# Solving nonlinear problems by Ostrowski-Chun type parametric families *

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**Abstract**

In this paper, by using a generalization of Ostrowski' and Chun's methods two bi-parametric families of predictor-corrector iterative schemes, with order of convergence 4 for solving system of nonlinear equations, are presented. The predictor of the first family is Newton's method, and the predictor of the second one is Steffensen's scheme. One of them is extended to the multidimensional case. Some numerical tests are performed to compare proposed methods with existing ones and to confirm the theoretical results. We check the obtained results by solving the Molecular Interaction Problem.

**Key Words:** Iterative schemes, Nonlinear equation, system of nonlinear equations, divided differences, optimal, efficiency index.

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## 1 Introduction

Solving nonlinear equations and systems is an important task in theory and practice, not only for Applied Mathematics, but also for many branches of Science and Engineering. A glance at the survey [1] and the references therein show a high level of contemporary interest. In case of problems coming from Chemistry, nonlinear equations regularly appear: in the reaction-diffusion equations that arise in autocatalytic chemical reactions (see [2]), iterative methods can be applied; also in the analysis of electronic structure of the hydrogen atom inside strong magnetic fields (see [3]). Moreover, numerical performance of some chemical problems allows us to check the models of observable phenomena [4]. Even more, many problems from Chemistry consist in finding chemical potentials that are basic for studying other thermodynamic properties: the modeling of such potential leads to nonlinear integral equations that can be reduced to a set of nonlinear algebraic equations (see [5] for example).

Let us consider the problem of finding a simple zero of the nonlinear function $f : I \subset \mathbb{R} \to \mathbb{R}$, that is, a solution $\xi \in I$ of the nonlinear equation $f(x) = 0$. The most used iterative techniques to determine these roots can be classified as: (a) methods that require only functional evaluations of $f$, and (b) schemes whose formula require evaluations of the function and its derivatives. There are two simple and effective known methods that represent these classes: Steffensen's scheme [6]

$$x_{k+1} = x_k - \frac{f(x_k)}{f[\omega_k, x_k]}, \tag{1}$$

where $\omega_k = x_k + f(x_k)$ and $f[\omega_k, x_k] = \dfrac{f(\omega_k) - f(x_k)}{\omega_k - x_k}$, and Newton's procedure (see [7])

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \tag{2}$$

where $f'(x)$ is the first derivative of function $f(x)$. The order of convergence of both methods is two.

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Multipoint methods have been developed as a result of the search for iterative methods to solve nonlinear equations with fast convergence and small number of operations or functional assessments per iteration. The most important class of multistep schemes are the optimal methods in the sense of Kung-Traub conjecture [8].

The problem of solving a system of nonlinear equations is avoided as far as possible. Generally, the nonlinear system is approximated by a system of linear equations. When this is not satisfactory, the problem must be confronted directly. The direct way is to adapt the methods designed for the scalar case to several variables. A scalar variable is replaced by a vector incorporating all the variables. Hence arises the greatest difficulty to get new iterative methods for nonlinear systems, since not always the methods of nonlinear equations are extensible to systems directly.

Recently, the weight-function procedure has been used, with some restrictions, in the development of high order iterative methods for systems: see, for example the papers of Sharma et al. ([10, 11]) and Abad et al. [12], where the authors apply the designed method to the software improvement of the Global Positioning System.

On the other hand, a common way to generate new schemes is the direct composition of known methods with a later treatment to reduce the number of functional evaluations (see [13, 14, 15, 16], for example). A variant of this technique is the so called Pseudocomposition, introduced in [17] and [18].

The aim of this work is to design new parametric families of iterative methods for nonlinear equations by using some of the known methods and subsequently extend one of them to systems of nonlinear equations. For this purpose we have used Ostrowski' [19] and Chun's [20] methods with iterative schemes

$$x_{k+1} = y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)}, \tag{3}$$

$$x_{k+1} = y_k - \frac{f(x_k) + 2f(y_k)}{f(x_k)} \frac{f(y_k)}{f'(x_k)}, \tag{4}$$

respectively, where $y_k$ is the step of Newton's method. These methods will be denoted by OM1 and CM1, respectively.

The paper is organized as follows: we start in Section 2 with the design of the families of iterative methods for nonlinear equations, with and without derivatives. Section 3 is devoted to the extension of the obtained family with derivatives to systems of nonlinear equations by using the divided difference operator. By means of standard test functions and the problem of molecular interaction, in Section 4, we confirm the theoretical results. We finalize the paper with some concluding remarks in Section 5.

## 2 Design of the families for nonlinear equations

We propose a new family as a generalization of Ostrowski' and Chun's methods in the form:

$$\begin{aligned} y_k &= x_k - \alpha \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \left[ \frac{f(x_k)}{a_1 f(x_k) + a_2 f(y_k)} + \frac{b_1 f(x_k) + b_2 f(y_k)}{f(x_k)} \right] \frac{f(y_k)}{f'(x_k)}, \end{aligned} \tag{5}$$

where $\alpha$, $a_1$, $a_2$, $b_1$ and $b_2$ are real parameters. In the following result we show which values of the parameters are necessary to guarantee the order of convergence is at least 4.

**Theorem 1** *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a sufficiently differentiable function in an open interval $I$, such that $\xi \in I$ is a simple solution of the nonlinear equation $f(x) = 0$. Then, the sequence $\{x_k\}_{k\geq 0}$ obtained by using expression (5) converges to $\xi$ with local order of convergence at least four if $\alpha = 1$, $a_2 = a_1^2(b_2 - 2)$, $b_1 = 1 - \frac{1}{a_1}$ and for all $a_1$ and $b_2 \in \mathbb{R}$ with $a_1 \neq 0$. Then, the error equation is*

$$e_{k+1} = \left((5 - a_1(b_2 - 2)^2)c_2^3 - c_2 c_3\right) e_k^4 + \mathcal{O}[e_k^5],$$

*where $e_k = x_k - \xi$ and $c_q = \left(\dfrac{1}{q!}\right) \frac{f^{(q)}(\xi)}{f'(\xi)}$, $q \geq 2$.*

**Proof.** To prove the local convergence of our iterative process to the solution of $f(x) = 0$ we use the Taylor series expansion of the functions involved around the solution

$$f(x_k) = f'(\xi)[e_k + c_2 e_k^2 + c_3 e_k^3 + c_4 e_k^4] + \mathcal{O}[e_k^5] \tag{6}$$

$$f'(x_k) = f'(\xi)[1 + 2c_2 e_k + 3c_3 e_k^2 + 4c_4 e_k^3 + 5c_5 e_k^4] + \mathcal{O}[e_k^5]. \tag{7}$$

By direct division of (6) and (7) and substituting the obtained result in the first step of the proposed iterative method (5) we obtain:

$$y_k = \xi - (1-\alpha)e_k + \alpha c_2 e_k^2 - 2\alpha(c_2^2 - c_3)e_k^3 - \alpha(-4c_2^3 + 7c_2c_3 - 3c_4)e_k^4 + \mathcal{O}[e_k^5].$$

By using again the Taylor series expansion we obtain:

$$f(y_k) = A_1 e_k + A_2 e_k^2 + A_3 e_k^3 + A_4 e_k^4 + \mathcal{O}[e_k]^5,$$

where $A_1 = 1-\alpha$, $A_2 = \left(1-\alpha+\alpha^2\right)c_2$, $A_3 = -2\alpha^2 c_2^2 + (1-\alpha+3\alpha^2-\alpha^3)c_3$ and $A_4 = (1-\alpha+6\alpha^2-4\alpha^3+\alpha^4)c_4 + 5\alpha^2 c_2^3 - \alpha^2(10-3\alpha)c_2c_3$. Hence, substituting $f(x_k)$, $f'(x_k)$ and $f(y_k)$ in (5) we obtain the following error equation for the new family:

$$e_{k+1} = B_1 e_k + B_2 e_k^2 + B_3 e_k^3 + B_4 e_k^4 + \mathcal{O}[e_k^5],$$

where $B_1 = (1-\alpha)\left(1 - b_1 - b_2 + b_2\alpha - \dfrac{1}{a_1 + a_2 - a_2\alpha}\right)$. If $\alpha = 1$ then $B_1 = 0$ and the error equation for the iterative method (5) takes the form:

$$e_{k+1} = B_2' e_k^2 + B_3' e_k^3 + B_4' e_k^4 + \mathcal{O}[e_k^5],$$

where $B_2' = \left(1 - \dfrac{1}{a_1} - b_1\right)c_2$. In this case, if $b_1 = \dfrac{a_1 - 1}{a_1}$, then $B_2' = 0$ and we obtain for the error equation the following expression:

$$e_{k+1} = B_3'' e_k^3 + B_4'' e_k^4 + \mathcal{O}[e_k^5],$$

where $B_3'' = \dfrac{\left(a_2 - a_1^2(-2+b_2)\right)c_2^2}{a_1^2}$. We see that if $a_2 = a_1^2(b_2 - 2)$, then $B_3'' = 0$ and

$$e_{k+1} = \left(5 - a_1(b_2-2)^2)c_2^3 - c_2c_3\right)e_k^4 + \mathcal{O}[e_k^5], \tag{8}$$

so the order of convergence is at least four. ∎

Therefore, we obtain the following iterative formula for the bi-parametric family

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{1}{a_1}\left[\frac{f(x_k)}{f(x_k) + a_1(b_2-2)f(y_k)} + \frac{(a_1-1)f(x_k) + a_1 b_2 f(y_k)}{f(x_k)}\right]\frac{f(y_k)}{f'(x_k)}, \end{aligned} \tag{9}$$

We present some particular cases of (9):

1. If $b_2 = 2$, the parameter $a_1$ disappears and the resulting scheme is Chun's method.
2. When $a_1 = 1$, the iterative formula takes the form:

$$x_{k+1} = y_k - \left[\frac{f(x_k)}{f(x_k) + (b_2-2)f(y_k)} + \frac{b_2 f(y_k)}{f(x_k)}\right]\frac{f(y_k)}{f'(x_k)}$$

   and we have a one parametric family including the original methods as particular cases: (a) if $b_2 = 2$, as we have said, we have Chun's method (4) and (b) if $b_2 = 0$, we get Ostrowski's scheme (3).

3. For any $a_1 \neq 0$ and $b_2 = 0$, the iterative formula is:

$$x_{k+1} = y_k - \frac{f(x_k) - 2(a_1-1)f(y_k)}{f(x_k) - 2a_1 f(y_k)}\frac{f(y_k)}{f'(x_k)}.$$

   If we denote $-2(a_1 - 1) = \beta$, then $-2a_1 = \beta - 2$ and we get King's family [21]

$$x_{k+1} = y_k - \frac{f(x_k) + \beta f(y_k)}{f(x_k) + (\beta-2)f(y_k)}\frac{f(y_k)}{f'(x_k)}.$$

4. For any $a_1 \neq 0$ and $b_2 = 1$, the iterative formula takes the form:

$$x_{k+1} = y_k - \frac{1}{a_1}\left[\frac{f(x_k)}{f(x_k) - a_1 f(y_k)} + \frac{(a_1-1)f(x_k) + a_1 f(y_k)}{f(x_k)}\right]\frac{f(y_k)}{f'(x_k)}.$$

At this point, can we get a similar family by approximating the derivatives by divided differences and preserving the order of convergence? The answer is given in the following result, where a technique describe in [22].

**Theorem 2** *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a sufficiently differentiable function in an open interval $I$, such that $\xi \in I$ is a simple solution of the nonlinear equation $f(x) = 0$. Then, the sequence $\{x_k\}_{k\geq 0}$ obtained by using the expression*

$$\begin{aligned} y_k &= x_k - \alpha\frac{f(x_k)}{f[z_k, x_k]}, \\ x_{k+1} &= y_k - \left[\frac{f(x_k)}{a_1 f(x_k) + a_2 f(y_k)} + \frac{b_1 f(x_k) + b_2 f(y_k)}{f(x_k)}\right]\frac{f(y_k)}{f[z_k, x_k]}, \end{aligned} \tag{10}$$

*where $z_k = x_k + f(x_k)^2$ and $f[z_k, x_k] = \dfrac{f(z_k) - f(x_k)}{z_k - x_k}$, converges to $\xi$ with order of convergence at least four if $\alpha = 1$, $a_2 = a_1^2(b_2 - 2)$, $b_1 = 1 - \frac{1}{a_1}$ and for all $a_1$ and $b_2 \in \mathbb{R}$, with $a_1 \neq 0$. The error equation is*

$$e_{k+1} = \left((5 - a_1(-2 + b_2)^2)c_2^3 - c_2 c_3 + \gamma c_2^2\right)e_k^4 + \mathcal{O}[e_k^5],$$

*where $e_k = x_k - \xi$, $\gamma = f'(\xi)^2$ and $c_q = \left(\dfrac{1}{q!}\right)\frac{f^{(q)}(\xi)}{f'(\xi)}$, $q \geq 2$.*

**Proof.** By using the Taylor series expansion of the function $f(x_k)$ around $\xi$ (6), we obtain the following expressions:

$$\begin{aligned} z_k &= e_k + \gamma[e_k^2 + 2c_2 e_k^3 + (c_2^2 + 2c_3)e_k^4 + 2(c_2 c_3 + c_4)e_k^5 + (c_3^2 + 2c_2 c_4 + 2c_5)e_k^6 + \mathcal{O}[e_k^7], \\ f(z_k) &= f'(\xi)[e_k + (c_2 + \gamma)e_k^2 + (c_4 + \gamma(5c_2^2 + 5c_3 + \gamma c_2))e_k^4 \\ &\quad + (c_5 + \gamma(2c_2^3 + 12c_2 c_3 + 6c_4 + \gamma(4c_2^2 + 3c_3)e_k^5 + \mathcal{O}[e_k^6], \\ f[z_k, x_k] &= f'(\xi)[1 + 2c_2 e_k + (\gamma c_2 + 3c_3)e_k^2 + (4c_4 + \gamma(2c_2^2 + 3c_3))e_k^3 + O[e_k]^4. \end{aligned}$$

Hence, substituting these expressions in (10), we obtain the following result for $y_k$:

$$y_k = (1-\alpha)e_k + \alpha(2c_3 - 2c_2^2 + \gamma c_2)e_k^2 + \alpha(4c_2^3 - 7c_2 c_3 - \gamma c_2^2 + 3c_4 + 3\gamma c_3)e_k^4 + \mathcal{O}[e_k^5].$$

By using the Taylor series expansion again, we obtain the following expression:

$$f(y_k) = A_1 e_k + A_2 e_k^2 + A_3 e_k^3 + A_4 e_k^4 + \mathcal{O}[e_k^5],$$

where $A_1 = 1-\alpha$, $A_2 = (1-\alpha+\alpha^2)c_2$, $A_3 = \alpha(\gamma - 2\alpha c_2)c_2 + (1 - \alpha + 3\alpha^2 - \alpha^3)c_3$ and $A_4 = (1 - \alpha + 6\alpha^2 - 4\alpha^3 + \alpha^4)c_4 + \alpha(3\gamma c_3 + \gamma(1-2\alpha)c_2^2 + 5\alpha c_2^3 + \alpha(3\alpha - 10)c_2 c_3)$. Through these results we get the following error equation for the iterative scheme (10):

$$e_{k+1} = B_1 e_k + B_2 e_k^2 + B_3 e_k^3 + B_4 e_k^4 + \mathcal{O}[e_k^5],$$

where $B_1 = (1-\alpha)\left(1 - b_1 - b_2 + b_2\alpha - \dfrac{1}{a_1 + a_2 - a_2\alpha}\right)$.

If $\alpha = 1$, then $B_1 = 0$ and the error equation takes the form:

$$e_{k+1} = B_2' e_k^2 + B_3' e_k^3 + B_4' e_k^4 + \mathcal{O}[e_k^5],$$

where $B_2' = \left(1 - \dfrac{1}{a_1} - b_1\right)c_2$. In this case, if $b_1 = \dfrac{a_1 - 1}{a_1}$, $B_2' = 0$ and we obtain for the error equation the following expression:

$$e_{k+1} = B_3'' e_k^3 + B_4'' e_k^4 + \mathcal{O}[e_k^5],$$

where $B_3'' = \dfrac{\left(a_2 - a_1^2(-2+b_2)\right) c_2^2}{a_1^2}$. We see that if $a_2 = a_1^2(b_2-2)$, then $B_3''=0$ and

$$e_{k+1} = ((5 - a_1(-2+b_2)^2)c_2^3 - c_2c_3 + \gamma c_2^2)e_k^4 + \mathcal{O}[e_k^5], \tag{11}$$

so the order of convergence is at least four. ■

Then, we obtain the following iterative formula for the bi-parametric family

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k, x_k]}, \\ x_{k+1} &= y_k - \frac{1}{a_1}\left[\frac{f(x_k)}{f(x_k) + a_1(b_2-2)f(y_k)} + \frac{(a_1-1)f(x_k) + a_1 b_2 f(y_k)}{f(x_k)}\right]\frac{f(y_k)}{f[z_k, x_k]}, \end{aligned} \tag{12}$$

and we define the following particular cases of the (12):

1. If $b_2 = 2$, then parameter $a_1$ is canceled in the iterative expression, that corresponds to the derivative-free Chun's scheme (CM2).
2. When $a_1 = 1$, the iterative formula takes the form:

   $$x_{k+1} = y_k - \left[\frac{f(x_k)}{f(x_k) + (b_2-2)f(y_k)} + \frac{b_2 f(y_k)}{f(x_k)}\right]\frac{f(y_k)}{f[z_k, x_k]}$$

   and we have a one parametric family that includes the derivative-free versions of original schemes: (a) if $b_2 = 2$, we have derivative-free Chun's method, whose iterative expression is

   $$x_{k+1} = y_k - \frac{f(x_k) + 2f(x_k)}{f(x_k)}\frac{f(y_k)}{f[z_k, x_k]},$$

   and (b) if $b_2 = 0$, we obtain derivative-free Ostrowski's method (OM2), with the iterative expression

   $$x_{k+1} = y_k - \frac{f(x_k)}{f(x_k) - 2f(x_k)}\frac{f(y_k)}{f[z_k, x_k]}.$$

3. When $a_1 \neq 0$ and $b_2 = 0$, the iterative formula takes the form:

   $$x_{k+1} = y_k - \frac{f(x_k) - 2(a_1-1)f(y_k)}{f(x_k) - 2a_1 f(y_k)}\frac{f(y_k)}{f[z_k, x_k].}$$

   If we denote $-2(a_1-1) = \beta$, then $-2a_1 = \beta - 2$ and we get the derivative-free King's family

   $$x_{k+1} = y_k - \frac{f(x_k) + \beta f(y_k)}{f(x_k) + (\beta-2)f(y_k)}\frac{f(y_k)}{f[z_k, x_k]}.$$

4. When $a_1 \neq 0$ and $b_2 = 1$, the resulting iterative formula is:

   $$x_{k+1} = y_k - \frac{1}{a_1}\left[\frac{f(x_k)}{f(x_k) - a_1 f(y_k)} + \frac{(a_1-1)f(x_k) + a_1 f(y_k)}{f(x_k)}\right]\frac{f(y_k)}{f[z_k, x_k]}.$$

# 3 Extension to systems of nonlinear equations

The objective of this section is to give a generalization to several variables of one of the families obtained in Section 2, preserving the local order of convergence. In order to get this aim, we are going to use the divided difference operator.

Let us consider a sufficiently differentiable function $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ in a convex set $\Omega \subset \mathbb{R}^n$ and let $\xi \in \Omega$ be a solution of the nonlinear system $F(x) = 0$. The divided difference operator of $F$ on $\mathbb{R}^n$ is a mapping $[\cdot,\cdot; F] \ : \ \Omega \times \Omega \subset \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n)$ (see [7]) such that

$$[x, y; F](x-y) = F(x) - F(y), \quad \text{for any } x, y \in \Omega.$$

In the proof of the following result, we will use the Genochi-Hermite formula (see [7])

$$[x,y;F] = \int_0^1 F'(x+t(x-y))dt,$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

The extension to multivariate case of family (5) requires to rewrite the iterative expression in such a way that no functional evaluation of the nonlinear function remain at the denominator, as they will become vectors in the multivariate case. To get this aim, let us consider that, being the first step of the iterative process $y_k = x_k - \alpha \frac{f(x_k)}{f'(x_k)}$, $f(x_k)$ can be expressed as $f(x_k) = \frac{1}{\alpha}(x_k - y_k)f'(x_k)$. By using this, we can rewrite the quotient $\frac{f(y_k)}{f(x_k)}$ as

$$\frac{f(y_k)}{f(x_k)} = 1 - \alpha \frac{f[x_k, y_k]}{f'(x_k)}.$$

By using this transformation, the proposed family (5) is fully extensible to several variables,

$$\begin{aligned} y^{(k+1)} &= x^{(k)} - \alpha[F'(x^{(k)})]^{-1}F(x^{(k)}) \\ x^{(k+1)} &= y^{(k)} - \left(G_1(x^{(k)}, y^{(k)}) + G_2(x^{(k)}, y^{(k)})\right)[F'(x^{(k)})]^{-1}F(y^{(k)}), \\ G_1(x^{(k)}, y^{(k)}) &= \left[(a_1+a_2)I - \alpha a_2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right]^{-1}, \\ G_2(x^{(k)}, y^{(k)}) &= (b_1+b_2)I - \alpha b_2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F], \end{aligned} \tag{13}$$

where

$$\begin{aligned} G_1(x^{(k)}, y^{(k)}) &= \left[(a_1+a_2)I - \alpha a_2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right]^{-1}, \\ G_2(x^{(k)}, y^{(k)}) &= (b_1+b_2)I - \alpha b_2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F], \end{aligned}$$

and $[x^{(k)}, y^{(k)}; F]$ denotes the divided difference operator of $F$ on $x^{(k)}$ and $y^{(k)}$, identity matrix is denoted by $I$ and $F'(x^{(k)})$ is the Jacobian matrix of the system. In the proof of the following result we are going to use the notation introduced in [23].

**Theorem 3** *Let $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in a convex set $\Omega$ and $\xi \in \Omega$ be a solution of $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular at $\xi$. Then, the sequence $\{x^{(k)}\}_{k\geq 0}$ obtained by using the iterative scheme (13), converges to $\xi$ with order of convergence at least four if $\alpha = 1$, $a_2 = a_1^2(b_2-2)$, $b_1 = 1 - \frac{1}{a_1}$ and for all $a_1$ and $b_2 \in \mathbb{R}$ with $a_1 \neq 0$. The error equation is*

$$e_{k+1} = -[(a_1(b_2-2)^2 - 5)C_2^3 + C_2C_3]e_k^4 + \mathcal{O}[e_k^5],$$

*where $e_k = x^{(k)} - \xi$ and $C_q = \left(\frac{1}{q!}\right)[f'(\xi)]^{-1}F^{(q)}(\xi)$, $q \geq 2$.*

**Proof.** By using Taylor expansion around $\xi$, we obtain:

$$F(x^{(k)}) = F'(\xi)(e_k + C_2e_k^2 + C_3e_k^3 + C_4e_k^4) + \mathcal{O}[e_k^5],$$

$$F'(x^{(k)}) = F'(\xi)(I + 2C_2e_k + 3C_3e_k^2 + 4C_4e_k^3) + \mathcal{O}[e_k^4].$$

Let us consider

$$[F'(x^{(k)})]^{-1} = (I + X_2e_k + X_3e_k^2 + X_4e_k^3)[F'(\xi)]^{-1} + \mathcal{O}(e_k^4).$$

Forcing $[F'(x^{(k)})]^{-1}F'(x^{(k)}) = I$, we get $X_2 = -2C_2$, $X_3 = 2C_2^2 - 3C_3$ and $X_4 = -4C_4 + 6C_3C_2 - 4C_2^3 + 6C_2C_3$. These expressions allow us to obtain

$$y^{(k)} = x^{(k)} - \alpha[F'(x^{(k)})]^{-1}F(x^{(k)}) = \xi + (1-\alpha)e^{(k)} - \alpha(A_2e_k^2 + A_3e_k^3 + A_4e_k^4) + \mathcal{O}[e_k^5], \tag{14}$$

where $A_2 = -C_2 - X_2$, $A_3 = -C_3 - C_2X_2 - X_3$ and $A_4 = -C_4 - C_3X_2 - C_2X_3 + X_4$. By using (14) and the Taylor series expansion around $\xi$ we obtain

$$F(y^{(k)}) = F'(\xi)(B_1e_k + B_2e_k^2 + B_3e_k^3 + B_4e_k^4) + \mathcal{O}[e_k^5],$$

where $B_1 = \beta$, $B_2 = (\alpha+\beta^2)C_2$, $B_3 = -\alpha A_3 + 2\alpha\beta C_2 A_3 + 3\alpha\beta^2 C_3 C_2 + \beta^4 C_4$, $B_4 = -\alpha A_4 + \alpha^2 C_2^3 - 2\alpha\beta C_2 A_3 + 3\alpha\beta^2 C_3 C - 2 + \beta^4 C_4$ and $\beta = 1-\alpha$. We calculate the Taylor expansion of $[x^{(k)}, y^{(k)}; F]$ by using (14),

$$[x^{(k)}, y^{(k)}; F] = F'(\xi)\left[I + D_2 e_k + D_3 e_k^2 + D_4 e_k^3\right] + \mathcal{O}[e_k^4],$$

where $D_2 = (2-\alpha)C_2$, $D_3 = \alpha C_2^2 + (3-3\alpha+\alpha^2)C_3$ and $D_4 = 2\alpha C_2 C_3 + \alpha(3-2\alpha)C_3 C_2 - (4-6\alpha+4\alpha^2-\alpha^3)C_4$. Then,

$$M = (a_1+a_2)I - \alpha a_2 [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F] = a_1 + E_2 e_k + E_3 e_k^2 + E_4 e_k^3 + \mathcal{O}[e_k^4],$$

where $E_2 = \alpha a_2 C_2$, $E_3 = \alpha C_2^3 + \alpha(\alpha-3)C_3$ and $E_4 = 6\alpha C_2 C_3 - 2\alpha C_2^3 - 4C_4 + 5\alpha(2-\alpha)C_3 C_2 + (4 - 6\alpha + 4\alpha^2 - \alpha^3)C_3 C_4$. So, we obtain $G_1(x^{(k)}, y^{(k)})$ as the inverse of matrix $M$:

$$G_1(x^{(k)}, y^{(k)}) = I + Y_2 e_k + Y_3 e_k^2 + Y_4 e_k^3 + \mathcal{O}[e_k^4],$$

where $Y_2 = \dfrac{\alpha a_2}{a_1} C_2$, $Y_3 = \dfrac{\alpha a_2}{a_1^2}[(\alpha a_2 - 3)C_2^2 + (\alpha-3)C_3$, $Y_4 = \dfrac{\alpha a_2}{a_1^3}[(8a_1 + 3\alpha a_1 a_2 + 3\alpha a_2 - \alpha^2 a_2^2)C_2^3]$ and

$$G_2(x^{(k)}, y^{(k)}) = b_1 + F_2 e_k + F_3 e_k^2 + F_4 e_k^3 + F_5 e_k^4 + \mathcal{O}[e_k^5],$$

where $F_2 = \alpha b_2 C_2$, $F_3 = -\alpha b_2[3C_2^2 - (\alpha-3)C_3]$ and $F_4 = b_2[\alpha(6-4\alpha+\alpha^2)C_4 - 6\alpha(2-\alpha)C_3 C_2 + 4(\alpha+1)C_2^3 - 6(\alpha+1)C_2 C_3]$.

Thus, we obtain the error equation of the proposed method

$$e_{k+1} = H_1 e_k + H_2 e_k^2 + H_3 e_k^3 + H_4 e_k^4 + \mathcal{O}[e_k^5],$$

where $H_1 = \dfrac{1}{a_1}(1 + a_1(b_1 - 1))(\alpha - 1)$. If $\alpha = 1$, then $H_1 = 0$ and the error equation takes the form:

$$e_{k+1} = H_2' e_k^2 + H_3' e_k^2 + H_4' e_k^4 + \mathcal{O}[e_k^5],$$

where $H_1' = -\dfrac{1}{a_1}(1 + a_1(b_1-1))C_2$. We note that if $b_1 = \dfrac{a_1 - 1}{a_1}$, then $H_2' = 0$. We introduce this value of $b_1$ and obtain the new form of the error equation

$$e_{k+1} = H_3''' e_k^3 + H_4''' e_k^4 + \mathcal{O}[e_k^5],$$

where $H_3''' = \dfrac{a_2 - a_1^2(b_2-2)}{a_1^2} C_2^2$. Finally, if $a_2 = a_1^2(b_2 - 2)$, the error equation is:

$$e_{k+1} = -[(a_1(b_2-2)^2 - 5)C_2^3 + C_2 C_3]e_k^4 + \mathcal{O}[e_k^5] \tag{15}$$

and this shows that the proposed method has order of convergence at least four. ■

Under the assumptions made in the previous result, the iterative scheme of the bi-parametric family (13) takes the form:

$$\begin{aligned}
y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}),\\
x^{(k+1)} &= y^{(k)} - G(x^{(k)}, y^{(k)})[F'(x^{(k)})]^{-1}F(y^{(k)}),\\
G(x^{(k)}, y^{(k)}) &= \frac{1}{a_1}\left[(1 + a_1 b_2 - 2a_1)I - a_1(b_2-2)[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right]^{-1}\\
&\quad + \frac{1}{a_1}\left((a_1 + a_1 b_2 - 1)I - b_2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right).
\end{aligned} \tag{16}$$

In the following we propose some particular cases:

1. As in the scalar case, if $b_2 = 2$,

$$G(x^{(k)}, y^{(k)}) = 3I - 2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]$$

and the resulting scheme is the extended Chun's method for nonlinear systems (CM3).

2. When $a_1 = 1$,

$$\begin{aligned} G(x^{(k)}, y^{(k)}) &= [(b_2-1)I - (b_2-2)[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]]^{-1} \\ &+ b_2 I - b_2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F] \end{aligned}$$

and we have a parametric family. Some particular cases of this class are the following:

(a) If $b_2 = 2$, we have Chun's method transferred to systems

$$x^{(k+1)} = y^{(k)} - \left(I - 2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right)[F'(x^{(k)})]^{-1}F(y^{(k)}).$$

(b) If $b_2 = 0$, we get Ostrowski's method transferred to systems (OM3)

$$x^{(k+1)} = y^{(k)} - \left(-I + 2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right)^{-1}[F'(x^{(k)})]^{-1}F(y^{(k)}).$$

3. For any $a_1 \neq 0$ and $b_2 = 0$,

$$G(x^{(k)}, y^{(k)}) = \frac{a_1-1}{a_1}I + \left[a_1(1-2a_1)I + 2a_1^2[F'(x^{(k)}]^{-1}[x^{(k)}, y^{(k)}; F]\right]^{-1}.$$

4. For any $a_1 \neq 0$ and $b_2 = 1$,

$$\begin{aligned} x^{(k+1)} &= y^{(k)} - \frac{1}{a_1}\left[(1-a_1)I + a_1[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right]^{-1}[F'(x^{(k)})]^{-1}F(y^{(k)}) \\ &+ \frac{1}{a_1}\left[(2a_1-1)I - [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right][F'(x^{(k)})]^{-1}F(y^{(k)}). \end{aligned}$$

# 4 Numerical results

In this section we show the numerical behavior of the proposed methods on some standard equations and systems and also on an applied problem. In the tests made, variable precision arithmetics has been used, with 4000 digits of mantissa (in the numerical tests for nonlinear equations) and 1000 digits of mantissa (in the numerical tests for systems of nonlinear equations) in MATLAB R2013a. These tests have been made by using the stopping criterium $\|F(x^{(k+1)})\| < 10^{-700}$ or $\|x^{(k+1)} - x^{(k)}\| < 10^{-700}$. We will also use the approximated computational order of convergence $\rho$ (usually called ACOC), defined by Cordero and Torregrosa in [25]

$$\rho = \frac{\ln(\|x^{(k+1)} - x^{(k)}\|/\|x^{(k)} - x^{(k-1)}\|)}{\ln(\|x^{(k)} - x^{(k-1)}\|/\|x^{(k-1)} - x^{(k-2)}|)}.$$

## 4.1 Academic test functions

Firstly, to check the behavior of the proposed methods with derivatives for solving nonlinear equations, we use the following elements of the family of obtained methods:

1. **MA1**: $a_1 = \frac{5}{4}$ and $b_2 = 0$

$$x_{k+1} = y_k - \frac{f(x_k) - \frac{1}{2}f(y_k)}{f(x_k) - \frac{5}{2}f(y_k)}\frac{f(y_k)}{f'(x_k)},$$

2. **MB1**: $a_1 = 1$ and $b_2 = 1$

$$x_{k+1} = y_k - \left(\frac{f(x_k)}{f(x_k) - f(y_k)} + \frac{f(y_k)}{f(x_k)}\right)\frac{f(y_k)}{f'(x_k)},$$

3. **MC1**: $a_1 = 1$ and $b_2 = 3$

$$x_{k+1} = y_k - \left(\frac{f(x_k)}{f(x_k) + f(y_k)} + \frac{3f(y_k)}{f(x_k)}\right)\frac{f(y_k)}{f'(x_k)},$$

where $y_k$ is Newton's step. In these numerical experiments, we compare MA1, MB1 and MC1 with Newton's method (NM), Ostrowski's method (OM) (3), Chun's method (CM) (4) and Jarratt's method (JM) [24]

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)} \\ x_{k+1} &= x_k - \frac{1}{2}\frac{3f'(x_k)+f'(y_k)}{3f'(x_k)-f'(y_k)}\frac{f(x_k)}{f'(x_k)}. \end{aligned}$$

Tables 1 to 4 show, for each initial estimation $x_0$ and every method, the approximated computational order of convergence $\rho$, the number of iterations, and two measures of the error, specifically, $\|x^{(k+1)} - x^{(k)}\|$ and $\|F(x^{(k+1)})\|$.

At the sight of the results in Table 1, we conclude that the new methods have an excellent behavior, giving the best error estimations in all cases.

| $f_1(x) = \sin x - x^2 + 1$, $x_0 = 1$ and $\xi \approx 1.409624004002596$ | | | | |
|---|---|---|---|---|
| Method | $\rho$ | iter | $\lvert x_{k+1} - x_k\rvert$ | $\lvert f(x_{k+1})\rvert$ |
| **NM** | 2.0000 | 10 | 1.867e-273 | 5.205e-546 |
| **JM** | 4.0000 | 5 | 7.315e-139 | 1.307e-553 |
| **OM** | 4.0000 | 6 | 3.774e-196 | 4.751e-782 |
| **CM** | 4.0000 | 5 | 4.093e-139 | 1.268e-554 |
| **MA1** | 4.0000 | 5 | 1.389e-178 | 5.588e-716 |
| **MB1** | 4.0000 | 7 | 2.005e-959 | 2.697e-2008 |
| **MC1** | 4.0000 | 5 | 3.938e-090 | 4.497e-358 |
| $f_2(x) = x^2 - \exp(x) - 3x + 2$, $x_0 = 0.8$ and $\xi \approx 0.257530285439861$ | | | | |
| Method | $\rho$ | iter | $\lvert x_{k+1} - x_k\rvert$ | $\lvert f(x_{k+1})\rvert$ |
| **NM** | 2.0000 | 8 | 4.472e-190 | 7.062e-380 |
| **JM** | 4.0000 | 5 | 1.756e-258 | 1.622e-1033 |
| **OM** | 4.0000 | 5 | 7.970e-271 | 1.909e-1083 |
| **CM** | 4.0000 | 5 | 3.475e-286 | 3.363e-1114 |
| **MA1** | 4.0000 | 5 | 8.287e-257 | 9.502e-1027 |
| **MB1** | 4.0000 | 6 | 8.005e-1065 | 0.0 |
| **MC1** | 4.0000 | 5 | 4.385e-266 | 2.889e-1064 |
| $f_3(x) = \cos x - x$, $x_0 = 1$ and $\xi \approx 0.739085133215161$ | | | | |
| Method | $\rho$ | iter | $\lvert x_{k+1} - x_k\rvert$ | $\lvert f(x_{k+1})\rvert$ |
| **NM** | 2.0000 | 8 | 7.118e-167 | 1.872e-333 |
| **JM** | 4.0000 | 5 | 4.214e-296 | 1.350e-1183 |
| **OM** | 4.0000 | 5 | 1.102e-268 | 1.693e-1073 |
| **CM** | 4.0000 | 5 | 1.632e-299 | 2.793e-1197 |
| **MA1** | 4.0000 | 5 | 1.594e-309 | 1.599e-1237 |
| **MB1** | 4.0000 | 6 | 1.026e-1093 | 1.349e-2008 |
| **MC1** | 4.0000 | 5 | 2.233e-273 | 2.409e-1092 |
| $f_4(x) = \cos x - x\exp x + x^2$, $x_0 = 0.5$ and $\xi \approx 0.639154096332008$ | | | | |
| Method | $\rho$ | iter | $\lvert x_{k+1} - x_k\rvert$ | $\lvert f(x_{k+1})\rvert$ |
| **NM** | 2.0000 | 9 | 1.068e-243 | 2.168e-486 |
| **JM** | 4.0000 | 5 | 4.140e-293 | 1.019e-1170 |
| **OM** | 4.0000 | 5 | 3.505e-182 | 7.589e-726 |
| **CM** | 4.0000 | 5 | 5.909e-289 | 4.679e-1154 |
| **MA1** | 4.0000 | 5 | 3.485e-254 | 1.219e-1014 |
| **MB1** | 4.0000 | 5 | 1.929e-770 | 1.349e-2008 |
| **MC1** | 4.0000 | 5 | 2.034e-196 | 6.597e-783 |

Table 1: Test functions and numerical results for methods with derivatives

Table 2: Test functions and numerical results for methods without derivatives

| $f_1(x) = \sin x - x^2 + 1$, $x_0 = 1$ and $\xi \approx 1.409624004002596$ | | | | |
|---|---|---|---|---|
| Method | $\rho$ | iter | $\lvert x_{k+1} - x_k\rvert$ | $\lvert f(x_{k+1})\rvert$ |
| **ST** | 2.0000 | 10 | 3.249e-274 | 2.615e-547 |
| **LZM** | 4.0000 | 5 | 5.953e-239 | 5.950e-954 |
| **CT4** | 4.0000 | 5 | 6.200e-151 | 1.855e-601 |
| **OM2** | 4.0000 | 6 | 5.935e-277 | 6.793e-1105 |
| **CM2** | 4.0000 | 5 | 1.012e-91 | 3.771e-364 |
| **MA2** | 4.0000 | 6 | 8.754e-246 | 3.492e-980 |
| **MB2** | 4.0000 | 6 | 2.311e-240 | 5.332e-959 |
| **MC2** | 4.0000 | 5 | 3.938e-90 | 4.497e-358 |

| $f_2(x) = x^2 - \exp(x) - 3x + 2$, $x_0 = 0.8$ and $\xi \approx 0.257530285439861$ | | | | |
|---|---|---|---|---|
| Method | $\rho$ | iter | $\lvert x_{k+1} - x_k\rvert$ | $\lvert f(x_{k+1})\rvert$ |
| **ST** | 2.0000 | 9 | 3.979e-175 | 1.554e-349 |
| **LZM** | 4.0000 | 5 | 4.687e-163 | 6.775e-651 |
| **CT4** | 4.0000 | 5 | 1.336e-166 | 4.202e-665 |
| **OM2** | - | n.c. | - | - |
| **CM2** | 4.0000 | 10 | 4.367e-111 | 1.697e-442 |
| **MA2** | - | n.c. | - | - |
| **MB2** | 4.0000 | 5 | 4.436e-266 | 3.025e-1064 |
| **MC2** | 4.0000 | 5 | 4.385e-266 | 2.889e-1064 |

| $f_3(x) = \cos x - x$, $x_0 = 1$ and $\xi \approx 0.739085133215161$ | | | | |
|---|---|---|---|---|
| Method | $\rho$ | iter | $\lvert x_{k+1} - x_k\rvert$ | $\lvert f(x_{k+1})\rvert$ |
| **ST** | 2.0000 | 8 | 4.380e-178 | 4.776e-356 |
| **LZM** | 4.0000 | 4 | 1.190e-84 | 1.460e-338 |
| **CT4** | 4.0000 | 5 | 6.809e-309 | 4.167e-1235 |
| **OM2** | 4.0000 | 5 | 2.492e-238 | 7.160e-952 |
| **CM2** | 4.0000 | 5 | 1.281e-286 | 3.064e-1145 |
| **MA2** | 4.0000 | 5 | 1.433e-231 | 8.582e-925 |
| **MB2** | 4.0000 | 5 | 1.154e-237 | 1.717e-1093 |
| **MC2** | 4.0000 | 5 | 2.233e-273 | 2.409e-1092 |

| $f_4(x) = \cos x - x\exp x + x^2$, $x_0 = 0.5$ and $\xi \approx 0.639154096332008$ | | | | |
|---|---|---|---|---|
| Method | $\rho$ | iter | $\lvert x_{k+1} - x_k\rvert$ | $\lvert f(x_{k+1})\rvert$ |
| **ST** | 2.0000 | 9 | 1.412e-219 | 5.402e-438 |
| **LZM** | 4.0000 | 5 | 1.071e-256 | 1.266e-1024 |
| **CT4** | 4.0000 | 5 | 1.242e-276 | 1.671e-1104 |
| **OM2** | 4.0000 | 5 | 2.656e-209 | 4.187e-834 |
| **CM2** | 4.0000 | 5 | 3.987e-220 | 9.449e-878 |
| **MA2** | 4.0000 | 5 | 2.132e-207 | 1.980e-826 |
| **MB2** | 4.0000 | 5 | 3.318e-193 | 4.676e-770 |
| **MC2** | 4.0000 | 5 | 2.034e-196 | 6.597e-783 |

Now, the elements of the family of derivative-free methods that we are going to use are:

1. **MA2**: $a_1 = \frac{5}{4}$ and $b_2 = 0$

$$x_{k+1} = y_k - \frac{f(x_k) - \frac{1}{2}f(y_k)}{f(x_k) - \frac{5}{2}f(y_k)} \frac{f(y_k)}{f[z_k, x_k]}$$

2. **MB2**: $a_1 = 1$ and $b_2 = 1$

$$x_{k+1} = y_k - \left(\frac{f(x_k)}{f(x_k) - f(y_k)} + \frac{f(y_k)}{f(x_k)}\right) \frac{f(y_k)}{f[z_k, x_k]}$$

3. **MC2**: $a_1 = 1$ and $b_2 = 3$

$$x_{k+1} = y_k - \left( \frac{f(x_k)}{f(x_k) + f(y_k)} + \frac{3f(y_k)}{f(x_k)} \right) \frac{f(y_k)}{f[z_k, x_k]}$$

where $y_k = x_k - \dfrac{f(x_k)}{f[z_k, x_k]}$, $f[z_k, x_k] = \dfrac{f(z_k) - f(x_k)}{z_k - x_k}$ and $z_k = x_k + f(x_k)^2$. In this case, we compare our schemes with Steffensen's method (SM) [6], LZM [26]

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)^2}{f(z_k) - f(x_k)}, \; z_k = x_k + f(x_k), \\ x_{k+1} &= y_k - \frac{f[x_k, y_k] - f[y_k, z_k] + f[x_k, z_k]}{f[x_k, y_k]^2} f(y_k), \end{aligned}$$

and CT4 [27] (with $\gamma = 1$, $a = 1$, $b = 1$, $c = 1$ and $d = 0$)

$$\begin{aligned} y_k &= x_k - \frac{\gamma f(x_k)^2}{f(z_k) - f(x_k)}, \; z_k = x_k + \gamma f(x_k), \\ x_{k+1} &= y_k - \frac{f(y_k)}{\frac{af(y_k) - bf(z_k)}{y_k - z_k} + \frac{cf(y_k) - df(x_k)}{y_k - x_k}}. \end{aligned}$$

From the results shown in Table 2, it can be stated that the proposed schemes are quite competitive respect to the known ones, being best ones in some cases.

| $F_1(x_1, x_2) = (\exp x_1 \exp x_2 + x_1 \cos x_2,\ x_1 + x_2 - 1$<br>$x^{(0)} = (3,\ -2)$<br>and $\xi_1 \approx 3.4675009642402,\ \xi_2 \approx -2.4675009642402$ | | | | |
|---|---|---|---|---|
| Method | $\rho$ | iter | $\|x^{(k+1)} - x^{(k)}\|$ | $\|F(x^{(k+1)})\|$ |
| **NM** | 2.0000 | 9 | 1.141e-397 | 4.802e-795 |
| **TM** | 3.0000 | 6 | 2.905e-336 | 5.671e-1008 |
| **JM** | 4.0000 | 5 | 5.597e-254 | 5.671e-1008 |
| **OM3** | 4.0000 | 5 | 3.978e-266 | 5.671e-1008 |
| **CM3** | 4.0000 | 5 | 9.701e-261 | 5.671e-1008 |
| **MA3** | 4.0000 | 5 | 3.749e-268 | 9.301e-1072 |
| **MB3** | 4.0000 | 5 | 7.966e-262 | 3.762e-1046 |
| **MC3** | 4.0000 | 5 | 7.972e-262 | 3.773e-1046 |
| $F_2(x_1, x_2, x_3, x_4) = (x_1x_3 + x_4(x_2 + x_3), x_1x_3 + x_4(x_1 + x_3),$<br>$x_1x_2 + x_4(x_1 + x_2), x_1x_2 + x_1x_3 + x_2x_3 - 1),\ x_0 = (1,\ 1,\ 1,\ -0.5)$<br>and $\xi_1 = \xi_2 = \xi_3 = 0.5773502691896257,$<br>$\xi_4 = -0.2886751345948129$ | | | | |
| Method | $\rho$ | iter | $\|x^{(k+1)} - x^{(k)}\|$ | $\|F(x^{(k+1)})\|$ |
| **NM** | 2.0000 | 11 | 4.407e-586 | 3.007e-1008 |
| **TM** | 3.0000 | 7 | 3.003e-341 | 2.835e-1008 |
| **JM** | 4.0000 | 6 | 4.407e-586 | 2.835e-1008 |
| **OM3** | 4.0000 | 6 | 4.407e-586 | 2.835e-1008 |
| **CM3** | 4.0000 | 6 | 9.920e-425 | 0.0 |
| **MA3** | 4.9996 | 5 | 7.717e-340 | 4.202e-1697 |
| **MB3** | 4.0000 | 6 | 6.486e-447 | 1.508e-1785 |
| **MC3** | 4.0000 | 6 | 1.555e-442 | 4.982e-1768 |

Table 3: Test functions and results for nonlinear systems, $F_1$ and $F_2$

In Tables 3 and 4, we show the results obtained by using the following elements of the family (16), for the following values of $a_1$ and $b_2$:

1. **MA3**: $a_1 = \frac{5}{4}$ and $b_2 = 0$
2. **MB3**: $a_1 = 1$ and $b_2 = 1$

| $F_3(x_1,x_2) = (x_1^2 - x_1 - x_2^2 - 1,\ sinx_1 + x_2)$, $x_0 = (-0.15,\ -0.15)$ and $\xi_1 \approx -0.8452567390376772,\ \xi_2 \approx -0.7481414932526368$ | | | | |
|---|---|---|---|---|
| Method | $\rho$ | iter | $\lVert x^{(k+1)} - x^{(k)}\rVert$ | $\lVert F(x^{(k+1)})\rVert$ |
| **NM** | 1.9995 | 11 | 3.892e-594 | 0.0 |
| **TM** | 2.9972 | 7 | 4.061e-266 | 6.803e-798 |
| **JM** | 3.9754 | 6 | 2.257e-476 | 5.845e-1008 |
| **OM3** | 3.9874 | 6 | 8.591e-480 | 5.845e-1008 |
| **CM3** | 3.9770 | 6 | 2.545e-240 | 3.598e-960 |
| **MA3** | 3.9831 | 5 | 6.832e-184 | 1.298e-734 |
| **MB3** | 4.0078 | 6 | 1.531e-274 | 2.237e-1096 |
| **MC3** | 4.0097 | 6 | 3.831e-244 | 6.833e-974 |

| $F_4(x_1,x_2,x_3) = (x_1^2 + x_2^2 + x_3^2 - 9,\ x_1x_2x_3 - 1,\ x_1 + x_2 - x_3^2)$, $x_0 = (2,\ -1.5\ -0.5)$ and $\xi_1 \approx 2.140258122005175\ \xi_2 \approx -2.090294642255235$, $\xi_3 \approx -02235251210713019$ | | | | |
|---|---|---|---|---|
| Method | $\rho$ | iter | $\lVert x^{(k+1)} - x^{(k)}\rVert$ | $\lVert F(x^{(k+1)})\rVert$ |
| **NM** | 2.0002 | 11 | 4.822e-478 | 3.078e-955 |
| **TM** | 3.0000 | 8 | 1.534e-311 | 3.709e-933 |
| **JM** | 4.0009 | 6 | 3.163e-477 | 4.454e-1007 |
| **OM3** | 4.0010 | 6 | 8.695e-479 | 2.286e-1007 |
| **CM3** | 3.9996 | 7 | 2.695e-475 | 2.273e-1007 |
| **MA3** | 3.9964 | 6 | 7.193e-566 | 2.696e-2008 |
| **MB3** | 3.9998 | 7 | 2.890e-628 | 2.224e-2007 |
| **MC3** | 4.0000 | 10 | 3.285e-288 | 1.729e-1150 |

Table 4: Test functions and results for nonlinear systems $F_3$ and $F_4$

3. **MC3**: $a_1 = 1$ and $b_2 = 3$

In these numerical experiments, we compare the extension for systems of Ostrowski's method(OM3) and Chun's method (CM3), MA3, MB3 and MC3 with Newton's method (NM), Jarratt's method (JM) and Traub's method (TM):

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}) \\ x^{(k+1)} &= y^{(k)} - [F'(x^{(k)})]^{-1}F(y^{(k)}). \end{aligned}$$

In order to preserve the local order of convergence we use in our computations a symmetric divided difference operator.

In general, numerical results confirm theoretical ones. The proposed methods for systems behave better or equal to Jarratt's scheme, that is widely used as fourth-order method for systems. Moreover, the transferred Ostrowski' and Chun's methods for solving nonlinear systems have also a good performance.

## 4.2 Molecular interaction problem

To solve the equation of molecular interaction, (see [28])

$$\begin{aligned} u_{xx} + u_{yy} = u^2, &\quad (x,y) \in [0,1]\times[0,1] \\ u(x,0) = 2x^2 - x + 1, &\quad u(x,1) = 2 \\ u(0,y) = 2y^2 - y + 1, &\quad u(1,y) = 2. \end{aligned} \tag{17}$$

we need to deal with a boundary value problem with a nonlinear partial differential equation of second order. To estimate its solution numerically, we have used central divided differences in order to transform the problem in a nonlinear system of equations, which is solved by using the proposed methods of order four and five.

The discretization process yields to the nonlinear system of equations,

$$u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2u_{i,j}^2 = 0, \quad i = 1,\dots,nx, \quad j = 1,\dots,ny, \tag{18}$$

where $u_{i,j}$ denotes the estimation of the unknown $u(x_i, y_j)$, $x_i = ih$ with $i = 0, 1, \ldots, nx$, $y_j = jk$ with $j = 0, 1, \ldots, ny$, are the nodes in both variables, being $h = \frac{1}{nx}$, $k = \frac{1}{ny}$ and $nx = ny$.

In this case, we fix $nx = ny = 4$, so a mesh of $5 \times 5$ is generated. As the boundary conditions give us the value of the unknown function at the nodes $(x_0, y_j), (x_4, y_j)$ for all $j$ and also at $(x_i, y_0), (x_i, y_4)$ for all $i$, we have only nine unknowns, that are renamed as:

$$x_1 = u_{1,1}, \quad x_2 = u_{2,1}, \quad x_3 = u_{3,1}, x_4 = u_{1,2}, \quad x_5 = u_{2,2}, \quad x_6 = u_{3,2}, x_7 = u_{1,3}, \quad x_8 = u_{2,3}, \quad x_9 = u_{3,3}.$$

So, the system can be expressed as

$$F(x) = Ax + \phi(x) - b = 0,$$

where

$$A = \begin{pmatrix} M & -I & 0 \\ -I & M & -I \\ 0 & -I & M \end{pmatrix}, \qquad \text{being} \quad M = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad \phi(x) = h^2(x_1^2, x_2^2, \ldots, x_9^2)^T,$$

$I$ is the $3 \times 3$ identity matrix and $b = \left(\frac{7}{4}, 1, \frac{27}{8}, 1, 0, 2, \frac{27}{8}, 2, 4\right)^T$. In this case, $F'(x) = A + 2h^2 diag(x_1, x_2, \ldots, x_9)$.

Now, we will check the performance of the methods by means of some numerical tests, by using variable precision arithmetics of 1000 digits of mantissa. These tests have been made by using the stopping criterium $\|F(x^{(k+1)})\| < 10^{-700}$ or $\|x^{(k+1)} - x^{(k)}\| < 10^{-700}$. In Table 5 , we show the numerical results obtained for the problem of molecular interaction (18). We show, the approximated computational order of convergence, the number of iterations, the difference between the two last iterations and the residual of the function at the last iteration.

| $x^{(0)} = (1, \ldots, 1)^T$ | | | | |
|---|---|---|---|---|
| Method | $\rho$ | iter | $\|x^{(k+1)} - x^{(k)}\|$ | $\|F(x^{(k+1)})\|$ |
| **NM** | 1.9999 | 9 | 1.482e-413 | 6.448e-828 |
| **TM** | 2.9988 | 6 | 1.153e-355 | 2.545e-1007 |
| **JM** | 3.9954 | 5 | 1.482e-413 | 1.976e-1007 |
| **OM3** | 3.9964 | 5 | 1.482e-413 | 1.618e-1007 |
| **CM3** | 3.9959 | 5 | 1.998e-353 | 1.618e-1007 |
| **MA3** | 4.0519 | 5 | 5.362e-510 | 1.707e-2007 |
| **MB3** | 3.9960 | 5 | 7.123e-362 | 1.049e-1449 |
| **MC3** | 3.9960 | 5 | 3.110e-362 | 3.811e-1451 |

Table 5: Numerical results for molecular interaction problem

In Table 5 we can observe that all the new methods converge to the solution of the problem, that appears in Table 6. It can be noticed that the lowest error of the test corresponds to method MA3, duplicating the number of exact digits respect the other ones.

| | $\xi$ |
|---|---|
| $u_{1,1}$ | 1.0259117... |
| $u_{2,1}$ | 1.2097139... |
| $u_{3,1}$ | 1.5167030... |
| $u_{1,2}$ | 1.2097139... |
| $u_{2,2}$ | 1.3877038... |
| $u_{3,2}$ | 1.6258725... |
| $u_{1,3}$ | 1.5167030... |
| $u_{2,3}$ | 1.6258725... |
| $u_{3,3}$ | 1.7642995... |

Table 6: Approximated solution

# 5 Concluding remarks

We have presented two family of iterative methods for solving nonlinear equations with and without derivatives, respectively. In addition, by using the first family we obtain a class of iterative methods for finding the solution of nonlinear systems.

The numerical results obtained in Section 4 confirm the theoretical results. Summarizing, we can conclude that the novel iterative methods have a good performance for solving nonlinear equations and systems. In the applied example, the new methods show good stability and precision in the results.

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