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INTERPOLATION THEOREMS FOR SOME VARIANTS OF LTL

A b s t r a c t. It is known that Craig interpolation theorem does not hold for LTL. In this paper, Craig interpolation theorems are shown for some fragments and extensions of LTL. These theorems are simply proved based on an embedding-based proof method with Gentzen-type sequent calculi. Maksimova separation theorems (Maksimova principle of variable separation) are also shown for these LTL variants.

1. Introduction

Linear-time temporal logic (LTL) has been used as a base logic for verifying and specifying concurrent systems [4, 21]. By the virtue of the simple linear-time formalism, LTL is known to be one of the most useful modal logics in Computer Science. A number of model checking tools have been

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constructed based on LTL [4]. Some Gentzen-type sequent calculi, which are a useful basis for automatic theorem proving, have been developed for LTL (see e.g., [2, 8, 12]). A sequent calculus LT_ω for LTL was introduced by Kawai [12], and a 2-sequent calculus $2\mathrm{S}\omega$ for LTL, which is a natural extension of the usual sequent calculus, was introduced by Baratella and Masini [2]. A direct syntactical equivalence between Kawai's LT_ω and Baratella-Masini's $2\mathrm{S}\omega$ was shown by introducing the translation functions that preserve cut-free proofs of these calculi [7].

It is known that LTL can naturally be defined as a Kripke semantics and the completeness theorem with respect to the semantics holds (see e.g.[2, 12] for the completeness). But, Craig interpolation theorem is usually discussed as a syntactical property for the provability of logic. Thus, in this paper, we define some variants of LTL by a syntactical way using Gentzen-type sequent calculi, and discuss the syntactically stated Craig interpolation theorem for these variants.

Craig interpolation theorem for classical logic was originally shown by Craig [3], and this theorem and its variants have been studied by many researchers for a number of non-classical logics. Craig interpolation theorems have many applications such as modular ontologies and model checking. A strong version of Craig interpolation theorems is known to be useful for extracting modular ontologies from a given large-scale ontology [13]. Craig interpolation theorems for some temporal logics including some variants of LTL have been well-studied for applications to model checking [19]. On the other hand, it was discussed in [17, 5] that Craig interpolation theorems do not hold for LTL and some of its fragments. It was proved by Gheerbrant and ten Cate [5] that Craig interpolation theorems hold for the next-time only fragment of LTL and for an extended LTL with a fixpoint operator. The proof by them was semantical.

Firstly in the present paper, an alternative embedding-based proof of the Craig interpolation theorem for the next-time only fragment (called here a *next-time LTL*) is given using a theorem for embedding the next-time LTL into classical logic. Next, it is shown that Craig interpolation theorem holds for an extended LTL with both infinitary conjunction and infinitary disjunction (called here an *infinitary LTL*). This theorem is proved using a theorem for embedding the infinitary LTL into the countable fragment of infinitary logic. Moreover, it is shown that Craig interpolation theorem holds for some paraconsistent variants of the next-time LTL and the infinitary LTL. *Maksimova separation theorems* (*Maksimova principle of variable separation*) [16] are also shown for these LTL variants. The proofs of these results for Craig interpolation and Maksimova separation are given based on an embedding-based proof method with Gentzen-type sequent calculi.

Some remarks are addressed as follows. It was shown in [8] that LT_ω is embeddable into a sequent calculus LK_ω for countable infinitary logic. An embedding-based cut-elimination proof for LT_ω and its infinitary extension ILT_ω (a sequent calculus for the infinitary LTL) was shown in [8]. It was proved in [14] that Craig interpolation theorem holds for the countable infinitary logic. It was also shown in [18] that Craig interpolation theorem does not hold for other (uncountable) infinitary logics. The sequent calculus (logic) ILT_ω (inifinitary LTL), which is regarded as a natural and simple extension of LT_ω, is a very expressive (undecidable) logic, which not only extends the linear-time μ-calculus, but also characterizes ω-words up to isomorphism.

The contents of this paper are then summarized as follows.

In Section 2, Kawai's sequent calculus LT_ω for LTL and Gentzen's sequent calculus LK for classical logic are presented, and the Craig interpolation theorem for LK is reviewed.

In Section 3, it is shown that Craig interpolation theorem holds for the next-time only fragment LT_x of LT_ω. This theorem is proved using a theorem for embedding LT_x into LK.

In Section 4, it is shown that Craig interpolation theorem holds for an infinitary extension ILT_ω of LT_ω. This theorem is proved using a theorem for embedding ILT_ω into a sequent calculus LK_ω for the countable infinitary logic.

In Section 5, it is shown that Craig interpolation theorem holds for a paraconsistent extension PLT_x of LT_x. PLT_x is regarded as a modified fragment of the sequent calculus for the paraconsistent LTL proposed in [11]. The Craig interpolation theorem for PLT_x is proved using a theorem for embedding PLT_x into LT_x, and based on a proof method proposed in [9] for a paraconsistent logic.

In Section 6, it is shown that Craig interpolation theorem holds for a paraconsistent extension PILT_ω of ILT_ω. This theorem is proved, in a similar way as in Section 5, using a theorem for embedding PILT_ω into ILT_ω.

In Section 7, it is shown that Maksimova separation theorem holds for the constant-free fragments of LT_x, ILT_ω, PLT_x and PILT_ω.

In Section 8, it is remarked that Craig interpolation theorem holds for a bounded-time version $\mathrm{BLT}[l]$ of LT_ω.

In Section 9, this paper is concluded, and some remarks are given.

2. Preliminaries

Formulas of LTL are constructed from countably many propositional variables, $\top$ (truth constant), $\bot$ (falsity constant), $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), G (globally), F (eventually) and X (next). Lower-case letters $p, q, ...$ are used to denote propositional variables, Greek lower-case letters $\alpha, \beta, ...$ are used to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) sets of formulas. For any $\sharp \in \{\mathrm{G}, \mathrm{F}, \mathrm{X}\}$, an expression $\sharp\Gamma$ is used to denote the set $\{\sharp\gamma \mid \gamma \in \Gamma\}$. The symbol $\equiv$ is used to denote the equality of symbols. The symbol ω is used to represent the set of natural numbers. Lower-case letters i, j and k are used to denote any natural numbers. An expression $\mathrm{X}^i\alpha$ is defined inductively by $\mathrm{X}^0\alpha \equiv \alpha$ and $\mathrm{X}^{i+1}\alpha \equiv \mathrm{X}^i\mathrm{X}\alpha$. An expression of the form $\Gamma \Rightarrow \Delta$ is called a *sequent*. An expression $L \vdash S$ is used to denote the fact that a sequent S is provable in a sequent calculus L. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: for any instance

$$\frac{S_1 \cdots S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$.

Kawai's sequent calculus LT_ω [12] for LTL is presented below. This formulation has some small modifications from the original one (see [7] for the detail).

Definition 2.1 (LT_ω)**.** The initial sequents of LT_ω are of the form: for any propositional variable p,

$$\mathrm{X}^i p \Rightarrow \mathrm{X}^i p \qquad \Rightarrow \mathrm{X}^i\top \qquad \mathrm{X}^i\bot \Rightarrow .$$

The structural rules of LT_ω are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\mathrm{cut})$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ \text{(we-left)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ \text{(we-right)}.$$

The logical inference rules of LT_ω are of the form:

$$\frac{\Gamma \Rightarrow \Sigma, \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Delta \Rightarrow \Pi}{\mathrm{X}^i(\alpha{\rightarrow}\beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ ({\rightarrow}\text{left}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha{\rightarrow}\beta)}\ ({\rightarrow}\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left1}) \qquad \frac{\mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \wedge \beta)}\ (\wedge\text{right}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \vee \beta)}\ (\vee\text{right1}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \vee \beta)}\ (\vee\text{right2})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Gleft}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}).$$

Some remarks are given as follows.

1. (Gright) and (Fleft) have infinite premises.

2. The sequents of the form: $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ for any formula α are provable in LT_ω. This fact can be proved by induction on α.

3. Cut-elimination theorem holds for LT_ω [12], and Craig interpolation theorem does not hold for LT_ω [17, 5].

A sequent calculus LK for classical logic can be defined as a subsystem of LT_ω. Cut-elimination and Craig interpolation theorems hold for LK (see e.g., [23, 3, 3]).

Definition 2.2 (LK)**.** LK is obtained from LT_ω by deleting {(Gleft), (Gright), (Fleft), (Fright)} and replacing X^i with X^0. The modified inference rules for LK by replacing X^i with X^0 are denoted by using "LK" as a superscript, e.g., $({\rightarrow}\text{left}^{LK})$.

An expression $V(\alpha)$ denotes the set of all propositional variables in a formula α

Proposition 2.3 (Craig interpolation theorem for LK)**.** *For any formulas α and β, if* LK $\vdash \alpha \Rightarrow \beta$, *then there exists a formula γ such that*

1. LK $\vdash \alpha \Rightarrow \gamma$ *and* LK $\vdash \gamma \Rightarrow \beta$,
2. $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

3. Next-time LTL

The next-time only fragment LT_x of LT_ω is introduced below.

Definition 3.1 (LT_x)**.** The $\{\mathrm{G}, \mathrm{F}\}$-free fragment LT_x of LT_ω is obtained from LT_ω by deleting {(Gleft), (Gright), (Fleft), (Fright)}.

Definition 3.2. We fix a countable non-empty set Φ of propositional variables and define the sets $\Phi_i := \{p_i \mid p \in \Phi\}$ $(i \in \omega)$ of propositional variables where $p_0 := p \in \Phi$, i.e., $\Phi_0 = \Phi$. The language $\mathcal{L}_{\mathrm{LT}_x}$ of LT_x is defined using $\Phi, \top, \bot, \neg, \rightarrow, \wedge, \vee$ and X. The language $\mathcal{L}_{\mathrm{LK}}$ of LK is defined using $\bigcup_{i\in\omega} \Phi_i$, $\top, \bot, \neg, \rightarrow, \wedge$ and $\vee$.

A mapping f from $\mathcal{L}_{\mathrm{LT}_x}$ to $\mathcal{L}_{\mathrm{LK}}$ is defined by the following clauses:

1. $f(\mathrm{X}^i p) := p_i \in \Phi_i$ for any $p \in \Phi$ (esp. $f(p) := p \in \Phi$),
2. $f(\mathrm{X}^i \sharp) := \sharp$ where $\sharp \in \{\top, \bot\}$,
3. $f(\mathrm{X}^i \neg\alpha) := \neg f(\mathrm{X}^i\alpha)$,
4. $f(\mathrm{X}^i(\alpha \sharp \beta)) := f(\mathrm{X}^i\alpha) \sharp f(\mathrm{X}^i\beta)$ where $\sharp \in \{\rightarrow, \wedge, \vee\}$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

Lemma 3.3. *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{LT}_x}$, and f be the mapping defined in Definition 3.2. Then:*

1. *if* $\mathrm{LT}_x \vdash \Gamma \Rightarrow \Delta$, *then* LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$,
2. *if* LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$, *then* LT_x $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$.

Proof. Straightforward. Similar to the proof for the corresponding embedding theorem of LT_ω in [8]. □

The cut-elimination theorem for LT_x can be obtained using Lemma 3.3.

Theorem 3.4 (Cut-elimination for LT_x)**.** *The rule* (cut) *is admissible in cut-free* LT_x*.*

Proof. Suppose $\mathrm{LT}_x \vdash \Gamma \Rightarrow \Delta$. Then we have $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Lemma 3.3 (1), and hence $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. By Lemma 3.3 (2), we obtain $\mathrm{LT}_x - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$. □

Theorem 3.5 (Embedding from LT_x into LK)**.** *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{LT}_x}$, and f be the mapping defined in Definition 3.2. Then:*

$$\mathrm{LT}_x \vdash \Gamma \Rightarrow \Delta \text{ iff } \mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta).$$

Proof. $(\Longrightarrow)$: By Lemma 3.3 (1). $(\Longleftarrow)$: Suppose $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$. Then, we have $\mathrm{LK} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ by the cut-elimination theorem for LK. By Lemma 3.3 (2), we obtain $\mathrm{LT}_x - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$. Therefore we obtain $\mathrm{LT}_x \vdash \Gamma \Rightarrow \Delta$. □

Lemma 3.6. *Let f be the mapping defined in Definition 3.2. For any $i \in \omega$, any propositional variable p in $\mathcal{L}_{\mathrm{LT_x}}$ and any formula α in $\mathcal{L}_{\mathrm{LT_x}}$,*

$$p \in V(\mathrm{X}^i\alpha) \text{ iff } p_j \in V(f(\mathrm{X}^i\alpha)) \text{ for some } j \in \omega.$$

Proof. By induction on α.

• Base step. It is obvious since $p \in V(\mathrm{X}^i p)$ and $p_i = f(\mathrm{X}^i p) \in V(f(\mathrm{X}^i p))$ hold.

• Induction step. We show only the following cases.

1. Case $(\alpha \equiv \mathrm{X}\beta)$. By induction hypothesis, we have the required fact: $p \in V(\mathrm{X}^{i+1}\beta)$ iff $p_j \in V(f(\mathrm{X}^{i+1}\beta))$ for some $j \in \omega$.
2. Case $(\alpha \equiv \beta \wedge \gamma)$. We obtain: $p \in V(\mathrm{X}^i(\beta \wedge \gamma))$

   iff $p \in V(\mathrm{X}^i\beta)$ or $p \in V(\mathrm{X}^i\gamma)$

   iff $[p_j \in V(f(\mathrm{X}^i\beta))$ for some $j \in \omega]$ or $[p_k \in V(f(\mathrm{X}^i\gamma))$ for some $k \in \omega]$ (by induction hypothesis)

iff $p_l \in V(f(\mathrm{X}^i\beta) \wedge f(\mathrm{X}^i\gamma))$ with $l \in \{j, k\}$
iff $p_l \in V(f(\mathrm{X}^i(\beta \wedge \gamma)))$ for some $l \in \omega$ (by the definition of f).

□

Lemma 3.7. *Let f be the mapping defined in Definition 3.2. For any formulas α and β in $\mathcal{L}_{\mathrm{LT_x}}$,*

if $V(f(\alpha)) \subseteq V(f(\beta))$, then $V(\alpha) \subseteq V(\beta)$.

Proof. Suppose $p \in V(\alpha)$. Then, we obtain $p_j \in V(f(\alpha))$ for some $j \in \omega$ by Lemma 3.6 taking 0 for i. By the assumption, we obtain $p_j \in V(f(\beta))$ for some $j \in \omega$, and hence obtain $p \in V(\beta)$ by Lemma 3.6 taking 0 for i. □

Theorem 3.8 (Craig interpolation theorem for LT_x)**.** *For any formulas α and β, if $\mathrm{LT}_x \vdash \alpha \Rightarrow \beta$, then there exists a formula γ such that*

1. $\mathrm{LT}_x \vdash \alpha \Rightarrow \gamma$ *and* $\mathrm{LT}_x \vdash \gamma \Rightarrow \beta$,
2. $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

Proof. Suppose $\mathrm{LT}_x \vdash \alpha \Rightarrow \beta$. Then, we have $\mathrm{LK} \vdash f(\alpha) \Rightarrow f(\beta)$ by Theorem 3.5. By Proposition 2.3, we have the following: there exists a formula γ of LK such that

1. $\mathrm{LK} \vdash f(\alpha) \Rightarrow \gamma$ and $\mathrm{LK} \vdash \gamma \Rightarrow f(\beta)$,
2. $V(\gamma) \subseteq V(f(\alpha)) \cap V(f(\beta))$.

Let $\mathcal{L}^*_{\mathrm{LK}}$ be $\mathcal{L}_{\mathrm{LK}} - \bigcup_{i \in \omega - \{0\}} \Phi_i$. We now consider the following two cases for the formula γ:

1. γ is in $\mathcal{L}^*_{\mathrm{LK}}$,
2. γ is in $\bigcup_{i \in \omega - \{0\}} \Phi_i$.

We firstly consider the former case, and then, consider the latter case.

• Case (γ is in $\mathcal{L}^*_{\mathrm{LK}}$): In this case, we have the fact $\gamma = f(\gamma)$ for any $\gamma \in \mathcal{L}^*_{\mathrm{LK}} \subseteq \mathcal{L}_{\mathrm{LK}}$. This fact can be shown by induction on γ. Thus we have the following: there exists a formula γ in $\mathcal{L}^*_{\mathrm{LK}}$ such that

1. $\mathrm{LK} \vdash f(\alpha) \Rightarrow f(\gamma)$ and $\mathrm{LK} \vdash f(\gamma) \Rightarrow f(\beta)$,

2. $V(f(\gamma)) \subseteq V(f(\alpha)) \cap V(f(\beta))$.

By Theorem 3.5, we thus obtain the following: there exists a formula γ such that

1. $\mathrm{LT}_x \vdash \alpha \Rightarrow \gamma$ and $\mathrm{LT}_x \vdash \gamma \Rightarrow \beta$,
2. $V(f(\gamma)) \subseteq V(f(\alpha)) \cap V(f(\beta))$.

Now it is sufficient to show that $V(f(\gamma)) \subseteq V(f(\alpha)) \cap V(f(\beta))$ implies $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$. This can be shown using Lemma 3.7.

• Case (γ is in $\bigcup_{i\in\omega-\{0\}} \Phi_i$): In this case, γ is of the form p_i, and we have the fact $p_i = f(\mathrm{X}^i p)$ for any propositional variable $p_i \in \bigcup_{i\in\omega-\{0\}} \Phi_i \subseteq \mathcal{L}_{\mathrm{LK}}$. Thus we have the following: there exists a formula $p_i = f(\mathrm{X}^i p)$ in $\mathcal{L}_{\mathrm{LK}}$ such that

1. $\mathrm{LK} \vdash f(\alpha) \Rightarrow f(\mathrm{X}^i p)$ and $\mathrm{LK} \vdash f(\mathrm{X}^i p) \Rightarrow f(\beta)$,
2. $V(f(\mathrm{X}^i p)) \subseteq V(f(\alpha)) \cap V(f(\beta))$.

By Theorem 3.5, we thus obtain the following: there exists a formula $\mathrm{X}^i p$ such that

1. $\mathrm{LT}_x \vdash \alpha \Rightarrow \mathrm{X}^i p$ and $\mathrm{LT}_x \vdash \mathrm{X}^i p \Rightarrow \beta$,
2. $V(f(\mathrm{X}^i p)) \subseteq V(f(\alpha)) \cap V(f(\beta))$.

Now it is sufficient to show that $V(f(\mathrm{X}^i p)) \subseteq V(f(\alpha)) \cap V(f(\beta))$ implies $V(\mathrm{X}^i p) \subseteq V(\alpha) \cap V(\beta)$. This can be shown using Lemma 3.7. □

4. Infinitary LTL

It is known that the Craig interpolation theorem for LT_ω does not hold. According to this fact, in our method, we cannot show a similar fact presented in Lemma 4.4: $\gamma = f(\gamma)$ for any formula γ of LK_ω. The reason of the failure of this fact is that LT_ω is not an extension of LK_ω. We thus introduce a natural extension ILT_ω of both LK_ω and LT_ω. Formulas of ILT_ω are obtained from that of LT_ω by adding $\bigwedge$ (infinitary conjunction) and $\bigvee$ (infinitary disjunction). For $\bigwedge$ and $\bigvee$, if Θ is non-empty countable set of formulas, then $\bigwedge\Theta$ and $\bigvee\Theta$ are also formulas. Note that $\bigwedge\{\alpha\}$ and

$\bigvee\{\alpha\}$ are equivalent to α, and that $\wedge$ and $\vee$ are regarded as special cases of $\bigwedge$ and $\bigvee$, respectively.

A sequent calculus ILT_ω is introduced below.

Definition 4.1 (ILT_ω)**.** ILT_ω is obtained from LT_ω by replacing $\{(\wedge\mathrm{left1}), (\wedge\mathrm{left2}), (\wedge\mathrm{right}), (\vee\mathrm{left}), (\vee\mathrm{right1}), (\vee\mathrm{right2})\}$ by the inference rules of the form:

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta \quad (\alpha \in \Theta)}{\mathrm{X}^i(\bigwedge\Theta), \Gamma \Rightarrow \Delta}\ (\bigwedge\mathrm{left}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha\ \}_{\alpha\in\Theta}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\bigwedge\Theta)}\ (\bigwedge\mathrm{right})$$

$$\frac{\{\ \mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta\ \}_{\alpha\in\Theta}}{\mathrm{X}^i(\bigvee\Theta), \Gamma \Rightarrow \Delta}\ (\bigvee\mathrm{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad (\alpha \in \Theta)}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\bigvee\Theta)}\ (\bigvee\mathrm{right})$$

where Θ denotes a non-empty countable set of formulas.

A sequent calculus LK_ω for countable infinitary logic is introduced below.

Definition 4.2 (LK_ω)**.** LK_ω is obtained from ILT_ω by deleting $\{$(Gleft), (Gright), (Fleft), (Fright)$\}$ and replacing X^i with X^0. The modified inference rules for LK_ω by replacing X^i with X^0 are denoted by using "LK_ω" as a superscript.

Definition 4.3. We fix a countable non-empty set Φ of propositional variables and define the sets $\Phi_i := \{p_i \mid p \in \Phi\}$ $(i \in \omega)$ of propositional variables where $p_0 := p \in \Phi$. The language $\mathcal{L}_{\mathrm{ILT}_\omega}$ of ILT_ω is defined using Φ, $\top, \bot, \neg, \rightarrow, \bigwedge, \bigvee, \mathrm{G}, \mathrm{F}$ and X. The language $\mathcal{L}_{\mathrm{LK}_\omega}$ of LK_ω is defined using $\bigcup_{i\in\omega}\Phi_i$, $\top, \bot, \neg, \rightarrow, \bigwedge$ and $\bigvee$.

A mapping f from $\mathcal{L}_{\mathrm{ILT}_\omega}$ to $\mathcal{L}_{\mathrm{LK}_\omega}$ is defined by the following clauses:

1. $f(\mathrm{X}^i p) := p_i \in \Phi_i$ for any $p \in \Phi$ (esp. $f(p) := p \in \Phi$),
2. $f(\mathrm{X}^i\sharp) := \sharp$ where $\sharp \in \{\top, \bot\}$,
3. $f(\mathrm{X}^i\neg\alpha) := \neg f(\mathrm{X}^i\alpha)$,
4. $f(\mathrm{X}^i(\alpha\rightarrow\beta)) := f(\mathrm{X}^i\alpha)\rightarrow f(\mathrm{X}^i\beta)$,
5. $f(\mathrm{X}^i\sharp\Theta) := \sharp f(\mathrm{X}^i\Theta)$ where $\sharp \in \{\bigwedge, \bigvee\}$ and Θ: a non-empty countable set of formulas,

6. $f(\mathrm{X}^i\mathrm{G}\alpha) := \bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\}$,
7. $f(\mathrm{X}^i\mathrm{F}\alpha) := \bigvee\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\}$.

Lemma 4.4. *Let $\mathcal{L}_{\mathrm{LK}^*_\omega}$ be $\mathcal{L}_{\mathrm{LK}_\omega} - \bigcup_{i\in\omega-\{0\}} \Phi_i$. Let f be the mapping defined in Definition 4.3. For any formula β in $\mathcal{L}_{\mathrm{LK}^*_\omega}$ ($\subseteq \mathcal{L}_{\mathrm{ILT}_\omega}$), $f(\beta) = \beta$.*

Proof. By induction on β. Since $\beta \in \mathcal{L}_{\mathrm{LK}^*_\omega}$, it is sufficient to consider the following cases: $\beta \equiv p$ (p: propositional variable), $\beta \equiv \top$, $\beta \equiv \bot$, $\beta \equiv \beta_1 \to \beta_2$, $\beta \equiv \neg\beta_1$, $\beta \equiv \bigwedge\Theta$ and $\beta \equiv \bigvee\Theta$ (Θ: countable nonempty set of formulas). These cases are simply obtained from the definition of f by considering the special cases that i in X^i is 0. We show only the case $\beta \equiv \beta_1 \to \beta_2$ below. By the definition of f, we have $f(\beta_1 \to \beta_2) = f(\beta_1) \to f(\beta_2)$. By induction hypothesis, we have $f(\beta_1) = \beta_1$ and $f(\beta_2) = \beta_2$. We thus obtain the required fact $f(\beta_1 \to \beta_2) = \beta_1 \to \beta_2$. □

Lemma 4.5. *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{ILT}_\omega}$, and f be the mapping defined in Definition 4.3. Then:*

1. *if* $\mathrm{ILT}_\omega \vdash \Gamma \Rightarrow \Delta$, *then* $\mathrm{LK}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. *if* $\mathrm{LK}_\omega - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$, *then* $\mathrm{ILT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

Proof. • (1): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in ILT_ω. We distinguish the cases according to the last inference of P, and show some cases.

1. Case ($\mathrm{X}^i p \Rightarrow \mathrm{X}^i p$): The last inference of P is of the form: $\mathrm{X}^i p \Rightarrow \mathrm{X}^i p$. In this case, we obtain $\mathrm{LK}_\omega \vdash f(\mathrm{X}^i p) \Rightarrow f(\mathrm{X}^i p)$, i.e., $\mathrm{LK}_\omega \vdash p_i \Rightarrow p_i$ ($p_i \in \Phi_i$) by the definition of f.

2. Case ($\to$left): The last inference of P is of the form:
$$\frac{\Gamma \Rightarrow \Sigma, \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Delta \Rightarrow \Pi}{\mathrm{X}^i(\alpha\to\beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ (\to\text{left}).$$
By induction hypothesis, we obtain the required fact:
$$\frac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Sigma), f(\mathrm{X}^i\alpha)\end{array} \quad \begin{array}{c}\vdots\\ f(\mathrm{X}^i\beta), f(\Delta) \Rightarrow f(\Pi)\end{array}}{f(\mathrm{X}^i\alpha)\to f(\mathrm{X}^i\beta), f(\Gamma), f(\Delta) \Rightarrow f(\Sigma), f(\Pi)}\ (\to\text{left}^{LK_\omega})$$
where $f(\mathrm{X}^i\alpha)\to f(\mathrm{X}^i\beta)$ coincides with $f(\mathrm{X}^i(\alpha\to\beta))$ by the definition of f.

3. Case ($\bigwedge$right): The last inference of P is of the form:

$$\frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha\ \}_{\alpha\in\Theta}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\bigwedge\Theta)}\ (\bigwedge\text{right}).$$

By induction hypothesis, we have $\mathrm{LK}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^i\alpha)$ for all $\alpha \in \Theta$, i.e., for all $f(\mathrm{X}^i\alpha) \in f(\mathrm{X}^i\Theta)$. Then, we obtain:

$$\frac{\begin{array}{c}\vdots\\ \{\ f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^i\alpha)\ \}_{f(\mathrm{X}^i\alpha)\in f(\mathrm{X}^i\Theta)}\end{array}}{f(\Gamma) \Rightarrow f(\Delta), \bigwedge f(\mathrm{X}^i\Theta)}\ (\bigwedge\text{right}^{LK_\omega})$$

where $\bigwedge f(\mathrm{X}^i\Theta)$ coincides with $f(\mathrm{X}^i(\bigwedge\Theta))$ by the definition of f.

• (2): By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in LK_ω. We distinguish the cases according to the last inference of Q, and show only the following case.

Case ($\bigwedge$right$^{LK_\omega}$): The last inference of Q is of the form:

$$\frac{\{\ f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^i\alpha)\ \}_{f(\mathrm{X}^i\alpha)\in f(\mathrm{X}^i\Theta)}}{f(\Gamma) \Rightarrow f(\Delta), \bigwedge f(\mathrm{X}^i\Theta)}\ (\bigwedge\text{right}^{LK_\omega})$$

where $\bigwedge f(\mathrm{X}^i\Theta)$ coincides with $f(\mathrm{X}^i(\bigwedge\Theta))$ by the definition of f. By induction hypothesis, we have $\mathrm{ILT}_\omega \vdash \Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha$ for all $\mathrm{X}^i\alpha \in \mathrm{X}^i\Theta$, i.e., for all $\alpha \in \Theta$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \{\ \Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha\ \}_{\alpha\in\Theta}\end{array}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\bigwedge\Theta)}\ (\bigwedge\text{right}).$$

□

Theorem 4.6 (Cut-elimination for ILT_ω)**.** *The rule* (cut) *is admissible in cut-free* ILT_ω.

Proof. Similar to Theorem 3.4. We use Lemma 4.5. □

Theorem 4.7 (Embedding from ILT_ω into LK_ω)**.** *Let* Γ *and* Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{ILT}_\omega}$*, and* f *be the mapping defined in Definition 4.3. Then:*

$$\mathrm{ILT}_\omega \vdash \Gamma \Rightarrow \Delta \textit{ iff } \mathrm{LK}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta).$$

Proof. Similar to Theorem 3.5. We use Lemma 4.5. □

Lemma 4.8. *Let f be the mapping defined in Definition 4.3. For any $i \in \omega$, any propositional variable p in $\mathcal{L}_{\mathrm{ILT}_\omega}$ and any formula α in $\mathcal{L}_{\mathrm{ILT}_\omega}$,*

$$p \in V(\mathrm{X}^i\alpha) \textit{ iff } p_j \in V(f(\mathrm{X}^i\alpha)) \textit{ for some } j \in \omega.$$

Proof. Similar to Lemma 3.6. By induction on α. We show only the following case for the induction step.

Case ($\alpha \equiv \mathrm{G}\beta$). We obtain:

$$p \in V(\mathrm{X}^i\mathrm{G}\beta)$$

iff $p \in V(\mathrm{X}^i\beta)$

iff $p_j \in V(f(\mathrm{X}^i\beta))$ for some $j \in \omega$ (by induction hypothesis)

iff $p_j \in V(\bigwedge\{f(\mathrm{X}^{i+k}\beta) \mid k \in \omega\})$ for some $j \in \omega$

iff $p_j \in V(f(\mathrm{X}^i\mathrm{G}\beta))$ for some $j \in \omega$ (by the definition of f).

□

Lemma 4.9. *Let f be the mapping defined in Definition 4.3. For any formulas α and β in $\mathcal{L}_{\mathrm{ILT}_\omega}$,*

if $V(f(\alpha)) \subseteq V(f(\beta))$, then $V(\alpha) \subseteq V(\beta)$.

Proof. Similar to Lemma 3.7. We use Lemma 4.8. □

Theorem 4.10 (Craig interpolation theorem for ILT_ω)**.** *For any formulas α and β, if $\mathrm{ILT}_\omega \vdash \alpha \Rightarrow \beta$, then there exists a formula γ such that*

1. $\mathrm{ILT}_\omega \vdash \alpha \Rightarrow \gamma$ *and* $\mathrm{ILT}_\omega \vdash \gamma \Rightarrow \beta$,
2. $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

Proof. Similar to Theorem 3.8. We use Theorem 4.7, Lemmas 4.9 and 4.4, and the Craig interpolation theorem for LK_ω. □

5. Paraconsistent next-time LTL

We introduce a paraconsistent extension PLT_x of LT_x. The logic PLT_x is regarded as a modified fragment of the sequent calculus for the paraconsistent LTL proposed in [11]. The language of PLT_x is obtained from that of LT_x by adding a paraconsistent negation connective $\sim$ similar to the strong negation connective in Nelson's paraconsistent logic N4 [1]. The negation connective $\sim$ in N4 and PLT_x is regarded as paraconsistent, i.e., the formula of the form $(\sim\alpha \wedge \alpha) \rightarrow \beta$ is not an axiom scheme of N4 and PLT_x.

Definition 5.1 (PLT_x)**.** PLT_x is obtained from LT_x by adding the initial sequents of the form: for any propositional variable p,

$$\mathrm{X}^i{\sim}p \Rightarrow \mathrm{X}^i{\sim}p \qquad \mathrm{X}^i{\sim}\top \Rightarrow \qquad \Rightarrow \mathrm{X}^i{\sim}\bot$$

and adding the logical inference rules of the form:

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i{\sim}{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{\sim}\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}{\sim}\alpha}\ ({\sim}{\sim}\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i{\sim}(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\rightarrow}\text{left1}) \qquad \frac{\mathrm{X}^i{\sim}\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i{\sim}(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\rightarrow}\text{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}(\alpha\rightarrow\beta)}\ ({\sim}{\rightarrow}\text{right})$$

$$\frac{\mathrm{X}^i{\sim}\alpha, \Gamma \Rightarrow \Delta \quad \mathrm{X}^i{\sim}\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i{\sim}(\alpha\wedge\beta), \Gamma \Rightarrow \Delta}\ ({\sim}\wedge\text{left})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}(\alpha\wedge\beta)}\ ({\sim}\wedge\text{right1}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}(\alpha\wedge\beta)}\ ({\sim}\wedge\text{right2})$$

$$\frac{\mathrm{X}^i{\sim}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i{\sim}(\alpha\vee\beta), \Gamma \Rightarrow \Delta}\ ({\sim}\vee\text{left1}) \qquad \frac{\mathrm{X}^i{\sim}\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i{\sim}(\alpha\vee\beta), \Gamma \Rightarrow \Delta}\ ({\sim}\vee\text{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}(\alpha\vee\beta)}\ ({\sim}\vee\text{right})$$

$$\frac{\mathrm{X}^i{\sim}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i{\sim}\neg\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\neg\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\neg\alpha}\ ({\sim}\neg\text{right})$$

$$\frac{\mathrm{X}^i{\sim}\alpha, \Gamma \Rightarrow \Delta}{{\sim}\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\mathrm{X}\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\alpha}{\Gamma \Rightarrow \Delta, {\sim}\mathrm{X}^i\alpha}\ ({\sim}\mathrm{X}\text{right}).$$

The sequents of the form $X^i\alpha \Rightarrow X^i\alpha$ for any formula α are provable in cut-free PLT_x.

An expression $\alpha \leftrightarrow \beta$ means $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. Then, the following sequents are provable in cut-free PLT_x: for any formulas α and β,

1. ${\sim}{\sim}\alpha \leftrightarrow \alpha$,
2. ${\sim}(\alpha \wedge \beta) \leftrightarrow {\sim}\alpha \vee {\sim}\beta$,
3. ${\sim}(\alpha \vee \beta) \leftrightarrow {\sim}\alpha \wedge {\sim}\beta$,
4. ${\sim}(\alpha{\rightarrow}\beta) \leftrightarrow \alpha \wedge {\sim}\beta$,
5. ${\sim}\neg\alpha \leftrightarrow \alpha$,
6. ${\sim}\mathrm{X}\alpha \leftrightarrow \mathrm{X}{\sim}\alpha$.

In the following, we introduce a translation of PLT_x into LT_x, and by using this translation, we show a theorem for embedding PLT_x into LT_x. A similar translation has been used by Vorob'ev [24], Gurevich [6], and Rautenberg [22] to embed Nelson's three-valued constructive logic [1, 20] into intuitionistic logic.

Definition 5.2. Let Φ be a non-empty set of propositional variables and Φ' be the set $\{p' \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{PLT}_x}$ (the set of formulas) of PLT_x is defined using Φ, $\top, \bot$, $\sim$, $\neg, \rightarrow, \wedge, \vee$ and X. The language $\mathcal{L}_{\mathrm{LT}_x}$ of LT_x is obtained from $\mathcal{L}_{\mathrm{PLT}_x}$ by adding Φ' and deleting $\sim$.

A mapping f from $\mathcal{L}_{\mathrm{PLT}_x}$ to $\mathcal{L}_{\mathrm{LT}_x}$ is defined inductively by

1. for any $p \in \Phi$, $f(p) := p$ and $f({\sim}p) := p' \in \Phi'$,
2. $f(\sharp) := \sharp$ where $\sharp \in \{\top, \bot\}$,
3. $f(\alpha \sharp \beta) := f(\alpha) \sharp f(\beta)$ where $\sharp \in \{\wedge, \vee, \rightarrow\}$,
4. $f(\sharp\alpha) := \sharp f(\alpha)$ where $\sharp \in \{\neg, \mathrm{X}\}$,
5. $f({\sim}\top) := \bot$,
6. $f({\sim}\bot) := \top$,
7. $f({\sim}{\sim}\alpha) := f(\alpha)$,

8. $f(\sim\neg\alpha) := f(\alpha)$,
9. $f(\sim \mathrm{X}\alpha) := \mathrm{X}f(\sim\alpha)$,
10. $f(\sim(\alpha \wedge \beta)) := f(\sim\alpha) \vee f(\sim\beta)$,
11. $f(\sim(\alpha \vee \beta)) := f(\sim\alpha) \wedge f(\sim\beta)$,
12. $f(\sim(\alpha\to\beta)) := f(\alpha) \wedge f(\sim\beta)$.

Lemma 5.3. *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{PLT}_x}$, and f be the mapping defined in Definition 5.2. Then:*

1. *if* $\mathrm{PLT}_x \vdash \Gamma \Rightarrow \Delta$, *then* $\mathrm{LT}_x \vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. *if* $\mathrm{LT}_x - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$, *then* $\mathrm{PLT}_x - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

Proof. We show only (1) below.

• (1) : By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in PLT_x. We distinguish the cases according to the last inference of P, and show some cases.

1. Case ($\mathrm{X}^i{\sim}p \Rightarrow \mathrm{X}^i{\sim}p$):
   The last inference of P is of the form: $\mathrm{X}^i{\sim}p \Rightarrow \mathrm{X}^i{\sim}p$. In this case, we obtain the required fact $\mathrm{LT}_x \vdash f(\mathrm{X}^i{\sim}p) \Rightarrow f(\mathrm{X}^i{\sim}p)$, since $f(\mathrm{X}^i{\sim}p)$ coincides with $\mathrm{X}^i p'$ by the definition of f.

2. Case ($\sim\to$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}(\alpha\to\beta)}\ ({\sim}{\to}\mathrm{left}).$$

   By induction hypothesis, we have: $\mathrm{LT}_x \vdash f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^i\alpha)$ and $\mathrm{LT}_x \vdash f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^i{\sim}\beta)$ where $f(\mathrm{X}^i\alpha)$ and $f(\mathrm{X}^i{\sim}\beta)$ respectively coincide with $\mathrm{X}^i f(\alpha)$ and $\mathrm{X}^i f(\sim\beta)$ by the definition of f. Then, we obtain:

$$\frac{\begin{matrix}\vdots \\ f(\Gamma) \Rightarrow f(\Delta), \mathrm{X}^i f(\alpha)\end{matrix} \quad \begin{matrix}\vdots \\ f(\Gamma) \Rightarrow f(\Delta), \mathrm{X}^i f(\sim\beta)\end{matrix}}{f(\Gamma) \Rightarrow f(\Delta), \mathrm{X}^i(f(\alpha) \wedge f(\sim\beta))}\ (\wedge\mathrm{left})$$

   where $\mathrm{X}^i(f(\alpha) \wedge f(\sim\beta))$ coincides with $f(\mathrm{X}^i{\sim}(\alpha\to\beta))$ by the definition of f.

3. Case ($\sim$Xleft): The last inference of P is of the form:

$$\frac{\mathrm{X}^i{\sim}\alpha, \Gamma \Rightarrow \Delta}{{\sim}\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\mathrm{Xleft}).$$

By induction hypothesis, we have: $\mathrm{LT}_x \vdash f(\mathrm{X}^i{\sim}\alpha), f(\Gamma) \Rightarrow f(\Delta)$ where $f(\mathrm{X}^i{\sim}\alpha)$ coincides with $f({\sim}\mathrm{X}^i\alpha)$ by the definition of f.

□

Theorem 5.4 (Cut-elimination for PLT_x)**.** *The rule* (cut) *is admissible in cut-free* PLT_x*.*

Proof. By using Lemma 5.3. □

Theorem 5.5 (Embedding from PLT_x into LT_x)**.** *Let* Γ *and* Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{PLT}_x}$*, and* f *be the mapping defined in Definition 5.2. Then:*

$\mathrm{PLT}_x \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LT}_x \vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. By using Lemma 5.3. □

Lemma 5.6. *Let* f *be the mapping defined in Definition 5.2. For any propositional variable* p *in* $\mathcal{L}_{\mathrm{PLT}_x}$*, and any formula* α *in* $\mathcal{L}_{\mathrm{PLT}_x}$*,*

1. $p \in V(\alpha)$ *iff* $q \in V(f(\alpha))$ *for some* $q \in \{p, p'\}$*,*
2. $p \in V({\sim}\alpha)$ *iff* $q \in V(f({\sim}\alpha))$ *for some* $q \in \{p, p'\}$*.*

Proof. By (simultaneous) induction on α.

• Base step. For the item 1, we have: $p \in V(p)$ and $p = f(p) \in V(f(p))$ by the definition of f. For the item 2, we have: $p \in V({\sim}p)$ and $p' = f({\sim}p) \in V(f({\sim}p))$ by the definition of f.

• Induction step. We show some cases.

1. Case ($\alpha \equiv \top$). For the item 1, this case holds since $p \in V(\top)$ and $q \in V(f(\top))$ do not hold. For the item 2, this case is similar to the case above.
2. Case ($\alpha \equiv {\sim}\beta$). For the item 1, we obtain: $p \in V({\sim}\beta)$ iff $q \in V(f({\sim}\beta))$ for some $q \in \{p, p'\}$ (by induction hypothesis for 2). For the item 2, we obtain:

$p \in V(\sim\sim\beta)$
iff $p \in V(\beta)$
iff $q \in V(f(\beta))$ for some $q \in \{p, p'\}$ (by induction hypothesis for 1)
iff $q \in V(f(\sim\sim\beta))$ for some $q \in \{p, p'\}$ (by the definition of f).

3. Case ($\alpha \equiv \neg\beta$). For the item 1, we obtain:

$p \in V(\neg\beta)$
iff $p \in V(\beta)$
iff $q \in V(f(\beta))$ for some $q \in \{p, p'\}$ (by induction hypothesis for 1)
iff $q \in V(\neg f(\beta))$ for some $q \in \{p, p'\}$
iff $q \in V(f(\neg\beta))$ for some $q \in \{p, p'\}$ (by the definition of f).

For the item 2, we obtain:

$p \in V(\sim\neg\beta)$
iff $p \in V(\beta)$
iff $q \in V(f(\beta))$ for some $q \in \{p, p'\}$ (by induction hypothesis for 1)
iff $q \in V(f(\sim\neg\beta))$ for some $q \in \{p, p'\}$ (by the definition of f).

4. Case ($\alpha \equiv \beta \wedge \gamma$). For the item 1, we obtain:

$p \in V(\beta \wedge \gamma)$
iff $p \in V(\beta)$ or $p \in V(\gamma)$
iff [$r \in V(f(\beta))$ for some $r \in \{p, p'\}$] or [$s \in V(f(\gamma))$ for some $s \in \{p, p'\}$] (by induction hypothesis for 1)
iff $q \in V(f(\beta) \wedge f(\gamma))$ for some $q \in \{p, p'\}$
iff $q \in V(f(\beta \wedge \gamma))$ for some $q \in \{p, p'\}$ (by the definition of f).

For the item 2, we obtain:

$p \in V(\sim(\beta \wedge \gamma))$
iff $p \in V(\sim\beta)$ or $p \in V(\sim\gamma)$
iff [$r \in V(f(\sim\beta))$ for some $r \in \{p, p'\}$] or [$s \in V(f(\sim\gamma))$ for some $s \in \{p, p'\}$] (by induction hypothesis for 2)
iff $q \in V(f(\sim\beta) \vee f(\sim\gamma))$ for some $q \in \{p, p'\}$

iff $q \in V(f(\sim(\beta \wedge \gamma)))$ for some $q \in \{p, p'\}$ (by the definition of f).

□

Lemma 5.7. *Let f be the mapping defined in Definition 5.2. For any formulas α and β in $\mathcal{L}_{\mathrm{PLT}_x}$, if $V(f(\alpha)) \subseteq V(f(\beta))$, then $V(\alpha) \subseteq V(\beta)$.*

Proof. Suppose $p \in V(\alpha)$. Then, we obtain $q \in V(f(\alpha))$ for some $q \in \{p, p'\}$ by Lemma 5.6. By the assumption, we obtain $q \in V(f(\beta))$ for some $q \in \{p, p'\}$, and hence obtain $p \in V(\beta)$ by Lemma 5.6. □

Theorem 5.8 (Craig interpolation theorem for PLT_x)**.** *For any formulas α and β, if $\mathrm{PLT}_x \vdash \alpha \Rightarrow \beta$, then there exists a formula γ such that*

1. $\mathrm{PLT}_x \vdash \alpha \Rightarrow \gamma$ *and* $\mathrm{PLT}_x \vdash \gamma \Rightarrow \beta$,
2. $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

Proof. Similar to Theorem 3.8. We use Theorems 5.5 and 3.8 and Lemma 5.7. □

6. Paraconsistent infinitary LTL

We introduce a paraconsistent extension PILT_ω of ILT_ω. The language of PILT_ω is obtained from that of ILT_ω by adding $\sim$.

Definition 6.1 (PILT_ω)**.** PILT_ω is obtained from ILT_ω by adding the initial sequents of the form: for any propositional variable p,

$$\mathrm{X}^i{\sim}p \Rightarrow \mathrm{X}^i{\sim}p \qquad \mathrm{X}^i{\sim}\top \Rightarrow \qquad \Rightarrow \mathrm{X}^i{\sim}\bot$$

adding the logical inference rules {(∼∼left),(∼∼right),(∼→left1),(∼→left2), (∼¬left), (∼¬right), (∼Xleft), (∼Xright)} in Definition 5.1, and adding the logical inference rules of the form:

$$\frac{\{\ \mathrm{X}^{i+j}{\sim}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\mathrm{X}^i{\sim}\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\sim\mathrm{Gleft}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+k}{\sim}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\mathrm{G}\alpha}\ (\sim\mathrm{Gright})$$

$$\frac{\mathrm{X}^{i+k}{\sim}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i{\sim}\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\sim\mathrm{Fleft}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}{\sim}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\mathrm{F}\alpha}\ (\sim\mathrm{Fright})$$

$$\frac{\{\ \mathrm{X}^i{\sim}\alpha, \Gamma \Rightarrow \Delta\ \}_{\alpha\in\Theta}}{\mathrm{X}^i{\sim}(\bigwedge\Theta), \Gamma \Rightarrow \Delta}\ ({\sim}\bigwedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\alpha\ \ (\alpha\in\Theta)}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}(\bigwedge\Theta)}\ ({\sim}\bigwedge\text{right})$$

$$\frac{\mathrm{X}^i{\sim}\alpha, \Gamma \Rightarrow \Delta\ \ (\alpha\in\Theta)}{\mathrm{X}^i{\sim}(\bigvee\Theta), \Gamma \Rightarrow \Delta}\ ({\sim}\bigvee\text{left}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}\alpha\ \}_{\alpha\in\Theta}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i{\sim}(\bigvee\Theta)}\ ({\sim}\bigvee\text{right})$$

where Θ denotes a non-empty countable set of formulas.

The sequents of the form $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ for any formula α are provable in cut-free PILT$_\omega$. An expression ${\sim}\Gamma$ means the set $\{{\sim}\gamma \mid \gamma \in \Gamma\}$. The following sequents are provable in cut-free PILT$_\omega$: for any formulas α, β, and any non-empty countable set Θ of formulas,

1. ${\sim}\mathrm{G}\alpha \leftrightarrow \mathrm{F}{\sim}\alpha$,
2. ${\sim}\mathrm{F}\alpha \leftrightarrow \mathrm{G}{\sim}\alpha$,
3. ${\sim}(\bigwedge\Theta) \leftrightarrow \bigvee({\sim}\Theta)$,
4. ${\sim}(\bigvee\Theta) \leftrightarrow \bigwedge({\sim}\Theta)$.

Definition 6.2. Let Φ be a non-empty set of propositional variables and Φ' be the set $\{p' \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{PILT}_\omega}$ (the set of formulas) of PILT$_\omega$ is defined using $\Phi, \top, \bot, {\sim}, \neg, \to, \bigwedge, \bigvee, \mathrm{G}, \mathrm{F}$ and X. The language $\mathcal{L}_{\mathrm{ILT}_\omega}$ of ILT$_\omega$ is obtained from $\mathcal{L}_{\mathrm{PILT}_\omega}$ by adding Φ' and deleting ${\sim}$.

A mapping f from $\mathcal{L}_{\mathrm{PILT}_\omega}$ to $\mathcal{L}_{\mathrm{ILT}_\omega}$ is defined inductively by

1. for any $p \in \Phi$, $f(p) := p$ and $f({\sim}p) := p' \in \Phi'$,
2. $f(\sharp) := \sharp$ where $\sharp \in \{\top, \bot\}$,
3. $f(\sharp\alpha) := \sharp f(\alpha)$ where $\sharp \in \{\neg, \mathrm{X}\}$,
4. $f(\alpha{\to}\beta) := f(\alpha){\to}f(\beta)$,
5. $f(\sharp\Theta) := \sharp f(\Theta)$ where $\sharp \in \{\bigwedge, \bigvee\}$ and Θ: a non-empty countable set of formulas,
6. $f({\sim}\top) := \bot$,
7. $f({\sim}\bot) := \top$,
8. $f({\sim}{\sim}\alpha) := f(\alpha)$,

9. $f({\sim}\neg\alpha) := f(\alpha)$,
10. $f({\sim}(\alpha{\to}\beta)) := f(\alpha) \wedge f({\sim}\beta)$,
11. $f({\sim}\bigwedge\Theta) := \bigvee f({\sim}\Theta)$ where Θ: a non-empty countable set of formulas,
12. $f({\sim}\bigvee\Theta) := \bigwedge f({\sim}\Theta)$ where Θ: a non-empty countable set of formulas,
13. $f({\sim}\mathrm{X}\alpha) := \mathrm{X}f({\sim}\alpha)$,
14. $f({\sim}\mathrm{G}\alpha) := \mathrm{F}f({\sim}\alpha)$,
15. $f({\sim}\mathrm{F}\alpha) := \mathrm{G}f({\sim}\alpha)$.

Lemma 6.3. *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{PILT}_\omega}$, and f be the mapping defined in Definition 6.2. Then:*

1. *if* $\mathrm{PILT}_\omega \vdash \Gamma \Rightarrow \Delta$, *then* $\mathrm{ILT}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. *if* $\mathrm{ILT}_\omega - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$, *then* $\mathrm{PILT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

Proof.

• (1) : By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in PILT_ω. We distinguish the cases according to the last inference of P, and show only the following case.

Case (∼Gleft): The last inference of P is of the form:

$$\frac{\{\ \mathrm{X}^{i+j}{\sim}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\mathrm{X}^{i}{\sim}\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\mathrm{Gleft}).$$

By induction hypothesis, we have: $\mathrm{ILT}_\omega \vdash f(\mathrm{X}^{i+j}{\sim}\alpha), f(\Gamma) \Rightarrow f(\Delta)$ for any $j \in \omega$, where $f(\mathrm{X}^{i+j}{\sim}\alpha)$ coincides with $\mathrm{X}^{i+j}f({\sim}\alpha)$ by the definition of f. Then, we obtain:

$$\frac{\begin{array}{c}\vdots\\ \{\ \mathrm{X}^{i+j}f({\sim}\alpha), f(\Gamma) \Rightarrow f(\Delta)\ \}_{j\in\omega}\end{array}}{\mathrm{X}^{i}\mathrm{F}f({\sim}\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\mathrm{Fleft})$$

where $\mathrm{X}^i\mathrm{F}f({\sim}\alpha)$ coincides with $f(\mathrm{X}^i{\sim}\mathrm{G}\alpha)$ by the definition of f.

• (2) : By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in ILT_ω. We distinguish the cases according to the last inference of Q, and show only the following case.

Case (Gleft): The last inference of Q is (Gleft).

1. Subcase (1): The last inference of Q is of the form:

$$\frac{\mathrm{X}^{i+k} f(\alpha), f(\Gamma) \Rightarrow f(\Delta)}{\mathrm{X}^i \mathrm{G} f(\alpha), f(\Gamma) \Rightarrow f(\Delta)} \text{ (Gleft)}$$

   where $\mathrm{X}^{i+k} f(\alpha)$ and $\mathrm{X}^i \mathrm{G} f(\alpha)$ respectively coincide with $f(\mathrm{X}^{i+k}\alpha)$ and $f(\mathrm{X}^i\mathrm{G}\alpha)$ by the definition of f. By induction hypothesis, we have: $\mathrm{PILT}_\omega \vdash \mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta$, and hence obtain the required fact:

$$\frac{\begin{array}{c}\vdots \\ \mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta\end{array}}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \Delta} \text{ (Gleft).}$$

2. Subcase (2): The last inference of Q is of the form:

$$\frac{\mathrm{X}^{i+k} f({\sim}\alpha), f(\Gamma) \Rightarrow f(\Delta)}{\mathrm{X}^i \mathrm{G} f({\sim}\alpha), f(\Gamma) \Rightarrow f(\Delta)} \text{ (Gleft)}$$

   where $\mathrm{X}^{i+k} f({\sim}\alpha)$ and $\mathrm{X}^i \mathrm{G} f({\sim}\alpha)$ respectively coincide with $f(\mathrm{X}^{i+k}{\sim}\alpha)$ and $f(\mathrm{X}^i{\sim}\mathrm{F}\alpha)$ by the definition of f. By induction hypothesis, we have: $\mathrm{PILT}_\omega \vdash \mathrm{X}^{i+k}{\sim}\alpha, \Gamma \Rightarrow \Delta$, and hence obtain the required fact:

$$\frac{\begin{array}{c}\vdots \\ \mathrm{X}^{i+k}{\sim}\alpha, \Gamma \Rightarrow \Delta\end{array}}{\mathrm{X}^i{\sim}\mathrm{F}\alpha, \Gamma \Rightarrow \Delta} \text{ (}{\sim}\text{Fleft).}$$

□

Theorem 6.4 (Cut-elimination for PILT_ω)**.** *The rule* (cut) *is admissible in cut-free* PILT_ω.

Proof. By using Lemma 6.3. □

Theorem 6.5 (Embedding from PILT_ω into ILT_ω)**.** *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{PILT}_\omega}$, and f be the mapping defined in Definition 6.2. Then:*

$$\mathrm{PILT}_\omega \vdash \Gamma \Rightarrow \Delta \textit{ iff } \mathrm{ILT}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta).$$

Proof. By using Lemma 6.3. □

Lemma 6.6. *Let f be the mapping defined in Definition 6.2. For any propositional variable p in $\mathcal{L}_{\mathrm{PILT}_\omega}$, and any formula α in $\mathcal{L}_{\mathrm{PILT}_\omega}$,*

1. *$p \in V(\alpha)$ iff $q \in V(f(\alpha))$ for some $q \in \{p, p'\}$,*
2. *$p \in V(\sim\alpha)$ iff $q \in V(f(\sim\alpha))$ for some $q \in \{p, p'\}$.*

Proof. Similar to Lemma 5.6. □

Lemma 6.7. *Let f be the mapping defined in Definition 6.2. For any formulas α and β in $\mathcal{L}_{\mathrm{PLT}_x}$, if $V(f(\alpha)) \subseteq V(f(\beta))$, then $V(\alpha) \subseteq V(\beta)$.*

Proof. Similar to Lemma 5.7. We use Lemma 6.6. □

Theorem 6.8 (Craig interpolation theorem for PILT_ω)**.** *For any formulas α and β, if $\mathrm{PILT}_\omega \vdash \alpha \Rightarrow \beta$, then there exists a formula γ such that*

1. *$\mathrm{PILT}_\omega \vdash \alpha \Rightarrow \gamma$ and $\mathrm{PILT}_\omega \vdash \gamma \Rightarrow \beta$,*
2. *$V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.*

Proof. Similar to Theorem 4.10. We use Theorems 6.5 and 4.10 and Lemma 6.7. □

7. Maksimova separation

We can show, in a similar way as in the previous sections, the following Craig interpolation theorem for the $\{\top, \bot\}$-free fragments of LT_x, ILT_ω, PLT_x and PILT_ω.

Theorem 7.1 (Craig interpolation theorem for the $\{\top, \bot\}$-free fragments)**.** *Let L be the $\{\top, \bot\}$-free fragment of LT_x, the $\{\top, \bot\}$-free fragment of ILT_ω, the $\{\top, \bot\}$-free fragment of PLT_x or the $\{\top, \bot\}$-free fragment of PILT_ω. Suppose $L \vdash \alpha \Rightarrow \beta$ for any $\{\top, \bot\}$-free formulas α and β. If $V(\alpha) \cap V(\beta) \neq \emptyset$, then there exists a formula γ such that*

1. $L \vdash \alpha \Rightarrow \gamma$ *and* $L \vdash \gamma \Rightarrow \beta$,

2. $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

If $V(\alpha) \cap V(\beta) = \emptyset$, *then*

3. $L \vdash \ \Rightarrow \neg\alpha$ *or* $L \vdash \ \Rightarrow \beta$.

Using this theorem, we can show the following Maksimova separation theorem for LT_x, ILT_ω, PLT_x and PILT_ω.

Theorem 7.2 (Maksimova separation theorem for the LTL variants)**.** *Let* L *be the* $\{\top, \bot\}$*-free fragment of* LT_x*, the* $\{\top, \bot\}$*-free fragment of* ILT_ω*, the* $\{\top, \bot\}$*-free fragment of* PLT_x *or the* $\{\top, \bot\}$*-free fragment of* PILT_ω*. Suppose* $V(\alpha_1, \alpha_2) \cap V(\beta_1, \beta_2) \neq \emptyset$ *for any* $\{\top, \bot\}$*-free formulas* $\alpha_1, \alpha_2, \beta_1$ *and* β_2*. If* $L \vdash \alpha_1 \wedge \beta_1 \Rightarrow \alpha_2 \vee \beta_2$*, then either* $L \vdash \alpha_1 \Rightarrow \alpha_2$ *or* $L \vdash \beta_1 \Rightarrow \beta_2$*.*

Proof. Suppose $V(\alpha_1, \alpha_2) \cap V(\beta_1, \beta_2) \neq \emptyset$ and $L \vdash \alpha_1 \wedge \beta_1 \Rightarrow \alpha_2 \vee \beta_2$. Then, we have: $L \vdash \alpha_1, \beta_1 \Rightarrow \alpha_2, \beta_2$, and hence have: $L \vdash \alpha_1, \neg\alpha_2 \Rightarrow \neg\beta_1, \beta_2$. Thus, we obtain: $L \vdash \alpha_1 \wedge \neg\alpha_2 \Rightarrow \neg\beta_1 \vee \beta_2$. By Theorem 7.1 (3), we obtain:

$$L \vdash \ \Rightarrow \neg(\alpha_1 \wedge \neg\alpha_2) \text{ or } L \vdash \ \Rightarrow \neg\beta_1 \vee \beta_2.$$

We thus obtain the required fact:

$$L \vdash \alpha_1 \Rightarrow \alpha_2 \text{ or } L \vdash \beta_1 \Rightarrow \beta_2$$

by:

$$\dfrac{\Rightarrow \neg(\alpha_1 \wedge \neg\alpha_2) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\alpha_1 \Rightarrow \alpha_1}}{\alpha_1 \Rightarrow \alpha_2, \alpha_1} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\alpha_2 \Rightarrow \alpha_2}}{\alpha_2, \alpha_1 \Rightarrow \alpha_2}}{\alpha_1 \Rightarrow \alpha_2, \neg\alpha_2}}{\alpha_1 \Rightarrow \alpha_2, \alpha_1 \wedge \neg\alpha_2}}{\neg(\alpha_1 \wedge \neg\alpha_2), \alpha_1 \Rightarrow \alpha_2}}{\alpha_1 \Rightarrow \alpha_2}\ (\text{cut})$$

or

$$\dfrac{\Rightarrow \neg\beta_1 \vee \beta_2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\beta_1 \Rightarrow \beta_1}}{\beta_1 \Rightarrow \beta_1, \beta_2}}{\neg\beta_1, \beta_1 \Rightarrow \beta_2} \qquad \dfrac{\dfrac{\vdots}{\beta_2 \Rightarrow \beta_2}}{\beta_2, \beta_1 \Rightarrow \beta_2}}{\neg\beta_1 \vee \beta_2, \beta_1 \Rightarrow \beta_2}}{\beta_1 \Rightarrow \beta_2}\ (\text{cut})$$

□

8. Remarks

In the following, it is explained that Craig interpolation theorem holds for a bounded-time version BLT[l] of LT_ω. The system BLT[l] was called BLTL (*bounded linear-time temporal logic*) in [10]. A paraconsistent extension PBLT[l] of BLT[l] can be defined similarly, and the Craig interpolation theorem for PBLT[l] can be shown in a similar way. It can also be shown that Maksimova separation theorem holds for the constant-free versions of these logics. The detail of these results is not explained in the following since such results can be obtained similarly as in the previous sections.

Let l be a fixed positive integer, and ω_l be the set $\{i \in \omega \mid i \leq l\}$. The system BLT[$l$] is obtained from LT_ω by replacing the inference rules {(Gleft), (Gright), (Fleft), (Fright)} with the inference rules of the form: for any $k \in \omega_l$,

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^{i}\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\mathrm{Gleft}^{l}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega_l}}{\Gamma \Rightarrow \Delta, \mathrm{X}^{i}\mathrm{G}\alpha}\ (\mathrm{Gright}^{l})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega_l}}{\mathrm{X}^{i}\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\mathrm{Fleft}^{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^{i}\mathrm{F}\alpha}\ (\mathrm{Fright}^{l})$$

and adding the inference rules of the form:

$$\frac{\mathrm{X}^{l}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^{i+l}\alpha, \Gamma \Rightarrow \Delta}\ (\mathrm{Xleft}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{l}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+l}\alpha}\ (\mathrm{Xright}).$$

The inference rules presented above correspond to the following Hilbert-style axioms:

1. $\mathrm{G}\alpha \leftrightarrow \alpha \wedge \mathrm{X}\alpha \wedge \mathrm{X}^2\alpha \wedge \cdots \wedge \mathrm{X}^l\alpha$,
2. $\mathrm{F}\alpha \leftrightarrow \alpha \vee \mathrm{X}\alpha \vee \mathrm{X}^2\alpha \vee \cdots \vee \mathrm{X}^l\alpha$,
3. $\mathrm{X}^{i+l}\alpha \leftrightarrow \mathrm{X}^l\alpha$.

Since the axioms 1 and 2 correspond to the finite versions of the following axioms in ILT_ω:

1. $\mathrm{G}\alpha \leftrightarrow \bigwedge_{i\in\omega} \mathrm{X}^i\alpha$,
2. $\mathrm{F}\alpha \leftrightarrow \bigvee_{i\in\omega} \mathrm{X}^i\alpha$.

Thus, BLT[l] is regarded as a finite approximation of LT_ω. Note that BLT[l] is embeddable into LK since G and F in BLT[l] are expressed using $\wedge$ and $\vee$ in LK based on a modified mapping f. By using this fact, we can obtain the Craig interpolation theorem for BLT[l].

9. Conclusions

In this paper, the Craig interpolation theorem for the next-time only fragment LT_x of a Gentzen-type sequent calculus LT_ω for LTL was proved using a theorem for embedding LT_x into a sequent calculus LK for classical logic. The Craig interpolation theorem for the infinitary extension ILT_ω of LT_ω was also proved using a theorem for embedding ILT_ω into a sequent calculus LK_ω for countable infinitary logic. Moreover, the Craig interpolation theorem for the paraconsistent extensions PLT_x and $PILT_\omega$ of LT_x and ILT_ω, respectively, was proved using some theorems for embedding PLT_x and $PILT_\omega$ into LT_x and ILT_ω, respectively. The Maksimova separation theorem for (the constant-free fragments of) LT_x, ILT_ω, PLT_x and $PILT_\omega$ was obtained as a corollary of the (constant-free version of) Craig interpolation theorem.

The result for LT_x, i.e., the next-time LTL, is not a new result of this paper, but the results for ILT_ω, PLT_x and $PILT_\omega$ are new results of this paper. The proposed embedding-based proof method for the logics under consideration is also a new contribution of this paper. It is known that Maehara's method [15, 23] is useful to obtain a syntactical proof of Craig interpolation theorem. Maehara's method may not work for ILT_ω since an infinite partition of a finite sequent in ILT_ω cannot be considered. It is also known that Maehara's method requires cut-elimination. But, the proposed method does not require cut-elimination. This is a merit of the proposed method.

A theorem for semantically embedding the semantics for LT_x (ILT_ω) into the semantics for classical logic (the countable infinitary logic, respectively) can similarly be shown. Thus, an embedding-based "semantical" proof of the Craig interpolation theorems for LT_x and ILT_ω can also be obtained. Moreover, the proposed method can straightforwardly be applied to the first-order versions and to the intuitionistic versions, although such a result is omitted here. In conclusion, our new method is useful for showing Craig interpolation and Maksimova separation theorems for some LTL variants.

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