Maximal ℓ_p-regularity of multi-term fractional equations with delay

Ivan Girona

Facultat de Matemàtiques, Universitat de València, 46022 València, Spain. e-mail: giloi@alumni.uv.es

Marina Murillo-Arcila

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022 València, Spain. e-mail: mamuar1@upv.es

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Abstract

We provide a characterization for the existence and uniqueness of solutions in the space of vector-valued sequences $\ell_p(\mathbb{Z}, X)$ for the multi-term fractional delayed model in the form:

$$\Delta^\alpha u(n) + \lambda \Delta^\beta u(n) = Au(n) + u(n-\tau) + f(n),\ n \in \mathbb{Z}, \alpha, \beta \in \mathbb{R}_+, \tau \in \mathbb{Z},\ \lambda \in \mathbb{R}$$

where X is a Banach space, A is a closed linear operator with domain $D(A)$ defined on X, $f \in \ell_p(\mathbb{Z}, X)$ and Δ^γ denotes the Grünwald-Letkinov fractional derivative of order $\gamma > 0$. We also give some conditions to ensure the existence of solutions when adding nonlinearities. Finally, we illustrate our results with an example given by a general abstract nonlinear model that includes the fractional Fisher equation with delay.

Keywords: Maximal ℓ_p-regularity; multi-term fractional, delay, Grünwald-Letnikov derivative.
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1. Introduction

The study of time delay equations has been considered by many authors due to their applications in many fields of sciences such as biology for describing resource regeneration times, maturity periods, or physics for describing reaction times; see for example [5], [20], [40], [35], [36], [37] , [31] and the references therein. On the other hand, fractional calculus has attracted the attention of many researchers thanks to the advantages of fractional derivatives which can describe non local processes in their nature, see for instance, the recent work of Wu, Baleanu and Xie [39] and Huang, Baleanu, Wu and Zeng in [19] where they analysed the chaotic behavior of the fractional discrete logistic map.

It is a well-known fact that the analysis of maximal regularity is a really useful tool for solving semilinear and quasilinear problems. Maximal regularity of evolution equations using operator-valued Fourier multipliers was first studied by H. Amman [2] and L. Weis [38]. Other authors as Arendt and Bu [3] studied maximal regularity of periodic problems for abstract evolution equations in UMD-spaces. Some other references concerning this topic are the works by Bu [10], [9] and [8]. Concerning delay equations, there is an increasing number of researchers working on this topic. For instance, Poblete [32] analysed maximal regularity on vector-valued

Hölder spaces. The fractional study was considered by Ponce in [33]. See also [16] and [12] for the analysis of the well-posedness for a class of third order time evolution equations with infinite delay in Lebesgue, Besov and Triebel-Lizorkin vector-valued Banach spaces.

In the discrete case, the first works of maximal regularity are due to Blunck ([6], [7]). Other authors who continued this research line by studying the existence and uniqueness of solutions for discrete systems that belong to the Lebesgue space of vector-valued sequences are [11, 13, 21, 22, 24, 26, 34]. See also the recent work [14] and the references therein where temporal regularity for second order difference equations is analyzed.

The first reference in the context of discrete maximal regularity for fractional equations was given by Lizama in [25] where he handles this study for fractional difference differential equations using methods of functional analysis and operator theory. Following this research line the following references [24, 26, 27, 29] correspond to studies of maximal ℓ_p-regularity in the context of fractional equations with time variable both in $\mathbb{N}$ and $\mathbb{Z}$. See also [23, 30] where a connection between maximal ℓ_p-regularity and non-local time steppings is established.

In this work we analyse for the first time in the literature the study of the existence and uniqueness of solutions that belong to the Lebesgue space of vector-valued sequences $\ell_p(\mathbb{Z}, X)$ for multi-term fractional difference differential equations with delay and time variable on $\mathbb{Z}$. See [28] where these models without delay were considered. More concretely, we succeed obtaining a maximal ℓ_p-regularity characterization for the multifractional model:

$$\Delta^\alpha u(n) + \lambda\Delta^\beta u(n) = Au(n) + u(n-\tau) + f(n),\ n \in \mathbb{Z}, \alpha, \beta \in \mathbb{R}_+, \tau \in \mathbb{Z},\ \lambda \in \mathbb{R} \qquad (1.1)$$

where X is a Banach space, A is a closed linear operator with domain $D(A)$ defined on X, $f \in \ell_p(\mathbb{Z}, X)$ and Δ^γ denotes the Grünwald-Letkinov fractional derivative of order $\gamma > 0$. It is worthwhile to point out that the model (1.1) includes a delayed version of the Basset equation [4] taking $X = \mathbb{C}, A = bI, \alpha = 2$ and $\beta = 3/2$ whereas it also includes a delayed version of the linearized Klein-Gordon equation [18] choosing $X = L^2(\Omega), A = \partial_{xx} - bI, \alpha = 2$ and $\lambda = 0$.

This paper is organized as follows: In Section 2, we recall the notions of UMD-spaces, R-bounded sets and the discrete time Fourier transform defined over the space of distributions. We also recall Blunck's Fourier multiplier theorems for operator-valued symbols on UMD-spaces [6]. In Section 3, we prove among others the main result of the paper. More concretely, we show that if:

$$\{(1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau}\}_{t\in(-\pi,\pi)} \subset \rho(A),$$

where $\rho(A)$ denotes the resolvent set of A then, the following assertions are equivalent:

(1) The equation $\Delta^\alpha u(n) + \lambda\Delta^\beta u(n) = Au(n) + u(n-\tau) + f(n)$ has a unique solution in $\ell_p(\mathbb{Z}, X)$ for each $f \in \ell_p(\mathbb{Z}, X)$.
(2) $M(t) := \big((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau} - A\big)^{-1}$ is an ℓ_p-multiplier from X to $[D(A)]$.
(3) $\{M(t) : t \in (-\pi, \pi)\}$ is R-bounded.

When X is a Hilbert space we obtain that maximal ℓ_p-regularity is equivalent to the fact that

$$\sup_{t\in(-\pi,\pi)} ||M(t)|| < \infty.$$

Finally, in Section 4 we provide sufficient conditions to ensure the existence of solutions in $\ell_p(\mathbb{Z}, X)$ for the nonlinear abstract model

$$\Delta^\alpha u(n) + \lambda\Delta^\beta u(n) = Au(n) + u(n-\tau) + G(u)(n) + g(n), \quad n \in \mathbb{Z},\ 0 < \rho < 1, \qquad (1.2)$$

where $g \in \ell_p(\mathbb{Z}, X)$ and $G : \ell_p(\mathbb{Z}, X) \longrightarrow \ell_p(\mathbb{Z}, X)$ are given. We prove that if condition (3) holds and G is continuously Fréchet differentiable at $u = 0$ and $G(0) = G'(0) = 0$, the nonlinear abstract equation (1.2) has at least a solution in $\ell_p(\mathbb{Z}, X)$. Finally, in order to illustrate our results we show that for all $m < -(2^\alpha + \lambda 2^\beta + 1)$ we can find $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$, there exists $u^\epsilon \in \ell_p(\mathbb{Z}, L^2(\mathbb{R}))$ that solves the equation:

$$\Delta^\alpha u(n,t) + \lambda \Delta^\beta u(n,t) = u_{xx}(n,t) + mu(n,t)(1 - u(n,t)) + u(n-\tau, t) + \epsilon f(n,t), t \in \mathbb{R}, n \in \mathbb{Z}$$

where $\lambda \in \mathbb{R}$, $m \in \mathbb{R}$ and $0 < \epsilon < 1$. This model includes the discrete time fractional Fisher equation with delay.

2. Preliminaries

In this section, we introduce the concepts of R-boundedness, the discrete time Fourier transform in the space of p-summable vector-valued sequences and ℓ_p-multipliers. We also recall Fourier multiplier theorems for operator valued symbols due to Blunck.

Definition 2.1. A Banach space X is called UMD if, for each $1 < p < +\infty$, it satisfies that

$$\left|\left| u_0 + \sum_{j=1}^{N} \varepsilon_k (u_k - u_{k-1}) \right|\right|_{L_p(\Omega,\Sigma,\nu;X)} \leq c_p \left|\left| u_N \right|\right|_{L_p(\Omega,\Sigma,\nu;X)},$$

for some constant $c_p > 0$, for all $N \in \mathbb{Z}_+$, $(\varepsilon_j)_{j\geq 1} \subset (-1, 1)$ and all $(u_j)_{j\geq 0} \subset L^p(\Omega, \Sigma, \nu; X)$.

For more details on UMD spaces see [2, p.141–147].

Definition 2.2. Let $\mathcal{B}(X, Y)$ denote the space of bounded linear operators from Banach spaces X to Y endowed with the uniform operator topology. A set $\mathcal{T} \subseteq \mathcal{B}(X, Y)$ is called R-bounded if there exists a constant $c > 0$ such that

$$||(\mathcal{T}_1 x_1, \ldots, \mathcal{T}_n x_n)||_R \leq c||(x_1, \ldots, x_n)||_R,$$

for all $\{\mathcal{T}_1, \ldots, \mathcal{T}_n\} \subset \mathcal{T}$, $\{x_1, \ldots, x_n\} \subset X$, $n \in \mathbb{N}$, where

$$||(x_1, \ldots, x_n)||_R := \frac{1}{2^n} \sum_{\varepsilon_j \in \{-1,1\}^n} \left|\left| \sum_{j=1}^{n} \varepsilon_j x_j \right|\right| \tag{2.1}$$

for $x_1, \ldots, x_n \in X$.

Some properties about R-bounded sets are analysed in [1, p.21-27].

Let us now recall the discrete time Fourier transform in $\ell_p(\mathbb{Z}, X)$, where X denotes a Banach space. Let $\mathcal{S}(\mathbb{Z}, X)$ denote the space of rapidly decreasing sequences. A sequence $f : \mathbb{Z} \longrightarrow X$ belongs to $\mathcal{S}(\mathbb{Z}, X)$ if for each $k \in \mathbb{N}_0$, there exists a constant $C_k > 0$ satisfying

$$p_k(f) := \sup_{n\in\mathbb{Z}} |n|^k ||f(n)|| \leq C_k.$$

The space $\mathcal{S}(\mathbb{Z}, X)$ is norm dense in $\ell_p(\mathbb{Z}, X)$ for all $1 \leq p < +\infty$. Let us consider the space $C^n_{per}(\mathbb{R}, X)$ defined by 2π-periodic functions $f : \mathbb{R} \longrightarrow X$ which are n times continuously differentiable.

In what follows, we will denote $\mathbb{T} := (-\pi, \pi)$ and $\mathbb{T}_0 := (-\pi, \pi) \setminus \{0\}$. It is well-known that $C^{\infty}_{per}(\mathbb{T}, X)$ endowed with the following countable family of seminorms:

$$q_k(\varphi) = \max_{k \in \mathbb{N}_0} \sup_{t \in \mathbb{T}} ||\varphi^{(k)}(t)||,$$

becomes a Fréchet space. If $X = \mathbb{C}$, we simply denote $C^{\infty}_{per}(\mathbb{T}, X) = C^{\infty}_{per}(\mathbb{T})$ and $\mathcal{S}(\mathbb{Z}, X) = \mathcal{S}(\mathbb{Z})$.

Let us now introduce the spaces of vector-valued distributions:

$$\mathcal{S}'(\mathbb{Z}, X) := \{T : \mathcal{S}(\mathbb{Z}) \longrightarrow X : T \text{ is linear and continuous}\}$$

and

$$\mathcal{D}'(\mathbb{T}, X) := \{T : C^{\infty}_{per}(\mathbb{T}) \longrightarrow X : T \text{ is linear and continuous}\}.$$

Remark 2.3. For each $f \in \ell_p(\mathbb{Z}, X)$, we can consider

$$T_f(\varphi) := \sum_{n \in \mathbb{Z}} f(n)\varphi(n)$$

for all $\varphi \in \mathcal{S}(\mathbb{Z})$. It is clear that $T_f \in \mathcal{S}'(\mathbb{Z}, X)$. We can identify the space $\ell_p(\mathbb{Z}, X)$ as a subspace of $\mathcal{S}'(\mathbb{Z}, X)$, that is, every $f \in \ell_p(\mathbb{Z}, X)$ will be identified with $T_f \in \mathcal{S}'(\mathbb{Z}, X)$. There also exists a natural mapping that identifies the space $C^{\infty}_{per}(\mathbb{T}, X)$ with a subspace of $\mathcal{D}'(\mathbb{T}, X)$. Indeed, we can define for each $S \in C^{\infty}_{per}(\mathbb{T}, X)$, the linear map

$$\mathcal{L}_S(\varphi) :=< \mathcal{L}_S, \varphi >:= \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t)S(t)dt$$

for all $\varphi \in C^{\infty}_{per}(\mathbb{T})$ and $\mathcal{L}_S \in \mathcal{D}'(\mathbb{T}, X)$.

The above considerations suggest the following definition:

Definition 2.4. The discrete time Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{Z}, X) \to C^{\infty}_{per}(\mathbb{T}, X)$ defined by

$$\mathcal{F}\varphi(t) \equiv \widehat{\varphi}(t) := \sum_{j=-\infty}^{\infty} e^{-ijt}\varphi(j), \qquad t \in (-\pi, \pi]$$

is an isomorphism whose inverse is given by

$$\mathcal{F}^{-1}\varphi(n) \equiv \check{\varphi}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t)e^{int}dt, \quad n \in \mathbb{Z}, \tag{2.2}$$

where $\varphi \in C^{\infty}_{per}(\mathbb{T}, X)$. In particular, we have $\varphi \in C^{\infty}_{per}(\mathbb{T}) \implies \check{\varphi} \in \mathcal{S}(\mathbb{Z})$.

This isomorphism, let us define the discrete time Fourier transform (DTFT) between the spaces of distributions $\mathcal{S}'(\mathbb{Z}, X)$ and $\mathcal{D}'(\mathbb{T}, X)$ as follows

$$\langle \mathcal{F}T, \psi \rangle \equiv \mathcal{F}(T)(\psi) := \widehat{T}(\psi) \equiv \langle T, \check{\psi} \rangle, \quad T \in \mathcal{S}'(\mathbb{Z}, X), \quad \psi \in C^{\infty}_{per}(\mathbb{T}), \tag{2.3}$$

whose inverse $\mathcal{F}^{-1} : \mathcal{D}'(\mathbb{T}, X) \to \mathcal{S}'(\mathbb{Z}, X)$ is given by

$$\langle \mathcal{F}^{-1}L, \psi \rangle \equiv \mathcal{F}^{-1}(L)(\psi) := \check{L}(\psi) \equiv \langle L, \widehat{\psi} \rangle, \quad L \in \mathcal{D}'(\mathbb{T}, X), \quad \psi \in \mathcal{S}(\mathbb{Z}).$$

In particular, given $f \in \ell_p(\mathbb{Z}, X)$, the following equality holds:

$$\langle \mathcal{F}T_f, \varphi \rangle = \langle T_f, \check{\varphi} \rangle = \sum_{n \in \mathbb{Z}} f(n)\check{\varphi}(n), \quad \varphi \in C^{\infty}_{per}(\mathbb{T}), \quad f \in \ell_p(\mathbb{Z}, X). \tag{2.4}$$

Definition 2.5. Given $u \in \ell_p(\mathbb{Z}, X)$ and $v \in \ell_1(\mathbb{Z})$ we define the convolution product

$$(u * v)(n) := \sum_{j=-\infty}^{n} u(n-j)v(j) = \sum_{j=0}^{\infty} u(j)v(n-j), \quad n \in \mathbb{Z}.$$

The convolution of a distribution $T \in \mathcal{S}'(\mathbb{Z}, X)$ with a function $a \in \ell_1(\mathbb{Z})$ is defined by

$$\langle T * a, \varphi \rangle := \langle T, a \circ \varphi \rangle, \quad \varphi \in \mathcal{S}(\mathbb{Z}), \tag{2.5}$$

where

$$(a \circ \varphi)(n) := \sum_{j=0}^{\infty} a(j)\varphi(j+n).$$

For any $\alpha \in \mathbb{R}$, we set

$$k^{\alpha}(n) := \begin{cases} \dfrac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!} & n \in \mathbb{Z}_+, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that if $\alpha \in \mathbb{R} \setminus \{-1, -2, \dots\}$, we easily obtain that $k^{\alpha}(n) = \dfrac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}$, where Γ is the Euler gamma function.

Definition 2.6. Given $\alpha \in \mathbb{R}^+$ and a sequence $f : \mathbb{Z} \longrightarrow X$, the fractional sum of order α is defined as

$$\Delta^{-\alpha} f(n) := (k^{\alpha} * f)(n) := \sum_{j=-\infty}^{n} k^{\alpha}(n-j)f(j), \quad n \in \mathbb{Z}.$$

The fractional difference of order α is defined as

$$\Delta^{\alpha} f(n) := (k^{-\alpha} * f)(n) := \sum_{j=-\infty}^{n} k^{-\alpha}(n-j)f(j) = \sum_{j=0}^{+\infty} k^{-\alpha}(j)f(n-j), \quad n \in \mathbb{Z}.$$

We have the generation formula

$$\sum_{j=0}^{\infty} k^{\beta}(j)z^j = \frac{1}{(1-z)^{\beta}}, \quad \beta \in \mathbb{R}, \quad |z| < 1,$$

see [41, p.42 formulae (1) and (8)]. In particular, for all $\alpha \in \mathbb{R}_+$ we have that the radial limit exists and

$$\widehat{k^{-\alpha}}(\omega) = \widetilde{k^{-\alpha}}(\omega) = \sum_{j=0}^{\infty} k^{-\alpha}(j)e^{-i\omega j} = \frac{1}{(1-e^{-i\omega})^{-\alpha}} = \left(1 - e^{-i\omega}\right)^{\alpha}, \quad \omega \in \mathbb{T}. \tag{2.6}$$

We recall the following lemma which will be an important tool for proving the characterization of maximal ℓ_p-regularity.

Lemma 2.7. *[27, Lemma 2.2] Let $u, v \in \ell_p(\mathbb{Z}, X)$ be given and $a \in \ell_1(\mathbb{Z}_+)$ which is defined by 0 for negative values of n. The following assertions are equivalent:*

(i) $a * v \in \ell_p(\mathbb{Z}, X)$ and $(a * v)(n) = u(n)$ for all $n \in \mathbb{Z}$.

(ii) $\langle u, \check{\varphi} \rangle = \langle v, (\varphi \cdot \widehat{a}_-)\check{}\,\rangle$ for all $\varphi \in C^\infty_{per}(\mathbb{T})$,

where

$$(\varphi \cdot \widehat{a}_-)\check{}\,(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{a}(-t)\varphi(t)e^{int}dt, \quad n \in \mathbb{Z}.$$

To finish this section we introduce the notion of ℓ_p-multiplier and we recall Blunck's theorems [6] that establish a relation between the concepts of R-boundedness and ℓ_p-multipliers.

Definition 2.8. Let X, Y be Banach spaces, $1 < p < \infty$. A function $M \in C^\infty_{per}(\mathbb{T}, \mathcal{B}(X, Y))$ is an ℓ_p-multiplier (from X to Y) if there exists a bounded operator $T : \ell_p(\mathbb{Z}, X) \to \ell_p(\mathbb{Z}, Y)$ such that

$$\sum_{n \in \mathbb{Z}} (Tf)(n)\check{\varphi}(n) = \sum_{n \in \mathbb{Z}} (\varphi \cdot M_-)\check{}\,(n) f(n) \tag{2.7}$$

for all $f \in \ell_p(\mathbb{Z}, X)$ and all $\varphi \in C^\infty_{per}(\mathbb{T})$. Here

$$(\varphi \cdot M_-)\check{}\,(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int}\varphi(t)M(-t)dt, \quad n \in \mathbb{Z}.$$

Theorem 2.9. *[6, Theorem 1.3] Let $p \in (1, \infty)$ and let X, Y be UMD spaces. Let $M \in C^\infty_{per}(\mathbb{T}, \mathcal{B}(X, Y))$ such that the sets*

$$\left\{M(t), (1 - e^{it})(1 + e^{it})M'(t) : \ t \in \mathbb{T}\right\}$$

are both R-bounded. Then M is an ℓ_p-multiplier (from X to Y) for $1 < p < \infty$.

The converse of Blunck's theorem also holds without any restriction on the Banach spaces X, Y as follows:

Theorem 2.10. *[6, Proposition 1.4] Let $p \in (1, \infty)$ and let X, Y be Banach spaces. Let $M \in L_{1,loc}(\mathbb{T}, \mathcal{B}(X, Y))$. Suppose that there is a bounded operator $T_M : l_p(\mathbb{Z}, X) \to l_p(\mathbb{Z}, Y)$ such that (2.7) holds. Then the set*

$$\{M(t) : t \in \mathbb{T}\}$$

is R-bounded.

3. A characterization of maximal ℓ_p-regularity

Let $\alpha, \beta \in \mathbb{R}_+$, $\tau \in \mathbb{Z}$, $\lambda \in \mathbb{R}$ and A a closed linear operator defined on a Banach space X. For each vector-valued sequence $f : \mathbb{Z} \longrightarrow X$, let us consider the following abstract linear discrete equation:

$$\Delta^\alpha u(n) + \lambda \Delta^\beta u(n) = Au(n) + u(n - \tau) + f(n), \quad n \in \mathbb{Z}. \tag{3.1}$$

In this section we will provide a characterization for the existence and uniqueness of solutions of the previous model (3.1).

Definition 3.1. Let $1 < p < +\infty$ be given. The equation (3.1) has maximal ℓ_p-regularity if for each $f \in \ell_p(\mathbb{Z}, X)$, there exists a unique solution $u \in \ell_p(\mathbb{Z}, [D(A)])$ of (3.1), where $[D(A)]$ denotes the domain of A endowed with the graph norm.

Let us first show an equivalence between the R-boundedness of the symbol of the convolution operator associated with the solution of the equation (3.1), given by

$$\left((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau} - A\right)^{-1},$$

and the fact of being an ℓ_p-multiplier on UMD spaces.

Theorem 3.2. *Let X be a UMD space, $1 < p < \infty$, $\alpha,\ \beta \in \mathbb{R}_+,\ \lambda \in \mathbb{R}$ and $\tau \in \mathbb{Z}$. Suppose that*

$$\{(1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau}\}_{t\in\mathbb{R}} \subset \rho(A)$$

and denote $M(t) := \left((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau} - A\right)^{-1}$.
Then, the following assertions are equivalent:

1. *$M(t)$ is an ℓ_p-multiplier from X to $[D(A)]$.*
2. *$\{M(t) : t \in \mathbb{T}\}$ is R-bounded.*

Proof. We first show (2) $\Rightarrow$ (1). Let $f_\alpha(t) := (1-e^{-it})^\alpha$, $f_\beta^\lambda(t) := \lambda(1-e^{-it})^\beta$ and $\delta_\tau(t) := e^{-it\tau}$. Thus, we have that $M(t)$ can be denoted as follows:

$$M(t) = (f_\alpha(t) + f_\beta^\lambda(t) - \delta_\tau(t) - A)^{-1}.$$

By Theorem 2.9, it is enough to prove that $\{M'(t)(1+e^{it})(1-e^{it})\}_{t\in\mathbb{T}}$ is R-bounded. An easy calculus shows that:

$$M'(t) = -M^2(t)\left(\frac{\partial f_\alpha}{\partial t}(t) + \frac{\partial f_\beta^\lambda}{\partial t}(t) - \frac{\partial \delta_\tau}{\partial t}(t)\right),$$

where

$$\frac{\partial f_\alpha}{\partial t}(t) = i\alpha(1-e^{-it})^{\alpha-1}e^{-it} = i\alpha f_\alpha(t)e^{-it}\frac{1}{1-e^{-it}} = i\alpha f_\alpha(t)\frac{1}{e^{it}-1},$$

$$\frac{\partial f_\beta^\lambda}{\partial t}(t) = i\beta f_\beta^\lambda(t)\frac{1}{e^{it}-1},$$

$$\frac{\partial \delta_\tau}{\partial t}(t) = -i\tau\delta_\tau(t).$$

Thus,

$$M'(t) = M^2(t)\left(i\frac{1}{1-e^{it}}\left(\alpha f_\alpha(t) + \beta f_\beta^\lambda(t)\right) + i\tau\delta_\tau(t)\right),$$

and consequently,

$$(1+e^{it})(1-e^{it})M'(t) = i\alpha M^2(t)(1+e^{it})f_\alpha(t) + i\beta M^2(t)(1+e^{it})f_\beta^\lambda(t) + i\tau M^2(t)\delta_\tau(t).$$

From [1, Proposition 2.2.5], we get that $\{M'(t)(1+e^{it})(1-e^{it})\}_{t\in\mathbb{T}}$ is R-bounded. Finally, we conclude that $M(t)$ is a ℓ_p-multiplier by Theorem 2.9.

To conclude, (1) $\Rightarrow$ (2) follows immediately as a consequence of Theorem 2.10. $\square$

Next theorem will be useful for proving the main result of the paper.

Theorem 3.3. *Let X be a UMD space. Let $1 < p < +\infty$, $\tau \in \mathbb{Z}$, $\alpha,\ \beta \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$. Suppose that*

$$\{(1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - e^{-it\tau}\} \subset \rho(A)$$

and the set $\{((1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - e^{-it\tau} - A)^{-1}\}_{t\in\mathbb{T}}$ is R-bounded. Then, the sets

$$N(t) := (1-e^{-it})^{\alpha}((1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - e^{-it\tau} - A)^{-1},$$

$$S(t) := \lambda(1-e^{-it})^{\beta}((1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - e^{-it\tau} - A)^{-1}$$

and

$$Q(t) := e^{-it\tau}((1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - e^{-it\tau} - A)^{-1},$$

are ℓ_p-multipliers.

Proof. Proceeding in a similar way as in Theorem 3.2, we obtain that $N(t) = f_{\alpha}(t)M(t)$, $S(t) = f_{\beta}^{\lambda}(t)M(t)$ and $Q(t) = \delta_{\tau}(t)M(t)$ and the R-boundedness of $N(t)$, $S(t)$ and $Q(t)$ follows. After some computations we get that

$$\begin{aligned}
&(1-e^{it})(1+e^{it})N'(t) = -i\alpha N(t)(1+e^{it}) + i\alpha N(t)^2(1+e^{it}) + i\beta N(t)S(t)(1+e^{it})\\
&(1-e^{it})(1+e^{it})S'(t) = -i\beta S(t)(1+e^{it}) + i\beta S(t)^2(1+e^{it}) + i\alpha N(t)S(t)(1+e^{it})\\
&(1-e^{it})(1+e^{it})Q'(t) = -i\tau(1+e^{it})(1-e^{it})Q(t) + i\alpha Q(t)N(t)(1+e^{it})\\
&+ i\beta Q(t)S(t)(1+e^{it}) + i\tau Q(t)^2,
\end{aligned}$$

and they are R-bounded since $N(t)$, $S(t)$ and $Q(t)$ are R-bounded. Finally, according to Theorem 2.9, $N(t)$, $S(t)$ and $Q(t)$ are ℓ_p-multipliers and the proof is finished. $\square$

We now show one of the main results of the paper that provides a characterization of maximal ℓ_p-regularity for the equation (3.1) in terms of the R-boundedness of the symbol.

Theorem 3.4. *Let A be a closed linear operator defined on a UMD space X. Let $\alpha,\ \beta \in \mathbb{R}_+$, $\tau \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$. Suppose that*

$$\{(1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - e^{-it\tau}\}_{t\in\mathbb{T}} \subset \rho(A)$$

and define $M(t) := \big((1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - e^{-it\tau} - A\big)^{-1}$.

Then, the following assertions are equivalent:

1. *The equation*
$$\Delta^{\alpha}u(n) + \lambda\Delta^{\beta}u(n) = Au(n) + u(n-\tau) + f(n)$$
has maximal ℓ_p-regularity.
2. *$M(t)$ is an ℓ_p-multiplier from X to $[D(A)]$.*
3. *$\{M(t) : t \in \mathbb{T}\}$ is R-bounded.*

Proof. By Theorem 3.2, (2) $\Leftrightarrow$ (3) follows immediately.
We now prove (1) $\Rightarrow$ (2). Given $f \in \ell_p(\mathbb{Z}, X)$, by assumption, there exists a unique solution $u_f \in \ell_p(\mathbb{Z}, [D(A)])$ of the equation (3.1). Let $T_{\alpha,\lambda,\beta,\tau} : \ell_p(\mathbb{Z}, X) \to \ell_p(\mathbb{Z}, [D(A)])$ be defined by $T_{\alpha,\lambda,\beta,\tau}(f) = u_f$. It is not difficult to see that this operator is bounded as a consequence of the closed graph theorem. Our aim is proving that the following identity $< T_{\alpha,\lambda,\beta,\tau}(f), \check{\psi} >=< f, (\psi \cdot M_-)\check{}>$, holds for all $f \in \ell_p(\mathbb{Z}, X)$ and $\psi \in C^{\infty}_{per}(\mathbb{T})$.

Indeed, let $\psi \in C^{\infty}_{per}(\mathbb{T})$, $f \in \ell_p(\mathbb{Z}, X)$ and $u := T_{\alpha,\lambda,\beta,\tau}(f)$. Since $k^{-\alpha} \in \ell_1(\mathbb{Z})$ by [41, p. 42, formula (2)], we have

$$\begin{aligned}(k^{-\alpha} \circ \check{S})(n) &= \sum_{j=0}^{\infty} k^{-\alpha}(j)\check{S}(j+n) = \sum_{j=0}^{\infty} k^{-\alpha}(j)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n+j)t}S(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\Big(\sum_{j=0}^{\infty} e^{ijt}k^{-\alpha}(j)\Big)S(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\widehat{k}^{-\alpha}(-t)S(t)dt = (\widehat{k}^{-\alpha}_{-} \cdot S\check{)}(n), \qquad (3.2)\end{aligned}$$

for all $S \in C^{\infty}_{per}(\mathbb{T}, \mathcal{B}(X, Y))$. Now, using the hypothesis and the fact that $M \in C^{\infty}_{per}(\mathbb{T}, \mathcal{B}(X, [D(A)]))$, we obtain

$$\begin{aligned}&< T_{\alpha,\lambda,\beta,\tau}(f), \check{\psi} >=< u, \check{\psi} >= \sum_{n\in\mathbb{Z}} u(n)\check{\psi}(n) = \sum_{n\in\mathbb{Z}} u(n)\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(t)e^{int}dt \\ &= \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-e^{it})^{\alpha}e^{int}\psi(t)((1-e^{it})^{\alpha} + \lambda(1-e^{it})^{\beta} - e^{it\tau} - A)^{-1}u(n)dt \\ &+ \lambda\sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-e^{it})^{\beta}e^{int}\psi(t)((1-e^{it})^{\alpha} + \lambda(1-e^{it})^{\beta} - e^{it\tau} - A)^{-1}u(n)dt \\ &- \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{it\tau}e^{int}\psi(t)((1-e^{it})^{\alpha} + \lambda(1-e^{it})^{\beta} - e^{it\tau} - A)^{-1}u(n)dt \\ &- \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}Ae^{int}\psi(t)((1-e^{it})^{\alpha} + \lambda(1-e^{it})^{\beta} - e^{it\tau} - A)^{-1}u(n)dt \\ &= \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{k}^{-\alpha}(-t)\psi(t)M(-t)u(n)dt + \lambda\sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{k}^{-\beta}(-t)\psi(t)M(-t)u(n)dt \\ &- \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{\delta}_{\tau}(-t)\psi(t)M(-t)u(n)dt - \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(t)M(-t)Au(n)dt \\ &=< u, (\widehat{k}^{-\alpha}_{-} \cdot \psi \cdot M_-)\check{}> +\lambda < u, (\widehat{k}^{-\beta}_{-} \cdot \psi \cdot M_-)\check{}> - < u, (\widehat{\delta}_{\tau -} \cdot \psi \cdot M_-)\check{}> - < Au, (\psi \cdot M_-)\check{}> \\ &= \langle u, k^{-\alpha} \circ (\varphi \cdot M_-\check{)}\rangle + \lambda\langle u, k^{-\beta} \circ (\varphi \cdot M_-\check{)}\rangle - \langle u, \delta_{\tau} \circ (\varphi \cdot M_-\check{)}\rangle - \langle Au, (\varphi \cdot M_-\check{)}\rangle.\end{aligned}$$

where $\widehat{\delta}_{\tau}(t) = e^{-i\tau t}$ and in the last equality we have used (3.2) with $S = \psi \cdot M_-$. Moreover, from (2.5) and definition 2.6 we get:

$$\begin{aligned}&< u, \check{\psi} > =< k^{-\alpha} * u, (\psi \cdot M_-)\check{}> +\lambda < k^{-\beta} * u, (\psi \cdot M_-)\check{}> - < \delta_{\tau} * u, (\psi \cdot M_-)\check{}> \\ &\quad - < Au, (\psi \cdot M_-)\check{}>=< \Delta^{\alpha}u + \lambda\Delta^{\beta}u - u_{\tau} - Au, (\psi \cdot M_-)\check{}>=< f, (\psi \cdot M_-)\check{}>\end{aligned}$$

where $\delta_\tau * u(n) = u(n-\tau) := u_\tau(n)$ and we then conclude that $M(t)$ is an ℓ_p-multiplier.

We now show that (2) $\Rightarrow$ (1). Let $f \in \ell_p(\mathbb{Z}, X)$ be given. By assumption, there exists $u \in \ell_p(\mathbb{Z}, [D(A)])$ such that

$$\sum_{n\in\mathbb{Z}} u(n)\check{\psi}(n) = \sum_{n\in\mathbb{Z}} (\psi \cdot M_-)\check{\ }(n) f(n), \tag{3.3}$$

for all $\psi \in C^\infty_{per}(\mathbb{T})$. According to Theorem 3.3, we infer that there exist $v, w, s \in \ell_p(\mathbb{Z}, [D(A)])$ such that

$$\begin{aligned} \sum_{n\in\mathbb{Z}} v(n)\check{\phi}(n) &= \sum_{n\in\mathbb{Z}} (\phi \cdot N_-)\check{\ }(n) f(n) \\ \sum_{n\in\mathbb{Z}} w(n)\check{\xi}(n) &= \sum_{n\in\mathbb{Z}} (\xi \cdot S_-)\check{\ }(n) f(n) \\ \sum_{n\in\mathbb{Z}} s(n)\check{\eta}(n) &= \sum_{n\in\mathbb{Z}} (\eta \cdot Q_-)\check{\ }(n) f(n), \end{aligned} \tag{3.4}$$

for all $\phi, \xi, \eta \in C^\infty_{per}(\mathbb{T})$ with

$$\begin{aligned} N(t) &= (1-e^{-it})^\alpha \left((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau} - A\right)^{-1} = \hat{k}^{-\alpha}(t) M(t) \\ S(t) &= \lambda(1-e^{-it})^\beta \left((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau} - A\right)^{-1} = \lambda\hat{k}^{-\beta}(t) M(t) \\ Q(t) &= e^{-it\tau} \left((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau} - A\right)^{-1} = \hat{\delta}_\tau(t) M(t). \end{aligned} \tag{3.5}$$

We now obtain

$$\begin{aligned} (\phi \cdot N_-)\check{\ }(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(t)\hat{k}^{-\alpha}(-t) M(-t) e^{int} dt \\ (\xi \cdot S_-)\check{\ }(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \lambda\xi(t)\hat{k}^{-\beta}(-t) M(-t) e^{int} dt \\ (\eta \cdot Q_-)\check{\ }(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \eta(t)\hat{\delta}_\tau(-t) M(-t) e^{int} dt. \end{aligned}$$

Choosing $\psi(t) = \phi(t)\hat{k}^{-\alpha}(-t) \in C^\infty_{per}(\mathbb{T})$ in (3.3), we get

$$\begin{aligned} \sum_{n\in\mathbb{Z}} u(n)(\hat{k}^{-\alpha}_- \cdot \phi)\check{\ }(n) &= \sum_{n\in\mathbb{Z}} (\phi \cdot \hat{k}^{-\alpha}_- \cdot M_-)\check{\ }(n) f(n) \\ &= \sum_{n\in\mathbb{Z}} (\phi \cdot N_-)\check{\ }(n) f(n) = \sum_{n\in\mathbb{Z}} v(n)\check{\phi}(n), \end{aligned}$$

where we used (3.4) in the last equality. Hence, we have:

$$< u, (\hat{k}^{-\alpha}_- \cdot \phi)\check{\ } > = < v, \check{\phi} >$$

for all $\phi \in C^\infty_{per}(\mathbb{T})$. Proceeding similarly and choosing first $\psi(t) = \xi(t)\hat{k}^{-\beta}(-t) \in C^\infty_{per}(\mathbb{T})$ and later $\eta(t)\widehat{\delta}_\tau(-t) \in C^\infty_{per}(\mathbb{T})$ in (3.3) we get:

$$< u, (\hat{k}^{-\beta}_- \cdot \xi)\check{\ } > = < w, \check{\xi} >$$

$$< u, (\hat{\delta}_{\tau_-} \cdot \eta)\check{} >=< s, \check{\eta} >$$

for all $\xi, \eta \in C_{per}^{\infty}(\mathbb{T})$. Then, according to Lemma 2.7 we arrive to the following identities:

$$\begin{aligned} \Delta^{\alpha} u(n) &= (k^{-\alpha} * u)(n) = v(n) \\ \Delta^{\beta} u(n) &= (k^{-\beta} * u)(n) = w(n) \\ u(n-\tau) &= (\delta_{\tau} * u)(n) = s(n). \end{aligned} \tag{3.6}$$

Since $N(t) + S(t) = Q(t) + AM(t) + I$, after multiplying by $e^{int}\psi(t)$ and integrating between $-\pi$ and π, we get

$$(N_- \cdot \psi)\check{}(n) + (S_- \cdot \psi)\check{}(n) = (Q_- \cdot \psi)\check{}(n) + A(M_- \cdot \psi)\check{}(n) + \check{\psi}(n)I,$$

for all $\psi \in C_{per}^{\infty}(\mathbb{T})$. As a consequence, we obtain

$$< f, (N_- \cdot \psi)\check{} > + < f, (S_- \cdot \psi)\check{} >=< f, (Q_- \cdot \psi)\check{} > + < f, A(M_- \cdot \psi)\check{} > + < f, \check{\psi} >, \tag{3.7}$$

and replacing (3.3) and (3.5) in the last equation (3.7), we arrive to

$$\sum_{n\in\mathbb{Z}} v(n)\check{\psi}(n) + \lambda \sum_{n\in\mathbb{Z}} w(n)\check{\psi}(n) = \sum_{n\in\mathbb{Z}} s(n)\check{\psi}(n) + \sum_{n\in\mathbb{Z}} Au(n)\check{\psi}(n) + \sum_{n\in\mathbb{Z}} f(n)\check{\psi}(n) \tag{3.8}$$

for all $\psi \in C_{per}^{\infty}(\mathbb{T})$. Finally, taking into account (3.6) we have succeed proving that u is a solution of the equation (3.1).

It only remains to prove uniqueness. Let $u : \mathbb{Z} \longrightarrow [D(A)]$ be a solution of the equation (3.1) with $f \equiv 0$. For all $\varphi \in C_{per}^{\infty}(\mathbb{T})$, and (3.4) we arrive to the following identity:

$$\langle u, \check{\varphi} \rangle = \langle \Delta^{\alpha} u + \lambda \Delta^{\beta} u - u_{\tau} - Au, (\varphi \cdot M_-\check{)} \rangle = 0.$$

Taking $\psi_k(t) := e^{-ikt}$, $k \in \mathbb{Z}$, we obtain $u \equiv 0$, and the proof is concluded.

□

Next corollary is a direct consequence of Theorem 3.4 and the closed graph theorem.

Corollary 3.5. *If the hypothesis of Theorem 3.4 hold, we have that u, $\Delta^{\alpha} u$, $\Delta^{\beta} u$, $Au \in \ell_p(\mathbb{Z}, X)$ and there exists a constant $C > 0$ such that:*

$$||\Delta^{\alpha} u||_{\ell_p(\mathbb{Z},X)} + |\lambda| ||\Delta^{\beta} u||_{\ell_p(\mathbb{Z},X)} + ||u||_{\ell_p(\mathbb{Z},X)} + ||Au||_{\ell_p(\mathbb{Z},X)} \leq C||f||_{\ell_p(\mathbb{Z},X)}. \tag{3.9}$$

The following corollary follows immediately from Theorem 3.4 since R-boundedness is equivalent to norm boundedness for Hilbert spaces.

Corollary 3.6. *Let H be a Hilbert space, α, $\beta \in \mathbb{R}_+$, $\tau \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$. Suppose that $\{(1 - e^{-it})^{\alpha} + \lambda(1 - e^{-it})^{\beta} - e^{-it\tau}\}_{t\in\mathbb{T}} \subset \rho(A)$. Then, the following assertions are equivalent:*

1. *For all $f \in \ell_p(\mathbb{Z}, H)$, there exists a unique solution $u \in \ell_p(\mathbb{Z}, H)$ of (3.1) such that $u(n) \in D(A)$ for all $n \in \mathbb{Z}$;*
2. $\sup_{t\in\mathbb{T}} || \left((1 - e^{-it})^{\alpha} + \lambda(1 - e^{-it})^{\beta} - e^{-it\tau} - A\right)^{-1} || < \infty$.

4. Maximal ℓ_p-regularity of nonlinear multi-term fractional delayed equations

In this section we provide a positive answer for the problem of the existence of solutions for multi-term fractional delayed equations when adding a nonlinear term. More concretely, we consider the following equation:

$$\Delta^\alpha u(n) + \lambda \Delta^\beta u(n) = Au(n) + u(n-\tau) + G(u)(n) + \rho f(n), \quad n \in \mathbb{Z}, \tag{4.1}$$

where $0 < \rho < 1$, $f \in \ell_p(\mathbb{Z}, X)$ and $G : \ell_p(\mathbb{Z}, X) \longrightarrow \ell_p(\mathbb{Z}, X)$ are given.

Next theorem shows that the nonlinear equation (4.1) has a solution, under some hypothesis on the nonlinear term G and the valued operator symbol of the equation defined by $\big((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau} - A\big)^{-1}$.

Theorem 4.1. *Let X be a UMD space, $1 < p < +\infty, \tau \in \mathbb{Z}, \lambda \in \mathbb{R}, \alpha, \beta \in \mathbb{R}_+$. Suppose that*

$$\big\{(1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau}\big\}_{t\in\mathbb{T}} \subset \rho(A),$$

and

(1) The set $\big\{((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - e^{-it\tau} - A)^{-1}\big\}_{t\in\mathbb{T}}$ is R-bounded.
(2) G is continuously Fréchet differentiable at $u = 0$ and $G(0) = G'(0) = 0$.

Then, there exists $\rho^* > 0$ such that the equation (4.1) has a solution in $\ell_p(\mathbb{Z}, X)$ for each $\rho \in [0, \rho^*)$, denoted by $u := u_\rho$.

Proof. Let $\rho \in (0,1)$ be given and let us define the following one-parameter family:

$$H[u, \rho] = -\mathcal{A}u + G(u) + \rho f,$$

with $\mathcal{A}u(n) := \Delta^\alpha u(n) + \lambda\Delta^\beta u(n) - u(n-\tau) - Au(n)$ and $D(\mathcal{A}) := \ell_p(\mathbb{Z}, [D(A)])$. By (2), we obtain that $H[0,0] = 0$, H is continuously differentiable at $(0,0)$ and the partial Fréchet derivative is $H^1_{(0,0)} = -\mathcal{A}$, which is invertible. Indeed, the space $\ell_p(\mathbb{Z}, [D(A)])$ endowed with the norm

$$|||u||| := ||\Delta^\alpha u||_{\ell_p(\mathbb{Z},X)} + |\lambda|||\Delta^\beta u||_{\ell_p(\mathbb{Z},X)} + ||u||_{\ell_p(\mathbb{Z},X)} + ||Au||_{\ell_p(\mathbb{Z},X)},$$

becomes a Banach space. Thus, according to the inequality (3.9) in Corollary 3.5, $|||u||| \le C||\mathcal{A}u||$ holds. Moreover $||\mathcal{A}u|| \le |||u|||$ trivially. Therefore, $\mathcal{A}$ is an isomorphism. By assumption and Theorem 3.4, the operator A has maximal ℓ_p-regularity and then $\mathcal{A}$ is an onto isomorphism.

An application of the implicit function theorem (see [17, Theorem 17.6]) let us conclude that there exists $\rho^* > 0$ such that the equation (4.1) is solvable in $\ell_p(\mathbb{Z}, X)$ for all $\rho \in [0, \rho^*)$. □

Finally, we illustrate the previous theorem with the following example.

Example 4.2. We consider the following nonlinear abstract model:

$$\Delta^\alpha u(n,t) + \lambda\Delta^\beta u(n,t) = u_{xx}(n,t) + mu(n,t)(1-u(n,t)) + u(n-\tau,t) + \rho f(n,t), n \in \mathbb{Z}, t \in \mathbb{R}, \tag{4.2}$$

where $\lambda \in \mathbb{R}$, $m \in \mathbb{R}$ and $0 < \rho < 1$. Note that if $\alpha = 1$ and $\lambda = 0$, the equation (4.2) corresponds to the discrete time Fisher equation with delay [42].

Equation (4.2) labels into the scheme of (4.1) for $Au = u'' + mu$ defined on $L^2(\mathbb{R})$ and $G(u,n) = -mu(n)^2$. It is clear that the operator $Bu = u''$ with domain $D(B) = H_0^2(\mathbb{R})$ generates a contraction C_0-semigroup on $L^2(\mathbb{R})$. Then, from [15, Theorem 3.5] the following inequality

$$||(\nu - A)^{-1}|| \leq \frac{1}{\Re(\nu) - m}, \tag{4.3}$$

holds whenever $\Re(\nu) - m > 0$. On the other hand, we get that

$$\begin{aligned}
&\Re((1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - e^{-it\tau}) \\
&\qquad = (2-2cos(t))^{\frac{\alpha}{2}} \cos\left(\alpha \arctan\left(\frac{\sin(t)}{1-\cos(t)}\right)\right) \\
&\qquad + \lambda(2-2\cos(t))^{\frac{\beta}{2}} \cos\left(\beta \arctan\left(\frac{\sin(t)}{1-\cos(t)}\right)\right) - \cos(t\tau) \\
&\qquad > (2-2\cos(t))^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{2}\right) + \lambda(2-2\cos(t))^{\frac{\beta}{2}} \cos\left(\frac{\beta\pi}{2}\right) - 1 \\
&\qquad > -(2^{\alpha} + \lambda 2^{\beta} + 1).
\end{aligned} \tag{4.4}$$

By inequality (4.3), if $m < -(2^{\alpha} + \lambda 2^{\beta} + 1)$, we have that

$$\sup_{t\in\mathbb{T}} \left|\left|((1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - e^{-it\tau} - A)^{-1}\right|\right| \leq \frac{1}{-m - (2^{\alpha} + \lambda 2^{\beta} + 1)} < \infty.$$

Moreover, it is clear that G is a Fréchet differentiable function at $u = 0$ and it satisfies that $G(0) = G'(0) = 0$. Finally, the assumptions of Theorem 4.1 hold, and there exists a constant $\rho^* > 0$, such that the problem (4.2) has a solution in $\ell_p(\mathbb{Z}, L^2(\mathbb{R}))$ for all $\rho \in [0, \rho^*)$.

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Conflict of interests

This work does not have any conflicts of interest.

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