REPORTS ON MATHEMATICAL LOGIC
49 (2014), 3–21
doi:10.4467/20842589RM.14.001.2271

Norihiro KAMIDE

AN EIGHT-VALUED PRACONSISTENT LOGIC

A b s t r a c t. It is known that many-valued paraconsistent logics are useful for expressing uncertain and inconsistency-tolerant reasoning in a wide range of Computer Science. Some four-valued and sixteen-valued logics have especially been well-studied. Some four-valued logics are not so fine-grained, and some sixteen-valued logics are enough fine-grained, but rather complex. In this paper, a natural eight-valued paraconsistent logic rather than four-valued and sixteen-valued logics is introduced as a Gentzen-type sequent calculus. This eight-valued logic is enough fine-grained and simpler than sixteen-valued logic. A triplet valuation semantics is introduced for this logic, and the completeness theorem for this semantics is proved. The cut-elimination theorem for this logic is proved, and this logic is shown to be decidable.

Received 8 September 2013
This paper is based on the results of the conference presentation [7]

1. Introduction

Many-valued paraconsistent logics are of growing importance in Computer Science since these are useful for expressing uncertain and inconsistency-tolerant reasoning. Some 4-valued and 16-valued logics have especially been well-studied [2, 3, 4, 5, 13, 15, 16, 18]. Some 4-valued logics are not so fine-grained, and some 16-valued logics are enough fine-grained, but rather complex. A many-valued paraconsistent logic rather than 4-valued and 16-valued logics is required for developing a fine-grained and simple reasoning system. In this paper, such a natural 8-valued paraconsistent logic, L_8, is introduced as a Gentzen-type sequent calculus. A triplet valuation semantics, which has three kinds of valuations v^n, v^t and v^f, is introduced for L_8, and the completeness theorem for this semantics is proved using some theorems for embedding L_8 into positive classical logic. The cut-elimination theorem for this logic is proved using such an embedding theorem. This logic is also shown to be decidable and paraconsistent.

The proposed logic L_8 adopts the following logical connectives: $\rightarrow$ (classical implication), $\sim_t$ (negation w.r.t. truth order), $\sim_f$ (negation w.r.t. falsity-order), $\wedge_t$ (classical conjunction or conjunction w.r.t. truth-order), $\vee_t$ (classical disjunction or disjunction w.r.t. truth-order), $\wedge_f$ (conjunction w.r.t. falsity-order) and $\vee_f$ (disjunction w.r.t. falsity-order). The logical connectives $\sim_f, \wedge_f$ and $\vee_f$ were originally introduced in Shramko-Wansing's 16-valued logics [15, 16] based on the trilattice $SIXTEEN_3$. Some Shramko-Wansing's 16-valued logics with the full set of connectives including the classical implication was axiomatized by Odintsov [13].

The $\{\wedge_t, \vee_t, \sim_t\}$-fragment of L_8 is a sequent calculus for Dunn's and Belnap's 4-valued logic [4, 5] and is a classical extension of a sequent calculus for Nelson's paraconsistent 4-valued logic [1]. Thus, L_8 may be viewed as a natural extension of Dunn's and Belnap's logic and Nelson's logic. The $\{\wedge_t, \vee_t, \sim_t\}$-fragment of L_8 is also a modified extension of a sequent calculus for Arieli-Avron's 4-valued bilattice logic [2, 3]. Moreover, L_8 is regarded as an 8-valued simplification of some Shramko-Wansing's 16-valued trilattice logics [15, 16].

The above mentioned 4-valued logics are known to be useful for a number of Computer Science applications, and then more expressive many-valued logics have been required for representing more fine-grained situations. Shramko-Wansing's 16-valued logics are an answer to this expressiveness issue, i.e., more fine-grained situations can be expressed using these 16-valued logics. But, these 16-valued logics are rather complex, e.g., some previously proposed sequent calculi [19, 10] and semantics [10] for these logics need some complex definitions. The aim of this paper is thus to construct an 8-valued logic which is a natural extension of the 4-valued logics and is also a simplification of Shramko-Wansing's 16-valued logics.

Suppose that an expression $A \leftrightarrow B$ roughly means a bi-consequence relation (i.e., $A \models B$ and $B \models A$) or the classical bi-implication connective (i.e., $A{\rightarrow}B$ and $B{\rightarrow}A$). Then, Shramko-Wansing's 16-valued logics have the axiom: $\sim_t\sim_f\alpha \leftrightarrow \sim_f\sim_t\alpha$ which implies 16-valued logics based on a semantics with quadruplet valuations v^n (classical valuation), v^t (concerning $\sim_t$), v^f (concerning $\sim_f$) and v^b (concerning $\sim_t\sim_f$) [10, 13]. Instead of this axiom, the logic L_8 adopts the axioms: $\sim_t\sim_f\alpha \leftrightarrow \alpha$ and $\sim_f\sim_t\alpha \leftrightarrow \alpha$ which imply an 8-valued logic based on a semantics with triplet valuations v^n, v^t and v^f.

As far as we know, there is only one previously introduced "natural" 8-valued logic. An 8-valued logic based on the tetralattice $EIGHT_4$ was introduced by Zaitsev [20]. As a base for further generalization of the 4-valued logics, a set $3 := \{\text{a, d, u}\}$ was chosen, where the initial values are a: incoming data is asserted, d: incoming data is denied, and u: incoming data is neither asserted nor denied, that corresponds to the answer "don't know." In [20], an adequate Hilbert-style axiomatization for Zeitsev's logic was proposed. The following axioms for two negation connectives $\sim_a$ and $\sim_d$ are included in this logic: $\sim_d\sim_a\sim_d\alpha \leftrightarrow \sim_a\sim_d\sim_a\alpha$ and $\sim_a\sim_d\sim_a\alpha \leftrightarrow \sim_d\sim_a\sim_d\alpha$ instead of Shramko-Wansing's axioms: $\sim_d\sim_a\alpha \leftrightarrow \sim_a\sim_d\alpha$. Zeitsev's 8-valued logic is philosophically plausible, but it has no Gentzen-type sequent calculus or alternative simple triplet valuation semantics.

The structure of this paper is then summarized as follows. In Section 2, the logic L_8 is introduced as a Gentzen-type sequent calculus, and the cut-elimination theorem for L_8 is shown using a theorem for syntactically embedding L_8 into a sequent calculus LK for positive classical logic. L_8 is also shown to be decidable and paraconsistent. In Section 3, a triplet valuation semantics for L_8 is introduced, and the completeness theorem for this semantics is shown using two theorems for syntactically and semantically embedding L_8 into positive classical logic. In Section 4, this paper is concluded, and some remarks are addressed.

2. Sequent calculus

The following list of symbols is adopted for the language used in this paper: countably many propositional variables $p_0, p_1, ...$, logical connectives $\to$, $\wedge_t$, $\vee_t$, $\wedge_f$, $\vee_f$, $\sim_t$ and $\sim_f$. The connectives $\to$, $\wedge_t$ and $\vee_t$ are just the classical implication, conjunction and disjunction, respectively. Greek lower-case letters $\alpha, \beta, ...$ are used to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) sets of formulas. An expression of the form $\Gamma \Rightarrow \Delta$ is called a *sequent*. An expression $L \vdash S$ (or $\vdash S$) is used to denote the fact that a sequent S is provable in a sequent calculus L. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: for any instance

$$\frac{S_1 \cdots S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$.

Definition 2.1. (L_8) The initial sequents of L_8 are of the form: for any propositional variable p,

$$p \Rightarrow p \qquad \sim_t p \Rightarrow \sim_t p \qquad \sim_f p \Rightarrow \sim_f p.$$

The structural inference rules of L_8 are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (cut)} \quad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (w-l)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \text{ (w-r)}.$$

The normal logical inference rules of L_8 are of the form:

$$\frac{\Gamma \Rightarrow \Sigma, \alpha \quad \beta, \Delta \Rightarrow \Pi}{\alpha{\to}\beta, \Gamma, \Delta \Rightarrow \Sigma, \Pi} \text{ (}{\to}\text{l)} \quad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha{\to}\beta} \text{ (}{\to}\text{r)}$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge_t \beta, \Gamma \Rightarrow \Delta} \text{ (}\wedge_t\text{l)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge_t \beta} \text{ (}\wedge_t\text{r)}$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee_t \beta, \Gamma \Rightarrow \Delta} \text{ (}\vee_t\text{l)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee_t \beta} \text{ (}\vee_t\text{r)}$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge_f \beta, \Gamma \Rightarrow \Delta} \text{ (}\wedge_f\text{l)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge_f \beta} \text{ (}\wedge_f\text{r)}$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \vee_f \beta, \Gamma \Rightarrow \Delta} \text{ (}\vee_f\text{l)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee_f \beta} \text{ (}\vee_f\text{r)}.$$

The double-negation-elimination inference rules of L_8 are of the form: for any $\sim_d \in \{\sim_t\sim_t, \sim_f\sim_f, \sim_t\sim_f, \sim_f\sim_t\}$,

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim_d\alpha, \Gamma \Rightarrow \Delta}\ (\sim_d\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim_d\alpha}\ (\sim_d\text{r}).$$

The $\sim_t$-prefixed logical inference rules of L_8 are of the form:

$$\frac{\alpha, \sim_t\beta, \Gamma \Rightarrow \Delta}{\sim_t(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta}\ (\sim_t{\rightarrow}\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \sim_t\beta}{\Gamma \Rightarrow \Delta, \sim_t(\alpha{\rightarrow}\beta)}\ (\sim_t{\rightarrow}\text{r})$$

$$\frac{\sim_t\alpha, \Gamma \Rightarrow \Delta \quad \sim_t\beta, \Gamma \Rightarrow \Delta}{\sim_t(\alpha\wedge_t\beta), \Gamma \Rightarrow \Delta}\ (\sim_t\wedge_t\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_t\alpha, \sim_t\beta}{\Gamma \Rightarrow \Delta, \sim_t(\alpha\wedge_t\beta)}\ (\sim_t\wedge_t\text{r})$$

$$\frac{\sim_t\alpha, \sim_t\beta, \Gamma \Rightarrow \Delta}{\sim_t(\alpha\vee_t\beta), \Gamma \Rightarrow \Delta}\ (\sim_t\vee_t\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_t\alpha \quad \Gamma \Rightarrow \Delta, \sim_t\beta}{\Gamma \Rightarrow \Delta, \sim_t(\alpha\vee_t\beta)}\ (\sim_t\vee_t\text{r})$$

$$\frac{\sim_t\alpha, \sim_t\beta, \Gamma \Rightarrow \Delta}{\sim_t(\alpha\wedge_f\beta), \Gamma \Rightarrow \Delta}\ (\sim_t\wedge_f\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_t\alpha \quad \Gamma \Rightarrow \Delta, \sim_t\beta}{\Gamma \Rightarrow \Delta, \sim_t(\alpha\wedge_f\beta)}\ (\sim_t\wedge_f\text{r})$$

$$\frac{\sim_t\alpha, \Gamma \Rightarrow \Delta \quad \sim_t\beta, \Gamma \Rightarrow \Delta}{\sim_t(\alpha\vee_f\beta), \Gamma \Rightarrow \Delta}\ (\sim_t\vee_f\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_t\alpha, \sim_t\beta}{\Gamma \Rightarrow \Delta, \sim_t(\alpha\vee_f\beta)}\ (\sim_t\vee_f\text{r}).$$

The $\sim_f$-prefixed logical inference rules of L_8 are of the form:

$$\frac{\Gamma \Rightarrow \Sigma, \sim_f\alpha \quad \sim_f\beta, \Delta \Rightarrow \Pi}{\sim_f(\alpha{\rightarrow}\beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ (\sim_f{\rightarrow}\text{l}) \quad \frac{\sim_f\alpha, \Gamma \Rightarrow \Delta, \sim_f\beta}{\Gamma \Rightarrow \Delta, \sim_f(\alpha{\rightarrow}\beta)}\ (\sim_f{\rightarrow}\text{r})$$

$$\frac{\sim_f\alpha, \sim_f\beta, \Gamma \Rightarrow \Delta}{\sim_f(\alpha\wedge_t\beta), \Gamma \Rightarrow \Delta}\ (\sim_f\wedge_t\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_f\alpha \quad \Gamma \Rightarrow \Delta, \sim_f\beta}{\Gamma \Rightarrow \Delta, \sim_f(\alpha\wedge_t\beta)}\ (\sim_f\wedge_t\text{r})$$

$$\frac{\sim_f\alpha, \Gamma \Rightarrow \Delta \quad \sim_f\beta, \Gamma \Rightarrow \Delta}{\sim_f(\alpha\vee_t\beta), \Gamma \Rightarrow \Delta}\ (\sim_f\vee_t\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_f\alpha, \sim_f\beta}{\Gamma \Rightarrow \Delta, \sim_f(\alpha\vee_t\beta)}\ (\sim_f\vee_t\text{r})$$

$$\frac{\sim_f\alpha, \sim_f\beta, \Gamma \Rightarrow \Delta}{\sim_f(\alpha\wedge_f\beta), \Gamma \Rightarrow \Delta}\ (\sim_f\wedge_f\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_f\alpha \quad \Gamma \Rightarrow \Delta, \sim_f\beta}{\Gamma \Rightarrow \Delta, \sim_f(\alpha\wedge_f\beta)}\ (\sim_f\wedge_f\text{r})$$

$$\frac{\sim_f\alpha, \Gamma \Rightarrow \Delta \quad \sim_f\beta, \Gamma \Rightarrow \Delta}{\sim_f(\alpha\vee_f\beta), \Gamma \Rightarrow \Delta}\ (\sim_f\vee_f\text{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_f\alpha, \sim_f\beta}{\Gamma \Rightarrow \Delta, \sim_f(\alpha\vee_f\beta)}\ (\sim_f\vee_f\text{r}).$$

The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in cut-free L_8. This fact can be shown by induction on α.

An expression $\alpha \Leftrightarrow \beta$ represents two sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$.

Proposition 2.2. *The following sequents are provable in cut-free* L_8*: for any formulas* α *and* β*, and any* $\sim_d \in \{\sim_t\sim_t, \sim_f\sim_f, \sim_t\sim_f, \sim_f\sim_t\}$*,*

1. $\sim_d\alpha \Leftrightarrow \alpha$,
2. $\sim_t(\alpha{\rightarrow}\beta) \Leftrightarrow \alpha \wedge {\sim_t}\beta$,
3. $\sim_t(\alpha{\wedge_t}\beta) \Leftrightarrow {\sim_t}\alpha{\vee_t}{\sim_t}\beta$,
4. $\sim_t(\alpha{\vee_t}\beta) \Leftrightarrow {\sim_t}\alpha{\wedge_t}{\sim_t}\beta$,
5. $\sim_t(\alpha \circ \beta) \Leftrightarrow {\sim_t}\alpha \circ {\sim_t}\beta$ *where* $\circ \in \{\wedge_f, \vee_f\}$,
6. $\sim_f(\alpha \circ \beta) \Leftrightarrow {\sim_f}\alpha \circ {\sim_f}\beta$ *where* $\circ \in \{\rightarrow, \wedge_t, \vee_t\}$,
7. $\sim_f(\alpha{\wedge_f}\beta) \Leftrightarrow {\sim_f}\alpha{\vee_f}{\sim_f}\beta$,
8. $\sim_f(\alpha{\vee_f}\beta) \Leftrightarrow {\sim_f}\alpha{\wedge_f}{\sim_f}\beta$.

Proof. We show some cases.
(1): $\mathrm{L}_8 \vdash {\sim_d}\alpha \Leftrightarrow \alpha$ is shown as follows:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha \Rightarrow \alpha\end{array}}{{\sim_d}\alpha \Rightarrow \alpha}\ ({\sim_d}\mathrm{l}) \qquad \dfrac{\begin{array}{c}\vdots\\ \alpha \Rightarrow \alpha\end{array}}{\alpha \Rightarrow {\sim_d}\alpha}\ ({\sim_d}\mathrm{r}).$$

(7): $\mathrm{L}_8 \vdash {\sim_f}(\alpha{\wedge_f}\beta) \Leftrightarrow {\sim_f}\alpha{\vee_f}{\sim_f}\beta$ is shown as follows:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ {\sim_f}\alpha \Rightarrow {\sim_f}\alpha\end{array}}{{\sim_f}\alpha, {\sim_f}\beta \Rightarrow {\sim_f}\alpha}\ (\text{w-l}) \quad \dfrac{\begin{array}{c}\vdots\\ {\sim_f}\beta \Rightarrow {\sim_f}\beta\end{array}}{{\sim_f}\alpha, {\sim_f}\beta \Rightarrow {\sim_f}\beta}\ (\text{w-l})}{{\sim_f}\alpha, {\sim_f}\beta \Rightarrow {\sim_f}\alpha{\vee_f}{\sim_f}\beta}\ ({\vee_f}\mathrm{r})}{{\sim_f}(\alpha{\wedge_f}\beta) \Rightarrow {\sim_f}\alpha{\vee_f}{\sim_f}\beta}\ ({\sim_f}{\wedge_f}\mathrm{l})$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ {\sim_f}\alpha \Rightarrow {\sim_f}\alpha\end{array}}{{\sim_f}\alpha, {\sim_f}\beta \Rightarrow {\sim_f}\alpha}\ (\text{w-l}) \quad \dfrac{\begin{array}{c}\vdots\\ {\sim_f}\beta \Rightarrow {\sim_f}\beta\end{array}}{{\sim_f}\alpha, {\sim_f}\beta \Rightarrow {\sim_f}\beta}\ (\text{w-l})}{{\sim_f}\alpha, {\sim_f}\beta \Rightarrow {\sim_f}(\alpha{\wedge_f}\beta)}\ ({\sim_f}{\wedge_f}\mathrm{r})}{{\sim_f}\alpha{\vee_f}{\sim_f}\beta \Rightarrow {\sim_f}(\alpha{\wedge_f}\beta)}\ ({\vee_f}\mathrm{l}).$$

□

In order to construct an embedding of L_8 into the propositional positive classical logic, a sequent calculus LK is introduced below.

Definition 2.3. (LK) A sequent calculus LK for the propositional positive classical logic is the $\{\to, \wedge_t, \vee_t\}$-fragment of L_8.

It is known that LK enjoys cut-elimination.

The following translation is regarded as an extension of the translation of Nelson's logics [1, 12] into (positive) intuitionistic logic. For the translation of Nelson's logics, see [6, 14, 17, 18].

Definition 2.4. We fix a countable non-empty set Φ of propositional variables, and define the sets $\Phi_x := \{p_x \mid p \in \Phi\}$ ($x \in \{t, f\}$) of propositional variables. The language $\mathcal{L}^8$ of L_8 is defined using Φ, $\to, \wedge_t, \vee_t, \wedge_f, \vee_f$, $\sim_t$ and $\sim_f$. The language $\mathcal{L}$ of LK is defined using $\Phi \cup \Phi_t \cup \Phi_f$, $\to$, $\wedge_t$ and $\vee_t$.

A mapping f from $\mathcal{L}^8$ to $\mathcal{L}$ is defined inductively as follows.

1. for any $p \in \Phi$, $f(p) := p \in \Phi$ and $f(\sim_x p) := p_x \in \Phi_x$ where $x \in \{t, f\}$,
2. $f(\sim_d \alpha) := f(\alpha)$ where $d \in \{tt, ff, tf, ft\}$,
3. $f(\alpha \circ \beta) := f(\alpha) \circ f(\beta)$ where $\circ \in \{\to, \wedge_t, \vee_t\}$,
4. $f(\alpha \wedge_f \beta) := f(\alpha) \vee_t f(\beta)$,
5. $f(\alpha \vee_f \beta) := f(\alpha) \wedge_t f(\beta)$,
6. $f(\sim_t(\alpha \to \beta)) := f(\alpha) \wedge_t f(\sim_t \beta)$,
7. $f(\sim_t(\alpha \wedge_t \beta)) := f(\sim_t \alpha) \vee_t f(\sim_t \beta)$,
8. $f(\sim_t(\alpha \vee_t \beta)) := f(\sim_t \alpha) \wedge_t f(\sim_t \beta)$,
9. $f(\sim_t(\alpha \wedge_f \beta)) := f(\sim_t \alpha) \vee_t f(\sim_t \beta)$,
10. $f(\sim_t(\alpha \vee_f \beta)) := f(\sim_t \alpha) \wedge_t f(\sim_t \beta)$,
11. $f(\sim_f(\alpha \circ \beta)) := f(\sim_f \alpha) \circ f(\sim_f \beta)$ where $\circ \in \{\to, \wedge_t, \vee_t\}$,
12. $f(\sim_f(\alpha \wedge_f \beta)) := f(\sim_f \alpha) \wedge_t f(\sim_f \beta)$,
13. $f(\sim_f(\alpha \vee_f \beta)) := f(\sim_f \alpha) \vee_t f(\sim_f \beta)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

We then obtain a weak theorem for syntactically embedding L_8 into LK.

Theorem 2.5. (Weak syntactical embedding) *Let* Γ *and* Δ *be sets of formulas in* $\mathcal{L}^8$*, and* f *be the mapping defined in Definition 2.4. Then:*

1. *If* $\mathrm{L}_8 \vdash \Gamma \Rightarrow \Delta$*, then* $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$*.*

2. *If* $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$*, then* $\mathrm{L}_8 - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$*.*

Proof. • (1): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in L_8. We distinguish the cases according to the last inference of P, and show some cases.

Case ($\sim_x p \Rightarrow \sim_x p$ where $x \in \{t, f\}$): The last inference of P is of the form: $\sim_x p \Rightarrow \sim_x p$ with $x \in \{t, f\}$. In this case, we obtain $f(\sim_x p) \Rightarrow f(\sim_x p)$, i.e., $p_x \Rightarrow p_x$ ($p_x \in \Phi_x$), which is an initial sequent of LK.

Case ($\sim_d$→l): The last inference of P is of the form:

$$\frac{\alpha, \Gamma' \Rightarrow \Delta}{\sim_d \alpha, \Gamma' \Rightarrow \Delta}\ (\sim_d \mathrm{l}).$$

By induction hypothesis, we have $\mathrm{LK} \vdash f(\alpha), f(\Gamma') \Rightarrow f(\Delta)$. We then obtain the required fact since $f(\alpha)$ coincides with $f(\sim_d \alpha)$ by the definition of f.

Case ($\sim_f$→l): The last inference of P is of the form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \sim_f \alpha \quad \sim_f \beta, \Gamma_2 \Rightarrow \Delta_2}{\sim_f (\alpha \to \beta), \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\sim_f \to \mathrm{l}).$$

By induction hypothesis, we have $\mathrm{LK} \vdash f(\Gamma_1) \Rightarrow f(\Delta_1), f(\sim_f \alpha)$ and $\mathrm{LK} \vdash f(\sim_f \beta),\ f(\Gamma_2) \Rightarrow f(\Delta_2)$. Then, we obtain:

$$\frac{\begin{array}{c}\vdots \\ f(\Gamma_1) \Rightarrow f(\Delta_1), f(\sim_f \alpha)\end{array} \quad \begin{array}{c}\vdots \\ f(\sim_f \beta), f(\Gamma_2) \Rightarrow f(\Delta_2)\end{array}}{f(\sim_f \alpha) \to f(\sim_f \beta), f(\Gamma_1), f(\Gamma_2) \Rightarrow f(\Delta_1), f(\Delta_2)}\ (\to \mathrm{l})$$

where $f(\sim_f \alpha) \to f(\sim_f \beta)$ coincides with $f(\sim_f (\alpha \to \beta))$ by the definition of f.

Case ($\sim_t$→r): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta', \alpha \quad \Gamma \Rightarrow \Delta', \sim_t \beta}{\Gamma \Rightarrow \Delta', \sim_t (\alpha \to \beta)}\ (\sim_t \to \mathrm{r}).$$

By induction hypothesis, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta'), f(\alpha)$ and LK $\vdash f(\Gamma) \Rightarrow f(\Delta'), f(\sim_t\beta)$. Then, we obtain:

$$\dfrac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta'), f(\alpha)\end{array} \quad \begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta'), f(\sim_t\beta)\end{array}}{f(\Gamma) \Rightarrow f(\Delta'), f(\alpha)\wedge_t f(\sim_t\beta)}\ (\wedge_t\mathrm{r})$$

where $f(\alpha)\wedge_t f(\sim_t\beta)$ coincides with $f(\sim_t(\alpha\rightarrow\beta))$ by the definition of f.

• (2): By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in LK $-$ (cut). We distinguish the cases according to the last inference of Q, and show some cases.

Case ($\wedge_t$left):

Subcase (1): The last inference of Q is of the form:

$$\dfrac{f(\alpha), f(\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\alpha\wedge_t\beta), f(\Gamma') \Rightarrow f(\Delta)}\ (\wedge_t\mathrm{l})$$

where $f(\alpha\wedge_t\beta)$ coincides with $f(\alpha)\wedge_t f(\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{L}_8 \vdash \alpha, \beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha, \beta, \Gamma' \Rightarrow \Delta\end{array}}{\alpha\wedge_t\beta, \Gamma' \Rightarrow \Delta}\ (\wedge_t\mathrm{l}).$$

Subcase (2): The last inference of Q is of the form:

$$\dfrac{f(\alpha), f(\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\alpha\vee_f\beta), f(\Gamma') \Rightarrow f(\Delta)}\ (\wedge_t\mathrm{l})$$

where $f(\alpha\vee_f\beta)$ coincides with $f(\alpha)\wedge_t f(\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{L}_8 \vdash \alpha, \beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha, \beta, \Gamma' \Rightarrow \Delta\end{array}}{\alpha\vee_f\beta, \Gamma' \Rightarrow \Delta}\ (\vee_f\mathrm{l}).$$

Subcase (3): The last inference of Q is of the form:

$$\dfrac{f(\alpha), f(\sim_t\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim_t(\alpha\rightarrow\beta)), f(\Gamma') \Rightarrow f(\Delta)}\ (\wedge_t\mathrm{l})$$

where $f(\sim_t(\alpha{\rightarrow}\beta))$ coincides with $f(\alpha){\wedge_t}f(\sim_t\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{L}_8 \vdash \alpha, \sim_t\beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha, \sim_t\beta, \Gamma' \Rightarrow \Delta\end{array}}{\sim_t(\alpha{\rightarrow}\beta), \Gamma' \Rightarrow \Delta}\ (\sim_t{\rightarrow}\mathrm{l}).$$

Subcase (4): The last inference of Q is of the form:

$$\dfrac{f(\sim_t\alpha), f(\sim_t\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim_t(\alpha{\vee_t}\beta)), f(\Gamma') \Rightarrow f(\Delta)}\ (\wedge_t\mathrm{l})$$

where $f(\sim_t(\alpha{\vee_t}\beta))$ coincides with $f(\sim_t\alpha){\wedge_t}f(\sim_t\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{L}_8 \vdash \sim_t\alpha, \sim_t\beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\dfrac{\begin{array}{c}\vdots\\ \sim_t\alpha, \sim_t\beta, \Gamma' \Rightarrow \Delta\end{array}}{\sim_t(\alpha{\vee_t}\beta), \Gamma' \Rightarrow \Delta}\ (\sim_t{\vee_t}\mathrm{l}).$$

Subcase (5): The last inference of Q is of the form:

$$\dfrac{f(\sim_f\alpha), f(\sim_f\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim_f(\alpha{\wedge_t}\beta)), f(\Gamma') \Rightarrow f(\Delta)}\ (\wedge_t\mathrm{l})$$

where $f(\sim_f(\alpha{\wedge_t}\beta))$ coincides with $f(\sim_f\alpha){\wedge_t}f(\sim_f\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{L}_8 \vdash \sim_f\alpha, \sim_f\beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\dfrac{\begin{array}{c}\vdots\\ \sim_f\alpha, \sim_f\beta, \Gamma' \Rightarrow \Delta\end{array}}{\sim_f(\alpha{\wedge_t}\beta), \Gamma' \Rightarrow \Delta}\ (\sim_f{\wedge_t}\mathrm{l}).$$

Subcase (6): The last inference of Q is of the form:

$$\dfrac{f(\sim_t\alpha), f(\sim_t\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim_t(\alpha{\wedge_f}\beta)), f(\Gamma') \Rightarrow f(\Delta)}\ (\wedge_t\mathrm{l})$$

where $f(\sim_t(\alpha{\wedge_f}\beta))$ coincides with $f(\sim_t\alpha){\wedge_t}f(\sim_t\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{L}_8 \vdash \sim_t\alpha, \sim_t\beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\dfrac{\begin{array}{c}\vdots\\ \sim_t\alpha, \sim_t\beta, \Gamma' \Rightarrow \Delta\end{array}}{\sim_t(\alpha{\wedge_f}\beta), \Gamma' \Rightarrow \Delta}\ (\sim_t{\wedge_f}\mathrm{l}).$$

Subcase (7): The last inference of Q is of the form:

$$\frac{f(\sim_f\alpha), f(\sim_f\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim_f(\alpha\wedge_t\beta)), f(\Gamma') \Rightarrow f(\Delta)}\ (\wedge_t\mathrm{l})$$

where $f(\sim_f(\alpha\wedge_t\beta))$ coincides with $f(\sim_f\alpha)\wedge_t f(\sim_f\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{L}_8 \vdash \sim_f\alpha, \sim_f\beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\frac{\begin{array}{c}\vdots\\ \sim_f\alpha, \sim_f\beta, \Gamma' \Rightarrow \Delta\end{array}}{\sim_f(\alpha\wedge_t\beta), \Gamma' \Rightarrow \Delta}\ (\sim_f\wedge_t\mathrm{l}).$$

Subcase (8): The last inference of Q is of the form:

$$\frac{f(\sim_f\alpha), f(\sim_f\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim_f(\alpha\wedge_f\beta)), f(\Gamma') \Rightarrow f(\Delta)}\ (\wedge_t\mathrm{l})$$

where $f(\sim_f(\alpha\wedge_f\beta))$ coincides with $f(\sim_f\alpha)\wedge_t f(\sim_f\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{L}_8 \vdash \sim_f\alpha, \sim_f\beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\frac{\begin{array}{c}\vdots\\ \sim_f\alpha, \sim_f\beta, \Gamma' \Rightarrow \Delta\end{array}}{\sim_f(\alpha\wedge_f\beta), \Gamma' \Rightarrow \Delta}\ (\sim_f\wedge_f\mathrm{l}).$$

□

Using Theorem 2.5, we obtain the following cut-elimination theorem for L_8.

Theorem 2.6 (Cut-elimination)**.** *The rule* (cut) *is admissible in cut-free* L_8.

Proof. Suppose $\mathrm{L}_8 \vdash \Gamma \Rightarrow \Delta$. Then, we have $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 2.5 (1), and hence $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the well-known cut-elimination theorem for LK. By Theorem 2.5 (2), we obtain $\mathrm{L}_8 - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$. □

Using Theorem 2.5 and the cut-elimination theorem for LK, we obtain the following (strong) syntactical embedding theorem.

Theorem 2.7. (Syntactical embedding) *Let* Γ *and* Δ *be sets of formulas in* $\mathcal{L}^8$, *and* f *be the mapping defined in Definition 2.4. Then:*

1. $\mathrm{L}_8 \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$.

2. $\mathrm{L}_8 - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. • (1). ($\Longrightarrow$): By Theorem 2.5 (1). ($\Longleftarrow$): Suppose $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$. Then we have $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the well-known cut-elimination theorem for LK. We thus obtain $\mathrm{L}_8 - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ by Theorem 2.5 (2). Therefore we have $\mathrm{L}_8 \vdash \Gamma \Rightarrow \Delta$.

• (2). ($\Longrightarrow$): Suppose $\mathrm{L}_8 - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$. Then we have $\mathrm{L}_8 \vdash \Gamma \Rightarrow \Delta$. We then obtain $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 2.5 (1). Therefore we obtain $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. ($\Longleftarrow$): By Theorem 2.5 (2). □

Theorem 2.8. (Decidability) L_8 *is decidable.*

Proof. By decidability of LK, for each α, it is possible to decide if $f(\alpha)$ is provable in LK. Then, by Theorem 2.7, L_8 is decidable. □

Definition 2.9. Let $\sharp$ be a negation (-like) connective. A sequent calculus L is called *explosive* with respect to $\sharp$ if for any formulas α and β, the sequent $\alpha, \sharp\alpha \Rightarrow \beta$ is provable in L. It is called *paraconsistent* with respect to $\sharp$ if it is not explosive with respect to $\sharp$.

Theorem 2.10. (Paraconsistency) *Let* $\sharp$ *be* $\sim_t$ *or* $\sim_f$. *Then,* L_8 *is paraconsistent with respect to* $\sharp$.

Proof. Consider a sequent $p, \sharp p \Rightarrow q$ where p and q are distinct propositional variables. Then, the unprovability of this sequent is guaranteed by using Theorem 2.6. □

3. Semantics

Definition 3.1. (Semantics for L_8) Triplet valuations v^n, v^t and v^f are mappings from the set of all propositional variables to the set $\{t, f\}$ of truth values. The triplet valuations v^n, v^t and v^f are extended to mappings from the set of all formulas to $\{t, f\}$ by the following clauses.

1. $v^n(\alpha \to \beta) = t$ iff $v^n(\alpha) = f$ or $v^n(\beta) = t$,
2. $v^n(\alpha \wedge_t \beta) = t$ iff $v^n(\alpha) = t$ and $v^n(\beta) = t$,
3. $v^n(\alpha \vee_t \beta) = t$ iff $v^n(\alpha) = t$ or $v^n(\beta) = t$,
4. $v^n(\alpha \wedge_f \beta) = t$ iff $v^n(\alpha) = t$ or $v^n(\beta) = t$,
5. $v^n(\alpha \vee_f \beta) = t$ iff $v^n(\alpha) = t$ and $v^n(\beta) = t$,
6. $v^n(\sim_t \alpha) = t$ iff $v^t(\alpha) = t$,
7. $v^n(\sim_f \alpha) = t$ iff $v^f(\alpha) = t$,
8. $v^t(\alpha \to \beta) = t$ iff $v^n(\alpha) = t$ and $v^t(\beta) = t$,
9. $v^t(\alpha \wedge_t \beta) = t$ iff $v^t(\alpha) = t$ or $v^t(\beta) = t$,
10. $v^t(\alpha \vee_t \beta) = t$ iff $v^t(\alpha) = t$ and $v^t(\beta) = t$,
11. $v^t(\alpha \wedge_f \beta) = t$ iff $v^t(\alpha) = t$ and $v^t(\beta) = t$,
12. $v^t(\alpha \vee_f \beta) = t$ iff $v^t(\alpha) = t$ or $v^t(\beta) = t$,
13. $v^t(\sim_t \alpha) = t$ iff $v^n(\alpha) = t$,
14. $v^t(\sim_f \alpha) = t$ iff $v^n(\alpha) = t$,
15. $v^f(\alpha \to \beta) = t$ iff $v^f(\alpha) = f$ or $v^f(\beta) = t$,
16. $v^f(\alpha \wedge_t \beta) = t$ iff $v^f(\alpha) = t$ and $v^f(\beta) = t$,
17. $v^f(\alpha \vee_t \beta) = t$ iff $v^f(\alpha) = t$ or $v^f(\beta) = t$,
18. $v^f(\alpha \wedge_f \beta) = t$ iff $v^f(\alpha) = t$ or $v^f(\beta) = t$,
19. $v^f(\alpha \vee_f \beta) = t$ iff $v^f(\alpha) = t$ and $v^f(\beta) = t$,
20. $v^f(\sim_t \alpha) = t$ iff $v^n(\alpha) = t$,
21. $v^f(\sim_f \alpha) = t$ iff $v^n(\alpha) = t$.

A formula α is called L_8-*valid* if $v^n(\alpha) = t$ holds for any triplet valuations v^n, v^t and v^f.

Note that v^n behaves classically with respect to the classical connectives $\wedge_t$, $\vee_t$ and $\rightarrow$. Moreover, note that the following conditions hold: For any $* \in \{n, t, f\}$,

1. $v^*(\alpha \wedge_t \beta) = v^*(\alpha \vee_f \beta)$,
2. $v^*(\alpha \vee_t \beta) = v^*(\alpha \wedge_f \beta)$,
3. $v^n(\alpha) = v^t({\sim_t}\alpha) = v^f({\sim_f}\alpha) = v^t({\sim_f}\alpha) = v^f({\sim_t}\alpha)$,
4. $v^t(\alpha) = v^n({\sim_t}\alpha)$,
5. $v^f(\alpha) = v^n({\sim_f}\alpha)$.

This semantics implies an 8-valued semantics since the following eight $(= 2^3)$ cases can be considered for the triplet valuations v^n, v^t and v^f: for any formula α,

1. $v^n(\alpha) = t$, $v^t(\alpha) = t$, $v^f(\alpha) = t$,
2. $v^n(\alpha) = t$, $v^t(\alpha) = t$, $v^f(\alpha) = f$,
3. $v^n(\alpha) = t$, $v^t(\alpha) = f$, $v^f(\alpha) = t$,
4. $v^n(\alpha) = t$, $v^t(\alpha) = f$, $v^f(\alpha) = f$,
5. $v^n(\alpha) = f$, $v^t(\alpha) = t$, $v^f(\alpha) = t$,
6. $v^n(\alpha) = f$, $v^t(\alpha) = t$, $v^f(\alpha) = f$,
7. $v^n(\alpha) = f$, $v^t(\alpha) = f$, $v^f(\alpha) = t$,
8. $v^n(\alpha) = f$, $v^t(\alpha) = f$, $v^f(\alpha) = f$.

In order to show a theorem for semantically embedding L_8 into LK, we present the standard semantics for LK.

Definition 3.2. (Semantics for LK) A valuation v is a mapping from the set of all propositional variables to the set $\{t, f\}$ of truth values. The valuation v is extended to the mapping from the set of all formulas to $\{t, f\}$ by

1. $v(\alpha \wedge_t \beta) = t$ iff $v(\alpha) = t$ and $v(\beta) = t$,

2. $v(\alpha \vee_t \beta) = t$ iff $v(\alpha) = t$ or $v(\beta) = t$,

3. $v(\alpha \rightarrow \beta) = t$ iff $v(\alpha) = f$ or $v(\beta) = t$.

A formula α is called *LK-valid* if $v(\alpha) = t$ holds for any valuations v.

The following completeness theorem for LK is well-known: A formula α is LK-valid iff LK $\vdash\ \Rightarrow \alpha$.

Lemma 3.3. *Let f be the mapping defined in Definition 2.4. For any triplet valuations v^n, v^t and v^f, we can construct a valuation v such that for any formula α,*

1. *$v^n(\alpha) = t$ iff $v(f(\alpha)) = t$,*

2. *$v^t(\alpha) = t$ iff $v(f(\sim_t \alpha)) = t$,*

3. *$v^f(\alpha) = t$ iff $v(f(\sim_f \alpha)) = t$.*

Proof. Let Φ be a set of propositional variables, and Φ_x be the sets $\{p_x \mid p \in \Phi\}$ $(x \in \{t, f\})$ of propositional variables. Suppose that v^n, v^t and v^f are triplet valuations. Suppose that v is a mapping from $\Phi \cup \Phi_t \cup \Phi_f$ to $\{t, f\}$ such that

1. $v^n(p) = t$ iff $v(p) = t$,

2. $v^t(p) = t$ iff $v(p_t) = t$,

3. $v^f(p) = t$ iff $v(p_f) = t$.

Then, the lemma is proved by (simultaneous) induction on α.

• Base step:

Case $\alpha \equiv p$ where p is a propositional variable: For v^n, $v^n(p) = t$ iff $v(p) = t$ (by the assumption) iff $v(f(p)) = t$ (by the definition of f). For v^t, $v^t(p) = t$ iff $v(p_t) = t$ (by the assumption) iff $v(f(\sim_t p)) = t$ (by the definition of f). For v^f, $v^f(p) = t$ iff $v(p_f) = t$ (by the assumption) iff $v(f(\sim_f p)) = t$ (by the definition of f).

• Induction step:

Case $\alpha \equiv \sim_t \beta$: For v^n, $v^n(\sim_t \beta) = t$ iff $v^t(\beta) = t$ iff $v(f(\sim_t \beta)) = t$ (by induction hypothesis). For v^t, $v^t(\sim_t \beta) = t$ iff $v^n(\beta) = t$ iff $v(f(\beta)) = t$ (by induction hypothesis) iff $v(f(\sim_t \sim_t \beta)) = t$ (by the definition of f). For

v^f, $v^f(\sim_t\beta) = t$ iff $v^n(\beta) = t$ iff $v(f(\beta)) = t$ (by induction hypothesis) iff $v(f(\sim_f\sim_t\beta)) = t$ (by the definition of f).

Case $\alpha \equiv \sim_f\beta$: Similar to Case $\alpha \equiv \sim_t\beta$.

Case $\alpha \equiv \beta\wedge_t\gamma$: For v^n, $v^n(\beta\wedge_t\gamma) = t$ iff $v^n(\beta) = t$ and $v^n(\gamma) = t$ iff $v(f(\beta)) = t$ and $v(f(\gamma)) = t$ (by induction hypothesis) iff $v(f(\beta)\wedge_t f(\gamma)) = t$ iff $v(f(\beta\wedge_t\gamma))) = t$ (by the definition of f). For v^t, $v^t(\beta\wedge_t\gamma) = t$ iff $v^t(\beta) = t$ or $v^t(\gamma) = t$ iff $v(f(\sim_t\beta)) = t$ or $v(f(\sim_t\gamma)) = t$ (by induction hypothesis) iff $v(f(\sim_t\beta)\vee_t f(\sim_t\gamma)) = t$ iff $v(f(\sim_t(\beta\wedge_t\gamma))) = t$ (by the definition of f). For v^f, $v^f(\beta\wedge_t\gamma) = t$ iff $v^f(\beta) = t$ and $v^f(\gamma) = t$ iff $v(f(\sim_f\beta)) = t$ and $v(f(\sim_f\gamma)) = t$ (by induction hypothesis) iff $v(f(\sim_f\beta)\wedge_t f(\sim_f\gamma)) = t$ iff $v(f(\sim_f(\beta\wedge_t\gamma))) = t$ (by the definition of f).

Case $\alpha \equiv \beta\vee_t\gamma$: Similar to Case $\alpha \equiv \beta\wedge_t\gamma$.

Case $\alpha \equiv \beta\wedge_f\gamma$: Similar to Case $\alpha \equiv \beta\vee_t\gamma$.

Case $\alpha \equiv \beta\vee_f\gamma$: Similar to Case $\alpha \equiv \beta\wedge_t\gamma$.

Case $\alpha \equiv \beta\rightarrow\gamma$: For v^n, $v^n(\beta\rightarrow\gamma) = t$ iff $v^n(\beta) = f$ or $v^n(\gamma) = t$ iff $v(f(\beta)) = f$ or $v(f(\gamma)) = t$ (by induction hypothesis) iff $v(f(\beta)\rightarrow f(\gamma)) = t$ iff $v(f(\beta\rightarrow\gamma)) = t$ (by the definition of f). For v^t, $v^t(\beta\rightarrow\gamma) = t$ iff $v^n(\beta) = t$ and $v^t(\gamma) = t$ iff $v(f(\beta)) = t$ and $v(f(\sim_t\gamma)) = t$ (by induction hypothesis) iff $v(f(\beta)\wedge_t f(\sim_t\gamma)) = t$ iff $v(f(\sim_t(\beta\rightarrow\gamma))) = t$ (by the definition of f). For v^f, $v^f(\beta\rightarrow\gamma) = t$ iff $v^f(\beta) = f$ or $v^f(\gamma) = t$ iff $v(f(\sim_f\beta)) = f$ or $v(f(\sim_f\gamma)) = t$ (by induction hypothesis) iff $v(f(\sim_f\beta)\rightarrow f(\sim_f\gamma)) = t$ iff $v(f(\sim_f(\beta\rightarrow\gamma))) = t$ (by the definition of f). □

Lemma 3.4. *Let f be the mapping defined in Definition 2.4. For any valuations v, we can construct triplet valuations v^n, v^t and v^f such that for any formula α,*

1. $v^n(\alpha) = t$ *iff* $v(f(\alpha)) = t$,
2. $v^t(\alpha) = t$ *iff* $v(f(\sim_t\alpha)) = t$,
3. $v^f(\alpha) = t$ *iff* $v(f(\sim_f\alpha)) = t$.

Proof. Similar to the proof of Lemma 3.3. □

Theorem 3.5. (Semantical embedding) *Let f be the mapping defined in Definition 2.4. For any formula α,*

α is L_8-valid iff $f(\alpha)$ is LK-valid.

Proof. By Lemmas 3.3 and 3.4. □

Theorem 3.6. (Completeness) *For any formula* α*,*

$\mathrm{L}_8 \vdash \Rightarrow \alpha$ *iff* α *is* L_8*-valid.*

Proof. We have:

α is L_8-valid

iff $f(\alpha)$ is LK-valid (by Theorem 3.5)

iff LK $\vdash \Rightarrow f(\alpha)$ (by the completeness theorem for LK)

iff $\mathrm{L}_8 \vdash \Rightarrow \alpha$ (by Theorem 2.7).

□

4. Concluding remarks

In this paper, the 8-valued paraconsistent logic L_8 instead of the standard 4-valued and 16-valued logics was introduced as a Gentzen-type sequent calculus. The logic L_8 is an extension of Belnap's and Dunn's 4-valued logics, and is a simplification of Shramko-Wansing's 16-valued logics. A triplet valuation semantics, which has three kinds of valuations v^n, v^t and v^f, was introduced for L_8, and the completeness theorem for this semantics was proved using two theorems for syntactically and semantically embedding L_8 into positive classical logic. The cut-elimination theorem for this logic was proved using a theorem for syntactically embedding L_8 into positive classical logic. This logic L_8 was also shown to be decidable and paraconsistent.

Some related results which have been developed by us are briefly reviewed below. A constructive and paraconsistent temporal logic was introduced in [8]. This paper [8] introduces some Gentzen-type and display sequent calculi for the proposed temporal logic. Some sequent calculi for Nelson's paraconsistent 4-valued logic N4 were studied in [11]. This paper [11] shows that a unified embedding-based method is useful for proving some theorems for N4. A paraconsistent 4-valued linear-time temporal logic in a similar setting as in N4 was studied in [9]. The 4-valued temporal logic introduced in [9] can be modified to the 8-valued setting proposed in the present paper.

Acknowledgments. We would like to thank the anonymous referees for their valuable comments.

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Teikyo University,
Faculty of Science and Engineering,
Department of Human Information Systems,
Toyosatodai 1-1, Utsunomiya-shi, Tochigi 320-8551, Japan.
drnkamide08@kpd.biglobe.ne.jp