$\begin{aligned} B_I &= \frac{\lambda_0^{-1}\nabla\lambda_0\lambda_0^{-1}\bar{Y}_n}{X} + \frac{\nabla X_0}{X_0^2}\lambda_0^{-1}\bar{Y}_n, \qquad B_{II} = \frac{\lambda_0^{-1}\bar{\lambda}_n\lambda_0^{-1}\nabla Y_0}{X_0} + \frac{\bar{X}}{X_0^2}\lambda_0^{-1}\nabla Y_0 \\ B_{III} &= 2\frac{\lambda_0^{-1}\nabla\bar{Y}}{X_0}\,. \end{aligned} \tag{6.71}$$

Then we have the following inequality

$$a(\varphi_n, \varphi_n) + \int_{\Omega_{\rho_0}} 2B_I^t \lambda_0 B_I \,\mathrm{d}\mu_0 \geq \int_{\Omega_{\rho_0}} \frac{1}{4} B_{III}^t \lambda_0 B_{III} \,\mathrm{d}\mu_0. \tag{6.72}$$

where B_I can be written as

$$B_I = \frac{\lambda_0^{-1}}{\sqrt{X_0}} \left(\nabla \lambda_0 \lambda_0^{-1} + \frac{\nabla X_0}{X_0} I_{2\times 2} \right) \bar{Y}_n = \frac{\lambda_0^{-1}}{\sqrt{X_0}} M \bar{Y}_n \,. \tag{6.73}$$

By part 4 of Definition 67 we have

$$|M|^2 \leq 2 \left| \nabla \lambda_0 \lambda_0^{-1} \right|^2 + 2 \left| \nabla \ln X_0 \right|^2 \leq C \rho^{-2} \tag{6.74}$$

and we have

$$\int_{\Omega_{\rho_0}} 2B_I^t \lambda_0 B_I \,\mathrm{d}\mu_0 \;\; \leq \;\; \int_{\Omega_{\rho_0}} \frac{2}{X_0} |M|^2 \bar{Y}_n^t \lambda_0^{-1} \bar{Y}_n \,\mathrm{d}\mu_0 \leq 2C \int_{\Omega_{\rho_0}} \rho^{-4} \bar{Y}_n^t \lambda_0^{-1} \bar{Y}_n \,\mathrm{d}\mu_0 \tag{6.75}$$

Then by inequities (6.72) and (6.75) we have

$$a(\varphi_n, \varphi_n) + 4 \int_{\Omega_{\rho_0}} \rho^{-4} \bar{Y}_n^t \lambda_0^{-1} \bar{Y}_n t d\mu_0 \geq \frac{1}{4} \int_{\Omega_{\rho_0}} \rho^{-2} \nabla \bar{Y}_n^t \lambda_0^{-1} \nabla \bar{Y}_n \,\mathrm{d}\mu_0. \tag{6.76}$$

Now we take the limit of above equation and use the equation (6.68) to find

$$\lim_{n\to\infty} \int_{\Omega_{\rho_0}} \rho^{-2} \nabla \bar{Y}_n^t \lambda_0^{-1} \nabla \bar{Y}_n \mathrm{d}\mu_0 = 0. \tag{6.77}$$

Thus

$$\lim_{n\to\infty} \left\| \bar{Y}_n \right\|_{\mathcal{H}_3} = 0. \tag{6.78}$$

We consider the first term in $F(0)$. Then

$$a(\varphi_n, \varphi_n) + \int_{\Omega_{\rho_0}} \left(\frac{\bar{Y}_n^t \lambda_0^{-1} \nabla Y_0}{X_0} \right)^2 \mathrm{d}\mu_0 \geq 8 \int_{\Omega_{\rho_0}} (\nabla \bar{v}_n)^2 \, \mathrm{d}\mu_0. \tag{6.79}$$

Since λ_0 is a positive definite symmetric metric it has unique square root $\lambda_0^{1/2}$. Now if we set $u = \lambda_0^{-1/2} \bar{Y}$ and $w = \lambda_0^{-1/2} \nabla Y_0$ we have

$$\begin{aligned} \int_{\Omega_{\rho_0}} \left(\frac{\bar{Y}_n^t \lambda_0^{-1} \nabla Y_0}{X_0} \right)^2 \mathrm{d}\mu_0 &\leq \int_{\Omega_{\rho_0}} \left(\frac{\bar{Y}_n^t \lambda_0^{-1} \bar{Y}_n}{X_0} \right) \left(\frac{\nabla Y_0^t \lambda_0^{-1} \nabla Y_0}{X_0} \right) \mathrm{d}\mu_0 \\ &\leq C \int_{\Omega_{\rho_0}} \rho^{-2} \bar{Y}_n^t \lambda_0^{-1} \bar{Y}_n r^{-4} \, \mathrm{d}\mu_0 \\ &\leq C \int_{\Omega_{\rho_0}} \rho^{-4} \bar{Y}_n^t \lambda_0^{-1} \bar{Y}_n \, \mathrm{d}\mu_0. \end{aligned} \tag{6.80}$$

The first inequality follows from the Cauchy-Schwarz inequality $u^t w \leq (u^t u)^{1/2} (w^t w)^{1/2}$. The second inequality is by part 1 and 3 of Definition 67. The third inequality is by the fact $\rho \leq r^2$. Then by inequality (6.80) and (6.79) we have

$$a(\varphi_n, \varphi_n) + C \int_{\Omega_{\rho_0}} \rho^{-4} \bar{Y}_n^t \lambda_0^{-1} \bar{Y}_n \, \mathrm{d}\mu_0 \geq 8 \int_{\mathbb{R}^3} |\nabla \bar{v}_n|^2 \, \mathrm{d}\mu_0 \geq 8 \int_{\mathbb{R}^3} |\nabla \bar{v}_n|^2 \, r^{-2} \, \mathrm{d}\mu_0 \tag{6.81}$$

The last inequality is by part (1) of Lemma 10. Now if we take the limit of inequality (6.81) and by the fact the right hand side is zero by (6.68), we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla \bar{v}_n|^2 \, r^{-2} \, \mathrm{d}\mu_0 = 0. \tag{6.82}$$

Thus by Lemma 69-(a) we have

$$\lim_{n \to \infty} \|\bar{v}_n\|_{\mathcal{H}_1} = 0. \tag{6.83}$$

Now we consider the last term of $F(0)$. We have the following inequality

$$a(\varphi_n,\varphi_n)+\int_{\Omega_{\rho_0}}\mathrm{Tr}\left[\left(\frac{\nabla Y_0\bar{Y}_n^t\lambda_0^{-1}}{X_0}\right)^2\right]\mathrm{d}\mu_0\geq\frac{1}{2}\int_{\Omega_{\rho_0}}\mathrm{Tr}\left[\left(\nabla\left(\bar{\lambda}_n\lambda_0^{-1}\right)\right)^2\right]\mathrm{d}\mu_0. \quad (6.84)$$

The integrand of the second term on the left hand side has vanishing determinant since $\det\left(\nabla Y_0\bar{Y}_n^t\lambda_0^{-1}\right)=\frac{\det\left(\nabla Y_0\bar{Y}_n^t\right)}{\rho^2}=0$. Thus by the matrix identity $\mathrm{Tr}(A^2)=(\mathrm{Tr}A)^2-2\det A$ and inequality (6.80) we have

$$\int_{\Omega_{\rho_0}}\mathrm{Tr}\left[\left(\frac{\nabla Y_0\bar{Y}_n^t\lambda_0^{-1}}{X_0}\right)^2\right]\mathrm{d}\mu_0 \;\leq\; C\int_{\Omega_{\rho_0}}\rho^{-4}\bar{Y}_n^t\lambda_0^{-1}\bar{Y}_n\,\mathrm{d}\mu_0. \quad (6.85)$$

By relation (6.46) the right hand side expands

$$\begin{aligned}\mathrm{Tr}\left[\left(\nabla\left(\bar{\lambda}_n\lambda_0^{-1}\right)\right)^2\right] &= 2\left[\nabla\bar{v}_n\right]^2+\mathrm{Tr}\left[\left(\nabla\bar{\lambda}_n'\lambda_0'^{-1}\right)^2\right]+\mathrm{Tr}\left[\left(\bar{\lambda}_n'\nabla\left(\lambda_0'^{-1}\right)\right)^2\right]\\ &+ 2\mathrm{Tr}\left[\nabla\bar{\lambda}_n'\left(\frac{\mathrm{adj}\bar{\lambda}_n'}{\rho^2}\right)\nabla\lambda_0'\lambda_0'^{-1}\right]. \quad (6.86)\end{aligned}$$

By integration we have

$$\begin{aligned}\int_{\Omega_{\rho_0}}\mathrm{Tr}\left[\left(\nabla\left(\bar{\lambda}_n\lambda_0^{-1}\right)\right)^2\right]\mathrm{d}\mu_0 &= \int_{\mathbb{R}^3}2\left|\nabla\bar{v}_n\right|^2\,\mathrm{d}\mu_0+\int_{\Omega_{\rho_0}}\mathrm{Tr}\left[\left(\nabla\bar{\lambda}_n'\lambda_0'^{-1}\right)^2\right]\mathrm{d}\mu_0\\ &\geq \int_{\mathbb{R}^3}2\left|\nabla\bar{v}_n\right|^2\,\mathrm{d}\mu_0+C_1^2\int_{\Omega_{\rho_0}}\left|\nabla\bar{\lambda}_n'\right|^2\rho^{-2}\,\mathrm{d}\mu_0. \quad (6.87)\end{aligned}$$

The equality is by identity (6.57). The inequality is by part 2 of Definition 67. Then by substitution of inequalities (6.87) and (6.85) in (6.84) we have

$$a(\varphi_n,\varphi_n)+C\int_{\Omega_{\rho_0}}\rho^{-4}\bar{Y}_n^t\lambda_0^{-1}\bar{Y}_n\,\mathrm{d}\mu_0\geq\int_{\mathbb{R}^3}\left|\nabla\bar{v}_n\right|^2\,\mathrm{d}\mu_0+\frac{C_1^2}{2}\int_{\Omega_{\rho_0}}\left|\nabla\bar{\lambda}_n'\right|^2\rho^{-2}\,\mathrm{d}\mu_0. \quad (6.88)$$

Now if we take the limit from both sides of this inequality and use equation (6.82) we have

$$\lim_{n\to\infty}\int_{\Omega_{\rho_0}}\left|\nabla\bar{\lambda}_n'\right|^2\rho^{-2}\,\mathrm{d}\mu_0 = 0. \tag{6.89}$$

Thus by Lemma 69-(b) we have

$$\lim_{n\to\infty}\left\|\bar{\lambda}_n'\right\|_{\mathcal{H}_2} = 0. \tag{6.90}$$

Thus (6.78), (6.83) and (6.90) contradict the fact that $\|\varphi_n\|_{\mathcal{H}} = 1$. Hence $\mu > 0$. □

6.3 Proof of Theorem 68

Proof. The proof is straightforward and similar to the proof of Theorem 1 of [51] and Chapter 40-B of [145].

(a) We have proved in Lemma 70 that $\mathcal{E}_\varphi''(t)$ is C^2 with respect to t. Also by Taylor's theorem we have

$$\mathcal{M}(u_0+\varphi) - \mathcal{M}(u_0) = \mathcal{E}_\varphi(1) - \mathcal{E}_\varphi(0) = \frac{\mathcal{E}_\varphi''(t)}{2} \quad \text{where } 0 < t < 1. \tag{6.91}$$

To prove this is positive we will show $\mathcal{E}_\varphi''(t) \geq 0$ and $\mathcal{E}_\varphi''(t) = 0$ implies $\varphi = 0$. By Lemma 70-(b) $\mathcal{E}_\varphi''(t)$ is uniformly continuous, that is for every $\epsilon > 0$ there exists $\eta(\epsilon)$ such that the following inequality holds

$$\left|\mathcal{E}_\varphi''(t) - \mathcal{E}_\varphi''(0)\right| \leq \epsilon\,\|\varphi\|_{\mathcal{H}}^2 \tag{6.92}$$

for every $\|\varphi\|_{\mathcal{H}} < \eta(\epsilon)$. From this inequality we have

$$\mathcal{E}_\varphi''(0) - \epsilon\,\|\varphi\|_{\mathcal{H}}^2 \leq \mathcal{E}_\varphi''(t). \tag{6.93}$$

By Lemma 72 we have

$$(\mu - \epsilon)\,\|\varphi\|^2_{\mathcal{H}} \leq \mathcal{E}''_{\varphi}(t) \tag{6.94}$$

Choosing $\eta(\epsilon)$ such that $0 < \epsilon < \mu$ the desired result follows.

(b) Let $u = u_0 + \varphi$ be the associated $t - \phi^i$ symmetric part of the initial data set (Σ, h, K) as in the statement of Theorem 68. It was proved that the ADM mass of this data set satisfies [3]

$$m \geq \mathcal{M}(u) = \mathcal{M}(u_0 + \varphi). \tag{6.95}$$

Then by part (a) we have

$$\mathcal{M}(u_0 + \varphi) > \mathcal{M}(u_0), \tag{6.96}$$

for nonzero φ. Since u_0 is an extreme data set, there exists a function f such that $\mathcal{M}(u_0) = f(J_1, J_2)$. Thus

$$m \geq f(J_1, J_2). \tag{6.97}$$

Clearly, by definition if the initial data set is extreme, then $m = f(J_1, J_2)$. Conversely, suppose the mass m of given initial data (Σ, h, K) satisfies $m = f(J_1, J_2) = \mathcal{M}(u_0)$. Hence $\varphi = 0$ and $u = u_0$ and from (6.95) and Remark 66 the initial data set is extreme. Thus $m = f(J_1, J_2)$ if and only if the data set belongs to the extreme class.

□

6.4 Summary

This chapter provided the local proof of the mass-angular momenta inequalities for $U(1)^2$-invariant black holes. The idea of the proof was as following. Consider the mass functional $\mathcal{M}$ with GB initial data $(\Sigma, \boldsymbol{h}, K)$. Then

1. Perturb the associated reduced data with $\varphi = (\bar{v}, \bar{Y}, \bar{\lambda}') \in B$

$$u_t = u_0 + t\varphi, \qquad \mathcal{E}_\varphi(t) = \mathcal{M}(u_t)\,. \tag{6.98}$$

2. Necessary conditions for the local minimum are

$$\mathcal{E}'_\varphi(0) := \frac{d}{dt}\mathcal{E}_\varphi(t)\Big|_{t=0} = 0, \qquad \mathcal{E}''_\varphi(0) := \frac{d^2}{dt^2}\mathcal{E}_\varphi(t)\Big|_{t=0} \geq 0. \tag{6.99}$$

3. Sufficient conditions for the local minimum are

   (i) For all $\varphi \in B$, there exists a fixed μ such that

$$\mathcal{E}''_\varphi(0) \geq \mu \left\|\varphi\right\|^2_{\mathcal{H}} \tag{6.100}$$

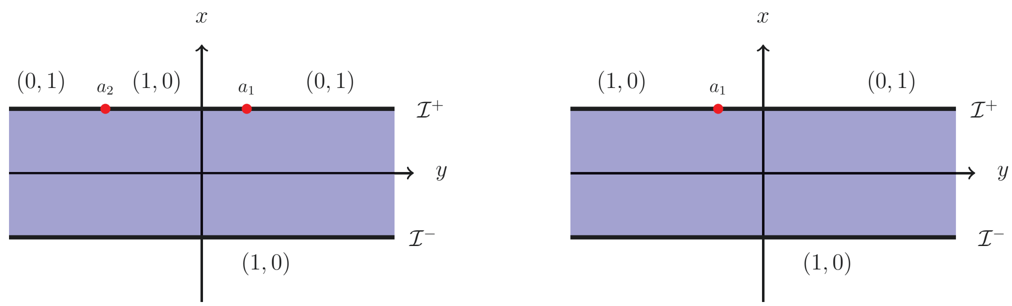
   (ii) Uniform continuity: For every $\epsilon > 0$ there exist $\eta(\epsilon)$ such that for $\left\|\varphi\right\|_B < \eta(\epsilon)$ we have

$$\left|\mathcal{E}''_\varphi(t) - \mathcal{E}''_\varphi(0)\right| \leq \epsilon \left\|\varphi\right\|^2_{\mathcal{H}}\,. \tag{6.101}$$

By Theorem 68 we showed that for a GB data sufficiently close to the extreme Myers-Perry data we have

$$M^3 \geq \frac{27\pi}{32}\left(|J_1| + |J_2|\right)^2\,. \tag{6.102}$$

The case of black ring is more complicated. Our analysis is for fixed orbit spaces and it does not compare extreme black ring and non-extreme black ring (see Figure 6.1).



(a) Orbit space of a non-extreme black ring slice on the infinite strip.

(b) Orbit space of an extreme black ring slice on infinite strip.

Figure 6.1: The doubling of extreme slice yield to non-extreme slice with double orbit space. Here $y = \log r$.

In particular, we show that for orbit space of extreme black ring we have the following inequality for initial data sufficiently close to the extreme black ring

$$M^3 \geq \frac{27\pi}{4} |J_1| \left(|J_2| - |J_1|\right) . \tag{6.103}$$

This local proof suggests the existence of global inequalities for some particular orbit spaces.

Chapter 7

Deformations of Extreme Myers-Perry Black Hole

In this chapter, we demonstrate the existence of a one-parameter family of initial data for the vacuum Einstein equations in five dimensions representing small deformations of the extreme Myers-Perry black hole. This initial data set has $t - \phi^i$ symmetry and preserves the angular momenta, asymptotic geometries, and cross section area of the extreme Myers-Perry event horizon but has strictly greater energy. The results of this chapter appeared in the journal article: (AA.3) General Relativity and Gravitation, 47 (2), 129(2015)[7].

7.1 Motivation and Main Result

An important problem in mathematical general relativity is construction of an initial data set with desired properties. This involves identifying the freely specifiable 'degrees of freedom' and then determining whether a corresponding solution of Einstein constraint equations (3.7) and (3.8) exists and is unique. A useful approach to achieve this is the conformal method ([34, 35, 55, 116]). In the special case of data with constant mean curvature the problem reduces to solving a conformally invariant system of equations for the conformal factor and a vector field which generates the extrinsic curvature. For spatially closed and asymptotically Euclidean initial data sets, one can prove existence using the conformal method [34] (for spacetime dimension $D \geq 4$). Subsequently, Maxwell [116] constructed asymptotically Euclidean initial data with apparent horizon boundary conditions (in particular, he treated the case with multiple disconnected apparent horizons). This case is naturally relevant to black holes.

While the above results are powerful in their generality, one can also consider the existence of initial data with very specific geometrical properties. This chapter will be concerned with initial data sets which have one Euclidean end and one cylindrical end. Roughly, the latter means that an initial data set (Σ, h) has an asymptotic end which is diffeomorphic to $\mathbb{R} \times N$ where N is a compact manifold. A systematic analysis of initial data on manifolds with cylindrical ends was performed in [36, 37]. In particular, existence of solutions of Lichnerowiscz's equation is proved using the powerful barrier method [100]. The purpose of our analysis, however, is to prove the existence of a rather specific class of perturbed initial data with additional properties (e.g. preserving angular momenta of the background data). We will make clear at the end of this section how our results are related to [36, 37].

The simplest example is a initial data set with cylindrical end geometry is the extreme $M = \sqrt{J}$ Kerr black hole [55]. These authors, using the conformal method alluded to above, proved that there exists a one-parameter family of axisymmetric initial data of the vacuum Einstein equations which preserve the asymptotic behaviour, angular momenta, and area of the cylindrical end (this area corresponds to the area of the spatial sections of the horizon of the Kerr black hole). In particular, as a consequence of the geometric inequalities, one can show the energy of any member of this family must be strictly greater than that of the extreme Kerr initial data. Note that the solutions satisfy weak regularity conditions (i.e. they belong to a certain Sobolev space) and in particular are not generically smooth, let alone analytic. This last distinction could be important when considering the evolution of this initial data. The extreme Kerr black hole is known to be the unique (analytic) vacuum, stationary, rotating asymptotically flat spacetime containing a single degenerate horizon [8, 41, 53, 67, 92]. Hence the evolution of the initial data sets discussed above could settle down to non-analytic asymptotically flat (possibly stationary) extreme black holes. Of course, we cannot address this issue without understanding the evolution.

It is natural to investigate the possibility of extending the result of [55] to extreme, five-dimensional black holes. The simplest candidate would be the extreme Myers-Perry black hole [125], which is qualitatively similar to Kerr. A maximal slice can be found with $U(1)^2$ isometry and has topology $\mathbb{R} \times S^3$ [7]. However there are two main differences as one moves from $n = 3$ to $n = 4$ spatial dimensions. First, it turns out that we will have to construct solutions of the constraint equations which belong to Bartnik's weighted Sobolev spaces $W'^{k,p}_{\delta}$ [15]. Our asymptotic fall-off conditions at the Euclidean end and cylindrical end require $kp > n$ (see Lemma A.1 in [55]). We only require weak differentiability to second order, so we take $(k, p, \delta) = (2, 3, -1)$[1] whereas in the analysis of [55], $(k, p, \delta) = (2, 2, -1/2)$. The latter spaces are weighted Hilbert spaces, which are extremely useful in the elegant construction given in [55]. Second, we require five scalar functions to characterize our data as opposed to two and our geometries have $U(1)^2$ symmetry which complicates the parameterization of the extrinsic curvature.

It is important to clarify what is new about this result and how it is related to the analysis of [36, 37]. In particular, Theorem 6.1 of [36] asserts the existence of a class of

[1]One could also take $(k, p, \delta) = (3, 2, -1)$ but this leads to a stronger regularity condition for a particular elliptic operator and the functions in the background metric do not satisfy this regularity.

solutions to Lichnerowicz's equation for complete initial data with non-negative scalar curvature and strictly positive scalar curvature on cylindrical ends. These results are quite powerful and general in that no symmetry assumptions are made on the data. However, if one wishes to impose additional conditions (e.g. axisymmetry) on the data, one might be interested if there exists special families of data with the same ADM energy, asymptotic end geometries, and conserved angular momenta and/or area of the cylindrical end. This work is concerned with finding a class of initial data suitably close to the extreme Myers-Perry data that preserves the angular momenta and area of its cylindrical end. This data set can be interpreted as perturbations of extreme Myers-Perry.

The complete properties of Myers-Perry initial data set is in Appendix A. However, we will review some of them here. The extreme Myers-Perry black hole has $t-\phi^i$ symmetric initial data $(\Sigma, \boldsymbol{h}, K)$ and the slice metric can be written as

$$\boldsymbol{h} = \frac{P}{r^2}\left(dr^2 + r^2 \mathrm{d}\theta^2\right) + \lambda_{ij}\mathrm{d}\phi^i \mathrm{d}\phi^j \tag{7.1}$$

where $r > 0$, $\theta \in (0, \pi/2)$, and $\phi^i \in (0, 2\pi)$ and

$$P = r^2 + ab + a^2\cos^2\theta + b^2\sin^2\theta, \qquad \lambda_{12} = \frac{ab\mu}{P}\sin^2\theta\cos^2\theta, \tag{7.2}$$

$$\lambda_{11} = \frac{a^2\mu}{P}\sin^4\theta + (r^2 + ab + a^2)\sin^2\theta, \tag{7.3}$$

$$\lambda_{22} = \frac{b^2\mu}{P}\cos^4\theta + (r^2 + ab + b^2)\cos^2\theta. \tag{7.4}$$

Now if we choose $\rho = \frac{1}{2}r^2\sin 2\theta$ and $z = \frac{1}{2}r^2\cos 2\theta$, then the conformal slice metric of the extreme Myers-Perry black hole can be written

$$\tilde{\boldsymbol{h}}_{ab} = \Phi_0^{-2}\boldsymbol{h}_{ab}, \qquad \tilde{\boldsymbol{h}} = e^{2U}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \lambda'_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j \tag{7.5}$$

where

$$\Phi_0^2 = \frac{\sqrt{\det \lambda}}{\rho}, \qquad \lambda'_{ij} = \Phi_0^{-2}\lambda_{ij}, \qquad e^{2U} = \Phi_0^{-2}\frac{P}{r^4}. \tag{7.6}$$

In general, the lapse and shift vectors are degrees of freedom for the initial data set (see Chapter 3). But since we want to preserve geometrical properties of the initial data under evolution, we compute the lapse of the extreme Myers-Perry spacetime and select the shift vector to be the product of r and the shift of extreme Myers-Perry metric.

$$N = \sqrt{\frac{r^4 P}{(P+\mu)r^4 + \mu^2\left(r^2+ab\right)}}, \tag{7.7}$$

$$X^{\varphi} = \frac{ra\mu(r^2+ab+b^2)}{(P+\mu)r^4 + \mu^2\left(r^2+ab\right)}, \qquad X^{\psi} = \frac{rb\mu(r^2+ab+a^2)}{(P+\mu)r^4 + \mu^2\left(r^2+ab\right)}. \tag{7.8}$$

In addition, in Chapter 4 we showed that the extrinsic curvature of a $t-\phi^i$ symmetric data can be generated from scalar potentials Y^i. In the coordinate system used above, these are

$$Y^1 = \frac{a(a^2-b^2)(r^2+ab+b^2)\cos^2\theta - r^2 a(2a^2+2ab+r^2)}{(a-b)^2} + \frac{a(r^2+ab+a^2)^2(r^2+ab+b^2)}{P(a-b)^2},$$

$$Y^2 = \frac{br^2((a+b)^2+r^2) - b(a^2-b^2)(r^2+ab+a^2)\cos^2\theta}{(a-b)^2} - \frac{b(r^2+ab+a^2)(r^2+ab+b^2)^2}{P(a-b)^2}.$$

Moreover, the area of event horizon cross-section of Myers-Perry initial data can be denoted by $A(r)$ which is the area of constant r, and we have

$$A_0 = \lim_{r\to 0} A(r) = 2\pi^2\mu^2\sqrt{ab}. \tag{7.9}$$

Then the initial data of the extreme Myers-Perry black hole is $(\Sigma, \boldsymbol{h}, K)$. Then we have the following result.

Theorem 73. *Let $(\Sigma, \boldsymbol{h}, K)$ be the GB ($t-\phi^i$-symmetric) initial data set constructed by extreme Myers-Perry described as above (see Appendix A) with angular momenta J_1*

and J_2 such that $J_1J_2 \geq 0$ and mass M. Then there is a small s_0 such that for $|s| < s_0$ there exists a family of initial data $(\Sigma, \boldsymbol{h}^s, K^s)$ (i.e. solutions of the constraints on Σ) such that:

(a) For $s = 0$ the family of initial data is extreme Myers-Perry initial data set, i.e. $(\Sigma, \boldsymbol{h}, K)$. The family is differentiable in s and it is close to extreme Myers-Perry with respect to an appropriate norm which involves two derivatives of the metric.

(b) The family of initial data has the same asymptotic geometry as the extreme Myers-Perry initial data. The angular momenta and the area of the cylindrical end in the family do not depend on s; they have same value as in extreme Myers-Perry initial data set, namely J_1, J_2 and A_0, respectively.

(c) The family of initial data is $U(1)^2$-invariant and maximal (i.e $Tr_{\boldsymbol{h}^s} K^s = 0$).

(d) The energy of this family of initial data is positive.

Before proving Theorem 73 we investigate the evolution of the family of initial data. Consider a member of the family of initial data set $(\Sigma, \boldsymbol{h}^s, K^s)$ for fixed $s \neq 0$. By an argument similar to that given in [55], the fall-off of the lapse and shift can always be selected so that the geometry of the cylindrical end and its area will be preserved, for sufficiently short times. If we consider a member of the family of initial data set $(\Sigma, \boldsymbol{h}^s, K^s)$ for fixed $s \neq 0$, then expanding these tensors for small time t we have

$$h^s_{ab}(t) \approx h^s_{ab}(0) + \dot{h}^s_{ab}(0)t \tag{7.10}$$

$$K^s_{ab}(t) \approx K^s_{ab}(0) + \dot{K}^s_{ab}(0)t \tag{7.11}$$

where $\dot{}$ denotes the derivative with respect to t. Here $\dot{h}^s_{ab}$ and $\dot{K}^s_{ab}$ are obtained from the evolution equation

$$\dot{h}^s_{ab} = 2NK^s_{ab} + \mathcal{L}_X h^s_{ab}, \tag{7.12}$$

$$\dot{K}^s_{ab} = \nabla_a\nabla_b N + \mathcal{L}_X K^s_{ab} + N\left\{2(h^s)^{cd}K_{ad}{}^s K^s_{bc} - (\mathrm{Tr}_{\boldsymbol{h}^s}K)\,K^s_{ab} - R^s_{ab}\right\} \tag{7.13}$$

where N and X are the lapse and shift vector of the foliation and R^s_{ab} is the Ricci curvature of the family. The lapse and shift can be calculated independently from the initial data. In the case of the extreme Myers-Perry black hole, we argue that if we choose lapse and shift with appropriate decay condition at the cylindrical end, then this fall-off condition at the cylindrical end will be preserved along the whole foliation. This process is similar to an asymptotic fall off condition. To preserve the cylindrical geometry under evolution we should have

$$\lim_{r\to 0}\dot{h}^s_{ab} = 0, \qquad \lim_{r\to 0}\dot{K}^s_{ab} = 0, \tag{7.14}$$

but this is equivalent to

$$\lim_{r\to 0} N = \lim_{r\to 0}\nabla N = \lim_{r\to 0}\nabla^2 N = 0, \tag{7.15}$$

$$\lim_{r\to 0} X^a = \lim_{r\to 0}\nabla X^a = 0, \tag{7.16}$$

where ∇ is a partial derivative with respect to the spatial coordinates. These conditions are satisfied for the lapse and shift of the extreme Myers-Perry initial data with lapse equation (7.7) and shift vector (7.8). The geometry of the cylindrical end will be preserved under small deformations s, provided we impose the fall-off conditions (7.15) and (7.16).

7.2 Construction of Perturbed Initial Data Via Conformal Method

In this section we construct a one parameter family of initial data $(\Sigma, \boldsymbol{h}^s, K^s)$ from the extreme Myers-Perry initial data via the conformal method. This family is a small perturbation of the extreme Myers-Perry initial data which preserves angular momenta, cylindrical end geometry, and area of the even horizon H. Let $(\Sigma, \boldsymbol{h}, K)$ be the maximal initial data (given in Appendix A) of the extreme Myers-Perry black hole. It is a vacuum, $t-\phi^i$ symmetric initial data set which satisfies in Einstein constraint equations (3.12). Firstly, we assume the following conformal rescaling for the initial data

$$h_{ab} = \Phi_0^2 \tilde{h}_{ab}, \qquad K_{ab} = \Phi_0^{-2} \tilde{K}_{ab}, \tag{7.17}$$

where $\Phi_0 = \log v_0$ and by equation (4.81) and Appendix A we have

$$\tilde{\boldsymbol{h}} = e^{2U}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \lambda'_{ij}\mathrm{d}\phi^i\phi^j, \qquad K_{ab} = 2S^t_{(a}\lambda^{-1}\Phi_{b)}, \tag{7.18}$$

where $S = (S^1, S^2)^t$, $\Phi = (\xi_{(1)}, \xi_{(2)})^t$ and

$$S^i_a = \frac{1}{2\rho^2}\iota_{\xi_{(2)}}\iota_{\xi_{(1)}} \star \mathrm{d}Y^i \tag{7.19}$$

where Y^i are twist potential of the initial data. By Corollary 5 and equation (3.12), the constraint equations for conformal initial data $(\Sigma, \tilde{\boldsymbol{h}}, \tilde{K})$ are

$$\Delta_{\tilde{h}}\Phi_0 - \frac{1}{6}\tilde{R}\Phi_0 + \frac{1}{6}|\tilde{K}|^2_{\tilde{\boldsymbol{h}}}\Phi_0^{-5} = 0. \tag{7.20}$$

$$\mathrm{div}\tilde{K} = 0. \tag{7.21}$$

Since $(\Sigma, \tilde{\boldsymbol{h}}, \tilde{K})$ is a vacuum, $t-\phi^i$ symmetric data set, by Section 4.1 $\tilde{K}_{ab}$ is a TT tensor, so the momentum constraint equation (7.21) is automatically satisfied and we need only consider the Lichnerowicz equation (7.20). The Laplace operator associated with the metric (7.18) (for any U, λ'_{ij}) in $t-\phi^i$ symmetry can be written as

$$\begin{aligned}\Delta_{\tilde{\boldsymbol{h}}}\Phi &= \frac{1}{\sqrt{\det\tilde{\boldsymbol{h}}}}\tilde{\nabla}_a\left(\sqrt{\det\tilde{\boldsymbol{h}}}\,\tilde{h}^{ab}\tilde{\nabla}_b\Phi\right) = \frac{e^{-2U}}{\rho}\nabla_a\left(\rho\delta_2^{ab}\nabla_b\Phi\right)\\ &= \frac{e^{-2U}}{\rho}\nabla_a\left(\rho r^{-2}\delta_4^{ab}\nabla_b\Phi\right) = \frac{e^{-2U}}{r^2}\Delta_4\Phi \end{aligned} \tag{7.22}$$

where Φ is an arbitrary function of only r and θ and Δ_4 and ∇ are the Laplace operator and covariant derivative with respect to δ_4, respectively. Secondly, we define $\tilde{R}_0$ and $\tilde{K}_0^2$ from the scalar curvature of the metric (7.18) and equation (4.20) as

$$\tilde{R} = e^{-2U}\left(-2\Delta_2 U + \frac{\det\nabla\lambda'}{2\rho^2}\right) \equiv e^{-2U}r^{-2}\tilde{R}_0, \tag{7.23}$$

$$|\tilde{K}|^2_{\tilde{\boldsymbol{h}}} = \frac{e^{-2U}}{2\rho^2}\nabla Y^t\lambda'^{-1}\nabla Y \equiv e^{-2U}r^{-2}\tilde{K}_0^2\,, \tag{7.24}$$

where Δ_2 is the Laplace operator with respect to $\delta_2 = \mathrm{d}\rho^2 + \mathrm{d}z^2$, that is

$$\Delta_2 = \frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^4}\left(r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2}\right). \tag{7.25}$$

Then the Lichnerowicz equation (7.20) for the conformal triple $(\Sigma, \tilde{\boldsymbol{h}}, \tilde{K})$ is

$$\Delta_4\Phi_0 - \frac{\tilde{R}_0}{6}\Phi_0 + \frac{\tilde{K}_0^2}{6\Phi_0^5} = 0. \tag{7.26}$$

Finally, we perturb equation (7.26) about the solution given by the maximal initial data for the extreme Myers-Perry black hole by taking

$$U \to U + s\bar{U}, \quad \lambda'_{ij} \to \lambda'_{ij} + s\bar{\lambda}'_{ij}, \quad Y^i \to Y^i + s\bar{Y}^i \tag{7.27}$$

for a fixed set of $U(1)^2$-invariant functions $\bar{U}$, $\bar{\lambda}'_{ij}$, $\bar{Y}^i$, and small s, and then seek a solution Φ of the form

$$\Phi = \Phi_0 + u, \tag{7.28}$$

where u is a function to be determined. Inserting (7.27) and (7.28) into (7.26), we have

$$\mathcal{T}(s,u) \equiv \Delta_4 (\Phi_0 + u) - \frac{1}{6}\tilde{R}_s (\Phi_0 + u) + \frac{\tilde{K}_s^2}{6(\Phi_0 + u)^5} = 0 \tag{7.29}$$

where $\tilde{R}_s$ and $\tilde{K}_s^2$ are obtained from $\tilde{R}_0$ and $\tilde{K}_0$ using the transformation (7.27). If we plug in $s = 0$ in equation (7.29), we have equation (7.26). But before proving the existence and uniqueness of the solution of the the operator $\mathcal{T}(s,u)$ we review some properties of extreme Myers-Perry initial data.

Lemma 74. *Let Φ_0, $\tilde{R}_0$, and $\tilde{K}_0^2$ be defined as in (7.6), (7.23), and (7.26), respectively and a and b have same sign. Then we have following bounds:*

1. $(ab\mu)^{1/4} \leq \left[(r^2+ab+b^2)(r^2+ab+a^2)\right]^{1/4} \leq r\Phi_0 \leq \left[(r^2+ab+b^2)(r^2+ab+a^2)+\mu^2\right]^{1/4}$.
2. $\left|\tilde{R}_0\right| \leq \frac{C}{r^4}$ *and* $\tilde{K}_0^2 \leq \frac{C}{r^6}$.
3. $|\Delta_4\Phi_0| \leq \frac{C}{r^6}$.

Proof. We will prove only part 1 here; the remaining bounds require lengthy algebraic manipulations and we used MAPLE software.

1. We have

$$\begin{aligned} r^2\Phi_0^2 &= \left[(r^2+ab+b^2)(r^2+ab+a^2) + \frac{\mu(r^2+ab)(a^2\cos^2\theta + b^2\sin^2\theta) + \mu a^2 b^2}{P}\right]^{1/2} \\ &\leq \left[(r^2+ab+b^2)(r^2+ab+a^2)+\mu^2\right]^{1/2} \end{aligned} \tag{7.30}$$

so if $r \to \infty$ then we have minimum of $r^2\Phi_0^2$

$$\sqrt{(r^2+ab+b^2)(r^2+ab+a^2)} \le r^2\Phi_0^2. \tag{7.31}$$

Therefore for $a, b > 0$ we have

$$\begin{aligned}(ab\mu)^{1/4} &\le \left[(r^2+ab+b^2)(r^2+ab+a^2)\right]^{1/4} \le r\Phi_0 \\ &\le \left[(r^2+ab+b^2)(r^2+ab+a^2)+\mu^2\right]^{1/4}\end{aligned} \tag{7.32}$$

□

Lemma 75. *The function* α *in equation* (7.75) *is nonnegative and*

$$\alpha = \frac{\tilde{K}_0^2}{2\Phi_0^6} + r^2|\nabla v|^2 = hr^{-6} \tag{7.33}$$

where h *is a bounded nonnegative function.*

Proof. First we know by conformal transformation $h_{ab} = \Phi^2\tilde{h}_{ab}$ the scalar curvature will be

$$\tilde{R} = R\Phi^2 + 6\left(\Delta_{\tilde{h}} v + |\tilde{\nabla} v|^2\right) \tag{7.34}$$

where $v = \log\Phi$. Since the extreme Myers-Perry initial data is a critical point of the mass functional, it satisfies in the field equation (5.87)

$$\Delta_{\tilde{h}} v = -\frac{1}{2\Phi^6}\tilde{K}_{ab}\tilde{K}^{ab}. \tag{7.35}$$

Consequently, we have

$$\begin{aligned}\tilde{R} &= K_{ab}K^{ab}\Phi^2 - 3\Phi^{-6}\tilde{K}_{ab}\tilde{K}^{ab} + 6|\tilde{\nabla} v|^2 \\ &= -2\tilde{K}_{ab}\tilde{K}^{ab}\Phi^{-6} + 6e^{-2U}|\nabla v|^2.\end{aligned} \tag{7.36}$$

Then by equations (7.23) we have

$$\tilde{R}_0 = -2\tilde{K}_0^2\Phi^{-6} + 6r^2|\nabla v|^2. \tag{7.37}$$

Therefore, by Lemma 74 the function α is

$$\alpha = \frac{\tilde{R}_0}{6} + \frac{5\tilde{K}_0^2}{6\Phi_0^6} = \frac{\tilde{K}_0^2}{2\Phi_0^6} + r^2(\nabla v)^2 = hr^{-6}. \tag{7.38}$$

□

Lemma 76. *If we transform metric functions by* (7.27) *for small* s *(i.e.*$-s_0 < s < s_0$*) and* $\bar{Y}^i, \bar{\lambda}'_{ij} \in C_c^\infty(\mathbb{R}^4\backslash\Gamma)$ *and* $\bar{U} \in C_c^\infty(\mathbb{R}^4\backslash\{0\})$ *then*

1. $\left\|\tilde{R}_s\right\|_{L'^3_{-3}} \leq C$
2. $\left\|D_1\tilde{R}_s\right\|_{L'^3_{-3}} \leq C$
3. $\left\|D_1\tilde{R}_{s_1} - D_1\tilde{R}_{s_2}\right\|_{L'^3_{-3}} \leq C\left|s_1 - s_2\right|$
4. $\left\|\tilde{K}_s^2\right\|_{L'^3_{-3}} \leq C$
5. $\left\|D_1\tilde{K}_s^2\right\|_{L'^3_{-3}} \leq C$
6. $\left\|D_1\tilde{K}_{s_1}^2 - D_1\tilde{K}_{s_2}^2\right\|_{L'^3_{-3}} \leq C\left|s_1 - s_2\right|$

where $D_1 = \frac{d}{ds}$.

Proof. The proof is straightforward. By definition of $\tilde{R}_\lambda$ we have

$$\begin{aligned}
\tilde{R}_s &= -r^2\Delta_2\left(U + s\bar{U}\right) + r^2\frac{\det\left(\nabla\lambda' + s\nabla\bar{\lambda}'\right)}{2\rho^2} \\
&= -r^2\Delta_2 U - r^2 s\Delta_2\bar{U} + \frac{r^2}{2\rho^2}\left[(\nabla\lambda'_{11} + s\nabla\bar{\lambda}'_{11})\cdot(\nabla\lambda'_{22} + s\mathrm{d}\bar{\lambda}'_{22}) - (\nabla\lambda'_{12} + s\nabla\bar{\lambda}'_{12})^2\right] \\
&= \tilde{R}_0 - r^2 s\Delta_2\bar{U} \\
&+ \frac{r^2}{2\rho^2}\left[s\nabla\bar{\lambda}'_{11}\cdot\nabla\lambda'_{22} + s(\nabla\lambda'_{11} + s\nabla\bar{\lambda}'_{11})\cdot\nabla\bar{\lambda}'_{22} - s(2\nabla\lambda'_{12} + s\nabla\bar{\lambda}'_{12})\cdot\nabla\bar{\lambda}'_{12}\right].
\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
D_1\tilde{R}_s &= -r^2\Delta_2\bar{U} + \frac{r^2}{2\rho^2}\Big[\nabla\bar{\lambda}'_{11}\cdot\nabla\lambda'_{22} + (\nabla\lambda'_{11} + s\nabla\bar{\lambda}'_{11})\cdot\nabla\bar{\lambda}'_{22} + s\nabla\bar{\lambda}'_{11}\cdot\nabla\bar{\lambda}'_{22} \\
&- (2\nabla\lambda'_{12} + s\nabla\bar{\lambda}'_{12})\cdot\nabla\bar{\lambda}'_{12} - s\nabla\bar{\lambda}'_{12}\cdot\nabla\bar{\lambda}'_{12}\Big] \\
&= -r^2\Delta_2\bar{U} + \frac{r^2}{2\rho^2}\mathrm{Tr}\left[\nabla\bar{\lambda}'\cdot(\nabla\lambda' + s\nabla\bar{\lambda}')\right], \qquad (7.39)
\end{aligned}
$$

and

$$
D_1\tilde{R}_{s_2} - D_1\tilde{R}_{s_1} = \frac{r^2}{2\rho^2}\mathrm{Tr}\left[\nabla\bar{\lambda}'\cdot\nabla\bar{\lambda}'\right](s_2 - s_1). \qquad (7.40)
$$

For part (1) by triangle inequality we have

$$
\begin{aligned}
\left\|\tilde{R}_s\right\|_{L'^3_{-3}} &\le \left\|\tilde{R}_0\right\|_{L'^3_{-3}} + |s|\left\|r^2\Delta_2\bar{U}\right\|_{L'^3_{-3}} + \left\|\frac{r^2}{2\rho^2}\left(s(2\nabla\lambda'_{12} + s\nabla\bar{\lambda}'_{12})\cdot\nabla\bar{\lambda}'_{12}\right)\right\|_{L'^3_{-3}} \\
&+ \left\|\frac{r^2}{2\rho^2}\left(s\nabla\bar{\lambda}'_{11}\cdot\nabla\lambda'_{22} + s(\nabla\lambda'_{11} + s\nabla\bar{\lambda}'_{11})\cdot\nabla\bar{\lambda}'_{22}\right)\right\|_{L'^3_{-3}} \le C. \qquad (7.41)
\end{aligned}
$$

We used inequality of Lemma 74-2 and the fact that functions $\bar{U}$ and $\bar{\lambda}'_{ij}$ have compact support outside the origin. We have similar result for $D_1\tilde{R}_s$ and $D_1\tilde{R}_s$. Moreover, by definition of full contraction of extrinsic curvature we have

$$
\begin{aligned}
\tilde{K}^2_s &= \frac{r^2}{2\rho^4}\left[\left(\nabla Y + s\nabla\bar{Y}\right)^t \mathrm{adj}\left(\lambda' + s\bar{\lambda}'\right)\left(\nabla Y + s\nabla\bar{Y}\right)\right] \\
&= \frac{r^2}{2\rho^4}\Big[\nabla Y^t \mathrm{adj}\lambda'\nabla Y + s\left(2\nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla Y + \nabla Y^t\mathrm{adj}\bar{\lambda}'\nabla Y\right) \\
&+ s^2\left(2\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla Y + \nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla\bar{Y}\right) + s^3\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla\bar{Y}\Big]
\end{aligned}
$$

$$= \tilde{K}_0^2 + \frac{r^2}{2\rho^4}\Big[s\left(2\nabla\bar{Y}^t \mathrm{adj}\lambda'\nabla Y + \nabla Y^t \mathrm{adj}\bar{\lambda}'\nabla Y\right) + s^3\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla\bar{Y}$$
$$+ s^2\left(2\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla Y + \nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla\bar{Y}\right)\Big]. \quad (7.42)$$

Then have

$$D_1\tilde{K}_s^2 = \frac{r^2}{2\rho^4}\Big[\left(2\nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla Y + \nabla Y^t\mathrm{adj}\bar{\lambda}'\nabla Y\right)$$
$$+ 3s^2\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla\bar{Y} + 2s\left(2\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla Y + \nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla\bar{Y}\right)\Big], \quad (7.43)$$

and

$$D_1\tilde{K}_{s_2}^2 - D_1\tilde{K}_{s_1}^2$$
$$= \frac{r^2}{2\rho^4}\left[2\left(2\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla Y + \nabla\bar{Y}^t\mathrm{adj}\lambda'\nabla\bar{Y}\right) + 3(s_2+s_1)\nabla\bar{Y}^t\mathrm{adj}\bar{\lambda}'\nabla\bar{Y}\right](s_2 - s_1).$$

Then by triangle inequality and the fact that $\bar{Y}^i$ and $\bar{\lambda}'$ have compact support outside the axis, one can show it is bounded. □

Now we prove the following important Lemma which shows existence and uniquenes of the solution of the operator $\mathcal{T}$ equation (7.29).

Lemma 77. *Let $\bar{Y}^i, \bar{\lambda}'_{ij} \in C_c^\infty(\mathbb{R}^4 \backslash \Gamma)$ and $\bar{U} \in C_c^\infty(\mathbb{R}^4\backslash\{0\})$. Then, there exists $s_0 > 0$ such that for all $s \in (-s_0, s_0)$*

1. *There exists a solution $u(s)$ of (7.29) belonging to $W_{-1}^{'2,3}$. (for clarity we suppress the $r-$ and $\theta-$ dependence of $u(s)$).*

2. *$u(s)$ is continuously differentiable in s and $\Phi(s) = \Phi_0 + u(s) > 0$.*

3. *$u(s)$ is the unique solution of (7.29) for small u and small s.*

7.2.1 Proof of Lemma 77

The main tool we use to establish the Lemma 77 is the implicit function theorem (see Theorem 17). The argument closely parallels that given in [55] and proceeds as follows. Firstly, we select appropriate Banach spaces X,Y, and Z as required for the implicit function theorem. Then we find neighbourhoods $O_x \subset X$ and $O_y \subset Y$ for which the map $\mathcal{T} : O_x \times O_y \to Z$ is well-defined. Care must be given to select Banach spaces that satisfy the fall-off conditions on the functions U, λ'_{ij}, and Y^i at infinity and singular behaviour at the origin of the function Φ_0. Since the solution need not be regular at the origin (we are working on $\mathbb{R}^4 - \{0\}$) we cannot select the standard weighted Sobolev spaces $W^{2,3}_{-1}$ defined in [15]. To begin we verify that $\mathcal{T} : O_x \times O_y \to Z$ is C^1 . Next we show that $D_2\mathcal{T}(0,0)$ (which is defined in equation (7.53)) is an isomorphism between Y and Z. The implicit function theorem is then used to conclude the existence of a unique u with the properties of the lemma.

$\mathcal{T}$ is well-defined

We choose $X = \mathbb{R}$, $Y = W'^{2,3}_{-1}$ and $Z = L'^{3}_{-3}$. Moreover, we choose $O_x = \mathbb{R}$ and $O_y = \{u \in W'^{2,3}_{-1} : \|u\|_{W'^{2,3}_{-1}} < \xi\}$ where ξ is computed as follows: by the inequality in Lemma 10-3 for $u \in O_y$ we have

$$r|u| \leq C_0\xi, \tag{7.44}$$

where C_0 is a constant. Also by Lemma 74, we have

$$r\Phi_0 \geq (ab\mu)^{1/4}. \tag{7.45}$$

Then , if we choose ξ such that

$$\frac{(ab\mu)^{1/4}}{C_0} > \xi > 0\,, \tag{7.46}$$

then for all $u \in O_y$, we have

$$0 < (ab\mu)^{1/4} - C_0\xi \leq r\left(\Phi_0 + u\right). \tag{7.47}$$

First we prove that $\mathcal{T} : \mathbb{R} \times O_y \to L'^3_{-3}$ is well-defined. That is, we need to show that for $s \in \mathbb{R}$ and $u \in O_y$ we have $\mathcal{T}(s, u) \in L'^3_{-3}$. By using the triangle inequality for equation (7.29), we have

$$\|\mathcal{T}(\lambda, u)\|_{L'^3_{-3}} \leq \underbrace{\|\Delta_4 u\|_{L'^3_{-3}}}_{I} + \underbrace{\|\Delta_4 \Phi_0\|_{L'^3_{-3}}}_{II} + \frac{1}{6}\underbrace{\left\|\tilde{R}_s(\Phi_0 + u)\right\|_{L'^3_{-3}}}_{III} + \underbrace{\left\|\frac{\tilde{K}_s^2}{6(\Phi_0 + u)^5}\right\|_{L'^3_{-3}}}_{IV} \tag{7.48}$$

We will show that each of these terms is bounded in L'^3_{-3}. To show this we will need the required properties of the functions $\bar{U}$ and $\bar{\lambda}'_{ij}$, and $\bar{Y}^i$ as well as the particular fall-off conditions on functions (i.e U, λ'_{ij}) of the conformal Myers-Perry metric.

(I) Since $u \in O_y$

$$\|\Delta_4 u\|_{L'^3_{-3}} \leq \|u\|_{W'^{2,3}_{-1}} \leq C \tag{7.49}$$

where C is a function of a and b. Henceforth, the notation C is a constant related only on metric parameters, i.e. a and b.

(II) In second term we use the bound on the Laplace operator Lemma 74-3:

$$\|\Delta_4 \Phi_0\|_{L'^3_{-3}} \leq \left\|\frac{C}{r^6}\right\|_{L'^3_{-3}} \leq C. \tag{7.50}$$

Finally, since $\bar{Y}^i$ and $\bar{\lambda}'_{ij}$ have compact support outside the axis and $\bar{U}$ and have compact support outside the origin, and by using (7.47) and Lemma 76 one can show that (III) and (IV)) are bounded. Thus $\mathcal{T} : \mathbb{R} \times O_y \to L'^3_{-3}$ is well-defined.

$\mathcal{T}$ is C^1

We denote by $D_1\mathcal{T}(\lambda, u)$ the partial Fréchet derivative of $\mathcal{T}$ with respect to the first argument evaluated at (s, u) and by $D_2\mathcal{T}(s, u)$ the partial Fréchet derivative of $\mathcal{T}$ with respect to the second argument u. These operators are formally obtained by directional derivatives of $\mathcal{T}$ and they are linear operators between the following spaces:

$$D_1\mathcal{T}(s, u) : \mathbb{R} \to L'^3_{-3}, \tag{7.51}$$

$$D_2\mathcal{T}(s, u) : W'^{2,3}_{-1} \to L'^3_{-3}. \tag{7.52}$$

We use the notation $D_1\mathcal{T}(s, u)[\zeta] \in L'^3_{-3}$ to denote the operator $D_1\mathcal{T}(s, u)$ acting on $\zeta \in \mathbb{R}$. Similarly, $D_2\mathcal{T}(s, u)[v] \in L'^3_{-3}$ denotes the operator $D_2\mathcal{T}(s, u)$ acting on $v \in W'^{2,3}_{-1}$. These linear operators will be

$$D_1\mathcal{T}(s, u)[\zeta] \;=\; \frac{d}{dt}\mathcal{T}(s + t\zeta, u)|_{t=0} = \frac{1}{6}\left(- D_1\tilde{R}_s(\Phi_0 + u) + \frac{D_1\tilde{K}_s^2}{(\Phi_0 + u)^5}\right)\zeta,$$

$$D_2\mathcal{T}(s, u)[v] \;=\; \frac{d}{dt}\mathcal{T}(s, u + tv)|_{t=0} = \Delta_4 v - \frac{1}{6}\left(\tilde{R}_s + \frac{5\tilde{K}_s^2}{(\Phi_0 + u)^6}\right)v. \tag{7.53}$$

Now, we will prove that the map $\mathcal{T} : \mathbb{R} \times O_y \to L'^3_{-3}$ is C^1. As a result of the properties of functions of the metric, we cannot use the chain rule. Alternatively, we will show that:

1. The linear operator $D_1\mathcal{T}(s,u)[\zeta]$ and $D_2\mathcal{T}(s,u)[v]$ are bounded. i.e.

$$\|D_1\mathcal{T}(s,u)[\zeta]\|_{L'^3_{-3}} \le C|\zeta|, \tag{7.54}$$

$$\|D_2\mathcal{T}(s,u)[v]\|_{L'^3_{-3}} \le C\,\|v\|_{W'^{2,3}_{-1}}. \tag{7.55}$$

2. The linear operator $D_1\mathcal{T}(s,u)[\zeta]$ and $D_2\mathcal{T}(s,u)[v]$ are continuous in (s,u) in the operator norms. That is, for every $\epsilon>0$ there exists $\delta>0$ such that

$$|s_1-s_2|<\delta \Longrightarrow \|D_1\mathcal{T}(s_1,u)-D_1\mathcal{T}(s_2,u)\|_{B(X,Z)}<\epsilon, \tag{7.56}$$

$$\|u_1-u_2\|_{W'^{2,3}_{-1}}<\delta \Longrightarrow \|D_2\mathcal{T}(s,u_1)-D_2\mathcal{T}(s,u_2)\|_{B(Y,Z)}<\epsilon. \tag{7.57}$$

3. The operators $D_1\mathcal{T}(s,u)[\zeta]$ and $D_2\mathcal{T}(s,u)[v]$ are the partial Fréchet derivatives of $\mathcal{T}$. That is

$$\lim_{\zeta\to 0}\frac{\|\mathcal{T}(s+\zeta,u)-\mathcal{T}(s,u)-D_1\mathcal{T}(s,u)[\zeta]\|_{L'^3_{-3}}}{|\zeta|}=0, \tag{7.58}$$

$$\lim_{v\to 0}\frac{\|\mathcal{T}(s,u+v)-\mathcal{T}(s,u)-D_2\mathcal{T}(s,u)[v]\|_{L'^3_{-3}}}{\|v\|_{W'^{2,3}_{-1}}}=0. \tag{7.59}$$

1. To prove inequality (7.54) we use triangle inequality, Lemma 76, and inequality (7.47) then

$$\begin{aligned}\|D_1\mathcal{T}(s,u)[\zeta]\|_{L'^3_{-3}} &\le \frac{|\zeta|}{6}\left\|D_1\tilde{R}_s(\Phi_0+u)\right\|_{L'^3_{-3}}+\frac{|\zeta|}{6}\left\|\frac{D_1\tilde{K}_s^2}{(\Phi_0+u)^5}\right\|_{L'^3_{-3}}\\ &\le C\,|\zeta|\,. \end{aligned}\tag{7.60}$$

Similarly, by definition of O_y and Lemma 76 we have

$$\begin{aligned}\|D_2\mathcal{T}(s,u)[v]\|_{L'^3_{-3}} &\leq \|\Delta_4 v\|_{L'^3_{-3}} + \frac{1}{6}\left\|\tilde{R}_s v\right\|_{L'^3_{-3}} + \left\|\frac{5\tilde{K}_s^2}{6(\Phi_0+u)^6}v\right\|_{L'^3_{-3}} \\ &\leq C\|v\|_{W'^{2,3}_{-1}}. \qquad (7.61)\end{aligned}$$

2. To show that $D_1\mathcal{T}(s,u)$ is continuous (it is in fact uniformly continuous), we use the triangle inequality, inequality (7.47), and Lemma 76 to obtain

$$\begin{aligned}\|D_1\mathcal{T}(s_1,u) - D_1\mathcal{T}(s_2,u)\|_{L'^3_{-3}} &\leq \frac{1}{6}\left\|(D_1\tilde{R}_{s_1} - D_1\tilde{R}_{s_2})(\Phi_0+u)\right\|_{L'^3_{-3}} \\ &+ \left\|\frac{D_1\tilde{K}_{s_1}^2 - D_1\tilde{K}_{s_2}^2}{6(\Phi_0+u)^5}\right\|_{L'^3_{-3}} \\ &\leq C\left|s_1 - s_2\right|. \qquad (7.62)\end{aligned}$$

To prove continuity in u consider the following identity for arbitrary x, y and integer p:

$$\frac{1}{x^p} - \frac{1}{y^p} = (y-x)\sum_{i=0}^{p-1} x^{i-p}y^{-1-i}. \qquad (7.63)$$

Then

$$r^{-7}\left(\frac{1}{(\Phi_0+u_1)^6} - \frac{1}{(\Phi_0+u_2)^6}\right) = (u_2-u_1)\,M, \qquad (7.64)$$

where

$$M = \sum_{i=0}^{5}\left(r\left(u+\Phi_0\right)\right)^{i-6}\left(r\Phi_0\right)^{-1-i}. \qquad (7.65)$$

Since $u_1, u_2 \in O_y$, and using the lower bound in equation (7.47) we have

$$M \leq C. \qquad (7.66)$$

Then by (7.66) and Lemma 74 we have

$$\begin{aligned}
\|D_2\mathcal{T}(s,u_1)[v] - D_2\mathcal{T}(s,u_2)[v]\|_{L'^3_{-3}} &= \left\| v\frac{5\tilde{K}_s^2}{6(\Phi_0+u_1)^6} - v\frac{5\tilde{K}_s^2}{6(\Phi_0+u_2)^6} \right\|_{L'^3_{-3}} \\
&\le C\left\|\frac{(u_1-u_2)\,v}{r}\right\|_{L'^3_{-3}} \qquad (7.67)
\end{aligned}$$

The right hand side of the above equation can be bounded as follows: (we write dx to represent the volume element for the Euclidean metric on $\mathbb{R}^4\backslash\{0\}$)

$$\begin{aligned}
\left\|\frac{(u_1-u_2)\,v}{r}\right\|_{L'^3_{-3}} &= \left(\int_{\mathbb{R}^4\backslash\{0\}} \frac{(u_1-u_2)^3\,v^3}{r^3} r^5 dx\right)^{1/3} \\
&= \left(\int_{\mathbb{R}^4\backslash\{0\}} \frac{(u_1-u_2)^3\,(rv)^3}{r} \mathrm{d}x\right)^{1/3} \\
&\le C\,\|v\|_{W'^{2,3}_{-1}} \left(\int_{\mathbb{R}^4\backslash\{0\}} \frac{(u_1-u_2)^3}{r} \mathrm{d}x\right)^{1/3} \\
&\le C\,\|v\|_{W'^{2,3}_{-1}}\,\|u_1-u_2\|_{W'^{2,3}_{-1}}. \qquad (7.68)
\end{aligned}$$

The first inequality follows from Lemma 10 and the second inequality from the definition of Sobolev norms. Therefore, we have

$$\|D_2\mathcal{T}(s,u_1)[v] - D_2T(s,u_2)[v]\|_{L'^3_{-3}} \le C\,\|v\|_{W'^{2,3}_{-1}}\,\|u_1-u_2\|_{W'^{2,3}_{-1}}. \qquad (7.69)$$

Thus, $D_2T(s,u)$ is a continuous operator.

3. Proving equation (7.58) is straightforward. We prove (7.59) as follows

$$\begin{aligned}
&\mathcal{T}(s,u+v) - \mathcal{T}(s,u) - D_2\mathcal{T}(s,u)[v] \\
= \ &\frac{\tilde{K}_s^2}{6}\left(\frac{1}{(\Phi_0+u+v)^5} - \frac{1}{(\Phi_0+u)^5} + \frac{5v}{(\Phi_0+u)^6}\right)
\end{aligned}$$

By simplifying we have

$$r^{-7}\left(\frac{1}{(\Phi_0+u+v)^5}-\frac{1}{(\Phi_0+u)^5}+\frac{5v}{(\Phi_0+u)^6}\right)=v^2M_1, \tag{7.70}$$

where

$$M_1=\frac{1}{\left(r\left(\Phi_0+u+v\right)\right)^5\left(r\left(\Phi_0+u\right)\right)^6}\sum_{\substack{i+j+k=4\\ \forall i,j,k\geq 0}} C_{ijk}\left(r\Phi_0\right)^i\left(ru\right)^j\left(rv\right)^k, \tag{7.71}$$

where C_{ijk} are numerical constants. To find the bound of M_1 we will use equation (7.44) and the fact that $u,v\in V$. Then we have

$$|M_1|\leq C\frac{(r^2+ab+b^2)(r^2+ab+a^2)+\mu^2}{\left([(r^2+ab+b^2)(r^2+ab+a^2)+\mu^2]^{1/4}-C_0\xi\right)^{11}}\leq C. \tag{7.72}$$

Then by Lemma 76 and above inequality we have

$$\|\mathcal{T}(s,u+v)-\mathcal{T}(s,u)-D_2\mathcal{T}(s,u)[v]\|_{L'^3_{-3}} \leq C\left\|\frac{v^2M_1}{r}\right\|_{L'^3_{-3}}\leq C\|v\|^2_{W'^{2,3}_{-1}}. \tag{7.73}$$

By steps similar to (7.68) we have the second inequality. Hence, we have proved statements (1), (2), and (3) and $\mathcal{T}(s,u):\mathbb{R}\times O_y\to L'^3_{-3}$ is C^1.

$D_2\mathcal{T}(0,0)$ is an isomorphism

We now verify that $D_2\mathcal{T}(0,0):W'^{2,3}_{-1}\to L'^3_{-3}$ is an isomorphism. We write this linear operator as

$$D_2\mathcal{T}(0,0)[v]=\Delta_4 v-\alpha v \tag{7.74}$$

where

$$\alpha=\frac{\tilde{R}_0}{6}+\frac{5\tilde{K}_0^2}{6\Phi_0^6}. \tag{7.75}$$

An important property of the function α by Lemma 75 is a nonnegative bounded function in $\mathbb{R}^4\backslash\{0\}$, that is $\alpha = hr^{-6}$ where $h \geq 0$. Therefore $\alpha \in L'^{3}_{-3}$. Hence by Theorem 14 when $M = \mathbb{R}^4\backslash\{0\}$ and $p = 3, \delta = -1$, the map $\Delta_4 - \alpha$ is an isomorphism from $W'^{2,3}_{-1} \to L'^{3}_{-3}$.

7.3 Proof of Theorem 73

Proof. In the previous section by conformal method we construct a family of initial data sets $(\Sigma, \boldsymbol{h}^s, K^s)$ such that

$$h^s_{ab} = \Phi^2 \tilde{h}^s_{ab}, \qquad K^s_{ab} = \Phi^{-2}\tilde{K}^s_{ab} \tag{7.76}$$

where the conformal data $(\Sigma, \tilde{\boldsymbol{h}}^s, \tilde{K}^s)$ constructed from fixed perturbation $\bar{Y}^i, \bar{\lambda}'_{ij} \in C^\infty_c(\mathbb{R}^4\backslash\Gamma)$ and $\bar{U} \in C^\infty_c(\mathbb{R}^4\backslash\{0\})$, of the extreme Myers-Perry conformal data $(\Sigma, \tilde{\boldsymbol{h}}, \tilde{K})$ with transformation (7.27) for small s. Then by Lemma 77, there exists $u(s) \in W'^{1,3}_{-1}$ such that $\Phi = \Phi_0 + u(s)$.

(a) By construction of the family when $s = 0$, then $(\Sigma, \boldsymbol{h}^s, K^s)$ equals the extreme Myers-Perry initial data $(\Sigma, \tilde{\boldsymbol{h}}, \tilde{K})$. Moreover, since $u(s) \in W'^{1,3}_{-1}$, Φ is a twice continuously differentiable function.

(b) The asymptotic geometry of the family $(\Sigma, \boldsymbol{h}^s, K^s)$ is described by the asymptotic behaviour of $u(s)$. Since $u(s) \in W'^{1,3}_{-1}$ and $\Phi_0 = o(r^{-1})$, the asymptotic behaviour of the family is same as the extreme Myers-Perry initial data. Moreover, we know the cross-section area of the event horizon is related to the h^s_{ij} and h^s_{zz} part of the data. Then by the choice $\bar{\lambda}'_{ij} \in C^\infty_c(\mathbb{R}^4\backslash\Gamma)$ and $\bar{U} \in C^\infty_c(\mathbb{R}^4\backslash\{0\})$, the perturbed data vanishes near event horizon and the area is preserved. Furthermore, by Proposition 51 the angular momenta obtained from the value of the twist potentials at axis Γ. Then the choice $\bar{Y}^i \in C_c^\infty(\mathbb{R}^4 \backslash \Gamma)$ implies preserved angular momenta, i.e. $J_1^s = J_1$ and $J_2^s = J_2$.

(c) The perturbed part of the data $(\bar{U}, \bar{\lambda}'_{ij}, \bar{Y}^i)$ is only a function of r and θ, this implies the family $(\Sigma, \boldsymbol{h}^s, K^s)$ has $U(1)^2$ symmetry. Moreover, the particular form of the extrinsic curvature

$$K^s_{ab} = 2S^t_{(a}(\lambda^s)^{-1}\Phi_{b)} \tag{7.77}$$

implies $\mathrm{Tr}_{\boldsymbol{h}^s} K^s = 0$, i.e. maximality.

(d) In Theorem 68 of the Chapter 6, we proved that the mass of extreme class is less than the mass of any perturbed data in small neighborhood. Hence, the mass of the family $(\Sigma, \boldsymbol{h}^s, K^s)$ is greater than or equals the mass of the extreme Myers-Perry black hole. Therefore, $(\Sigma, \boldsymbol{h}^s, K^s)$ has positive energy, i.e. $E = m_s - m > 0$.

□

7.4 Summary

In this chapter, with the use of the implicit function theorem we constructed a family of initial data with the same properties of the extreme Myers-Perry black hole initial data. These properties are: asymptotic and horizon geometry, angular momenta, and the cross-section area of the event horizon. Moreover, by Theorem 68 the energy of this family is strictly greater than the extreme Myers-Perry initial data. This suggests similar results for different orbit spaces such as orbit space of the extreme black ring, see Figure 6.1b.

Chapter 8

Conclusion and Open Problems

In this thesis, extending upon the work of S. Dain [53] for four-dimensional axisymmetric black holes we provide, for the first time, the main developments in the mass-angular momenta inequalities and positive mass theorem for five-dimensional $U(1)^2$-invariant spacetimes. The results of this thesis can be divided into three main steps. First, we defined generalized Brill (GB) initial data sets $(\Sigma, \boldsymbol{h}, K)$, which is a class of initial data with symmetries, and we proved some of the topological and geometrical characteristics of this class. Moreover, we constructed a one-parameter family of initial data with asymptotic geometry and angular momenta of the extreme Myers-Perry initial data set. Second, we constructed a mass functional $\mathcal{M}$ and proved some remarkable properties of this functional. Finally, we proved two geometrical inequalities: 1) positive mass theorem for GB initial data sets, 2) local mass angular momenta inequalities for GB initial data sets.

Although the local mass angular momenta inequality for a GB initial data set is an interesting result and encodes the information about stability of this solution under small perturbations, one would like to prove the global version of this inequality for all (or some particular) orbit spaces. In this chapter, we give an overview of the current project which is the global mass angular momenta inequality for black holes with horizon topology $H \cong S^3$ in five dimensions [2]. Moreover, we finish with an outline of some open problems.

Consider the GB initial data $(\Sigma, \boldsymbol{h}, K)$ with the trivial orbit space, see Figure A.2. Then we have the following global parametrization for the Killing part of the metric.

$$\lambda'_{11} = \left(\sqrt{\rho^2+z^2} - z\right) e^{p} \cosh W, \quad \lambda'_{22} = \left(\sqrt{\rho^2+z^2} + z\right) e^{-p} \cosh W, \tag{8.1}$$

$$\lambda'_{12} = \rho \sinh W \,, \tag{8.2}$$

and we define the functions

$$g \equiv \frac{1}{2} \log \rho, \qquad \bar{g} \equiv \frac{1}{2} \log \left(\frac{\sqrt{\rho^2+z^2} - z}{\sqrt{\rho^2+z^2} + z} \right) \tag{8.3}$$

which are harmonic over $\Omega \subset \mathbb{R}^3 \backslash \Gamma$, that is

$$\Delta_3 \log \rho = \Delta_3 \log \left(\frac{\sqrt{\rho^2+z^2} - z}{\sqrt{\rho^2+z^2} + z} \right) = 0 \qquad \text{on } \Omega \subset \mathbb{R}^3 \backslash \Gamma \tag{8.4}$$

where the Laplace operator Δ_3 is with respect to the following flat three dimensional metric

$$\delta_3 = \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2 \mathrm{d}\varphi^2 = r^2 \left(\mathrm{d}r^2 + \frac{r^2 \mathrm{d}x^2}{4(1-x^2)} + \frac{r^2(1-x^2)}{4} \mathrm{d}\varphi \right) . \tag{8.5}$$

Then the associated reduced data can be characterized by $\Psi = (v, p, W, Y^i)$. Moreover, the mass functional $\mathcal{M}$ has the following representation

$$\begin{aligned}\mathcal{M}(\Psi) \;&\equiv\; \frac{1}{16}\int_{\mathbb{R}^3}\left\{12\,|\nabla v|^2 + |\nabla p|^2 + \sinh^2 W\,|\nabla p + \nabla\bar{g}|^2 + \frac{e^{-6g-6v-\bar{g}-p}}{\cosh^2 W}\left|\nabla Y^1\right|^2\right.\\ &+\; \left. e^{-6g+\bar{g}+p-6v}\cosh^2 W\left|\nabla Y^2 - \tanh W\, e^{-p-\bar{g}}\nabla Y^1\right|^2\right\}\mathrm{d}\mu\,. \qquad (8.6)\end{aligned}$$

where $\mathrm{d}\mu = \rho\,\mathrm{d}\rho\mathrm{d}z\mathrm{d}\varphi$ and the covariant derivative is with respect to δ_3. Moreover, if we define the parameter of the Dirichlet energy E which is defined in equation (5.63) by transformation

$$y_1 = 2\log(\det\lambda), \quad y_2 = \frac{1}{2}\log\left(\frac{\lambda_{11}}{\lambda_{22}}\right), \qquad y_3 = \frac{1}{2}\log\left(\frac{\lambda_{11}\lambda_{22}}{\det\lambda}\right), \qquad (8.7)$$

$$y_4 = Y^1, \quad y_5 = Y^2\,, \qquad (8.8)$$

and define the data $\bar{\Psi} = (y_1, \ldots, y_5)$ as parameters of the energy functional

$$\begin{aligned}E(\bar{\Psi}) \;&=\; \frac{1}{32}\int_{\mathbb{R}^3}\left(12\,|\nabla y_1|^2 + \cosh^2 y_3\,|\nabla y_2|^2 + |\nabla y_3|^2 + e^{-(6y_1+y_2)}\,|\nabla y_4|^2\right.\\ &+\; \left. e^{-6y_1+y_2}\left|e^{-y_2}\tanh W\nabla y_4 - \nabla y_5\right|^2\right)\mathrm{d}\mu\,. \qquad (8.9)\end{aligned}$$

Note that the relation of Ψ and $\bar{\Psi}$ is

$$\bar{\Psi} = (y_1, y_2, y_3, y_4, y_5) \longleftrightarrow (g+v, p+\bar{g}, q, Y^1, Y^2) = \Psi \qquad (8.10)$$

Therefore, for $\Omega \subset \mathbb{R}^3\backslash\Gamma$, we have the following relation between mass functional and Dirichlet energy functional

$$E_\Omega(\bar{\Psi}) \;=\; \mathcal{M}_\Omega(\Psi) + \frac{1}{16}\oint_{\partial\Omega}(g+2v)\,\nabla g\cdot\nu\;\mathrm{d}S + \frac{1}{8}\oint_{\partial\Omega} p\nabla\bar{g}\cdot\nu\,\mathrm{d}S\,. \qquad (8.11)$$

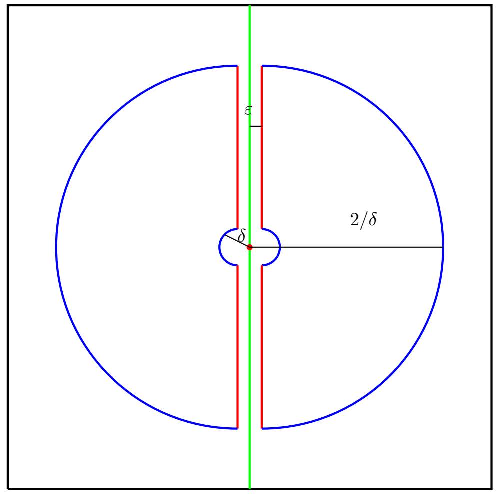


Figure 8.1: The green line is the axis. $\Omega_{\delta,\varepsilon}$ is the region between two blue curves and red lines. Moreover, the $A_{\delta,\varepsilon}$ is the cylindrical region between red lines.

Now the strategy of proof is as follows. Let $\Psi = (v, p, W, Y^1, Y^2)$ denote the given initial data, and $\Psi_0 = (v_0, p_0, W_0, Y_0^1, Y_0^2)$ denote the extreme MP data with the same angular momenta. Let $\delta, \varepsilon > 0$ be small parameters and set $\Omega_{\delta,\varepsilon} = \{\delta < r < 2/\delta; \rho > \varepsilon\}$ and $\mathcal{A}_{\delta,\varepsilon} = B_{2/\delta} \setminus \Omega_{\delta,\varepsilon}$, where $B_{2/\delta}$ is the ball of radius $2/\delta$ centered at the origin, see Figure 8.1. We will cut off the original initial data to obtain $\Psi_{\delta,\varepsilon}$ which satisfies

$$\text{supp}(v_{\delta,\varepsilon} - U_0) \subset B_{2/\delta}, \qquad \text{supp}(p_{\delta,\varepsilon} - p_0, W_{\delta,\varepsilon} - W_0, Y^1_{\delta,\varepsilon} - Y^1_0, Y^2_{\delta,\varepsilon} - Y^2_0) \subset \Omega_{\delta,\varepsilon}. \quad (8.12)$$

Moreover, we must show this cut-off data will converge to the original, allowing us to apply the convexity argument from Schoen and Zhou. This will allow us to prove the following result [2].

Proposition 78. *Let $\varepsilon \ll \delta \ll 1$ and suppose that Ψ satisfies all the asymptotics induced from GB data and extreme Myers-Perry initial data. Then $\Psi_{\delta,\varepsilon}$ satisfies* (8.12) *and*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \mathcal{M}(\Psi_{\delta,\varepsilon}) = \mathcal{M}(\Psi). \quad (8.13)$$

This is the most difficult step in the proof. Then we can prove the global mass-angular momenta inequality for this orbit space structure which appeared in the article, arXiv preprint arXiv:1510.06974, which was submitted to Journal of Communication in Mathematical Physics (CMP) in Oct 2015 [2].

Theorem 79. *Assume that* $(\Sigma, \boldsymbol{h}, K)$ *is GB initial data with mass* m *and fixed angular momenta* J_1 *and* J_2 *and fixed orbit space* $\mathcal{B} = \mathcal{B}_{MP}$ *such that* $\iota_{\xi_{(i)}} j = 0$ *and* $\mu \geq 0$*. Then if* $\Psi = (v, p, W, Y^i)$ *then*

$$\mathcal{M}(\Psi) - \mathcal{M}(\Psi_0) \geq 2C \left(\int_{\mathbb{R}^3} dist^6_{P(3,\mathbb{R})}(\Psi, \Psi_0) \right)^{1/3} \tag{8.14}$$

where $P(3,\mathbb{R}) = SL(3,\mathbb{R})/SO(3,\mathbb{R})$ *and* Ψ_0 *is the data set of the extreme Myers-Perry. It implies*

$$m^3 \geq \frac{27\pi}{32} \left(|J_1| + |J_2|\right)^2 \tag{8.15}$$

and the inequality is saturated if and only if the data is isomorphic to the slice extreme Myers-Perry black hole.

This result shows that the mass functional $\mathcal{M}$ is convex functional for black holes with horizon topology $H \cong S^3$ and it is a generalization of Schoen-Zhou result to five dimensional black holes [139].

There are several open questions concerning the properties of $\mathcal{M}$. For instance, since the ADM mass is conserved quantity under evolution of the Einstein equations, it is necessary to prove $\mathcal{M}$ is a conserved quantity under axisymmetric evolution of Einstein equations. Second, $\mathcal{M}$ is related to the recently constructed Hollands-Wald energy functional [95]. This energy functional shows the relation between thermal instability and dynamical instability for non-extreme black holes and black branes. Moreover, Dain used his mass functional and showed the axisymmetric linear perturbation of the extreme Kerr black hole is stable [56]. Therefore, we expect that the second variation of $\mathcal{M}$ must be related to the Hollands-Wald energy functional, and it should be possible to use $\mathcal{M}$ and prove linear axisymmetric perturbations of the extreme Myers-Perry black hole and (maybe) the extreme doubly spinning black rings are stable. Such methods should be useful for dealing with the challenging issue of non-linear stability.

Finally, in Table 8.1 we classify all the possible generalizations of the mass-angular momenta inequality for GB initial data sets.

		Mass-Charge-Angular Momenta	
	dim$\Sigma = n$	$n = 3$	$n = 4$
vacuum	$H \cong S^{n-1}$	Dain [53, 139]	Local (AA.3),global (AA.6)
	H =eqs. (3.33)	—	Local version (AA.3)
$T_{ab} \neq 0$	$H \cong S^{n-1}$	Chrusciel,Dain [44, 58]	Local (AA.5),global (AA.6)
	H =eqs. (3.33)	—	Local version (AA.5)
Charges	$H \cong S^{n-1}$	Chrusciel and Costa [41, 139]	Open problem
	H =eqs. (3.33)	—	Open problem
Multiple ends	$H \cong S^{n-1}$	Chrusciel,Li,Wienstein[44, 139]	Open problem
	H =eqs. (3.33)	—	Open problem
Non-Maximal	$H \cong S^{n-1}$	Cha and Khuri [30]	Open problem
	H =eqs. (3.33)	—	Open problem
Manifold with bdry	$H \cong S^{n-1}$	Open problem	Open problem
	H =eqs. (3.33)	—	Open problem

Table 8.1: Open problems: mass-charge-angular momenta inequalities

We hope to address these open problems in the near future.

Appendix A

Myers-Perry Black Hole

In 1985 Myers and Perry found an asymptotically flat solution of the Einstein vacuum equations which describe a black hole with spherical horizon topology in all dimensions[125]. Here we consider the five dimensional Myers-Perry black hole $(M, \boldsymbol{g})$ with the metric in coordinates $(t, \tilde{r}, \theta, \phi_1, \phi_2)$ and represented locally as [124]

$$\begin{aligned} ds^2 &= -dt^2 + \frac{\mu}{P}\left(dt + a\sin^2\theta d\phi_1 + b\cos^2\theta d\phi_2\right)^2 + \frac{\tilde{r}^2 P}{\Delta(\tilde{r})} d\tilde{r}^2 + P d\theta^2 \\ &+ \left(\tilde{r}^2 + b^2\right)\cos^2\theta d\phi_2^2 + \left(\tilde{r}^2 + a^2\right)\sin^2\theta d\phi_1^2, \end{aligned} \tag{A.1}$$

where

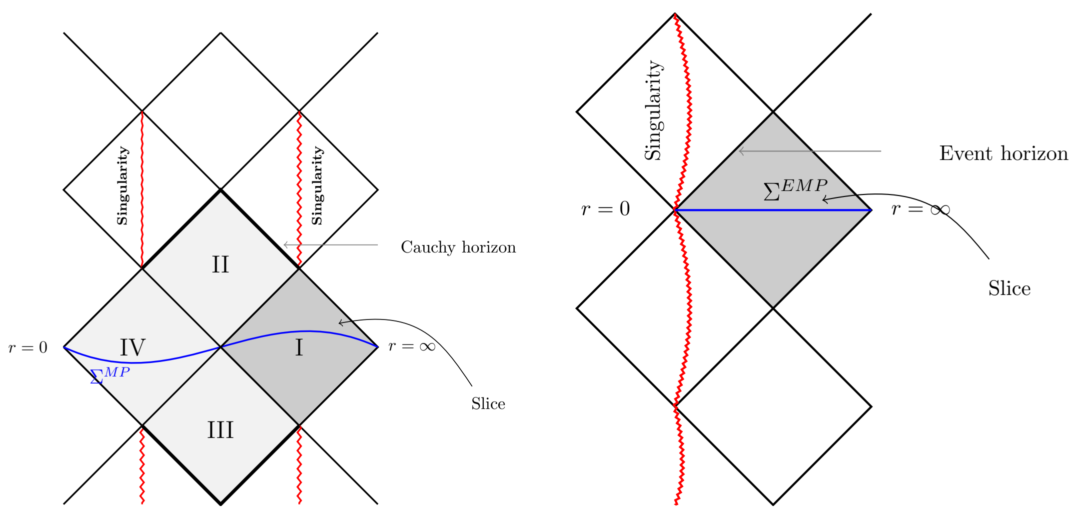
$$P = \tilde{r}^2 + b^2\sin^2\theta + a^2\cos^2\theta, \qquad \Delta = \left(\tilde{r}^2 + a^2\right)\left(\tilde{r}^2 + b^2\right) - \mu\tilde{r}^2. \tag{A.2}$$

The solution is parameterized by (μ, a, b) with orthogonally transitive isometry group $\mathbb{R} \times U(1)^2$, where $\mathbb{R}$ is the time translation symmetry and $U(1)^2$ is the rotational symmetry generated by ∂_{ϕ^1} and ∂_{ϕ^2}. Here $(\tilde{r}, \theta)$ parameterize the two-dimensional surfaces orthogonal to orbits of the isometry group. The horizons of this black hole are located at the roots of $\Delta(\tilde{r})$, denoted $\tilde{r}_{H\pm} = \pm\sqrt{\frac{\mu - a^2 - b^2 + \sqrt{(\mu - a^2 - b^2)^2 - 4a^2b^2}}{2}}$. The metric is written in a chart that covers the black hole exterior $\tilde{r}_{H+} < \tilde{r} < \infty$. In addition $0 < \theta < \pi/2$, and ϕ^1, ϕ^2 are rotational coordinate with period 2π. Moreover, the singularity of this metric for nonvanishing a and b with $a^2 \neq b^2$ is located at roots of P, i.e.

$$\sin^2\theta = \frac{\tilde{r}^2 - a^2}{b^2 - a^2} \tag{A.3}$$

where $b > a$. As it is well known, the solution is qualitatively similar to the Kerr solution. In the extreme limit, $\mu = (a+b)^2$ and $\Delta(\tilde{r}) = \left(\tilde{r}^2 - ab\right)^2$. The ADM mass and angular momenta of this metric are

$$M = \frac{3}{8}\pi\mu, \qquad J^1 = \frac{2}{3}Ma, \qquad J^2 = \frac{2}{3}Mb \tag{A.4}$$



(a) Maximal analytic extension of MP metric and Σ is slice of constant time. The singularity is at $\Sigma = 0$

(b) Maximal analytic extension of extreme MP black hole. The gray region is domain of outer communication=DOC

Figure A.1: Carter-Penrose diagram of Myers-Perry black hole in 5 dimensions

A.1 Myers-Perry Initial Data

Consider constant time slice of the Myers-Perry black hole in $(t, \tilde{r}, \theta, \phi^1, \phi^2)$. Then the metric for initial data is

$$\boldsymbol{h} = \frac{\tilde{r}^2 P}{\Delta(\tilde{r})} \mathrm{d}\tilde{r}^2 + P \mathrm{d}\theta^2 + \lambda_{ij} \mathrm{d}\phi^i \mathrm{d}\phi^j \tag{A.5}$$

where it covers the region $\tilde{r}_{H+} \leq \tilde{r} \leq \infty$ (see dark gray region in Figure A.1a) and the positive definite matrix λ has components

$$\lambda_{11} = \frac{a^2\mu}{P} \sin^4\theta + (\tilde{r}^2 + a^2)\sin^2\theta, \quad \lambda_{12} = \frac{ab\mu}{P}\sin^2\theta\cos^2\theta, \tag{A.6}$$
$$\lambda_{22} = \frac{b^2\mu}{P}\cos^4\theta + (\tilde{r}^2 + b^2)\cos^2\theta$$

The metric has a coordinate singular at inner boundary $r = r_{H+}$ and the slice is a Riemannian manifold with boundary $\Sigma_0^{MP} = [r_{H+}, \infty) \times S^3$. One can define a quasi-isotropic coordinate and extend the manifold to the doubling manifold or full slice Σ^{MP} (see Figure A.1a) as

$$\tilde{r}^2 = r^2 + \frac{1}{2}\left(\mu - a^2 - b^2\right) + \frac{\mu\,(\mu - 2a^2 - 2b^2) + (a^2 - b^2)^2}{16 r^2} \tag{A.7}$$

Note that the inner boundary at $\tilde{r}_{H+}$ is shifted to $r = 0$ and the metric is

$$\boldsymbol{h} = \frac{P}{r^2}\left(\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2\right) + \lambda_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j \tag{A.8}$$

where $0 < r < \infty$, $0 < \theta < \pi/2$, and $0 < \phi_1, \phi_2 < 2\pi$. The point $r = 0$ is another asymptotic infinity (see Figure A.2) and one can show this with computing the distance to $r = 0$ along a curve of constant (θ, ϕ_1, ϕ_2) from $r = r_0$, i.e.

$$\text{Distance} = \int_r^{r_0} \frac{\sqrt{P}}{r} dr \to \infty \qquad \text{as} \quad r \to 0 \tag{A.9}$$

Let $r_{\min} = \left[(\mu - (a+b)^2)(\mu - (a-b)^2)\right]^{1/4}$, then $r = r_{\min}$ is the stable minimal surface in Σ^{MP} which represents the event horizon of spacetime. In the extreme limit $\mu = (a+b)^2$ the quasi-isotropic coordinate simplifies to

$$\tilde{r}^2 = r^2 + ab \tag{A.10}$$

The conformal metric $\tilde{h}$ of the slice is

$$\boldsymbol{h} = \Phi^2 \tilde{\boldsymbol{h}}, \qquad \tilde{\boldsymbol{h}} = e^{2U}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \lambda'_{ij}\mathrm{d}\phi^i \mathrm{d}\phi^j \tag{A.11}$$

and can be determined by the relations

$$\Phi^2 = \frac{\sqrt{\det \lambda}}{\rho}, \qquad e^{2U} = \frac{P\rho}{r^4\sqrt{\det \lambda}}, \qquad \lambda'_{ij} = \Phi^{-2}\lambda_{ij} \tag{A.12}$$

where $\rho = \frac{1}{2}r^2 \sin 2\theta$ and $z = \frac{1}{2}r^2 \cos 2\theta$.

A.1.1 Non-extreme Myers-Perry Initial Data

The non-extreme Myers-Perry initial data set $(\Sigma^{MP}, \boldsymbol{h}, K)$ belongs to the class of $t - \phi^i$ symmetric initial data. The manifold is a complete Riemannian manifold $\Sigma^{MP} = \mathbb{R} \times S^3$ with two asymptotically flat ends (See Figure 3.4a). In this section we show behaviour of this data near two ends and on the axis. The asymptotic behaviours of data at infinity ($r \to \infty$) are given by

$$\Phi = 1 + \frac{\mu}{4r^2} + \mathcal{O}(r^{-4}), \qquad v = \frac{\mu}{4r^2} + o(r^{-4}) \tag{A.13}$$

$$\lambda'_{ij} = \sigma_{ij} + \frac{f_{il}\sigma_{lj}}{r^2} \qquad \mathrm{Tr} f_{ij} = 0 \tag{A.14}$$

$$V = \frac{(a^2 - b^2)\cos 2\theta}{r^2} + O(r^{-4}) \tag{A.15}$$

The region $r \to 0$ corresponds to another asymptotic region. In the non-extreme case, we have

$$\Phi = \frac{\sqrt[4]{(\mu - (a+b)^2)^2(\mu - (a-b)^2)^2}}{4r^2} + \mathcal{O}(1), \quad \Phi_{,r} = \mathcal{O}(r^{-3}) \quad r \to 0 \tag{A.16}$$

$$v = -2\log r + O(1) \tag{A.17}$$

$$V = \frac{4(a^2 - b^2)\cos 2\theta}{(\mu - (a-b)^2)(\mu - (a+b)^2)} r^2 + O(r^2) \tag{A.18}$$

We will consider explicitly the non-extreme case so the end is asymptotically flat. First we have the following expansion for v at origin and infinity

$$|\nabla v|^2 = -\frac{\mu}{2r^5} + \mathcal{O}(r^{-7}) \quad r \to \infty \qquad |\nabla v|^2 = -\frac{2}{r^3} + \mathcal{O}(r^{-1}) \quad r \to 0 \tag{A.19}$$

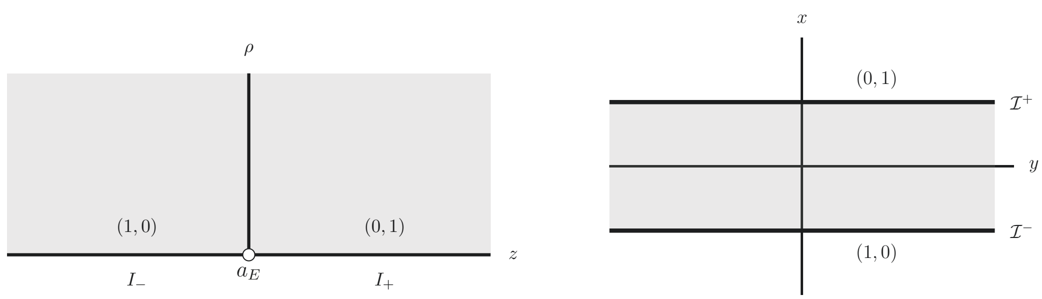
since the volume element is $\rho \mathrm{d}\rho \mathrm{d}z = r^5 \sin\theta \cos\theta \, \mathrm{d}r\mathrm{d}\theta$, $(dv)^2$ is bounded at origin and infinity. Now we consider term which is related to the scalar curvature in the mass functional $\mathcal{M}$:

$$\frac{\det \nabla\lambda'}{\det \lambda'} = \mathcal{O}(r^{-8}) \quad r \to \infty \qquad \frac{\det \nabla\lambda'}{\det \lambda'} = \mathcal{O}(1) \quad r \to 0 \tag{A.20}$$

This is clearly bounded. Now we check the last term which is related to the full contraction of extrinsic curvature:

$$\frac{\nabla Y^t \lambda'^{-1} \nabla Y}{2 \det \lambda'} = \mathcal{O}(r^{-10}) \quad r \to \infty \qquad e^{-6v} \frac{\nabla Y^t \lambda'^{-1} \nabla Y}{2 \det \lambda'} = \mathcal{O}(r^2) \quad r \to 0 \qquad \text{(A.21)}$$

An important piece of data in the mass functional is the twist potential column vector $Y = (Y^1, Y^2)^t$. Here we compute explicitly the twist potentials using equation (3.37) of non-extreme Myers-Perry initial data (black hole) and they are



(a) Orbit space of a Myers-Perry spatial slice on upper half plane

(b) Orbit space of a Myers-Perry spatial slice represented on infinite strip

Figure A.2: (a) and (b) are spacetime interval structures for the Myers-Perry black hole.

$$\begin{aligned}
Y^1 &= \frac{\left[C^2 + 256 r^4 \Sigma (a^2 - b^2) \cos^2\theta\right] \left(C - 16 r^2 \left(a^2 - b^2\right)\right) \mu a}{16^3 r^6 \Sigma (a^2 - b^2)^2} - \frac{C^2 - 32 r^4 H_1}{256 r^4 (a^2 - b^2)^2} \mu a \\
Y^2 &= -\frac{\left[C \left(C - 32 r^2 \left(a^2 - b^2\right)\right) + 256 r^4 (a^2 - b^2)(\Sigma \cos^2\theta + (a^2 - b^2))\right] C \mu b}{16^3 r^6 \Sigma (a^2 - b^2)^2} \\
&+ \frac{C^2 - 16 r^2 C \left(a^2 - b^2\right) + 32 r^4 H_2}{256 r^4 (a^2 - b^2)^2} \mu b \\
C &= 16 r^4 + 8(\mu + a^2 - b^2) r^2 + \left(\mu - (a - b)^2\right)\left(\mu - (a + b)^2\right) \\
H_1 &= 3(a^2 - b^2)^2 + \mu \left(3\mu - 6b^2 + 2a^2\right) \\
H_2 &= (a^2 - b^2)^2 + \mu \left(2a^2 + 2b^2 - 3\mu\right)
\end{aligned} \qquad \text{(A.22)}$$

A.1.2 Extreme Myers-Perry Initial Data

The extreme Myers-Perry initial data set $(\Sigma^{EMP}, \boldsymbol{h}, K)$ is a $t-\phi^i$ symmetric initial data set. The manifold is a complete Riemannian manifold $\Sigma^{EMP} = \mathbb{R}\times S^3$ with one asymptotically flat end and one cylindrical end (See Figure 3.4b). Similar to previous section, we show the asymptotic behaviour. But we start with twist potentials

$$\begin{aligned} Y^1 &= \frac{a(a^2-b^2)(r^2+ab+b^2)\cos^2\theta - r^2a(2a^2+2ab+r^2)}{(a-b)^2} \\ &+ \frac{a(r^2+ab+a^2)^2(r^2+ab+b^2)}{\Sigma(a-b)^2} \\ Y^2 &= \frac{br^2((a+b)^2+r^2) - b(a^2-b^2)(r^2+ab+a^2)\cos^2\theta}{(a-b)^2} \\ &- \frac{b(r^2+ab+a^2)(r^2+ab+b^2)^2}{\Sigma(a-b)^2} \end{aligned} \tag{A.23}$$

The expansion at infinity is

$$Y^1 = \frac{a^3(a+b)^2}{(a-b)^2} - \frac{4J_1}{\pi}\cos^2\theta(2-\cos^2\theta) + \mathcal{O}(r^{-2}) \tag{A.24}$$

$$Y^2 = -\frac{ab^2(a+b)^2}{(a-b)^2} - \frac{4J_2}{\pi}\cos^4\theta + \mathcal{O}(r^{-2}) \tag{A.25}$$

The asymptotic behaviour of the conformal factor at infinity is given by

$$\Phi = 1 + \frac{\mu}{4r^2} + \mathcal{O}(r^{-4}) \qquad r\to\infty \tag{A.26}$$

Hence this region is an asymptotically flat end. In the extreme case, however, one can check that

$$\Phi = \frac{(ab(a+b)^3)^{1/4}}{(a\cos^2\theta + b\sin^2\theta)r} + O(r) \qquad r\to 0 \tag{A.27}$$

By examining the behaviour of the metric h, one can see that the asymptotic region $r \to 0$ is a cylindrical end. In fact, explicit computation of U and λ_i' shows that the conformal metric $\tilde{h}$ approaches the metric of a cone over an S^3 equipped with an inhomogeneous metric,

$$\tilde{h} = \Omega^2 \left(dr^2 + r^2\gamma\right) \tag{A.28}$$

where $\Omega = \Omega(\theta) \neq 0$ and γ is conformal to the inhomogeneous metric on cross-sections of the horizon of the extreme Myers-Perry black hole. One can expand the function V at infinity and at the origin. As we discussed before we only consider the behaviour of V near $\rho = 0$. We find

$$V = \frac{(a^2 - b^2)\cos 2\theta}{4r^2} + \mathcal{O}(r^{-4}) \qquad r \to \infty \tag{A.29}$$

$$V_+ = \frac{2z + a^2 + ab}{\sqrt{4z^2 + 3a^2b^2 + 2a^2z + b^4 + 2b^2z + 4abz + a^3b + 3ab^3}} \qquad z \in I_+ \tag{A.30}$$

$$V_- = \frac{-2z + b^2 + ab}{\sqrt{a^4 + 3a^3b + 3a^2b^2 - 2a^2z + ab^3 - 4abz - 2b^2z + 4z^2}} \qquad z \in I_- \tag{A.31}$$

Thus V satisfies boundary conditions of GB data. In particular, we read off $\bar{V} = \frac{1}{4}(a^2 - b^2)x$ and it follows $\tilde{h}$ (see (A.12)) has vanishing ADM mass. In addition, when $z \to \pm\infty$ we have $V_\pm \to 1$ and $V_\pm$ are bounded continuous functions on rods $I_\pm$. Therefore, they are integrable. Let us consider boundedness of other terms in the mass functional $\mathcal{M}$. We will consider explicitly the extreme case so the end is asymptotically flat. First we have the following expansion for v at origin and infinity

$$(\mathrm{d}v)^2 = -\frac{\mu}{2r^5} + \mathcal{O}(r^{-7}) \quad r \to \infty \qquad (\mathrm{d}v)^2 = -\frac{C(\theta)}{r^4} + \mathcal{O}(r^{-3}) \quad r \to 0 \tag{A.32}$$

since the volume element is $\rho \mathrm{d}\rho \mathrm{d}z = r^5 \sin\theta\cos\theta\, \mathrm{d}r\mathrm{d}\theta$, $(dv)^2$ is bounded at origin and infinity. Now we consider a term which related to scalar curvature in mass functional

$\mathcal{M}$. We use identity (5.55) and we have

$$\frac{\det \nabla\lambda'}{\det \lambda'} = \mathcal{O}(r^{-8}) \quad r \to \infty \qquad \frac{\det \nabla\lambda'}{\det \lambda'} = \mathcal{O}(r^{-4}) \quad r \to 0 \tag{A.33}$$

This is clearly bounded. The only term remaining is related to the full contraction of extrinsic curvature and we have

$$\frac{\nabla Y^t \lambda'^{-1} \nabla Y}{2 \det \lambda'} = \mathcal{O}(r^{-10}) \quad r \to \infty \qquad e^{-6v} \frac{\nabla Y^t \lambda'^{-1} \nabla Y}{2 \det \lambda'} = \mathcal{O}(r^{-4}) \quad r \to 0 \tag{A.34}$$

Appendix B

Carter Identity In Dimension Five

In this section we first derive a five-dimensional version of Carter's identity. Assume we are in a five-dimensional vacuum spacetime with isometry group $\mathbb{R} \times U(1)^2$. The field equations can be expressed simply as the conservation of a current (see [67] for details) such that.

$$\mathrm{div} J = \mathrm{div}\left(\rho\, \Phi^{-1} \nabla \Phi\right) = 0 \tag{B.1}$$

This equation arises as critical points of action (3.38) and where

$$\Phi \equiv \Phi(X, Y, \lambda) = \begin{pmatrix} \frac{1}{X} & -\frac{Y^t}{X} \\ -\frac{Y}{X} & \lambda + \frac{YY^t}{X} \end{pmatrix} \qquad \Phi^{-1} = \begin{pmatrix} X + Y^t \lambda^{-1} Y & Y^t \lambda^{-1} \\ \lambda^{-1} Y & \lambda^{-1} \end{pmatrix} \tag{B.2}$$

and $\det \Phi = 1$, λ is a positive definite 2×2 symmetric matrix with $\det \lambda = X$ and Y is a column vector. One can derive the Mazur identity (for a detailed discussion see [29]) for two matrices $\Phi_{[1]}$ and $\Phi_{[2]}$ (not necessarily solutions) with corresponding currents $J_{[1]}, J_{[2]}$

$$\Delta \Psi - \mathrm{Tr}\left(\Phi_{[2]} \left(\mathrm{div} \mathring{J}\right) \Phi_{[1]}^{-1}\right) = \frac{1}{\rho^2} \mathrm{Tr}\left(\mathring{J}^t \Phi_{[2]} \mathring{J} \Phi_{[1]}^{-1}\right) \tag{B.3}$$

where Δ is Laplace operator with respect to flat metric δ_3 and

$$\Psi = \mathrm{Tr}\left(\Phi_{[2]}\Phi_{[1]}^{-1} - I\right) \qquad \mathring{J} = J_{[2]} - J_{[1]} \tag{B.4}$$

and

$$\mathrm{div}J = \begin{pmatrix} -G_X - Y^t G_Y & -XG_Y^t\lambda + Y^t G_X + Y^t G_\lambda + Y^t G_Y Y^t \\ -G_Y & G_\lambda + G_Y Y^t \end{pmatrix} \tag{B.5}$$

Note that this identity holds quite generally for any field theory which can be derived from a positive definite action with Lagrangian of the form $L \sim \mathrm{Tr}(\Phi^{-1}\mathrm{d}\Phi)^2$. The linearized version of this identity in four dimensions was originally found by Carter [27] and plays an important role in geometric inequalities in 3+1 dimensional spacetime [51, 53, 57, 59]. We will now derive a generalization of this identity for five dimensions. Assume we have $\Phi_{[1]}(X, Y, \lambda)$ and $\Phi_{[2]}(X_2, Y_2, \lambda_2)$ related by

$$\begin{aligned} X_2 &= X + t\dot{X} \qquad & Y_2 &= Y + t\dot{Y} \qquad & \lambda_2 &= \lambda + t\dot{\lambda} \\ G_{\lambda_2} &= G_\lambda + t\dot{G}_\lambda, \qquad & G_{X_2} &= G_X + t\dot{G}_X & & \end{aligned} \tag{B.6}$$

The overdot $\dot{}$ represents the linear order of expansion or first variation with respect to t (when taking variations of the products of several terms, we use the notation δ instead of dot for convenience of notation).

B.1 LHS of Carter identity

We start by expanding of the main terms in left hand side of Carter identity

$$\Phi_{[1]} = \Phi, \qquad \Phi_{[2]} = \Phi + t\dot{\Phi} + t^2\ddot{\Phi} + O(t^3), \qquad \mathrm{div}\mathring{J} = \mathrm{div}\dot{J}t + t\mathrm{div}\ddot{J}t^2 + O(t^3)\,, \tag{B.7}$$

where

$$\dot{\Phi} = \begin{pmatrix} -\frac{\dot{X}}{X^2} & \frac{\dot{X}Y^t}{X^2} - \frac{\dot{Y}^t}{X} \\ \frac{\dot{X}Y}{X^2} - \frac{\dot{Y}}{X} & \dot{\lambda} + \delta\left[\frac{YY^t}{X}\right] \end{pmatrix} \qquad \ddot{\Phi} = \begin{pmatrix} \frac{\dot{X}^2}{X^3} & -\frac{\dot{X}^2Y^t}{X^2} + \frac{\dot{X}\dot{Y}^t}{X^2} \\ -\frac{\dot{X}^2Y}{X^2} + \frac{\dot{X}\dot{Y}}{X^2} & \delta^2\left[\frac{YY^t}{X}\right] \end{pmatrix} \tag{B.8}$$

$$\mathrm{div}\dot{J} = \begin{pmatrix} -\dot{G}_X - \dot{Y}^tG_Y - Y^t\dot{G}_Y & \delta\left[-XG_Y^t\lambda + Y^tG_X + Y^tG_\lambda + Y^tG_YY^t\right] \\ -\dot{G}_Y & \dot{G}_\lambda + \dot{G}_YY^t + G_Y\dot{Y}^t \end{pmatrix} \tag{B.9}$$

The first term is straightforward

$$\begin{aligned}
\Psi &= \mathrm{Tr}\left(\left[\Phi + t\dot{\Phi} + t^2\ddot{\Phi} + O(t^3)\right]\Phi^{-1} - I\right) \\
&= \mathrm{Tr}\left(t\dot{\Phi}\Phi^{-1} + t^2\ddot{\Phi}\Phi^{-1}\right) + O(t^3) \\
&= t^2\mathrm{Tr}\left(\ddot{\Phi}\Phi^{-1}\right) + O(t^3) \\
&= t^2\left[\frac{\dot{X}^2}{X^3}\left(X + Y^t\lambda^{-1}Y\right) + 2\left(-\frac{\dot{X}^2Y^t}{X^2} + \frac{\dot{X}\dot{Y}^t}{X^2}\right)\lambda^{-1}Y\right. \\
&+ \left.\mathrm{Tr}\left(\delta^2\left[\frac{YY^t}{X}\right]\lambda^{-1}\right)\right] + O(t^3) \\
&= \left(\frac{\dot{Y}^t\lambda^{-1}\dot{Y}}{X} + \frac{\dot{X}^2}{X^2}\right)t^2 + O(t^3)
\end{aligned} \tag{B.10}$$

In the second term in equation (B.7) is

$$\begin{aligned}
&\mathrm{Tr}\left(\Phi_{[2]}\left(\mathrm{div}\mathring{J}\right)\Phi_{[1]}^{-1}\right) \\
=\ & t\mathrm{Tr}\left(\Phi\mathrm{div}\dot{J}\Phi^{-1}\right) + t^2\mathrm{Tr}\left(\dot{\Phi}\mathrm{div}\dot{J}\Phi^{-1}\right) + t^2\mathrm{Tr}\left(\Phi\mathrm{div}\ddot{J}\Phi^{-1}\right) + O(t^3) \\
=\ & t\mathrm{Tr}\left(\mathrm{div}\dot{J}\right) + t^2\mathrm{Tr}\left(\dot{\Phi}\mathrm{div}\dot{J}\Phi^{-1}\right) + t^2\mathrm{Tr}\left(\mathrm{div}\ddot{J}\right) + O(t^3) \\
=\ & t\delta\left[\mathrm{Tr}\left(\mathrm{div}J\right)\right] + t^2\mathrm{Tr}\left(\dot{\Phi}\mathrm{div}\dot{J}\Phi^{-1}\right) + t^2\delta^2\left[\mathrm{Tr}\left(\mathrm{div}J\right)\right] + O(t^3) \\
=\ & t^2\mathrm{Tr}\left(\dot{\Phi}\mathrm{div}\dot{J}\Phi^{-1}\right) + O(t^3)
\end{aligned} \tag{B.11}$$

Then after some lines of algebra we obtain

$$\begin{aligned} LHS &= \Delta\left(\frac{\dot{Y}^t\lambda^{-1}\dot{Y}}{X}+\frac{\dot{X}^2}{X^2}\right)+\frac{\dot{Y}^t\lambda^{-1}\dot{Y}}{X}G_X-\frac{\dot{X}}{X}\dot{G}_X-2\frac{\dot{X}}{X}G_Y^t\dot{Y} \\ &- 2\dot{Y}^t\lambda^{-1}\dot{\lambda}G_Y+\frac{\dot{Y}^t\lambda^{-1}G_\lambda^t\dot{Y}}{X}-\mathrm{Tr}\left(\lambda^{-1}\dot{\lambda}\dot{G}_\lambda^t\right)-2\dot{G}_Y^t\dot{Y} \end{aligned} \tag{B.12}$$

B.2 RHS of Carter identity

To find RHS we expand each term and we have

$$\begin{aligned} \mathring{J}_{11} &= \left[-\nabla\left(\frac{\dot{X}}{X}\right)-\frac{\dot{Y}^t\lambda^{-1}\nabla Y}{X}+\frac{Y^t\lambda^{-1}\dot{\lambda}\lambda^{-1}\nabla Y}{X}-\frac{Y^t\lambda^{-1}\nabla\dot{Y}}{X}\right. \\ &+ \left.\frac{\dot{X}Y^t\lambda^{-1}\nabla Y}{X^2}\right]t+O(t) \\ \mathring{J}_{22} &= \left[-\lambda^{-1}\dot{\lambda}\lambda^{-1}\nabla\lambda+\lambda^{-1}\nabla\dot{\lambda}-\frac{\lambda^{-1}\dot{\lambda}\lambda^{-1}\nabla YY^t}{X}+\frac{\lambda^{-1}\nabla\dot{Y}Y^t}{X}+\frac{\lambda^{-1}\nabla Y\dot{Y}^t}{X}\right. \\ &- \left.\frac{\dot{X}\lambda^{-1}\nabla YY^t}{X^2}\right]t+O(t) \\ \mathring{J}_{21} &= \left[\frac{\lambda^{-1}\dot{\lambda}\lambda^{-1}\nabla Y}{X}-\frac{\lambda^{-1}}{X^2}\left(X\nabla\dot{Y}-\dot{X}\nabla Y\right)\right]t+O(t) \\ \mathring{J}_{12} &= \left[-\nabla\dot{Y}^t+\frac{\dot{Y}^t\nabla X}{X}+Y^t\nabla\left(\frac{\dot{X}}{X}\right)+\dot{Y}^t\lambda^{-1}\nabla\lambda-Y^t\lambda^{-1}\dot{\lambda}\lambda^{-1}\nabla\lambda+Y^t\lambda^{-1}\nabla\dot{\lambda}\right. \\ &+ \frac{\dot{Y}^t\lambda^{-1}\nabla YY^t}{X}-\frac{Y^t\lambda^{-1}\dot{\lambda}\lambda^{-1}\nabla YY^t}{X}+\frac{Y^t\lambda^{-1}\nabla\dot{Y}Y^t}{X} \\ &+ \left.\frac{Y^t\lambda^{-1}\nabla Y\dot{Y}^t}{X}-\frac{\dot{X}Y^t\lambda^{-1}\nabla YY^t}{X^2}\right]t+O(t) \end{aligned} \tag{B.13}$$

Since we only want the order $O(t^2)$ we have $\Phi_{[2]} = \Phi_{[1]} = \Phi$ and

$$\begin{aligned}
& \mathrm{Tr}\left(\mathring{J}^t \Phi_{[2]} \mathring{J} \Phi_{[1]}^{-1}\right) \\
= \; & \mathrm{Tr}\Bigg[\left(\mathring{J}^t_{11}\Phi_{11} + \mathring{J}^t_{21}\Phi_{21}\right)\left(\mathring{J}_{11}\Phi_{11}^{-1} + \mathring{J}_{12}\Phi_{21}^{-1}\right) + \left(\mathring{J}^t_{11}\Phi_{12} + \mathring{J}^t_{21}\Phi_{22}\right)\left(\mathring{J}_{21}\Phi_{11}^{-1} + \mathring{J}_{22}\Phi_{21}^{-1}\right) \\
+ \; & \left(\mathring{J}^t_{12}\Phi_{11} + \mathring{J}^t_{22}\Phi_{21}\right)\left(\mathring{J}_{11}\Phi_{12}^{-1} + \mathring{J}_{12}\Phi_{22}^{-1}\right) + \left(\mathring{J}^t_{12}\Phi_{12} + \mathring{J}^t_{22}\Phi_{22}\right)\left(\mathring{J}_{21}\Phi_{12}^{-1} + \mathring{J}_{22}\Phi_{22}^{-1}\right)\Bigg] \\
= \; & I_1 + I_2 + I_3 + I_4 \qquad \text{(B.14)}
\end{aligned}$$

To compute this we divide it to four terms. First, we compute I_1:

$$\begin{aligned}
I_1 &= \mathrm{Tr}\left[\left(\mathring{J}^t_{11}\Phi_{11} + \mathring{J}^t_{21}\Phi_{21}\right)\left(\mathring{J}_{11}\Phi_{11}^{-1} + \mathring{J}_{12}\Phi_{21}^{-1}\right)\right] \\
&= \mathrm{Tr}\left[\frac{1}{X}\left(\mathring{J}^t_{11} - \mathring{J}^t_{21}Y\right)\left(\mathring{J}_{11}\left(X + Y^t\lambda^{-1}Y\right) + \mathring{J}_{12}\lambda^{-1}Y\right)\right] \\
&= \frac{1}{X}\mathrm{Tr}\left[XA_1^2 + A_1 Y^t \lambda^{-1} A_2 Y + A_3\right] \qquad \text{(B.15)}
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= -\nabla\left(\frac{\dot{X}}{X}\right) - \frac{\dot{Y}^t\lambda^{-1}\nabla Y}{X} \quad A_2 = \nabla\left(\dot{\lambda}\lambda^{-1}\right) + \frac{\nabla Y \dot{Y}^t \lambda^{-1}}{X} \\
A_3 &= -2\nabla\dot{Y}^t\lambda^{-1}Y + \frac{\dot{X}Y^t\lambda^{-1}\nabla Y}{X} + Y^t\lambda^{-1}\dot{\lambda}\lambda^{-1}\nabla Y \\
&+ \frac{\nabla X \dot{Y}^t \lambda^{-1} Y}{X} + \dot{Y}^t\lambda^{-1}\nabla\lambda\lambda^{-1}Y \qquad \text{(B.16)}
\end{aligned}$$

Second, by straightforward computation we get

$$
\begin{aligned}
I_2 &= \mathrm{Tr}\left[\left(\mathring{J}^t_{11}\Phi_{12}+\mathring{J}^t_{21}\Phi_{22}\right)\left(\mathring{J}_{21}\Phi^{-1}_{11}+\mathring{J}_{22}\Phi^{-1}_{21}\right)\right]\\
&= \mathrm{Tr}\left[\frac{1}{X}\left(-\mathring{J}^t_{11}Y^t-\mathring{J}^t_{21}\left(X\lambda+YY^t\right)\right)\left(\mathring{J}_{21}\left(X+Y^t\lambda^{-1}Y\right)+\mathring{J}_{22}\lambda^{-1}Y\right)\right]\\
&= \frac{1}{X}\mathrm{Tr}\left[\left(-A_1Y^t+B^t_1\right)\left(\lambda^{-1}A_2Y+\lambda^{-1}B_1\right)\right] \qquad \text{(B.17)}
\end{aligned}
$$

where $B_1=\frac{\dot{X}}{X}\nabla Y+\dot{\lambda}\lambda^{-1}\nabla Y-\nabla\dot{Y}$. Third, we obtain

$$
\begin{aligned}
I_3 &= \mathrm{Tr}\left[\left(\mathring{J}^t_{12}\Phi_{11}+\mathring{J}^t_{22}\Phi_{21}\right)\left(\mathring{J}_{11}\Phi^{-1}_{12}+\mathring{J}_{12}\Phi^{-1}_{22}\right)\right]\\
&= \mathrm{Tr}\left[\frac{1}{X}\left(\mathring{J}^t_{12}-\mathring{J}^t_{22}Y\right)\left(\mathring{J}_{11}Y^t\lambda^{-1}+\mathring{J}_{12}\lambda^{-1}\right)\right]\\
&= \frac{1}{X}\mathrm{Tr}\left[\left(-A_1Y+C_1\right)\left(Y^t\lambda^{-1}A_2+C^t_1\lambda^{-1}\right)\right] \qquad \text{(B.18)}
\end{aligned}
$$

where $C_1=\nabla\lambda\lambda^{-1}\dot{Y}+\frac{\nabla X}{X}\dot{Y}-\nabla\dot{Y}$. Finally, we have

$$
\begin{aligned}
I_4 &= \mathrm{Tr}\left[\left(\mathring{J}^t_{12}\Phi_{12}+\mathring{J}^t_{22}\Phi_{22}\right)\left(\mathring{J}_{21}\Phi^{-1}_{12}+\mathring{J}_{22}\Phi^{-1}_{22}\right)\right]\\
&= \mathrm{Tr}\left[\frac{1}{X}\left(-\mathring{J}^t_{12}Y^t+\mathring{J}^t_{22}\left(X\lambda+YY^t\right)\right)\left(\mathring{J}_{21}Y^t\lambda^{-1}+\mathring{J}_{22}\lambda^{-1}\right)\right]\\
&= \frac{1}{X}\mathrm{Tr}\left[\left(A_1YY^t+XA_2\lambda-\left(YB^t_1+C_1Y^t\right)\right)\left(\lambda^{-1}A_2\right)\right] \qquad \text{(B.19)}
\end{aligned}
$$

Then the right hand side will be

$$
\mathrm{RHS}=A^2_1+\frac{1}{X}B^t_1\lambda^{-1}B_1+\frac{1}{X}C^t_1\lambda^{-1}C_1+\mathrm{Tr}\left(A^2_2\right)\geq 0 \qquad \text{(B.20)}
$$

This is positive since λ is a positive definite matrix. Thus Carter identity is

$$
\begin{aligned}
& \Delta\left(\frac{\dot{Y}^t\lambda^{-1}\dot{Y}}{X}+\frac{\dot{X}^2}{X^2}\right)+\frac{\dot{Y}^t\lambda^{-1}\dot{Y}}{X}G_X-\frac{\dot{X}}{X}\dot{G}_X-2\frac{\dot{X}}{X}G_Y^t\dot{Y} \\
- \; & 2\dot{Y}^t\lambda^{-1}\dot{\lambda}G_Y+\frac{\dot{Y}^t\lambda^{-1}G_\lambda^t\dot{Y}}{X}-\mathrm{Tr}\left(\lambda^{-1}\dot{\lambda}\dot{G}_\lambda^t\right)-2\dot{G}_Y^t\dot{Y} \\
= \; & \left(\nabla\left(\frac{\dot{X}}{X}\right)+\frac{\dot{Y}^t\lambda^{-1}\nabla Y}{X}\right)^2+X\left(\dot{U}_2^t\lambda\dot{U}_2+\nabla U_1^t\lambda\nabla U_1\right) \\
+ \; & \mathrm{Tr}\left[\left(\nabla\left(\dot{\lambda}\lambda^{-1}\right)+\frac{\nabla Y\dot{Y}^t\lambda^{-1}}{X}\right)^2\right]
\end{aligned}
\tag{B.21}
$$

where

$$
U_1\equiv\frac{\lambda^{-1}\dot{Y}}{X}\qquad U_2\equiv\frac{\lambda^{-1}\nabla Y}{X}
\tag{B.22}
$$

This is the five-dimensional extension of Carter's identity which appeared in [27] .

Appendix C

Higher Homotopy Groups for Maximal Spatial Slices

This appendix appeared in the journal article (AA.1) Classical and Quantum Gravity, 31 (5), 055,004(2014)[7]. The computations below rely on the excision theorem for homotopy groups:

Theorem 80 (Excision for Homotopy Groups)**.** *[80, Thm 4.23] Let X be a CW-complex decomposed as the union of subcomplexes A and B with nonempty connected intersection $C = A \cap B$. If (A, C) is m-connected and (B, C) is n-connected, $m, n \geq 0$, then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.*

C.1 The doubly spinning black ring maximal spatial slice

In this part we verify that $\pi_3(\Sigma_{BR}) = \mathbb{Z}$ and $\pi_4(\Sigma_{BR}) \neq 0$. We know

$$\Sigma_{BR} \cong S^4 - \left(B^4 \cup \mathsf{Int}(R)\right), \tag{C.1}$$

where B^4 is a closed 4-dimensional ball in S^4, $R \cong S^1 \times B^3$ is a regular closed neighborhood of a locally flat embedded S^1 in S^4, and $B^4 \cap R = \emptyset$. Consider

$$S^4 = M \cup N \tag{C.2}$$

where

$$M = S^4 - \mathsf{Int}(R), \qquad N = S^4 - \mathsf{Int}(B), \quad \text{and} \quad \overline{\Sigma}_{BR} = M \cap N \tag{C.3}$$

Observe that Σ_{BR} and its closure in S^4 are homotopy equivalent. The following connectivity properties hold:

- *Claim 1.* The pair $\left(N, \overline{\Sigma}_{BR}\right)$ is 2-connected.
- *Claim 2.* The pair $\left(M, \overline{\Sigma}_{BR}\right)$ is 3-connected.

First we verify that $\pi_3(\Sigma_{BR}) = \mathbb{Z}$ and $\pi_4(\Sigma_{BR}) \neq 0$ assuming that both claims hold, and after the computation we will verify claims. The excision theorem for homopy groups [80, Thm 4.23] applied to S^4 as the union of M and N together with claims 1 and 2 imply that the maps induced by inclusion

$$\pi_4\left(M, \overline{\Sigma}_{BR}\right) \xrightarrow{\cong} \pi_4(S^4, N) \tag{C.4}$$

is an isomorphism, and

$$\pi_5\left(M, \overline{\Sigma}_{BR}\right) \longrightarrow \pi_5(S^4, N) \tag{C.5}$$

is surjective. In (4.93) we show that $M \cong S^2 \times D^2$; then the long exact sequence for the pair $(M, \overline{\Sigma}_{BR})$ yields

$$0 = \pi_{i+1}(M) \to \pi_{i+1}(M, \overline{\Sigma}_{BR}) \xrightarrow{\cong} \pi_i(\overline{\Sigma}_{BR}) \to \pi_i(M) = 0, \quad i \geq 3.$$

It follows that

$$\pi_3(\overline{\Sigma}_{BR}) \cong \pi_4\left(M, \overline{\Sigma}_{BR}\right) \quad \text{and} \quad \pi_4(\overline{\Sigma}_{BR}) \cong \pi_5\left(M, \overline{\Sigma}_{BR}\right). \tag{C.6}$$

Then the long exact sequence of homotopy groups for the pair (S^4, N) and the fact that N is contractible shows that

$$0 = \pi_{i+1}(N) \to \pi_{i+1}(S^4) \xrightarrow{\cong} \pi_{i+1}(S^4, N) \to \pi_i(N) = 0. \tag{C.7}$$

The isomorphims from (C.4), (C.6), and (C.7) imply that

$$\pi_3(\Sigma_{BR}) \xrightarrow{\cong} \pi_3(\overline{\Sigma}_{BR}) \xrightarrow{\cong} \pi_4(M, \overline{\Sigma}_{BR}) \xrightarrow{\cong} \pi_4(S^4, N) \xrightarrow{\cong} \pi_4(S^4) \xrightarrow{\cong} \mathbb{Z}$$

verifying that $\pi_3(\Sigma_{BR}) \cong \mathbb{Z}$. Analogously, from (C.5), (C.6), and (C.7) the composition

$$\pi_4(\Sigma_{BR}) \xrightarrow{\cong} \pi_4(\overline{\Sigma}_{BR}) \xrightarrow{\cong} \pi_5(M, \overline{\Sigma}_{BR}) \to \pi_5(S^4, N) \xrightarrow{\cong} \pi_5(S^4) \xrightarrow{\cong} \mathbb{Z}_2$$

is a surjective map, verifying that $\pi_4(\Sigma_{BR})$ is not trivial.

Verification of claim 1. Since Σ_{BR} is 1-connected and N is contractible, the long exact sequence for $(N, \overline{\Sigma}_{BR})$ yields

$$0 = \pi_{i+1}(N) \to \pi_{i+1}\left(N, \overline{\Sigma}_{BR}\right) \to \pi_i\left(\overline{\Sigma}_{BR}\right) \to \pi_i(N) = 0. \tag{C.8}$$

It follows that $\left(N, \overline{\Sigma}_{BR}\right)$ is 2-connected as claimed.

Verification of claim 2. Observe that the pair $(B, \partial B)$ is 2-connected. Indeed, since B is a 4-dimensional ball, the long exact sequence of homotopy groups for $(B, \partial B)$,

$$0 = \pi_{i+1}B \to \pi_{i+1}(B, \partial B) \to \pi_i(\partial B) \to \pi_i B = 0, \tag{C.9}$$

shows that

$$\pi_{i+1}(B, \partial B) \cong \pi_i(\partial B) \cong \pi_i(S^3). \tag{C.10}$$

Analogously, observe that the pair $(\overline{\Sigma}_{BR}, \partial B)$ is 1-connected. Indeed, since $\partial B \cong S^3$, the long exact sequence for $(\overline{\Sigma}_{BR}, \partial B)$ shows that

$$0 = \pi_1(S^3) \to \pi_1(\overline{\Sigma}_{BR}) \to \pi_1(\overline{\Sigma}_{BR}, \partial B) \to \pi_0(S^3) = 0. \tag{C.11}$$

Since $\pi_1(\Sigma_{BR}) \cong \pi_1(\overline{\Sigma}_{BR}) = 0$, it follows that $\pi_1(\overline{\Sigma}_{BR}, \partial B)$ is trivial. Since $(B, \partial B)$ is 2-connected and $(\overline{\Sigma}_{BR}, \partial B)$ is 1-connected, a direct application of the excision theorem for higher homotopy groups [80, Thm 4.23] applied to $M = \overline{\Sigma}_{BR} \cup B$ shows that the composition

$$\pi_i(B, \partial B) \xrightarrow{\cong} \pi_i(M, \overline{\Sigma}_{BR}), \quad i = 1, 2 \tag{C.12}$$

is an isomorphism, and the composition

$$\pi_3(B, \partial B) \to \pi_3(M, \overline{\Sigma}_{BR}) \tag{C.13}$$

is surjective. Then equations (C.10), (C.12) and (C.13) yield that $\pi_i(M, \overline{\Sigma}_{BR})$ is trivial for $0 \le i \le 3$, hence $(M, \overline{\Sigma}_{BR})$ is 3-connected as claimed.

An application to the Hurewicz map $\pi_3(\Sigma_{BR}) \to \mathcal{H}_3(\Sigma_{BR})$. An interesting application of the fact $\pi_3(\Sigma_{BR}) \cong \mathbb{Z}$ is that the Hurewicz map $\pi_3(\Sigma_{BR}) \to H_3(\Sigma_{BR})$ is an isomorphism. Indeed, the excision theorem for homology shows that $\mathcal{H}_i(M, B) \cong \mathcal{H}_i(\Sigma_{BR}, \partial B)$, and the long exact sequence in homology for the pair (M, B) shows that $\mathcal{H}_i M \cong \mathcal{H}_i(M, B)$. Therefore

$$\mathcal{H}_i(\Sigma_{BR}, \partial B) \cong \mathcal{H}_i M. \tag{C.14}$$

Since $\partial B \cong S^3$, the Hurewicz maps between the long exact sequences of homotopy and homology groups for the pair $(\Sigma_{BR}, \partial B)$ provide the commutative diagram,

$$\begin{array}{ccccccc} & & \pi_3(S^3) & \longrightarrow & \pi_3(\Sigma_{BR}) & & \\ & & \cong\downarrow & & \downarrow & & \\ 0 = \mathcal{H}_4(M) \cong \mathcal{H}_4(\Sigma_{BR}, \partial B) & \longrightarrow & \mathcal{H}_3(S^3) & \xrightarrow{\cong} & \mathcal{H}_3(\Sigma_{BR}) & \longrightarrow & \mathcal{H}_3(\Sigma_{BR}, \partial B) \cong H_3(M) = 0 \end{array}$$

Since $\pi_3(S^3) = \mathbb{Z}$, the diagram shows that $\pi_3(\Sigma_{BR})$ contains an infinite cyclic group, and maps onto an infinity cyclic group (this does not imply that $\pi_3(\Sigma_{BR})$ is an infinite cyclic group). Since we show that $\pi_3(\Sigma_{BR}) \cong \mathbb{Z}$, it follows that the Hurewicz map $\pi_3(\Sigma_{BR}) \to \mathcal{H}_3(\Sigma_{BR})$ is an isomorphism.

C.2 The Black Saturn maximal spatial slice

In this part we verify that $\pi_3(\Sigma_{BS}) = \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_4(\Sigma_{BS})$ is not trivial. The computation follows the same strategy as the previous computation; in this case the computation relies on facts about homotopy groups for S^4. From [7] ,

$$\Sigma_{BS} = S^4 - \left\{ B_1^4 \sqcup \mathsf{Int}\left(B_2^4\right) \sqcup \left[S^1 \times \mathsf{Int}(B^3)\right]\right\}$$

where B_1^4 and B_2^4 are disjoint closed ϵ-ball in S^4, $R \cong S^1 \times B^3$ is a regular closed neighborhood of an embedded S^1 in S^4 and $R \cap (B_1^4 \cup B_2^4) = \emptyset$. Consider

$$S^4 = M \cup N$$

where $M = S^4 - \mathsf{Int}(R) \cong S^2 \times D^2$ and $N = S^4 - \mathsf{Int}(B_1 \cup B_2) \cong S^3 \times \mathbb{R}$ and $M \cap N = \overline{\Sigma}_{BS}$. The following connectivity properties hold:

- *Claim 1.* The pair $\left(N, \overline{\Sigma}_{BS}\right)$ is 2-connected.
- *Claim 2.* The pair $\left(M, \overline{\Sigma}_{BR}\right)$ is 3-connected.

First we verify that $\pi_3(\Sigma_{BS}) = \mathbb{Z} \oplus \mathbb{Z}$ assuming that both claims hold, and after the computation we will verify the claims. By the excision theorem for homotopy groups applied to $S^4 = M \cup N$ we obtain that the map induced by inclusion

$$\pi_i(M, \overline{\Sigma}_{BS}) \to \pi_i(S^4, N) \tag{C.15}$$

is an isomorphism for $i \leq 4$, and it is surjective for $i = 5$. Since M is 2-connected, the long exact sequence for $(M, \overline{\Sigma}_{BS})$ yields that

$$\pi_{i+1}(M, \overline{\Sigma}_{BS}) \cong \pi_i(\overline{\Sigma}_{BS}), \quad i \geq 3. \tag{C.16}$$

Consider the long exact sequence for (S^4, N),

$$\pi_4(S^3) \to \pi_4(S^4) \to \pi_4(S^4, N) \to \pi_3(S^3) \to \pi_3(S^4). \tag{C.17}$$

Since $\pi_4(S^3) \cong \mathbb{Z}/2$ is finite, $\pi_4(S^4) \cong \mathbb{Z}$ and $\pi_3(S^4) = 0$ we have an exact sequence

$$0 \to \mathbb{Z} \to \pi_4(S^4, N) \to \mathbb{Z} \to 0. \tag{C.18}$$

Since $\pi_4(S^4, N)$ is an abelian group, it follows that

$$\pi_4(S^4, N) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{C.19}$$

Therefore the isomorphisms (C.15), (C.16) and (C.19) show that

$$\pi_3(\Sigma_{BS}) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{C.20}$$

Analogously, the long exact sequence for (S^4, N) provides an exact sequence

$$\pi_5(S^4, N) \to \pi_4(N) \to \pi_4(S^4). \tag{C.21}$$

Since N is homotopic to S^3, $\pi_4(S^3) \cong \mathbb{Z}/2$ is finite, and $\pi_4(S^4) \cong \mathbb{Z}$, the map

$$\pi_5(S^4, N) \to \pi_4(N) \cong \mathbb{Z}/2 \tag{C.22}$$

is surjective. In particular $\pi_5(S^4, N) \neq 0$. Then the isomorphism from (C.16) for $i = 4$, the surjections from (C.15) for $i = 5$ and (C.22) show that the composition

$$\pi_4(\Sigma_{BS}) \cong \pi_4(\overline{\Sigma}_{BS}) \xrightarrow{\cong} \pi_5(M, \overline{\Sigma}_{BS}) \to \pi_5(S^4, N) \to \pi_4(S^3) \cong \mathbb{Z}/2 \tag{C.23}$$

is surjective and hence $\pi_4(\Sigma_{BS})$ is not trivial.

Verification of Claim 1. Since N is homotopy equivalent to $S^3 \times \mathbb{R}$, the long exact sequence for $\big(N, \overline{\Sigma}_{BS}\big)$ provides an exact sequence

$$\pi_{i+1}(S^3) \to \pi_{i+1}(S^3 \times \mathbb{R}, \overline{\Sigma}_{BS}) \to \pi_i(\Sigma_{BS}) \to \pi_i(S^3). \tag{C.24}$$

It follows that

$$\pi_{i+1}(S^3 \times \mathbb{R}, \overline{\Sigma}_{BS}) \cong \pi_i(\Sigma_{BS}), \quad i = 0, 1. \tag{C.25}$$

Since Σ_{BS} is 1-connected, it follows that $(N, \overline{\Sigma}_{BS})$ is 2-connected.

Verification of Claim 2. Consider $S^2 \times D^2$ as the union of $\overline{\Sigma}_{BS}$ and $P = B_1 \cup B_2^4 \cup \gamma$ where γ is a simple path in $\overline{\Sigma}_{BS}$ from ∂B_1^4 to ∂B_2^4. Let ∂P denote the intersection of P and $\overline{\Sigma}_{BS}$. The introduction of γ is to guarantee that ∂P is connected; indeed observe that ∂P is homotopic to a wedge of a pair of 3-spheres $S^3 \vee S^3$.

First we show $(P, \partial P)$ is 3-connected. Since P is contractible, the long exact sequence for $(P, \partial P)$ implies that

$$\pi_{i+1}(P, \partial P) \cong \pi_i(S^3 \vee S^3) \quad i \geq 0. \tag{C.26}$$

Since S^3 is 3-connected, the wedge $S^3 \vee S^3$ is 2-connected by the main result in [87]. Then (C.26) implies that $(P, \partial P)$ is 3-connected.

Now we show that $(\overline{\Sigma}_{BS}, \partial P)$ is 1-connected. The long exact sequence for this pair provides the exact sequence (of sets)

$$0 = \pi_1(\partial P) \to \pi_1(\overline{\Sigma}_{BS}) \overset{\cong}{\to} \pi_1(\overline{\Sigma}_{BS}, \partial P) \to \pi_0(\partial P) = 0 \tag{C.27}$$

Since $\overline{\Sigma}_{BS}$ is 1-connected, it follows that $(\overline{\Sigma}_{BS}, \partial P)$ is 1-connected.

Then the excision theorem for homotopy groups applied to $M = \overline{\Sigma}_{BS} \cup P$ implies that

$$\pi_i(P, \partial P) \cong \pi_i(M, \Sigma_{BS}), \quad 0 \le i \le 3. \tag{C.28}$$

Since $(P, \partial P)$ is 3-connected, we have that (M, Σ_{BS}) is 3-connected.

Bibliography

[1] A. Aceña, S. Dain, and M. E. G. Clément. Horizon area–angular momentum inequality for a class of axially symmetric black holes. *Classical and Quantum Gravity*, 28(10):105014, 2011.

[2] A. Alaee, M. Khuri, and H. K. Kunduri. Proof of the mass-angular momentum inequality for bi-axisymmetric black holes with spherical topology. *arXiv preprint arXiv:1510.06974*, 2015.

[3] A. Alaee and H. K. Kunduri. Mass functional for initial data in 4+ 1-dimensional spacetime. *Physical Review D*, 90(12):124078, 2014.

[4] A. Alaee and H. K. Kunduri. Proof of the local mass-angular momenta inequality for $u(1)^2$ invariant black holes. *Classical and Quantum Gravity*, 32(16):165020, 2015.

[5] A. Alaee and H. K. Kunduri. Remarks on mass and angular momenta for $u(1)^2$ -invariant initial data. *arXiv preprint arXiv:1508.02337*, 2015.

[6] A. Alaee and H. K. Kunduri. Small deformations of extreme five dimensional myers–perry black hole initial data. *General Relativity and Gravitation*, 47(2):1–29, 2015.

[7] A. Alaee, H. K. Kunduri, and E. M. Pedroza. Notes on maximal slices of five-dimensional black holes. *Classical and Quantum Gravity*, 31(5):055004, 2014.

[8] A. J. Amsel, G. T. Horowitz, D. Marolf, and M. M. Roberts. Uniqueness of extremal kerr and kerr-newman black holes. *Physical Review D*, 81(2):024033, 2010.

[9] S. Aretakis. Stability and instability of extreme reissner-nordström black hole spacetimes for linear scalar perturbations i. *Communications in Mathematical Physics*, 307(1):17–63, 2011.

[10] S. Aretakis. Horizon instability of extremal black holes. *arXiv preprint arXiv:1206.6598*, 2012.

[11] R. Arnowitt, S. Deser, and C. W. Misner. Republication of: The dynamics of general relativity. *General Relativity and Gravitation*, 40(9):1997–2027, 2008.

[12] A. Ashtekar. Asymptotic structure of the gravitational field at spatial infinity. 1978.

[13] A. Ashtekar and B. Krishnan. Isolated and dynamical horizons and their applications. *Living Rev. Rel*, 7(10):2000, 2004.

[14] J. M. Bardeen, B. Carter, and S. W. Hawking. The four laws of black hole mechanics. *Communications in Mathematical Physics*, 31(2):161–170, 1973.

[15] R. Bartnik. The mass of an asymptotically flat manifold. *Communications on pure and applied mathematics*, 39(5):661–693, 1986.

[16] R. Bartnik. New definition of quasilocal mass. *Physical review letters*, 62(20):2346, 1989.

[17] R. Bartnik and J. Isenberg. The constraint equations. In *The Einstein equations and the large scale behavior of gravitational fields*, pages 1–38. Springer, 2004.

[18] E. P. Belasco and H. C. Ohanian. Initial conditions in general relativity: lapse and shift formulation. *Journal of Mathematical Physics*, 10(8):1503–1507, 1969.

[19] A. L. Besse. *Einstein manifolds*. Springer, 2008.

[20] I. Booth. Black-hole boundaries. *Canadian journal of physics*, 83(11):1073–1099, 2005.

[21] W. Brakes. Quickly unknotting topological spheres. *Proceedings of the American Mathematical Society*, pages 413–416, 1978.

[22] H. L. Bray et al. Proof of the riemannian penrose inequality using the positive mass theorem. *Journal of Differential Geometry*, 59(2):177–267, 2001.

[23] D. R. Brill. On the positive definite mass of the bondi-weber-wheeler time-symmetric gravitational waves. *Annals of Physics*, 7(4):466–483, 1959.

[24] J. D. Brown and J. W. York Jr. Quasilocal energy and conserved charges derived from the gravitational action. *Physical Review D*, 47(4):1407, 1993.

[25] G. Bunting. Proof of the uniqueness conjecture for black holes. *University of New England, Armidale NSW*, 1983.

[26] G. L. Bunting and A. Masood-ul Alam. Nonexistence of multiple black holes in asymptotically euclidean static vacuum space-time. *General relativity and gravitation*, 19(2):147–154, 1987.

[27] B. Carter. Axisymmetric black hole has only two degrees of freedom. *Physical Review Letters*, 26(6):331–333, 1971.

[28] B. Carter. Black hole equilibrium states. *Black holes*, pages 57–214, 1973.

[29] B. Carter. Bunting identity and mazur identity for non-linear elliptic systems including the black hole equilibrium problem. *Communications in Mathematical Physics*, 99(4):563–591, 1985.

[30] Y. S. Cha and M. A. Khuri. Deformations of axially symmetric initial data and the mass-angular momentum inequality. In *Annales Henri Poincaré*, volume 16, pages 841–896. Springer, 2014.

[31] Y. Choquet-Bruhat. *General relativity and the Einstein equations*. Oxford University Press, 2009.

[32] Y. Choquet-Bruhat and D. Christodoulou. Elliptic systems inh s, δ spaces on manifolds which are euclidean at infinity. *Acta Mathematica*, 146(1):129–150, 1981.

[33] Y. Choquet-Bruhat and R. Geroch. Global aspects of the cauchy problem in general relativity. *Communications in Mathematical Physics*, 14(4):329–335, 1969.

[34] Y. Choquet-Bruhat, J. Isenberg, and J. W. York Jr. Einstein constraints on asymptotically euclidean manifolds. *Physical Review D*, 61(8):84034, 2000.

[35] Y. Choquet-Bruhat and J. W. York Jr. The cauchy problem. In *General Relativity and Gravitation. Vol. 1. One hundred years after the birth of Albert Einstein. Edited by A. Held. New York, NY: Plenum Press, p. 99, 1980*, volume 1, page 99, 1980.

[36] P. Chrusciel and R. Mazzeo. Initial data sets with ends of cylindrical type: I. the lichnerowicz equation,. *arXiv preprint arXiv:1201.4937*, 2012.

[37] P. Chrusciel, R. Mazzeo, and S. Pocchiola. Initial data sets with ends of cylindrical type: Ii. the vector constraint equation,. *arXiv preprint arXiv:1203.5138*, 2012.

[38] P. T. Chruściel. On the invariant mass conjecture in general relativity. *Communications in mathematical physics*, 120(2):233–248, 1988.

[39] P. T. Chruściel. Mass and angular-momentum inequalities for axi-symmetric initial data sets i. positivity of mass. *Annals of Physics*, 323(10):2566–2590, 2008.

[40] P. T. Chrusciel and J. L. Costa. On uniqueness of stationary vacuum black holes. *arXiv preprint arXiv:0806.0016*, 2008.

[41] P. T. Chruściel and J. L. Costa. Mass, angular-momentum and charge inequalities for axisymmetric initial data. *Classical and Quantum Gravity*, 26(23):235013, 2009.

[42] P. T. Chrusciel, J. L. Costa, M. Heusler, et al. Stationary black holes: uniqueness and beyond. *Living Rev. Relativity*, 15(7), 2012.

[43] P. T. Chruściel and G. J. Galloway. Uniqueness of static black-holes without analyticity. *arXiv preprint arXiv:1004.0513*, 2010.

[44] P. T. Chruściel, Y. Li, and G. Weinstein. Mass and angular-momentum inequalities for axi-symmetric initial data sets. ii. angular momentum. *Annals of Physics*, 323(10):2591–2613, 2008.

[45] P. T. Chrusciel and R. M. Wald. On the topology of stationary black holes. *arXiv preprint gr-qc/9410004*, 1994.

[46] M. E. G. Clement, J. L. Jaramillo, and M. Reiris. Proof of the area-angular momentum-charge inequality for axisymmetric black holes. *arXiv preprint arXiv:1207.6761*, 2012.

[47] M. E. G. Clement, M. Reiris, and W. Simon. The area-angular momentum inequality for black holes in cosmological spacetimes. *arXiv preprint arXiv:1501.07243*, 2015.

[48] J. L. Costa. On the classification of stationary electro-vacuum black holes. *Classical and Quantum Gravity*, 27(3):035010, 2010.

[49] J. L. Costa. Proof of a dain inequality with charge. *Journal of Physics A: Mathematical and Theoretical*, 43(28):285202, 2010.

[50] M. Cvetic, H. Lü, D. N. Page, and C. Pope. New einstein-sasaki and einstein spaces from kerr-de sitter. *Journal of High Energy Physics*, 2009(07):082, 2009.

[51] S. Dain. Proof of the (local) angular momentum–mass inequality for axisymmetric black holes. *Classical and Quantum Gravity*, 23(23):6845, 2006.

[52] S. Dain. A variational principle for stationary, axisymmetric solutions of einstein's equations. *Classical and Quantum Gravity*, 23(23):6857, 2006.

[53] S. Dain. Proof of the angular momentum-mass inequality for axisymmetric black holes. *J. Differ. Geom*, 79(1):33–67, 2008.

[54] S. Dain. Geometric inequalities for axially symmetric black holes. *Classical and Quantum Gravity*, 29(7):073001, 2012.

[55] S. Dain and M. E. G. Clément. Small deformations of extreme kerr black hole initial data. *Classical and Quantum Gravity*, 28(7):075003, 2011.

[56] S. Dain and I. G. de Austria. On the linear stability of the extreme kerr black hole under axially symmetric perturbations. *arXiv preprint arXiv:1402.2848*, 2014.

[57] S. Dain, J. L. Jaramillo, and M. Reiris. Area-charge inequality for black holes. *arXiv preprint arXiv:1109.5602*, 2011.

[58] S. Dain, M. Khuri, G. Weinstein, and S. Yamada. Lower bounds for the area of black holes in terms of mass, charge, and angular momentum. *Physical Review D*, 88(2):024048, 2013.

[59] S. Dain and M. Reiris. Area-angular-momentum inequality for axisymmetric black holes. *Physical review letters*, 107(5):051101, 2011.

[60] O. Dearricott, R. H. Guzmán, and L. Hernández-Lamoneda. *Geometry of Manifolds with Non-negative Sectional Curvature*, volume 2110. Springer, 2014.

[61] C. DeWitt and B. S. DeWitt. *Black holes*, volume 23. CRC Press, 1973.

[62] H. Elvang and P. Figueras. Black saturn. *Journal of High Energy Physics*, 2007(05):050, 2007.

[63] R. Emparan and H. S. Reall. Generalized weyl solutions. *Physical Review D*, 65(8):084025, 2002.

[64] R. Emparan and H. S. Reall. A rotating black ring solution in five dimensions. *Physical Review Letters*, 88(10):101101, 2002.

[65] R. Emparan and H. S. Reall. Black holes in higher dimensions. *Living Rev. Rel*, 11(6):0801–3471, 2008.

[66] L. C. Evans. Partial differential equations (providence, ri: American mathematical society). Technical report, ISBN 0-8218-0772-2, 1998.

[67] P. Figueras and J. Lucietti. On the uniqueness of extremal vacuum black holes. *Classical and Quantum Gravity*, 27(9):095001, 2010.

[68] P. Figueras, K. Murata, and H. S. Reall. Black hole instabilities and local penrose inequalities. *Classical and Quantum Gravity*, 28(22):225030, 2011.

[69] J. L. Friedman, K. Schleich, and D. M. Witt. Topological censorship. *Physical Review Letters*, 71(10):1486, 1993.

[70] G. J. Galloway. A "finite infinity" version of topological censorship. *Classical and Quantum Gravity*, 13(6):1471, 1996.

[71] G. J. Galloway. Rigidity of marginally trapped surfaces and the topology of black holes. *arXiv preprint gr-qc/0608118*, 2006.

[72] G. J. Galloway, K. Schleich, D. M. Witt, and E. Woolgar. Topological censorship and higher genus black holes. *Physical Review D*, 60(10):104039, 1999.

[73] G. J. Galloway and R. Schoen. A generalization of hawking's black hole topology theorem to higher dimensions. *Communications in mathematical physics*, 266(2):571–576, 2006.

[74] R. Geroch. Energy extraction. *Annals of the New York Academy of Sciences*, 224(1):108–117, 1973.

[75] G. Gibbons, S. Hawking, G. T. Horowitz, and M. J. Perry. Positive mass theorems for black holes. *Communications in Mathematical Physics*, 88(3):295–308, 1983.

[76] G. Gibbons and G. Holzegel. The positive mass and isoperimetric inequalities for axisymmetric black holes in four and five dimensions. *Classical and Quantum Gravity*, 23(22):6459, 2006.

[77] G. W. Gibbons. The time symmetric initial value problem for black holes. *Communications in Mathematical Physics*, 27(2):87–102, 1972.

[78] H. Gluck et al. Unknotting s^1 in s^4. *Bulletin of the American Mathematical Society*, 69(1):91–94, 1963.

[79] J. B. Griffiths and J. Podolskỳ. *Exact space-times in Einstein's general relativity.* Cambridge University Press, 2009.

[80] A. Hatcher. Algebraic topology. *Cambridge UP, Cambridge*, 2002.

[81] S. W. Hawking. Gravitational radiation in an expanding universe. *Journal of Mathematical Physics*, 9(4):598–604, 1968.

[82] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*, volume 1. Cambridge university press, 1973.

[83] J. Hennig, M. Ansorg, and C. Cederbaum. A universal inequality between angular momentum and horizon area for axisymmetric and stationary black holes with surrounding matter. *arXiv preprint arXiv:0805.4320*, 2008.

[84] J. Hennig, C. Cederbaum, and M. Ansorg. A universal inequality for axisymmetric and stationary black holes with surrounding matter in the einstein-maxwell theory. *Communications in Mathematical Physics*, 293(2):449–467, 2010.

[85] M. Herzlich. The positive mass theorem for black holes revisited. *Journal of Geometry and Physics*, 26(1):97–111, 1998.

[86] M. Heusler. *Black hole uniqueness theorems*, volume 6. Cambridge University Press, 1996.

[87] P. J. Hilton. On the homotopy groups of the union of spheres. *J. London Math. Soc*, 30(154-172):71, 1955.

[88] M. W. Hirsch. Smooth regular neighborhoods. *Annals of Mathematics*, pages 524–530, 1962.

[89] M. W. Hirsch. *Differential topology*. 1976.

[90] S. Hollands. Horizon area–angular momentum inequality in higher-dimensional spacetimes. *Classical and Quantum Gravity*, 29(6):065006, 2012.

[91] S. Hollands, J. Holland, and A. Ishibashi. Further restrictions on the topology of stationary black holes in five dimensions. In *Annales Henri Poincare*, volume 12, pages 279–301. Springer, 2011.

[92] S. Hollands and A. Ishibashi. On the stationary implies axisymmetric theorem for extremal black holes in higher dimensions. *Communications in Mathematical Physics*, 291(2):443–471, 2009.

[93] S. Hollands and A. Ishibashi. Black hole uniqueness theorems in higher dimensional spacetimes. *Classical and Quantum Gravity*, 29(16):163001, 2012.

[94] S. Hollands, A. Ishibashi, and R. M. Wald. A higher dimensional stationary rotating black hole must be axisymmetric. *Communications in mathematical physics*, 271(3):699–722, 2007.

[95] S. Hollands and R. M. Wald. Stability of Black Holes and Black Branes. *Commun.Math.Phys.*, 321:629–680, 2013.

[96] S. Hollands and S. Yazadjiev. Uniqueness theorem for 5-dimensional black holes with two axial killing fields. *Communications in Mathematical Physics*, 283(3):749–768, 2008.

[97] G. T. Horowitz and M. J. Perry. Gravitational energy cannot become negative. *Physical Review Letters*, 48(6):371, 1982.

[98] W. Y. Hsiang. *Lectures on Lie groups*, volume 2. World Scientific, 2000.

[99] G. Huisken, T. Ilmanen, et al. The inverse mean curvature flow and the riemannian penrose inequality. *Journal of Differential Geometry*, 59(3):353–437, 2001.

[100] J. Isenberg. Constant mean curvature solutions of the einstein constraint equations on closed manifolds. *Classical and Quantum Gravity*, 12(9):2249, 1995.

[101] W. Israel. Event horizons in static electrovac space-times. *Communications in Mathematical Physics*, 8(3):245–260, 1968.

[102] J. L. Jaramillo and E. Gourgoulhon. Mass and angular momentum in general relativity. In *Mass and Motion in General Relativity*, pages 87–124. Springer, 2011.

[103] J. L. Jaramillo, M. Reiris, and S. Dain. Black hole area–angular-momentum inequality in nonvacuum spacetimes. *Physical Review D*, 84(12):121503, 2011.

[104] J. Jost. *Riemannian geometry and geometric analysis.* Springer Science & Business Media, 2008.

[105] M. Khuri and G. Weinstein. The positive mass theorem for multiple rotating charged black holes. *arXiv preprint arXiv:1502.06290*, 2015.

[106] S. Kobayashi and K. Nomizu. *Foundations of differential geometry*, volume 1. New York, 1963.

[107] H. K. Kunduri and J. Lucietti. A classification of near-horizon geometries of extremal vacuum black holes. *Journal of Mathematical Physics*, 50(8):082502, 2009.

[108] H. K. Kunduri and J. Lucietti. A supersymmetric black lens. *arXiv preprint arXiv:1408.6083*, 2014.

[109] J. Lee. *Introduction to smooth manifolds*, volume 218. Springer Science & Business Media, 2012.

[110] A. Lichnerowicz. L' integration des équations de la gravitation relativiste et le probleme des n corps. *J. Math. Pures Appl*, 23(37):4, 1944.

[111] H. Lü, D. N. Page, and C. Pope. New inhomogeneous einstein metrics on sphere bundles over einstein–kähler manifolds. *Physics Letters B*, 593(1):218–226, 2004.

[112] J. Lucietti, K. Murata, H. S. Reall, and N. Tanahashi. On the horizon instability of an extreme reissner-nordström black hole. *Journal of High Energy Physics*, 2013(3):1–44, 2013.

[113] D. Maison. Ehlers-harrison-type transformations for jordan's extended theory of gravitation. *General Relativity and Gravitation*, 10(8):717–723, 1979.

[114] J. Maldacena. The large-n limit of superconformal field theories and supergravity. *International journal of theoretical physics*, 38(4):1113–1133, 1999.

[115] M. Mars. Present status of the penrose inequality. *Classical and Quantum Gravity*, 26(19):193001, 2009.

[116] D. Maxwell. Solutions of the einstein constraint equations with apparent horizon boundaries. *Communications in mathematical physics*, 253(3):561–583, 2005.

[117] D. Maxwell. The conformal method and the conformal thin-sandwich method are the same. *Classical and Quantum Gravity*, 31(14):145006, 2014.

[118] R. C. McOwen. The behavior of the laplacian on weighted sobolev spaces. *Communications on Pure and Applied Mathematics*, 32(6):783–795, 1979.

[119] C. Moller. Les theories relativistes de la gravitation. *CRNS, Paris*, 1962.

[120] V. Moncrief and J. Isenberg. Symmetries of cosmological cauchy horizons. *Communications in Mathematical Physics*, 89(3):387–413, 1983.

[121] V. Moncrief and J. Isenberg. Symmetries of higher dimensional black holes. *Classical and Quantum Gravity*, 25(19):195015, 2008.

[122] K. Murata, H. S. Reall, and N. Tanahashi. What happens at the horizon (s) of an extreme black hole? *Classical and Quantum Gravity*, 30(23):235007, 2013.

[123] N. Ó. Murchadha. Total energy momentum in general relativity. *Journal of mathematical physics*, 27(8):2111–2128, 1986.

[124] R. C. Myers. Myers-perry black holes. *arXiv preprint arXiv:1111.1903*, 2011.

[125] R. C. Myers and M. Perry. Black holes in higher dimensional space-times. *Annals of Physics*, 172(2):304–347, 1986.

[126] P. Orlik and F. Raymond. Actions of the torus on 4-manifolds. *Trans. Amer. Math. Soc.*, 152:531–559, 1970.

[127] R. Osserman. The isoperimetric inequality. *Bulletin of the American Mathematical Society*, 84(6):1182–1238, 1978.

[128] R. Penrose. Naked singularities. *Annals of the New York Academy of Sciences*, 224(1):125–134, 1973.

[129] R. Penrose. Quasi-local mass and angular momentum in general relativity. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 381, pages 53–63. The Royal Society, 1982.

[130] E. Poisson. *A Relativist's Toolkit*. Cambridge University Press, 2004.

[131] A. Pomeransky and R. Sen'kov. Black ring with two angular momenta. *arXiv preprint hep-th/0612005*, 2006.

[132] I. Rácz. A simple proof of the recent generalizations of hawking's black hole topology theorem. *Classical and Quantum Gravity*, 25(16):162001, 2008.

[133] I. Rácz and R. M. Wald. Global extensions of spacetimes describing asymptotic final states of black holes. *Classical and Quantum Gravity*, 13(3):539, 1996.

[134] M. Reiris. Instability of the extreme kerr-newman black-holes. *arXiv preprint arXiv:1311.3156*, 2013.

[135] R. Schoen, K. Uhlenbeck, et al. A regularity theory for harmonic maps. *Journal of Differential Geometry*, 17(2):307–335, 1982.

[136] R. Schoen and S.-T. Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45–76, 1979.

[137] R. Schoen and S.-T. Yau. Proof of the positive mass theorem. ii. *Communications in Mathematical Physics*, 79(2):231–260, 1981.

[138] R. Schoen and S. T. Yau. Proof that the bondi mass is positive. *Physical Review Letters*, 48(6):369, 1982.

[139] R. Schoen and X. Zhou. Convexity of reduced energy and mass angular momentum inequalities. In *Annales Henri Poincaré*, volume 14, pages 1747–1773. Springer, 2013.

[140] A. Strominger and C. Vafa. Microscopic origin of the bekenstein-hawking entropy. *Physics Letters B*, 379(1):99–104, 1996.

[141] L. B. Szabados. Quasi-local energy-momentum and angular momentum in gr: A review article. *Living Rev. Relativity*, 7(4), 2004.

[142] F. R. Tangherlini. Schwarzschild field in n dimensions and the dimensionality of space problem. *Il Nuovo Cimento (1955-1965)*, 27(3):636–651, 1963.

[143] R. M. Wald. *General relativity*. University of Chicago press, 2010.

[144] E. Witten. A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80(3):381–402, 1981.

[145] E. Zeidler. *Nonlinear functional analysis and its applications*, volume III. Springer Science & Business Media, 1989.