REPORTS ON MATHEMATICAL LOGIC
41 (2006), 9–30

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# FINITELY EQUIVALENTIAL GENTZEN SYSTEMS ASSOCIATED WITH ARBITRARY FINITE ALGEBRAS

*In memory of Willem Blok*

A b s t r a c t. Given an arbitrary finite algebra, there exists a (many-sided) sequent calculus satisfying the cut elimination property and from which it is possible to define all finitely valued logics determined by a matrix on the algebra. In this paper we study some algebraic properties of these sequent calculi. Our starting point is the definition of a Gentzen system as the consequence relation determined by a sequent calculus over the set of (many-sided) sequents. For the Gentzen systems associated with an arbitrary finite algebra we characterize the algebraic reducts of their reduced matrices as the quasivariety generated by the algebra. To prove this result we define and study the basic properties of the *finitely equivalential* Gentzen systems. Throughout the paper different results illustrate how to bridge the gap between the proof-theoretical and the algebraic properties of a sequent calculus.

*Received 25 January 2005*
*Keywords: Many-valued propositional logic, Gentzen system, sequent calculus, completeness, quasivariety, reduced matrix, Leibniz congruence, finitely equivalential, finite algebra.*

## 1 Introduction and outline of the paper

A class of sequent calculi associated with finite algebras (and finitely valued logics) is defined in [5]. In this paper we continue the study of the algebraic properties of the Gentzen systems determined by these sequent calculi. This study started in [16], where, generalizing a result of [18], the algebraizable $m$-dimensional Gentzen systems were characterized. By using the notion of satisfaction of sequents defined in [20] and [5], we define, for any finite algebra, a semantical consequence relation on the set of $m$-sequents, where $m$ is the cardinality of the algebra. For the Gentzen systems determined by the sequent calculi associated with the finite linear MV-algebras, it was proved in [16], by using algebraic methods, that the semantical consequence relation was exactly the consequence relation determined by the sequent calculus. This result was called the *Strong Completeness Theorem* and was generalized in [15] to the Gentzen systems associated with *arbitrary* finite algebras. As a consequence, for every finite algebra only one Gentzen system is associated with it by means of the sequent calculi defined in [5] (see Definition 3.8). The computer system `MUltlog` ([4], [21] and [1]) provides a way to obtain automatically an axiomatization of these consequence relations such that the rules satisfy certain optimality conditions. A computer system called `MUltseq` (see [11] and [2]) was developed on the basis of `MUltlog` to obtain provers for equations and quasiequations in any finite algebra.

In this paper we define and study the basic properties of the *finitely equivalential Gentzen systems*. These systems generalize the definitions of *congruential* and *finitely equivalential* deductive systems introduced in [6, 8] and studied in [9] and play an essential role in this paper, because all the Gentzen systems associated with finite algebras mentioned in the previous paragraph turn out to be finitely equivalential.

The general properties of the finitely equivalential Gentzen systems, together with the *Strong Completeness Theorem* make it possible to prove that for each finite algebra, the class of algebraic reducts of the reduced matrices of the Gentzen system associated with this algebra is exactly the quasivariety generated by the algebra.

The paper is organized as follows: in Section 2 we recall the basic definitions concerning many–sided sequents and the sequent calculi **VL** associated with finite algebras. In Section 3 we introduce the notion of a Gentzen system and its matrices, considering also the Gentzen systems determined by a sequent calculus. Section 4 is devoted to the study of the Leibniz operator and the reduced matrices of a Gentzen system, and we obtain some results that show how the form of the rules of a calculus (and in particular the presence of the structural rules) may help to characterize the Leibniz congruence. In Section 5 we define the *finitely equivalential* Gentzen systems and prove the main result of the paper, which states that for a finitely equivalential Gentzen system $\mathcal{G}$ complete with respect to a class $\mathcal{K}$ of reduced $\mathcal{G}$–matrices, the class of the algebraic reducts of the reduced matrices of the system is precisely the quasivariety generated by the algebraic reducts of the matrices in $\mathcal{K}$. Finally in Section 6 we apply the general results to the matrices of the Gentzen systems associated with arbitrary finite algebras.

## 2 Preliminary definitions and results

Let $\mathcal{L}$ be a propositional language (i.e. a set of propositional connectives). By an $\mathcal{L}$-algebra we mean a structure $\mathbf{A} = \langle A, \{\varpi^{\mathbf{A}} : \varpi \in \mathcal{L}\}\rangle$, where $A$ is a non-empty set, called the universe of $\mathbf{A}$, and $\varpi^{\mathbf{A}}$ is an operation on $A$ of arity $k$ for each connective $\varpi$ of rank $k$.

We denote by $\mathbf{Fm}_{\mathcal{L}}$ the absolutely free algebra of type $\mathcal{L}$ freely generated by a countable infinite set of variables. Its elements are called **$\mathcal{L}$-formulas** (or **$\mathcal{L}$-terms**). If $\mathbf{A}$ is an $\mathcal{L}$-algebra, the set of homomorphisms from $\mathbf{Fm}_{\mathcal{L}}$ to $\mathbf{A}$ (also called **valuations**) will be denoted by $val(\mathbf{A})$. The set of homomorphisms from $\mathbf{Fm}_{\mathcal{L}}$ to $\mathbf{Fm}_{\mathcal{L}}$ (also called **substitutions**) will be denoted by $Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$.

### 2.1 $m$-sequents and $m$-sequent calculi

Let $\mathcal{L}$ be a propositional language. An **$m$-sequent**, also called $m$-dimensional sequent or $m$-sided sequent, is a sequence $(\Gamma_0, \Gamma_1, \dots, \Gamma_{m-1})$ where each $\Gamma_i$ is a finite sequence of $\mathcal{L}$-formulas, which is called the $i$-th **component** (or **place**) of the sequent. Those sequents have been taken into account in [20], [4], [5], [22] and [16]. As in these works we will write $\Gamma_0 \mid \Gamma_1 \mid \cdots \mid \Gamma_{m-1}$ for $(\Gamma_0, \Gamma_1, \dots, \Gamma_{m-1})$. We denote by $m$-$Seq_{\mathcal{L}}$ the set of $m$-sequents.

Thus in the 2-dimensional case we will write $\Gamma \mid \Delta$ instead of the more common notations $\Gamma \vdash \Delta$ or $\Gamma \to \Delta$. The use of the symbol $\mid$ as a separator of the components prevents us from thinking of entailment relations between the components of a sequent.

If we have two or more sequents, we will separate them by the symbol ";". In this way there will be no confusion between, for instance, the 3-sequent $\Gamma, x \mid \Delta, y \mid \Pi$ and the sequence of two 2-sequents $\Gamma, x \mid \Delta \;\; ; \;\; y \mid \Pi$. The comma will be reserved for the juxtaposition operation on sequences: that is, expressions such as $\Gamma, \delta$ will stand for $(\gamma_0, \dots, \gamma_{k-1}, \delta)$, where $\Gamma = (\gamma_0, \dots, \gamma_{k-1})$.

To increase the readability of some of the results of this paper in which we use simultaneously formulas, sequences of formulas, sequents and sets of sequents, we will use the following notation: lowercase letters from the end of the alphabet, possibly with subindex and superindex $(p, q, p_i^j, x, y, z, \dots)$

to denote propositional variables; lowercase Greek letters ($\varphi, \psi, \varphi_i^j, \ldots,$) to denote formulas; uppercase Greek letters ($\Gamma, \Delta, \ldots$) to denote sequences and sets of formulas; boldface uppercase Greek letters ($\mathbf{\Gamma}, \mathbf{\Delta}, \mathbf{\Delta}_i, \ldots$) to denote sequents, and *boldface slanted* uppercase letters of the end of the alphabet ($\boldsymbol{T}, \boldsymbol{S}, \ldots,$) to denote sets of sequents.

If $\mathbf{\Gamma}$ is an $m$-sequent and $i < m$, then $\mathbf{\Gamma}(i)$ denotes the $i$-th component of $\mathbf{\Gamma}$. If $\Delta$ is a finite sequence of formulas and $I \subseteq \{0, \ldots, m-1\}$, we denote by $[I : \Delta]$ the $m$-sequent whose $i$-th component is $\Delta$ if $i \in I$ and is empty otherwise, that is:

$$[I : \Delta](i) = \begin{cases} \Delta & \text{if } i \in I \\ \\ \emptyset & \text{if } i \notin I. \end{cases}$$

We will write $[i_1, \ldots, i_n : \Delta]$ for $[\{i_i, \ldots, i_n\} : \Delta]$.

If $\mathbf{\Gamma}$ and $\mathbf{\Pi}$ are $m$-sequents then we denote by $[\![\mathbf{\Gamma}, \mathbf{\Pi}]\!]$ the $m$-sequent

$$\mathbf{\Gamma}(0), \mathbf{\Pi}(0) \mid \cdots \mid \mathbf{\Gamma}(m-1), \mathbf{\Pi}(m-1).$$

Note that while $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2 \ldots$ are sequents, the expression $\mathbf{\Gamma}(i)$ stands for the $i$-th component of the sequent $\mathbf{\Gamma}$, which is a sequence of formulas.

If $I \subseteq \{0, \ldots, m-1\}$, then we will write $I^C$ for the set $\{j < m : j \notin I\}$.

For every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$

$$h(\gamma_0^0, \ldots, \gamma_0^{t_0-1} \mid \gamma_1^0, \ldots, \gamma_1^{t_1-1} \mid \cdots \mid \gamma_{m-1}^0, \ldots, \gamma_{m-1}^{t_{m-1}-1})$$

stands for the sequent

$$h(\gamma_0^0), \ldots, h(\gamma_0^{t_0-1}) \mid h(\gamma_1^0), \ldots, h(\gamma_1^{t_1-1}) \mid \cdots \mid h(\gamma_{m-1}^0), \ldots, h(\gamma_{m-1}^{t_{m-1}-1}).$$

An $m$-**rule of inference** is a set $(r)$ of ordered pairs of the form $\langle \boldsymbol{T}, \mathbf{\Gamma} \rangle$, where $\boldsymbol{T} \cup \{\mathbf{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$ and $\boldsymbol{T}$ is finite, such that it is closed under substitutions, i.e., for every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$, if $\langle \boldsymbol{T}, \mathbf{\Gamma} \rangle \in (r)$ then $\langle h(\boldsymbol{T}), h(\mathbf{\Gamma}) \rangle \in (r)$ where $h(\boldsymbol{T}) = \{h(\mathbf{\Delta}) : \mathbf{\Delta} \in \boldsymbol{T}\}$. Rules having all pairs of the form $\langle \emptyset, \mathbf{\Gamma} \rangle$ are called **axioms** and, in this case, $\mathbf{\Gamma}$ is called an **instance** of the axiom. Rules are often written in a schematic form; for instance,

$$\frac{\mathbf{\Gamma}}{[\![\mathbf{\Gamma}, [i : \varphi]]\!]}$$

denotes the rule $\{\langle \{\mathbf{\Gamma}\}, [\![\mathbf{\Gamma}, [i : \varphi]]\!] \rangle : \mathbf{\Gamma}$ is an $m$-sequent and $\varphi$ is a formula$\}$.

An $m$-**sequent calculus** is a set of $m$-rules of inference.

The following are called **structural rules** (as introduced in [22] and [5]), where $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ are arbitrary sequents and $\varphi$ and $\psi$ are arbitrary $\mathcal{L}$-formulas:

- **Axiom**: $[0, \ldots, m-1 \colon \varphi]$.

- **Weakening rule** $(w : i)$ for the place $i < m$:

$$\frac{\mathbf{\Gamma}}{[\![\mathbf{\Gamma}, [i\colon \varphi]]\!]}\ (w\colon i)$$

- **Contraction rule** $(c : i)$ for the place $i < m$:

$$\frac{[\![\mathbf{\Gamma}, [i\colon \varphi, \varphi]]\!]}{[\![\mathbf{\Gamma}, [i\colon \varphi]]\!]}\ (c\colon i)$$

- **Exchange rule** $(x : i)$ for the place $i < m$:

$$\frac{[\![\mathbf{\Gamma}, [i\colon \varphi, \psi], \mathbf{\Delta}]\!]}{[\![\mathbf{\Gamma}, [i\colon \psi, \varphi], \mathbf{\Delta}]\!]}\ (x\colon i)$$

- **Cut rule** $(cut : i, j)$ for the places $i < m, j < m,\ i \neq j$:

$$\frac{[\![\mathbf{\Gamma}, [i\colon \varphi]]\!] \quad [\![\mathbf{\Delta}, [j\colon \varphi]]\!]}{[\![\mathbf{\Gamma}, \mathbf{\Delta}]\!]}\ (cut\colon i, j)$$

Note that we have a structural rule of each kind for each component of the sequents (or pair of components, in the case of the cut rule). Note also that the rules $(cut\colon i, j)$ and $(cut\colon j, i)$ are equivalent in presence of the exchange rules.

### 2.2 The VL-sequent calculi

Each finite $\mathcal{L}$-algebra of cardinal $m$ induces a semantical interpretation on the set of $m$-sequents, in such a way that several $m$-sequent calculi are known to be complete with respect to this semantical interpretation. We will now recall some of the basic definitions involved.

**Definition 2.1.** Let $\mathbf{L}$ be a finite $\mathcal{L}$-algebra with universe

$$L = \{v_0, \ldots, v_{m-1}\}.$$

(i) Let $h \in val(\mathbf{L})$. $h$ **L-satisfies** an $m$-sequent $\Gamma(0) \mid \cdots \mid \Gamma(m-1)$ if there is an $i < m$ such that, for some formula $\gamma \in \Gamma(i)$, $h(\gamma) = v_i$. If $\mathbf{\Gamma}$ is an $m$-sequent, we denote by $s(\mathbf{\Gamma})$ the set of valuations that $\mathbf{L}$-satisfy the sequent $\mathbf{\Gamma}$.

(ii) $\mathbf{\Gamma} \in m\text{-}Seq_{\mathcal{L}}$ is **L-valid** $s(\mathbf{\Gamma}) = val(\mathbf{L})$.

The above definition of validity is the restriction to the propositional case of [5, Def. 3.2] and of the definitions given in [4] and [22]. It is always possible to find sequent calculi complete with respect to this definition of **L**-validity (see [5] and [22] for historical remarks). The calculi we will deal with were defined by some of the members of the *Vienna Group of Many Valued Logics* in [5], and will be denoted in this paper as **VL**-calculus, that is, preceding the name of the algebra with the letter **V** (which stands for Vienna). We will now recall the definition of the introduction rules of the **VL** calculi,

**Definition 2.2.** (cf. [5, Definition 3.3]) and [4]). Let **L** be a finite $\mathcal{L}$-algebra with universe $L = \{v_0, \ldots, v_{m-1}\}$. A **VL-introduction rule** $(\varpi : i)$ for a connective $\varpi$ at place $i$ is a schema of the form:

$$\frac{\{\Gamma_0, \Delta_0^j \mid \cdots \mid \Gamma_{m-1}, \Delta_{m-1}^j\}_{j \in I}}{\Gamma_0 \mid \cdots \mid \Gamma_i, \varpi(\varphi_0, \ldots, \varphi_{n-1}) \mid \cdots \mid \Gamma_{m-1}} \; \varpi : i \tag{1}$$

where $\Delta_l^j \subseteq \{\varphi_0, \ldots, \varphi_{n-1}\}$, for every $l < m$ and $j \in I$, $\varpi$ is a propositional connective of rank $n$, $I$ is a finite set, and, for each $h \in val(\mathbf{L})$, the following properties are equivalent:

(VL1) $h$ **L**-satisfies the sequent $\Delta_0^j \mid \cdots \mid \Delta_{m-1}^j$ for every $j \in I$.

(VL2) $h(\varpi(\varphi_0, \ldots, \varphi_{n-1})) = v_i$.

The existence of such rules for an arbitrary finite algebra is proved in [20, Lemma 1]. As pointed out in [5], it should be stressed that for any connective $\varpi$ and any $i < m$, there may be different rules that satisfy the definition of a **VL** introduction rule $(\varpi : i)$. In [5] there is a description of how to find these rules from the *partial normal forms* in the sense of [19] (see also [22, p. 8–9]).

A procedure to find rules that are minimal with respect to the number of premises and the number of formulas per premise has been implemented in the system `MUltlog` already mentioned; by applying this system to the truth tables of the connectives of the three-element MV-algebra $\mathbf{S}(3)$, the sequent calculus shown in Appendix A is obtained. To give the reader a better intuition of the rules we will briefly discuss the rule $(\to : 1)$ for the three-element MV-algebra $\mathbf{S}(3)$ where the elements are $v_0 = 0, v_1 = 1/2, v_2 = 1$, and the operation $\to$ is defined as $a \to b = \min\{1, b + 1 - a\}$, which yields the following truth table

| $\to$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1/2 | 1/2 | 1 | 1 |
| 1 | 0 | 1/2 | 1. |

The corresponding rule (→: 1) which has this form

$$\frac{\Gamma \mid \Delta, \varphi \mid \Pi, \varphi \quad \Gamma \mid \Delta, \varphi, \psi \mid \Pi \quad \Gamma, \psi \mid \Delta \mid \Pi, \varphi}{\Gamma \mid \Delta, \varphi \rightarrow \psi \mid \Pi} \ (\rightarrow: 1)$$

expresses (in a conjunctive normal form) under which values of the variables the expression $\varphi \rightarrow \psi$ equals 1/2. More precisely,

$a \rightarrow b = 1/2$ iff ($a = 1/2$ or $a = 1$) and ($a = 1/2$ or $b = 1/2$) and ($b = 0$ or $a = 1$).

So, if $h \in val(\mathbf{S}(3)), h(\varphi \rightarrow \psi) = 1/2$ iff $h$ satisfies the sequents $\emptyset \mid \varphi \mid \varphi$, $\emptyset \mid \varphi, \psi \mid \emptyset$ and $\psi \mid \emptyset \mid \varphi$.

**Definition 2.3.** (see [5]). Let $\mathbf{L}$ be a finite $\mathcal{L}$-algebra of cardinal $m$. A **VL**-sequent calculus consists of

(i) A **VL**-introduction rule $(\varpi : i)$ for every connective $\varpi \in \mathcal{L}$ and every place $i < m$,

(ii) All the structural rules, that is, the axiom and the rules $(w : i)$, $(c : i)$, $(x : i)$ for all $i < m$, and the rules $(cut : i, j)$ for all $i, j < m$, $i \neq j$.

This definition corresponds to the propositional fragment of the sequent calculi **LM** defined in [4, 22, 5]. Among the properties of the sequent calculi just defined we are interested in the restriction to the propositional case of the following result:

**Theorem 2.4 (Completeness and Cut Elimination).** *Let* $\mathbf{L}$ *be a finite algebra of cardinal* $m$*, then the following properties hold:*

(i) *If an* $m$*-sequent is provable in a* **VL***-sequent calculus, then it is* $\mathbf{L}$*-valid.*

(ii) *If an* $m$*-sequent is* $\mathbf{L}$*-valid, then it is provable without cuts in any* **VL***-sequent calculus.*

**Proof.** (i) See [5, Theorem 3.1]. (ii) See [5, Theorem 3.2]. □

This completeness theorem makes it possible to define all the (finitely valued) logics that can be obtained from the algebra $\mathbf{L}$ and a set of distinguished values; we proceed as follows:

**Definition 2.5.** Let $\mathbf{L}$ be a finite $\mathcal{L}$-algebra with universe

$$L = \{v_0, \ldots, v_{m-1}\}$$

and let $L^+ \subseteq L$ a set of **distinguished truth values**. The finitely valued logic $\models_{\langle \mathbf{L}, L^+ \rangle}$ is defined by the following condition: for any set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \models_{\langle \mathbf{L}, L^+ \rangle} \varphi$ iff for all $h \in val(\mathbf{L}), h(\Gamma) \subseteq L^+$ implies $h(\varphi) \in L^+$.

Next result is easy to prove:

**Proposition 2.6.** [13, Theorem 2] *Let* $\mathbf{L}$ *be a finite* $\mathcal{L}$*-algebra with universe* $L = \{v_0, \ldots, v_{m-1}\}$ *and let* $L^+ = \{v_i : i \in I\} \subseteq L$. *For any set of formulas* $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_{\langle \mathbf{L}, L^+ \rangle} \varphi \textit{ iff the sequent } [\![ [I^C : \Gamma], [I : \varphi] ]\!] \textit{ is } \mathbf{VL}\textit{-provable.} \qquad \square$$

## 3 Gentzen Systems

In order to study the consequence relations determined by the **VL**-sequent calculi, we first recall the abstract definition of an $m$-dimensional Gentzen system. These systems were introduced in [16] and can be seen as a generalization of the 2-dimensional Gentzen systems introduced in [18], and also as a generalization of the $m$-dimensional deductive systems in the sense of W. Blok and D. Pigozzi ([7]).

An $m$**-dimensional Gentzen system** is a pair $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ where $\vdash_{\mathcal{G}}$ is a finitary consequence relation on the set of $m$-sequents, $m$-$Seq_{\mathcal{L}}$, which is also **substitution invariant** [1] in the following sense: for every $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$, $\boldsymbol{T} \vdash_{\mathcal{G}} \boldsymbol{\Gamma}$ implies $h(\boldsymbol{T}) \vdash_{\mathcal{G}} h(\boldsymbol{\Gamma})$. A set of $m$-sequents $\boldsymbol{T}$ is called a $\mathcal{G}$**-theory** if $\boldsymbol{T} = \{\boldsymbol{\Gamma} \in m\text{-}Seq_{\mathcal{L}} : \boldsymbol{T} \vdash_{\mathcal{G}} \boldsymbol{\Gamma}\}$.

Note that every $m$-sequent calculus $LX$ determines a Gentzen system $\mathcal{G}_{LX} = \langle \mathcal{L}, \vdash_{LX} \rangle$ by using the rules of the calculus to derive sequents from sets of sequents, not just from the axiom alone, as stated in the following definition (cf. [18, p.14] and [3, p. 267]):

**Definition 3.1.** Given $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$, we say that $\boldsymbol{\Gamma}$ **follows from** $\boldsymbol{T}$ in $\mathcal{G}_{LX}$, in symbols $\boldsymbol{T} \vdash_{LX} \boldsymbol{\Gamma}$ iff there is a finite sequence of sequents $\boldsymbol{\Gamma}_0, \ldots, \boldsymbol{\Gamma}_{n-1}$, $n \geq 1$, called a **proof** of $\boldsymbol{\Gamma}$ from $\boldsymbol{T}$, such that $\boldsymbol{\Gamma}_{n-1} = \boldsymbol{\Gamma}$ and for each $i < n$ one of the following conditions holds:

(i) $\boldsymbol{\Gamma}_i$ is an instance of an axiom;

(ii) $\boldsymbol{\Gamma}_i \in \boldsymbol{T}$;

(iii) $\boldsymbol{\Gamma}_i$ is obtained from $\{\boldsymbol{\Gamma}_j : j < i\}$ by using a rule $(r)$ of $LX$, i.e., $\langle \boldsymbol{S}, \boldsymbol{\Gamma}_i \rangle \in (r)$ for some $\boldsymbol{S} \subseteq \{\boldsymbol{\Gamma}_j : j < i\}$.

**Definition 3.2.** Let $\mathbf{L}$ be a finite $\mathcal{L}$-algebra. A **VL-Gentzen system** is a Gentzen system determined by a **VL**-sequent calculus.

[1] This consequence relations are also called *structural*, but we reserve this expression for the *structural* rules.

As examples of these Gentzen system we can mention the Gentzen systems determined by a $\mathbf{VS}(m)$-sequent calculus, where $\mathbf{S}(m)$ is the linear MV-algebra of $m$ elements and which are studied in [16]. As a particular case we have the $\mathbf{VS}(3)$-Gentzen system determined by the sequent calculus given in Appendix A.

**Definition 3.3.** An $m$-dimensional Gentzen system satisfies an $m$-rule $(r)$ if $\boldsymbol{T} \vdash_{\mathcal{G}} \mathbf{\Gamma}$ for every $\langle \boldsymbol{T}, \mathbf{\Gamma} \rangle \in (r)$.

### 3.1 Matrices for Gentzen systems

Now we recall the definitions related with the concept of a matrix for a Gentzen system, together with some results obtained in [12, 15, 14].

Let $m \in \omega$ and let $A$ be a set. An $m$-**sequent on** $A$ is an $m$-tuple of finite sequences of elements of $A$. We will denote by $m\text{-}Seq(A)$ the set of all $m$-sequents on $A$ and we will use boldface lowercase letters to denote $m$-sequents on $A$. If $\mathbf{a}, \mathbf{b} \in m\text{-}Seq(A)$ we define $\mathbf{a}(i)$, $[\![\mathbf{a}, \mathbf{b}]\!]$, $[I : a]$ (for any $a \in A$) etc. in the same way as for $m$-sequents.

Let $\mathbf{A}$ be an $\mathcal{L}$-algebra and let $h \in val(\mathbf{A})$. If $\mathbf{\Gamma}$ is the sequent

$$\gamma_0^0, \ldots, \gamma_0^{t_0-1} \mid \cdots \mid \gamma_{m-1}^0, \ldots, \gamma_{m-1}^{t_{m-1}-1},$$

then $h(\mathbf{\Gamma})$ stands for

$$((h(\gamma_0^0), \ldots, h(\gamma_0^{t_0-1})), \ldots, (h(\gamma_{m-1}^0), \ldots, h(\gamma_{m-1}^{t_{m-1}-1}))) \in A^{t_0} \times \cdots \times A^{t_{m-1}}.$$

Let $\mathbf{\Gamma}(p_0, \ldots, p_{n-1}) \in m\text{-}Seq_{\mathcal{L}}$. If $h$ is a valuation such that $h(p_i) = a_i$, for all $i < n$, then we write $\mathbf{\Gamma}^{\mathbf{A}}(a_0, \ldots, a_{n-1}) = h(\mathbf{\Gamma}(p_0, \ldots, p_{n-1}))$.

An $m$-**relation on** $A$ is a set $R \subseteq m\text{-}Seq(A)$, that is, a set of $m$-tuples formed by finite sequences of elements of $A$; $\mathcal{R}_m(A)$ will be the set of all $m$-relations on $A$. If there is no risk of confusion we write $\mathcal{R}_m$ instead of $\mathcal{R}_m(A)$.

An $m$–**matrix**, or just a **matrix**, is a pair $\langle \mathbf{A}, R \rangle$ where $\mathbf{A}$ is an $\mathcal{L}$-algebra and $R$ is an $m$-relation on $A$. If $\langle \mathbf{A}, R \rangle$ is an $m$-matrix, let $\models_{\langle \mathbf{A}, R \rangle}$ be the consequence relation on the set $m\text{-}Seq_{\mathcal{L}}$ defined by: $\boldsymbol{T} \models_{\langle \mathbf{A}, R \rangle} \mathbf{\Gamma}$ iff for every $h \in val(\mathbf{A})$, $h(\boldsymbol{T}) \subseteq R$ implies $h(\mathbf{\Gamma}) \in R$. This consequence relation is always substitution invariant, but it may not be finitary.

Let $\mathcal{G}$ be an $m$-dimensional Gentzen system, $\langle \mathbf{A}, R \rangle$ an $m$-matrix and $(r)$ an $m$–rule of inference. $R$ is **closed under the rule** $(r)$ if for every pair $\langle \boldsymbol{T}, \mathbf{\Gamma} \rangle \in (r)$, $\boldsymbol{T} \models_{\langle \mathbf{A}, R \rangle} \mathbf{\Gamma}$.

A matrix $\langle \mathbf{A}, R \rangle$ is called a **matrix model** of $\mathcal{G}$ (or $\mathcal{G}$-**matrix**) if for every set of sequents $\boldsymbol{T} \cup \{\mathbf{\Gamma}\}$, $\boldsymbol{T} \vdash_{\mathcal{G}} \mathbf{\Gamma}$ implies $\boldsymbol{T} \models_{\langle \mathbf{A}, R \rangle} \mathbf{\Gamma}$. Then $R$ is called a $\mathcal{G}$-**filter**. We denote by $Fi_{\mathcal{G}}\mathbf{A}$ the set of $\mathcal{G}$-filters on $\mathbf{A}$. When $\mathcal{G}$ is defined by means of some axioms and inference rules, $R$ is a $\mathcal{G}$-filter iff $R$ contains all the interpretations of these axioms and is closed under each of these rules. The $\mathcal{G}$-filters on the algebra $\mathbf{Fm}_{\mathcal{L}}$ are just the $\mathcal{G}$-theories.

If $\mathcal{K}$ is a class of $m$-matrices, then $\models_{\mathcal{K}}$ denotes the consequence relation on the set of $m\text{-}Seq_{\mathcal{L}}$ defined by: $\boldsymbol{T} \models_{\mathcal{K}} \boldsymbol{\Gamma}$ iff $\boldsymbol{T} \models_{\langle \mathbf{A},R\rangle} \boldsymbol{\Gamma}$ for every $\langle \mathbf{A}, R\rangle \in \mathcal{K}$.

**Definition 3.4.** Let $\mathcal{G}$ be a Gentzen system and let $\mathcal{K}$ be a class of $\mathcal{G}$-matrices. We say that $\mathcal{G}$ is **complete** with respect to $\mathcal{K}$ if

$$\boldsymbol{T} \vdash_{\mathcal{G}} \boldsymbol{\Gamma} \Leftrightarrow \boldsymbol{T} \models_{\mathcal{K}} \boldsymbol{\Gamma},$$

for any $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$.

Now we are going to define a semantical consequence relation over the set of $m$-sequents based on the definition of $\mathbf{L}$-satisfaction. This consequence relation is defined from an $m$–matrix on the algebra $\mathbf{L}$. So we start by defining the following $m$–relation, which contains the interpretation of the valid sequents:

**Definition 3.5.** Let $\mathbf{L}$ be a finite algebra with universe

$$L = \{v_0, \ldots, v_{m-1}\}$$

of cardinal $m$, then

$$D_{\mathbf{L}} = \{\mathbf{a} \in m\text{-}Seq(L) : \text{ there is } i < m \text{ such that } v_i \in \mathbf{a}(i)\}.$$

The connection between the $m$-matrix $\langle \mathbf{L}, D_{\mathbf{L}}\rangle$ and the definition of $\mathbf{L}$-validity and $\mathbf{L}$-satisfaction is shown in the following

**Proposition 3.6.** *Let* $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$. *The following properties hold:*

(i) *If* $h \in val(\mathbf{L})$, *then* $h \in s(\boldsymbol{\Gamma})$ *iff* $h(\boldsymbol{\Gamma}) \in D_{\mathbf{L}}$.

(ii) $\emptyset \models_{\langle \mathbf{L}, D_{\mathbf{L}}\rangle} \boldsymbol{\Gamma} \Longleftrightarrow \boldsymbol{\Gamma}$ *is an* $\mathbf{L}$*-valid sequent.*

(iii) $\boldsymbol{T} \models_{\langle \mathbf{L}, D_{\mathbf{L}}\rangle} \boldsymbol{\Gamma} \Longleftrightarrow \bigcap_{\boldsymbol{\Pi} \in \boldsymbol{T}} s(\boldsymbol{\Pi}) \subseteq s(\boldsymbol{\Gamma})$.

**Proof.** Straightforward. □

Since the **VL** introduction rules for a given connective are not unique, for any finite $\mathcal{L}$-algebra $\mathbf{L}$ there may be several calculi that satisfy the definition of a **VL**-sequent calculus. Next result states that the consequence relation associated to any **VL**-sequent calculus and the semantical consequence relation $\models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$ are equal; so it provides, from the semantical point of view, a characterization of the Gentzen systems $\mathcal{G}_{\mathbf{VL}}$, and from the syntactical point of view an axiomatization of the relation $\models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$. This result is known in [15] as the *Strong Completeness Theorem.*

**Theorem 3.7.** *Let* **L** *be a finite algebra of cardinal* $m$ *and let* $\mathcal{G}$ *be any* **VL** *Gentzen system. Then* $\mathcal{G}$ *is complete with respect to the class* $\{\langle \mathbf{L}, D_{\mathbf{L}} \rangle\}$*, that is, if* $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$*, then*

$$\boldsymbol{T} \vdash_{\mathcal{G}} \boldsymbol{\Gamma} \quad \Longleftrightarrow \quad \boldsymbol{T} \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \boldsymbol{\Gamma}. \qquad \square$$

The proof of this result is based in the finitariety of the semantical operator $\models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle}$ and the fact that the **VL**-Gentzen systems and the Gentzen system $\langle \mathcal{L}, \models_{\langle \mathbf{L}, D_{\mathbf{L}} \rangle} \rangle$ satisfy the same particular case of the *deduction detachment theorem for Gentzen systems* (see [15, Thm. 4.1] for details).

So it follows from Theorem 3.7 that each **VL**-sequent calculus determines the same consequence relation over the set of $m$-sequents, that is, the same Gentzen system: the one defined by the $m$-matrix $\langle \mathbf{L}, D_{\mathbf{L}} \rangle$.

**Definition 3.8.** Let **L** be a finite algebra. **The Gentzen system associated** with **L** is the one determined by any **VL**–sequent calculus; this Gentzen system is denoted by $\mathcal{G}_{\mathbf{L}} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$.

Theorem 3.7 plays a central role in this paper and its following straightforward corollary will be repeatedly used in the last part of this paper:

**Corollary 3.9.** *Let* **L** *be a finite algebra of cardinal* $m$*. If* $\boldsymbol{T} \cup \{\boldsymbol{\Gamma}\} \subseteq m\text{-}Seq_{\mathcal{L}}$*, then*

$$\boldsymbol{T} \vdash_{\mathbf{L}} \boldsymbol{\Gamma} \Longleftrightarrow \bigcap_{\boldsymbol{\Delta} \in \boldsymbol{T}} s(\boldsymbol{\Delta}) \subseteq s(\boldsymbol{\Gamma}). \qquad \square$$

## 4 The Leibniz operator and reduced matrices

The Leibniz operator was first defined in [6] for deductive systems. In this section we extend this definition to the context of Gentzen systems.

**Definition 4.1.** Let $R \in \mathcal{R}_m(A)$ and $\theta \subseteq A^2$. We say that $\theta$ is **compatible** with $R$ if for all $(a, b) \in A^2$, if $(a, b) \in \theta$ then, for all $\mathbf{a}, \mathbf{a}' \in m\text{-}Seq(A)$,

$$[\![\mathbf{a}, [i : a], \mathbf{a}']\!] \in R \text{ implies } [\![\mathbf{a}, [i : b], \mathbf{a}']\!] \in R.$$

For all $R \in \mathcal{R}_m(A)$, we define $\Omega_{\mathbf{A}}(R)$ as the largest congruence of $\mathbf{A}$ compatible with $R$. Such a congruence always exists and will be called the **Leibniz congruence** associated with $R$. The map $\Omega_{\mathbf{A}}$, with domain $\mathcal{R}_m(A)$, is called the **($m$-dimensional) Leibniz operator**. We write $\Omega$ instead of $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$.

The following sequents play an essential role in the sequel:

**Definition 4.2.** Let $i < m$, then

$$\Xi_i(p, q) = [\![ [\{i\}^C : p], [i : q] ]\!]$$

All the components of the sequent $\Xi_i(p, q)$ are $p$, except the $i$-th one, which is $q$.

In the presence of some structural rules, the sequents $\Xi_i(p, q)$ make it possible to characterize the Leibniz congruence, as proved in the following results:

**Lemma 4.3.** *Let $\mathcal{G}$ be a Gentzen system which satisfies the cut rules (and contraction and exchange if $m > 2$). For all $\mathbf{\Gamma} \in m\text{-}Seq_{\mathcal{L}}$, all pair of formulas $\varphi$, $\psi$, and every $i < m$,*

$$[\![\mathbf{\Gamma}, [i : \varphi]]\!]\,;\, \Xi_i(\varphi, \psi) \vdash_{\mathcal{G}} [\![\mathbf{\Gamma}, [i : \psi]]\!].$$

**Proof.** By using $(cut : i, 0)$ we have that

$$[\![\mathbf{\Gamma}, [i : \varphi]]\!]\,;\, \Xi_i(\varphi, \psi) \vdash_{\mathcal{G}} [\![\mathbf{\Gamma}, [\{0, i\}^C : \varphi], [i : \psi]]\!].$$

Now, by $(cut : i, 1)$ we have that

$$[\![\mathbf{\Gamma}, [i : \varphi]]\!]\,;\, [\![\mathbf{\Gamma}, [\{0, i\}^C : \varphi], [i : \psi]]\!] \vdash_{\mathcal{G}} [\![\mathbf{\Gamma}, \mathbf{\Gamma}, [\{0, 1, i\}^C : \varphi], [i : \psi]]\!].$$

Applying $(cut : i, 2), \ldots (cut : i, i-1), (cut : i, i+1), \ldots, (cut : i, m-1)$ we obtain the sequent $[\![\mathbf{\Gamma}, \ldots, \mathbf{\Gamma}, [i : \psi]]\!]$. Finally the result follows by successive applications of the exchange and contraction rules. $\square$

Note that, if $m = 2$, by the cut rule we have that

$$\frac{\Gamma, \varphi \mid \Delta \quad \psi \mid \varphi}{\Gamma, \psi \mid \Delta}\ (cut : 0, 1) \qquad \text{and} \qquad \frac{\Gamma \mid \Delta, \varphi \quad \varphi \mid \psi}{\Gamma \mid \Delta, \psi}\ (cut : 1, 0)$$

Thus, if $m = 2$, Lemma 4.3 and the next proposition can be proved without using neither the contraction nor the exchange rules (as was done in [18, prop. 2.21]).

**Proposition 4.4.** *Let $\mathcal{G}$ be a Gentzen system that satisfies the axiom and cut rules (and contraction and exchange if $m > 2$). Then we have that*

(i) $\emptyset \vdash_{\mathcal{G}} \Xi_i(x,x)$.

(ii) $\{\Xi_i(x,y) : i < m\} \vdash_{\mathcal{G}} \Xi_j(y,x)$ *for all* $j < m$.

(iii) $\{\Xi_i(x,y);\ \Xi_i(y,z) : i < m\} \vdash_{\mathcal{G}} \Xi_j(x,z)$ *for all* $j < m$.

**Proof.** These properties follow easily from axiom and Lemma 4.3. □

**Proposition 4.5.** *Let $\mathcal{G}$ be a Gentzen system that satisfies the axiom and the exchange and the cut rules (and contraction if $m > 2$). If $\langle A, R\rangle$ is a $\mathcal{G}$–matrix, then the set*

$$\theta_R = \{(a,b) \in A^2 : \Xi_i^{\mathbf{A}}(a,b) \in R \textit{ for all } i < m\},$$

*satisfies the following properties*

(i) *$\theta_R$ is an equivalence relation and $\Omega_{\mathbf{A}}(R) \subseteq \theta_R$.*

(ii) *$\theta_R$ is compatible with $R$.*

(iii) *If $\theta_R$ is a congruence relation then $\Omega_{\mathbf{A}}(R) = \theta_R$.*

**Proof.** (i) It follows straightforward from the previous Proposition. To prove (ii) note that since $R$ is closed under the exchange and cut rules (and contraction if $m > 2$) then, by Lemma 4.3,

$$[\![\mathbf{a}, [i : a]]\!] \in R \text{ and } \Xi_i^{\mathbf{A}}(a,b) \in R \Rightarrow [\![\mathbf{a}, [i : b]]\!] \in R. \tag{2}$$

Let us see that $\theta_R$ is compatible with $R$. Let $[\![\mathbf{a}, [i : a], \mathbf{a}']\!] \in R$ and $(a,b) \in \theta_R$. Since $R$ is closed under exchange we get that $[\![\mathbf{a}, \mathbf{a}', [i : a]]\!] \in R$. As $\Xi_i^{\mathbf{A}}(a,b) \in R$, by using (2) we obtain that $[\![\mathbf{a}, \mathbf{a}', [i : b]]\!] \in R$, and by the exchange rules $[\![\mathbf{a}, [i : b], \mathbf{a}']\!] \in R$. The proof of (iii) is straightforward. □

**Corollary 4.6.** *Let $\mathcal{G}$ be a Gentzen system that satisfies the axiom and the exchange and cut rules (and contraction if $m > 2$). Then we have:*

(i) *$\boldsymbol{T}_1 \subseteq \boldsymbol{T}_2$ implies $\Omega\boldsymbol{T}_1 \subseteq \Omega\boldsymbol{T}_2$, for all $\boldsymbol{T}_1, \boldsymbol{T}_2 \in Th\,\mathcal{G}$.*

(ii) *If we also assume that for every $\boldsymbol{T} \in Th\,\mathcal{G}$, the equivalence relation*

$$\theta_{\boldsymbol{T}} = \{(\varphi,\psi) \in Fm_{\mathcal{L}}^2 : \Xi_i(\varphi,\psi) \in \boldsymbol{T} \textit{ and } \Xi_i(\psi,\varphi) \in \boldsymbol{T} \textit{ for all } i < m\}$$

*is a congruence on $\mathbf{Fm}_{\mathcal{L}}$ then*

$$\Omega\boldsymbol{T} = \theta_{\boldsymbol{T}} \textit{ for every } \boldsymbol{T} \in Th\,\mathcal{G}.$$

**Proof.** (i) Let $\boldsymbol{T}_1 \subseteq \boldsymbol{T}_2$. It is enough to see that $\Omega\boldsymbol{T}_1$ is compatible with $\boldsymbol{T}_2$. By Proposition 4.5.(i), $\Omega\boldsymbol{T}_1 \subseteq \theta_{\boldsymbol{T}_1}$; since $\boldsymbol{T}_1 \subseteq \boldsymbol{T}_2$ we have that $\theta_{\boldsymbol{T}_1} \subseteq \theta_{\boldsymbol{T}_2}$. Thus $\Omega\boldsymbol{T}_1 \subseteq \theta_{\boldsymbol{T}_2}$ and by Proposition 4.5(ii), $\theta_{\boldsymbol{T}_2}$ is compatible with $\boldsymbol{T}_2$. Hence $\Omega\boldsymbol{T}_1$ is compatible with $\boldsymbol{T}_2$. (ii) follows straightforward from Proposition 4.5. □

A Gentzen system that satisfies condition (i) of Corollary 4.6 is called **protoalgebraic** in [14], where the concept of a protoalgebraic Gentzen system was deeply investigated. Nevertheless, the analysis of the previous results makes it possible to state this

**Corollary 4.7.** *Let $\mathcal{G} = \langle \mathcal{G}, \vdash_{\mathcal{G}} \rangle$ be a Gentzen system and let $\{\mathbf{\Pi}_i(x,y) : i \in I\}$ be a finite a set of sequents in two variables such that*

(i) $\emptyset \vdash_{\mathcal{G}} \mathbf{\Pi}_i(x,x)$ *for every $i \in I$.*

(ii) $[\![\mathbf{\Gamma}, [j : \varphi], \mathbf{\Gamma}']\!]\,; \{\mathbf{\Pi}_i(\varphi,\psi) : i \in I\} \vdash_{\mathcal{G}} [\![\mathbf{\Gamma}, [j : \psi], \mathbf{\Gamma}']\!]$ *for every $j < m$, $\{\mathbf{\Gamma}, \mathbf{\Gamma}'\} \subseteq m\text{-}Seq_{\mathcal{L}}$ and $\{\varphi, \psi\} \subseteq \mathbf{Fm}_{\mathcal{L}}$.*

*Then $\mathcal{G}$ is protoalgebraic.* □

The matrices whose Leibniz congruence is the identity relation are called *reduced matrices* and play a very important role in Abstract Algebraic Logic (see [10] for a survey on this subject). This motivates the following definitions: if $\mathcal{G}$ is a Gentzen system, a $\mathcal{G}$–matrix $\langle \mathbf{A}, R \rangle$ is a **reduced $\mathcal{G}$-matrix** if $\Omega_{\mathbf{A}}(R) = \Delta_{\mathbf{A}} = \{(a,a) \in A^2\}$. If $\mathbf{A}$ is an $\mathcal{L}$-algebra, $\mathbf{A}$ is an **algebraic reduct** of a $\mathcal{G}$-reduced matrix if there exists $R \in m\text{-}Seq(A)$ such that $\langle \mathbf{A}, R \rangle$ is a reduced $\mathcal{G}$-matrix. We denote by $Matr\mathcal{G}$ the set of all $\mathcal{G}$-matrices and by $Matr^*\mathcal{G}$ the set of all $\mathcal{G}$-reduced matrices. If $\mathcal{K}$ is a class of matrices, $Alg(\mathcal{K})$ denotes the set of all algebraic reducts of the matrices in $\mathcal{K}$.

If $K$ is a class of algebras, $Q(K)$ will denote the quasivariety generated by the class $K$.

## 5 Finitely equivalential Gentzen systems

One of the objectives of the paper is to characterize the algebraic reducts of the reduced $\mathcal{G}_{\mathbf{L}}$-matrices. Although a direct proof of this characterization based on the properties of the sequents $\mathbf{\Xi}_i(p,q)$ and the rules (structural and *logical*) of the sequent calculi could be supplied (as was done in [12]), we present here a more general proof inspired in some results obtained by J. Czelakowski ([9, Thm. 3.2.2]) and H. Herrmann ([17]) concerning what is known as *equivalential or congruential deductive system.* So we start by providing the notion of *finitely equivalential* Gentzen system and then we define and characterize the algebraic reducts of the reduced matrices of a finitely equivalential Gentzen system. The results of this section are concerned more with the properties of the consequence relation than with the influence of the structural rules in the concept of and *equivalential Gentzen system.* This is because it seems not possible to derive the propertes of this kind of Gentzen systems only from the form of the rules of a calculus (as was done for protoalgebraic Gentzen systems in [14]).

**Definition 5.1.** A Gentzen system $\mathcal{G}$ is **finitely equivalential** if there exists a finite set of $m$-sequents in two variables $\{\mathbf{\Delta}_i(p,q) : i < n\}$ such that, for any $\mathcal{G}$-matrix $\langle A, R\rangle$,

$$\Omega_{\mathbf{A}}(R) = \{(a,b) \in A^2 : \mathbf{\Delta}_i^{\mathbf{A}}(a,b) \in R \text{ for all } i < n\}.$$

The set $\{\mathbf{\Delta}_i(p,q) : i < n\} \subseteq m\text{-}Seq_{\mathcal{L}}$ is called a **set of equivalence sequents** for $\mathcal{G}$.

**Lemma 5.2.** *Let $\mathcal{G}$ be a finitely equivalential Gentzen system with a set $\{\mathbf{\Delta}_i(p,q) : i < n\}$ of equivalence sequents. Then, if $\langle \mathbf{A}, R\rangle \in Matr^*\mathcal{G}$, $h \in val(\mathbf{A})$ and $\{\varphi,\psi\} \subseteq Fm_{\mathcal{L}}$, we have that*

$$h(\varphi) = h(\psi) \quad \Longleftrightarrow \quad \{h(\mathbf{\Delta}_i(\varphi,\psi) : i < n\} \subseteq R.$$

**Proof.** Note that

$$h(\varphi) = h(\psi) \text{ iff } (h(\varphi), h(\psi)) \in \Omega_{\mathbf{A}}(R) \text{ iff } \mathbf{\Delta}_i^{\mathbf{A}}(h(\varphi), h(\psi)) \in R,$$

for all $i < n$. Then observe that $\mathbf{\Delta}_i^{\mathbf{A}}(h(\varphi), h(\psi)) = h(\mathbf{\Delta}_i(\varphi,\psi))$. □

Now we will see how to establish a link between certain finitely equivalential Gentzen systems and certain classes of algebras. First we recall the concept of the relative equational consequence relation associated with a class of algebras. An **$\mathcal{L}$-equation** is a pair of $\mathcal{L}$–formulas $\{\varphi,\psi\}$, usually written as $\varphi \approx \psi$. If $K$ is a class of algebras, the **relative equational consequence relation** determined by $K$ is the consequence relation on the set of $\mathcal{L}$-equations defined as follows: $\Sigma \models_K \varphi \approx \psi$ iff for all $\mathbf{A} \in K$ and every $h \in val(\mathbf{A})$, if $h(\chi) = h(\eta)$ for every $\chi \approx \eta \in \Sigma$ then $h(\varphi) = h(\psi)$, for every set of $\mathcal{L}$–equations $\Sigma \cup \varphi \approx \psi$.

**Lemma 5.3.** *Let $\mathcal{G}$ be a Gentzen system complete with respect to a class $\mathcal{K}$ of reduced $\mathcal{G}$-matrices, such that $\mathcal{G}$ is finitely equivalential with $\{\mathbf{\Delta}_j(p,q) : j < n\} \subseteq m\text{-}Seq_{\mathcal{L}}$ a set of equivalence sequents. Then for every set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathcal{L}$–equations*

$$\Sigma \models_{Alg\mathcal{K}} \varphi \approx \psi \textit{ iff } \{\mathbf{\Delta}_j(\xi,\eta) : \xi \approx \eta \in \Sigma, j < n\} \vdash_{\mathcal{G}} \mathbf{\Delta}_l(\varphi,\psi), \textit{for all } l < n.$$

**Proof.** Since $\mathcal{G}$ is complete with respect to $\mathcal{K}$, we will show that $\Sigma \models_{Alg\mathcal{K}} \varphi \approx \psi$ iff $\{\mathbf{\Delta}_j(\xi,\eta) : \xi \approx \eta \in \Sigma, j < n\} \models_{\mathcal{K}} \mathbf{\Delta}_l(\varphi,\psi)$ for all $l < n$. And this is true because by Lemma 5.2 we have that for every matrix $\langle A, R\rangle \in \mathcal{K}$, $\Sigma \models_{\{A\}} \varphi \approx \psi$ iff $\{\mathbf{\Delta}_j(\xi,\eta) : \xi \approx \eta \in \Sigma, j < n\} \models_{\langle A,R\rangle} \mathbf{\Delta}_l(\varphi,\psi)$, for all $l < n$. □

In the terminology of [17] (for logics) and [16] (for Gentzen systems), we say that the sequents $\mathbf{\Delta}_j(p,q), j < n$ provide an interpretation of the consequence relation $\models_{Alg\mathcal{K}}$ into the consequence relation $\vdash_{\mathcal{G}}$.

Now we prove the main result of the paper that will be further applied to the algebraic reducts of the reduced $\mathcal{G}_{\mathbf{VL}}$-matrices.

**Theorem 5.4.** *Let $\mathcal{G}$ be a finitely eqivalential Gentzen system complete with respect to a class $\mathcal{K}$ of reduced $\mathcal{G}$-matrices. Then we have that*

$$Alg(Matr^*\mathcal{G}) = Q(Alg\mathcal{K}).$$

**Proof.** Let $\{\mathbf{\Delta}_j(p,q) : j < n\} \subseteq m\text{-}Seq_{\mathcal{L}}$ be a set of equivalence sequents for $\mathcal{G}$.

$\subseteq$ Let $\langle \mathbf{A}, R\rangle \in Matr^*\mathcal{G}$ and let $\Sigma \cup \{\varphi \approx \psi\}$ be a set of $\mathcal{L}$–equations. To show $\mathbf{A} \in Q(Alg\mathcal{K})$ we will prove that $\Sigma \models_{Alg\mathcal{K}} \varphi \approx \psi$ implies $\Sigma \models_{\mathbf{A}} \varphi \approx \psi$. Consider $h \in val(\mathbf{A})$ such that $h(\xi) = h(\eta)$ for any $\xi \approx \eta \in \Sigma$. By Lemma 5.2 $\{\mathbf{\Delta}_j^{\mathbf{A}}(h(\xi), h(\eta)) : j < n, \xi \approx \eta \in \Sigma\} \subseteq R$. Since by Lemma 5.3 we have that for all $l < n$ $\{\mathbf{\Delta}_j(\xi,\eta) : j < n, \xi \approx \eta \in \Sigma\} \vdash_{\mathcal{G}} \mathbf{\Delta}_l(\varphi,\psi)$, we obtain that if $l < n$, then $h(\mathbf{\Delta}_l(\varphi,\psi)) \in R$ and again by Lemma 5.2 $h(\varphi) = h(\psi)$.

$\supseteq$ Since $Alg\mathcal{K} \subseteq Alg(Matr^*\mathcal{G})$, it is enough to prove that the class $Alg(Matr^*\mathcal{G})$ is a quasivariety, i.e., it is closed under the operators $\mathbf{S}$ (subalgebra), $\mathbf{P}$ (product) and $\mathbf{P_U}$ (ultraproduct).[2]

(i) The fact that $Alg(Matr^*\mathcal{G})$ is closed under $\mathbf{S}$ follows from the following Lemma (similar to [9, Thm. 3.2.2] but replacing $\mathbf{A}$ by $m\text{-}Seq(A)$), whose proof is straightforward.

**Lemma 5.5.** *Let $\mathcal{G}$ be a finitely equivalential Gentzen system. If $\mathbf{A} \in \mathbf{S}(\mathbf{B})$ and $R \in Fi_{\mathcal{G}}\mathbf{B}$, then*

$$m\text{-}Seq(A) \cap R \in Fi_{\mathcal{G}}\mathbf{A} \text{ and } \Omega_{\mathbf{A}}(m\text{-}Seq(A) \cap R) = A^2 \cap \Omega_{\mathbf{B}}(R). \qquad \square$$

(ii) To prove that $Alg(Matr^*\mathcal{G})$ is closed under $\mathbf{P}$ we consider a set $\{\langle \mathbf{A}_i, R_i\rangle : i \in I\}$ of reduced $\mathcal{G}$-matrices. We consider now the matrix

$$\mathcal{M} = \left\langle \prod_{i\in I} \mathbf{A}_i, R \right\rangle.$$

[2]Note that, if we are only interested in the Gentzen systems $\mathcal{G}_{\mathbf{L}}$, there is no need to consider the operator $\mathbf{P_U}$, since we are dealing with the quasivariety generated by a single finite algebra.

where, by definition, for all $(n_0, \ldots, n_{m-1}) \in \omega^m$ and every $k < m, l < n_k$ and $a_k^l \in \prod_{i \in I} A_i$,

$$((a_0^0, \ldots, a_0^{n_0-1}), \ldots, (a_{m-1}^0, \ldots, a_{m-1}^{n_0-1})) \in R$$

if and only if for all $i \in I$,

$$((a_0^0(i), \ldots, a_0^{n_0-1}(i)), \ldots, (a_{m-1}^0(i), \ldots, a_{m-1}^{n_0-1}(i)) \in R_i.$$

Obviously $R$ is a $\mathcal{G}$-filter. To prove that $\mathcal{M}$ is reduced, notice that if

$$((a_i)_{i \in I}, (b_i)_{i \in I}) \in (\prod_{i \in I} A_i)^2,$$

then

$$\begin{array}{ll} ((a_i)_{i \in I}, (b_i)_{i \in I}) \in \Omega_{\prod_{i \in I} \mathbf{A}_i}(R) & \iff \\ \{\mathbf{\Delta}_j^{\prod_{\mathbf{i} \in \mathbf{I}} \mathbf{A_i}}((a_i)_{i \in I}, (b_i)_{i \in I}) : j < m\} \subseteq R & \iff \\ \{\mathbf{\Delta}_j^{\mathbf{A_i}}(a_i, b_i) : j < m\} \subseteq R_i \text{ for all } i \in I & \iff \\ a_i = b_i \text{ for all } i \in I, & \iff \\ (a_i)_{i \in I} = (b_i)_{i \in I}. & \end{array}$$

(iii) To prove that $Alg(Matr^*\mathcal{G})$ is closed under $\mathbf{P_U}$ we consider a set $\{\langle \mathbf{A}_i, R_i \rangle : i \in I\}$ of reduced $\mathcal{G}$-matrices and an ultrafilter $\mathcal{U}$ on $I$. We consider now the matrix

$$\mathcal{M} = \left\langle \prod_{i \in I} \mathbf{A}_i/_{\mathcal{U}}, D \right\rangle.$$

where, by definition, for all $(n_0, \ldots, n_{m-1}) \in \omega^m$ and every $k < m, l < n_k$ and $a_k^l \in \prod_{i \in I} A_i$,

$$((a_0^0/_{\mathcal{U}}, \ldots, a_0^{n_0-1}/_{\mathcal{U}}), \ldots, (a_{m-1}^0/_{\mathcal{U}}, \ldots, a_{m-1}^{n_0-1}/_{\mathcal{U}})) \in D$$

if and only if

$$\{i \in I : ((a_0^0(i), \ldots, a_0^{n_0-1}(i)), \ldots, (a_{m-1}^0(i), \ldots, a_{m-1}^{n_0-1}(i)) \in R_i\} \in \mathcal{U}.$$

Now, it is straightforward to prove that $\mathcal{M}$ is a reduced $\mathcal{G}$-matrix. □

## 6 The $\mathcal{G}_\mathbf{L}$-matrices

In order to apply Proposition 4.5 to the Gentzen systems $\mathcal{G}_\mathbf{L}$ we will prove that for any $\mathcal{G}_\mathbf{L}$-theory $\boldsymbol{T}$, the relation $\theta_{\boldsymbol{T}}$ is a congruence. First we state the following result that shows that the sequents defined in 4.2 satisfy a result similar to Lemma 5.2, when considering the matrix $\langle \mathbf{L}, D_\mathbf{L} \rangle$:

**Lemma 6.1.** *Let* $h \in val(\mathbf{L})$ *and* $\{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$*, then*

$$h(\varphi) = h(\psi) \iff h \in \bigcap_{i<m} s(\Xi_i(\varphi, \psi)) \iff \{h(\Xi_i(\varphi, \psi)) : i < m\} \subseteq D_{\mathbf{L}}.$$

**Proof.** For the first equivalence $\Rightarrow$) is straightforward and $\Leftarrow$) is proved as follows: if $h(\varphi) = v_j$ and since $h \in s(\{\Xi_j(\varphi, \psi)) = s([\![ [\{j\}^C : \varphi], [j : \psi] ]\!])$, we get that $h(\psi) = v_j$. The second equivalence follows from Lemma 3.6(i). □

**Theorem 6.2.** *Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra. Let* $R$ *be a* $\mathcal{G}_{\mathbf{L}}$*-filter on* $\mathbf{A}$*. Then*

$$\Omega_{\mathbf{A}}(R) = \{(a, b) : \Xi_i^{\mathbf{A}}(a, b) \in R \textit{ for all } i < m\}.$$

**Proof.** First we prove that if $\varpi$ is an $n$-ary connective, then for all $j < m$

$$\{\Xi_i(x_k, y_k) : i < m, k < n\} \vdash_{\mathbf{L}} \Xi_j(\varpi(x_0, \dots, x_{n-1}), \varpi(y_0, \dots, y_{n-1})) \quad (3)$$

Let $h \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$ such that $h \in \cap_{i<m,k<n} s(\Xi_i(x_k, y_k))$; by Lemma 6.1 we have that $h(x_k) = h(y_k)$ for all $k < n$. Hence

$$h(\varpi(x_0, \dots, x_{n-1})) = h(\varpi(y_0, \dots, y_{n-1})),$$

so

$$h \in s(\Xi_j(\varpi(x_0, \dots, x_{n-1}), \varpi(y_0, \dots, y_{n-1}))),$$

by Lemma 6.1 again. With (3) and Proposition 4.4 we have proved that the set

$$\theta_R = \{(a, b) : \Xi_i^{\mathbf{A}}(a, b) \in R \text{ for all } i < m\}$$

is a congruence relation, so by Proposition 4.5 we are done. □

**Theorem 6.3.** *Let* $\mathbf{L}$ *be a finite algebra of cardinal* $m$*. The Gentzen system* $\mathcal{G}_{\mathbf{L}}$ *is finitely equivalential and the set* $\{\Xi_i(p, q) : i < m\}$ *is a set of equivalence sequents for it.*

**Proof.** This is just Theorem 6.2. □

Now we finally come to the study of the algebraic reducts of the reduced $\mathcal{G}_{\mathbf{L}}$-matrices. First we note that the matrix $\langle \mathbf{L}, D_{\mathbf{L}} \rangle$ satisfies another interesting property:

**Proposition 6.4.** *For any finite algebra* $\mathbf{L}$*, the matrix* $\langle \mathbf{L}, D_{\mathbf{L}} \rangle$ *is a reduced* $\mathcal{G}_{\mathbf{L}}$*-matrix.*

**Proof.** $\langle \mathbf{L}, D_{\mathbf{L}} \rangle$ is a $\mathcal{G}_{\mathbf{L}}$-matrix by Theorem 3.7. Now let $a, b \in L$ and $h \in val(\mathbf{L})$ such that $h(p) = a$ and $h(q) = b$. We have the following equivalences:

$$\begin{aligned}
(a,b) \in \Omega_{\mathbf{L}}(D_{\mathbf{L}}) &\iff \Xi_i^{\mathbf{L}}(a,b) \in D_{\mathbf{L}} \text{ for all } i < m \text{ (by Theorem 6.2)} \\
&\iff h(\Xi_i(p,q)) \in D_{\mathbf{L}} \text{ for all } i < m \\
&\iff h(p) = h(q) \text{ (by Lemma 6.1)} \\
&\iff a = b,
\end{aligned}$$

which closes the proof. □

Now, as a corollary we obtain the following

**Theorem 6.5.** *Let* $\mathbf{L}$ *be a finite algebra. Then*

$$Alg(Matr^{*}\mathcal{G}_{\mathbf{L}}) = Q(\mathbf{L}).$$

**Proof.** By Theorem 5.4 and the fact that $\mathcal{G}_{\mathbf{L}}$ is complete with respect to the class $\{\langle \mathbf{L}, D_{\mathbf{L}} \rangle\}$ (Theorem 3.7) of reduced matrices (Proposition 6.4) and finitely equivalential (Theorem 6.3). □

## 7 Conclusions

Throughout the paper we have shown how the rules of a sequent calculus encode in a certain sense some properties that have a clear algebraic counterpart. In this sense this paper follows the results presented in [14] where the relation between the cut rule and the protoalgebraicity of a system was clearly established.

Some other results concerning the relative equational consequence relation determined by a finite algebra can be obtained. In the paper we have proved that given a finite algebra $\mathbf{L}$, the class of the algebraic reducts of the reduced matrices of the Gentzen system $\mathcal{G}_{\mathbf{L}}$ is the quasivariety generated by $\mathbf{L}$. As an easy corollary of Lemma 5.3 and Proposition 6.4, we have that

**Corollary 7.1.** *Let* $\mathbf{L}$ *be a finite algebra of cardinal* $m$*; for every set* $\Sigma \cup \{\varphi \approx \psi\}$ *of* $\mathcal{L}$*–equations,*

$$\Sigma \models_{Q(\mathbf{L})} \varphi \approx \psi \Leftrightarrow \{\Xi_i(\xi,\eta) : \xi \approx \eta \in \Sigma, i < m\} \vdash_{\mathbf{L}} \Xi_j(\varphi,\psi), \textit{for any } j < m. \tag{4}$$

which means that we have an interpretation of the relative equational consequence relation associated to the quasivariety generated by $\mathbf{L}$ in the Gentzen system $\mathcal{G}_{\mathbf{L}}$. But the Gentzen system satisfies the *Deduction Detachment Theorem for Gentzen Systems* ([15, Definition 3.1 and Thm. 3.4

and 3.6]), which means that checking the right hand side of the equivalence in (4) reduces to check if a certain set of sequents consists only of derivable sequents. If we take into account now that the sequent calculi **VL** enjoy the cut elimination property, we obtain, by using (4) a decision procedure for the relative equational consequence determined by the algebra **L**. Since the process that produces the rules of the corresponding sequent calculus is fully automatized (via the system `MUltlog`), the decision procedure (described in [11]) for the relative equational consequence relation is easy to implement, and in fact can be used through [2].

## A. A S(3)-sequent calculus

Let $\mathbf{S}(3) = (\{0,1,2\}, \neg, \vee, \wedge, \rightarrow)$ be the three element $MV$-algebra. By applying the system `MUltlog` to the truth tables of the connectives of $\mathbf{S}(3)$ we obtain the following $\mathbf{VS}(3)$-sequent calculus

- Introduction rules for the negation

$$\frac{\Gamma \mid \Delta \mid \Pi, \varphi}{\Gamma, \neg\varphi \mid \Delta \mid \Pi}\ (\neg : 0) \quad \frac{\Gamma \mid \Delta, \varphi \mid \Pi}{\Gamma \mid \Delta, \neg\varphi \mid \Pi}\ (\neg : 1) \quad \frac{\Gamma, \varphi \mid \Delta \mid \Pi}{\Gamma \mid \Delta \mid \Pi, \neg\varphi}\ (\neg : 2)$$

- Introduction rules for the disjunction

$$\frac{\Gamma, \varphi \mid \Delta \mid \Pi \quad \Gamma, \psi \mid \Delta \mid \Pi}{\Gamma, \varphi \vee \psi \mid \Delta \mid \Pi}\ (\vee : 0)$$

$$\frac{\Gamma \mid \Delta, \varphi, \psi \mid \Pi \quad \Gamma, \varphi \mid \Delta, \varphi \mid \Pi \quad \Gamma, \psi \mid \Delta, \psi \mid \Pi}{\Gamma \mid \Delta, \varphi \vee \psi \mid \Pi}\ (\vee : 1)$$

$$\frac{\Gamma \mid \Delta \mid \Pi, \varphi, \psi}{\Gamma \mid \Delta \mid \Pi, \varphi \vee \psi}\ (\vee : 2)$$

- Introduction rules for the conjunction

$$\frac{\Gamma, \varphi, \psi \mid \Delta \mid \Pi}{\Gamma, \varphi \wedge \psi \mid \Delta \mid \Pi}\ (\wedge : 0)$$

$$\frac{\Gamma \mid \Delta, \varphi, \psi \mid \Pi \quad \Gamma \mid \Delta, \varphi \mid \Pi, \varphi \quad \Gamma \mid \Delta, \psi \mid \Pi, \psi}{\Gamma \mid \Delta, \varphi \wedge \psi \mid \Pi}\ (\wedge : 1)$$

$$\frac{\Gamma \mid \Delta \mid \Pi, \varphi \quad \Gamma \mid \Delta \mid \Pi, \psi}{\Gamma \mid \Delta \mid \Pi, \varphi \wedge \psi}\ (\wedge : 2)$$

- Introduction rules for the implication

$$\frac{\Gamma \mid \Delta \mid \Pi, \varphi \quad \Gamma, \psi \mid \Delta \mid \Pi}{\Gamma, \varphi \to \psi \mid \Delta \mid \Pi} \ (\to: 0)$$

$$\frac{\Gamma \mid \Delta, \varphi \mid \Pi, \varphi \quad \Gamma \mid \Delta, \varphi, \psi \mid \Pi \quad \Gamma, \psi \mid \Delta \mid \Pi, \varphi}{\Gamma \mid \Delta, \varphi \to \psi \mid \Pi} \ (\to: 1)$$

$$\frac{\Gamma, \varphi \mid \Delta, \varphi \mid \Pi, \psi \quad \Gamma, \varphi \mid \Delta, \psi \mid \Pi, \psi}{\Gamma \mid \Delta \mid \Pi, \varphi \to \psi} \ (\to: 2)$$

- Structural rules: all of them (see Section 2.1).

## Acknowledgements

We are very grateful to Josep M. Font for his valuable comments on a draft of the paper. Also, we would like to thank an anonymous referee whose comments were really helpful. We acknowledge the support of Grant BFM2001-3329 from the Spanish DGICYT and Grant 2001SGR-00017 from Generalitat de Catalunya.

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