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THE LARGEST HIGHER COMMUTATOR SEQUENCE

A b s t r a c t. Given the congruence lattice $\mathbb{L}$ of a finite algebra $\mathbf{A}$ that generates a congruence permutable variety, we look for those sequences of operations on $\mathbb{L}$ that have the properties of higher commutator operations of expansions of $\mathbf{A}$. If we introduce the order of such sequences in the natural way the question is whether exists or not the largest one. The answer is positive. We provide a description of the largest element and as a consequence we obtain that the sequences form a complete lattice.

1. Introduction

In 1948 Birkhoff in [4] considered lattices expanded with one binary operation that satisfies the basic properties of the commutator. Such lattices

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have been studied later by Czelakowski in [6, 7, 8]. Here, if we restrict our investigation to lattices that are congruence lattices of an algebra that belongs to a congruence modular variety, then we know that there are at least one expansion of such a lattice with such a binary operation, namely the commutator operation defined for example in [10, p.256], [9] or [13]. One can introduce an order in the set of all possible such binary operations in a natural way using the lattice order, see Definition 2.2. According to this order there exists the largest binary operation that expands the given complete lattice and satisfies the isolated properties of the commutator that we are going to explain in the sequel: (HC1), (HC2), (HC4) and (HC7). Czelakowski has proved that it is the join of all such binary operations, see [6, Corollary 1.5].

Generalizing the binary commutator operation, A. Bulatov introduced multi-placed commutators for an algebra $\mathbf{A}$ [5, Definition 3]. For each $k \in \mathbb{N}$, and each k-tuple $(\alpha_1, \dots, \alpha_k) \in (\mathrm{Con}(\mathbf{A}))^k$, he defined a congruence $[\alpha_1, \dots, \alpha_k]_{\mathbf{A}}$ of $\mathbf{A}$ and named it the *k-ary commutator* of $\alpha_1, \dots, \alpha_k$. When $\mathbf{A}$ has a Mal'cev term, [12, 1] discuss several properties of these higher commutators.

As in [2], with each algebra $\mathbf{A}$, we can associate the *commutator structure* of $\mathbf{A}$. This is the structure $(\mathrm{Con}(\mathbf{A}), \wedge, \vee, (f_i)_{i\in\mathbb{N}})$, where

$$f_i : (\mathrm{Con}(\mathbf{A}))^i \to \mathrm{Con}(\mathbf{A}), (\alpha_1, \dots, \alpha_i) \mapsto [\alpha_1, \dots, \alpha_i]_{\mathbf{A}};$$

$f_1(\alpha_1) = [\alpha_1]_{\mathbf{A}}$ is defined to be α_1. The sequence $(f_i)_{i\in\mathbb{N}}$ is then called the *commutator sequence* of $\mathbf{A}$. If $\mathbf{A}$ belongs to a congruence permutable variety, then for all $n, k \in \mathbb{N}$ with $k \leq n$, and for all $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathrm{Con}(\mathbf{A})$ and $\{\rho_j \mid j \in J\} \subseteq \mathrm{Con}(\mathbf{A})$, we have

- (HC1) $f_n(\alpha_1, \dots, \alpha_n) \leq \bigwedge_{j=1}^n \alpha_j$.
- (HC2) if $\alpha_1 \leq \beta_1, \dots, \alpha_n \leq \beta_n$, then $f_n(\alpha_1, \dots, \alpha_n) \leq f_n(\beta_1, \dots, \beta_n)$.
- (HC3) $f_{n+1}(\alpha_1, \dots, \alpha_{n+1}) \leq f_n(\alpha_2, \dots, \alpha_{n+1})$.
- (HC4) $f_n(\alpha_1, \dots, \alpha_n) = f_n(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$ for all $\sigma \in S_n$.
- (HC7) $f_n(\alpha_1, \dots, \alpha_{k-1}, \bigvee_{j\in J} \rho_j, \alpha_{k+1}, \dots, \alpha_n)$
  $= \bigvee_{j\in J} f_n(\alpha_1, \dots, \alpha_{k-1}, \rho_j, \alpha_{k+1}, \dots, \alpha_n)$.
- (HC8) $f_k(\alpha_1, \dots, \alpha_{k-1}, f_{n-k+1}(\alpha_k, \dots, \alpha_n)) \leq f_n(\alpha_1, \dots, \alpha_n)$.

We note that the properties (HC3) and (HC8) do not make sense for the binary commutators, because they connect the commutators of distinct arities. Also, the properties (HC5) and (HC6) listed in [1] are missing from our list here. The property (HC5) relates higher commutators to a certain centralizing relation, and (HC6) relates the commutator operations of an algebra to the commutator operations of a homomorphic image. All the properties (HC1) to (HC8) for algebras in congruence permutable varieties has been proved in [1]. A. Moorhead has proved (HC1) to (HC7) for congruence modular varieties in [11].

Let us now consider an arbitrary sequence $(f_i)_{i\in\mathbb{N}}$ of operations on a complete lattice $\mathbb{L}$ such that for each $i \in \mathbb{N}$, the function f_i is an i-ary operation on $\mathbb{L}$. We shall call such a sequence an *operation sequence* on $\mathbb{L}$. An operation sequence $(f_i)_{i\in\mathbb{N}}$ is *admissible* if it satisfies the properties (HC1), (HC3), (HC4), (HC7) and (HC8). Such a sequence of operations satisfies also (HC2) by Proposition 2.1. In [2] has been proved that the number of such sequences even on a finite lattice can be infinite and the authors have constructed examples of infinitely many admissible sequences, see the proof of [2, Theorem 3.5]. For an algebra $\mathbf{A}$ in a congruence permutable variety, the commutator sequence is an admissible sequence on the lattice $\mathbb{L} := \mathrm{Con}(\mathbf{A})$ by [1]. The order among all such sequences of operations on $\mathbb{L}$ we induce in the following way. First, for all $n \in \mathbb{N}$ we introduce the order $\leq_o$ among all n-ary members of all operation sequences in the same way as it has been introduced for binary operations by Czelakowski. Then this order induces the natural order $\leq_s$ of all the operation sequences, see Definition 2.4. One can observe that the sequence of joins of operations of the same arity is not an admissible sequence, because the property (HC8) fails. In the present note we will investigate the following problem:

> Given a complete lattice $\mathbb{L}$, is there the largest admissible sequence on $\mathbb{L}$?

This question is natural from lattice theoretic point of view. Namely, in many algebraic structures, if one look at the set of properly chosen functions over it, endowed with operations that naturally arise from the starting structure then we obtain the same algebraic structure. For example, the set of all mappings over a vector space forms again a vector space if addition of mappings and multiplying a mapping by a scalar are both induced by corresponding operations from the given vector space. In this note, we start from the complete lattice and we want to know whether those sequences form again a complete lattice. The positive answers of the proposed questions are given in the Corollary 3.8 and Theorem 3.7. This theorem states that the set of all admissible sequences together with the order induced by the lattice order forms a complete lattice.

2. Preliminaries

The next proposition is an easy observation that generalizes the same fact for binary operations from [6, p.111].

Proposition 2.1. *Let $\mathbb{L}$ be a complete lattice and let $n \in \mathbb{N}$, $n \geq 2$. If $f : L^n \to L$ satisfies (HC7) then f satisfies (HC2).*

Proof. Let $n \geq 2$ and let $f : L^n \to L$ be such that it satisfies (HC7). First we prove that for all $i \in \{1, \dots, n\}$ and all $x_1, \dots, x_n, y_i \in L$ we have

$$x_i \leq y_i \Rightarrow f(x_1, \dots, x_n) \leq f(x_1, \dots, y_i, \dots, x_n). \tag{1}$$

We suppose $x_i \leq y_i$. Then we have $y_i = x_i \vee y_i$. Now using (HC7), we obtain $f(x_1, \dots, x_n) \leq f(x_1, \dots, x_i, \dots, x_n) \vee f(x_1, \dots, y_i, \dots, x_n) = f(x_1, \dots, x_i \vee y_i, \dots, x_n) = f(x_1, \dots, y_i, \dots, x_n)$. The property (HC2) follows from n applications of (1). □

Definition 2.2. Let $(L, \wedge, \vee)$ be a complete lattice and let $n \in \mathbb{N}$. If $f, g : L^n \to L$ then we write $f \leq_o g$ if $f(x_1, \dots, x_n) \leq g(x_1, \dots, x_n)$, for all $(x_1, \dots, x_n) \in L^n$.

Note that $\leq_o$ is a partial order on the set of all operations $\mathcal{O}_n$ on L of the same arity $n \in \mathbb{N}$.

Proposition 2.3. *If $(L, \wedge, \vee)$ is a complete lattice and $n \in \mathbb{N}$ then $(\mathcal{O}_n, \leq_o)$ is a complete lattice.*

Proof. We define $f \wedge g, f \vee g \in \mathcal{O}_n$ such that

$$(f \wedge g)(x_1, \dots, x_n) := f(x_1, \dots, x_n) \wedge g(x_1, \dots, x_n)$$

and

$$(f \vee g)(x_1, \dots, x_n) := f(x_1, \dots, x_n) \vee g(x_1, \dots, x_n)$$

for all $(x_1, \ldots, x_n) \in L^n$. Clearly the lattice is complete because we define arbitrary joins and meets in the same manner. $\square$

For a given lattice $(L, \wedge, \vee)$ we denote the set of all operation sequences by $Seq(\mathbb{L})$ and the set of all admissible operation sequences by $SeqComm(\mathbb{L})$. We observe that $\bigcup Seq(\mathbb{L}) = \bigcup_{n\in\mathbb{N}} \mathcal{O}_n$. If the lattice $\mathbb{L}$ is complete, we denote the smallest element of $\mathbb{L}$ by 0. For each $n \in \mathbb{N}$ the n-ary operation $0_n : L^n \to L$ is defined such that $0_n(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in L$. The sequence $(0_n)_{n\in\mathbb{N}}$ we call the zero sequence. Obviously, $(0_n)_{n\in\mathbb{N}} \in SeqComm(\mathbb{L})$ and therefore $SeqComm(\mathbb{L}) \neq \emptyset$. In the following definition we introduce a partial order $\leq_s$ on $Seq(\mathbb{L})$ that is naturally induced by the partial order $\leq_o$.

Definition 2.4. Let $(L, \wedge, \vee)$ be a complete lattice and $n \in \mathbb{N}$. If $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are sequences of operations on L such that f_n and g_n have the same arity n, for all $n \in \mathbb{N}$ then we write $(f_n)_{n\in\mathbb{N}} \leq_s (g_n)_{n\in\mathbb{N}}$ if $f_n \leq_o g_n$ for all $n \in \mathbb{N}$.

Proposition 2.5. *Let $(L, \wedge, \vee)$ be a complete lattice. Then $(Seq(\mathbb{L}), \leq_s)$ is a complete lattice.*

Proof. We define $((f \wedge g)_n)_{n\in\mathbb{N}}, ((f \vee g)_n)_{n\in\mathbb{N}} \in Seq(\mathbb{L})$ such that $(f \wedge g)_n := f_n \wedge g_n$ and $(f \vee g)_n := f_n \vee g_n$ for all $n \in \mathbb{N}$. Clearly the lattice is complete because we define arbitrary joins and meets in the same manner. $\square$

In the next proposition the binary operations on the lattice that satisfy the properties (HC1), (HC2), (HC4) and (HC7) are denoted by $[\bullet, \bullet]_i$. In the rest of the note we denote them by f_2^i.

Proposition 2.6. *[6, Lemma 1.4] Let $\mathbb{L}$ be a complete lattice and let $Comm(\mathbb{L}) := \{[\bullet, \bullet]_i \,|\, i \in \mathcal{I}\}$ be the family of all binary operations on L that satisfy (HC1), (HC2), (HC4) and (HC7). If $\emptyset \neq \mathcal{J} \subseteq \mathcal{I}$, then $\bigvee_{i\in\mathcal{J}} [\bullet, \bullet]_i \in Comm(\mathbb{L})$.*

3. The Result

Obviously for each complete lattice $(L, \wedge, \vee)$ we have $SeqComm(\mathbb{L}) \subseteq Seq(\mathbb{L})$. We denote elements of $SeqComm(\mathbb{L})$ by $(f_n^i)_{n\in\mathbb{N}}$, $i \in I$, where

I enumerates the elements of $SeqComm(\mathbb{L})$. The next proposition one can prove straightforward.

Proposition 3.1. *Let $(L, \wedge, \vee)$ be a complete lattice. Then $(SeqComm(\mathbb{L}), \leq_s)$ is a partial order with the zero sequence as the smallest element.*

In the next definition we construct the admissible sequence that is the largest element of $SeqComm(\mathbb{L})$. This is going to be proved in the rest of the section. Each permutation $\sigma \in S_n$ of the set $\{1, \ldots, n\}$ we write in the form $(i_1, \ldots, i_n)$, where $\sigma(k) = i_k$ for all $k \in \{1, \ldots, n\}$.

Definition 3.2. Let $\mathbb{L}$ be a complete lattice and let $\emptyset \neq J \subseteq I$. We define $(\lceil , \rceil_n^J)_{n\in\mathbb{N}} \in Seq(\mathbb{L})$ by: if $n \in \{1, 2\}$, then $\lceil , \rceil_n^J := \bigvee_{i\in J} f_n^i$ (the join of the n-ary members of the sequences) and for each $n \geq 3$ we have

$$\lceil x_1, \ldots, x_n \rceil_n^J :=$$
$$:= \bigvee_{k\in\{2,\ldots,n-1\}} \bigvee_{\substack{(i_1,\ldots,i_n)\in S_n \\ i_1\leq\cdots\leq i_k \\ i_{k+1}\leq\cdots\leq i_n}} \lceil \lceil x_{i_1}, \ldots, x_{i_k} \rceil_k^J, x_{i_{k+1}}, \ldots, x_{i_n} \rceil_{n+1-k}^J$$
$$\vee \bigvee_{i\in J} f_n^i(x_1, \ldots, x_n), \quad (2)$$

for all $(x_1, \ldots, x_n) \in L^n$. If $J = I$ we omit the superscript J with no confusion.

Example. For the case $n = 3$ we take $x_1, x_2, x_3 \in L$ and observe that there is only one possibility for k, so $k = 2$. Therefore we look for all permutations of the set $\{1, 2, 3\}$ such that the first two places do not make an inversion. Hence we take the permutations $(1, 2, 3), (1, 3, 2), (2, 3, 1)$ and obtain:

$$\lceil x_1, x_2, x_3 \rceil_3$$
$$= \lceil \lceil x_1, x_2 \rceil_2, x_3 \rceil_2 \vee \lceil \lceil x_1, x_3 \rceil_2, x_2 \rceil_2 \vee \lceil \lceil x_2, x_3 \rceil_2, x_1 \rceil_2 \vee \bigvee_{i\in I} f_3^i(x_1, x_2, x_3).$$

Lemma 3.3. *Let $\mathbb{L}$ be a complete lattice and let $\emptyset \neq J \subseteq I$. Then $\lceil , \rceil_n^J$ satisfies (HC1), (HC2), (HC4), and (HC7) for every $n \in \mathbb{N}$.*

Proof. We prove the statement by induction on n. For $n = 1$ the properties (HC1), (HC2), (HC4) and (HC7) follow directly from the fact that for each $i \in J$ f_1^i satisfies the same properties. For $n = 2$ we use Proposition 2.6. Let $n \geq 3$.

(HC1) Let $(x_1, \ldots, x_n) \in L^n$, let $k \in \{2, \ldots, n-1\}$ and let $(i_1, \ldots, i_n) \in S_n$ be such that $i_1 \leq \cdots \leq i_k$ and $i_{k+1} \leq \cdots \leq i_n$. Then by the induction hypothesis we have $\lceil x_{i_1}, \ldots, x_{i_k} \rceil_k^J \leq \bigwedge_{j=1}^k x_{i_j}$ and

$$\lceil \lceil x_{i_1}, \ldots, x_{i_k} \rceil_k^J, x_{i_{k+1}}, \ldots, x_{i_n} \rceil_{n+1-k}^J \leq \lceil x_{i_1}, \ldots, x_{i_k} \rceil_k^J \wedge \bigwedge_{j=k+1}^{n} x_{i_j}.$$

Hence

$$\lceil \lceil x_{i_1}, \ldots, x_{i_k} \rceil_k^J, x_{i_{k+1}}, \ldots, x_{i_n} \rceil_{n+1-k}^J \leq \bigwedge_{j=1}^{k} x_{i_j} \wedge \bigwedge_{j=k+1}^{n} x_{i_j} = \bigwedge_{j=1}^{n} x_j.$$

Since f_n^i satisfies (HC1) for all $i \in J$ we obtain $\lceil x_1, \ldots, x_n \rceil_n^J \leq \bigwedge_{j=1}^n x_j$.

(HC4) We are going to prove the property for the case of transposition. Let $(x_1, \ldots, x_n) \in L^n$, and let $m, p \in \{1, \ldots, n\}$ be such that $m < p$. We have to prove that

$$\lceil x_1, \ldots, x_p, \ldots, x_m, \ldots, x_n \rceil_n^J = \lceil x_1, \ldots, x_m, \ldots, x_p, \ldots, x_n \rceil_n^J.$$

If we fix $k \in \{2, \ldots, n-1\}$ then for each $(i_1, \ldots, i_n) \in S_n$ such that $i_1 \leq \cdots \leq i_k$ and $i_{k+1} \leq \cdots \leq i_n$ we have the following cases. If $m, p \in \{i_1, \ldots, i_k\}$ or $m, p \in \{i_{k+1}, \ldots, i_n\}$ we apply the induction hypothesis on $\lceil , \rceil_k^J$ or $\lceil , \rceil_{n+1-k}^J$, respectively. If neither $m, p \in \{i_1, \ldots, i_k\}$ nor $m, p \in \{i_{k+1}, \ldots, i_n\}$ we observe that for each permutation $(i_1, \ldots, i_n) \in S_n$ such that $i_1 \leq \cdots \leq i_k$, $i_{k+1} \leq \cdots \leq i_n$ and $p \in \{i_1, \ldots, i_k\}$ and $m \in \{i_{k+1}, \ldots, i_n\}$ there is the unique permutation $(j_1, \ldots, j_n) \in S_n$ such that $j_1 \leq \cdots \leq j_k$, $j_{k+1} \leq \cdots \leq j_n$ and $\{j_1, \ldots, j_k\} = \{i_1, \ldots, i_k\} \backslash \{p\} \cup \{m\}$ and $\{j_{k+1}, \ldots, j_n\} = \{i_{k+1}, \ldots, i_n\} \backslash \{m\} \cup \{p\}$. Therefore the both disjuncts occur in

$$\bigvee_{\substack{(i_1, \ldots, i_n) \in S_n \\ i_1 \leq \cdots \leq i_k \\ i_{k+1} \leq \cdots \leq i_n}} \lceil \lceil x_{i_1}, \ldots, x_{i_k} \rceil_k^J, x_{i_{k+1}}, \ldots, x_{i_n} \rceil_{n+1-k}^J.$$

Since all f^i_n satisfy (HC4) we obtain that the form (2) is the same for both $\lceil x_1,\dots,x_p,\dots,x_m,\dots,x_n\rceil^J_n$ and $\lceil x_1,\dots,x_m,\dots,x_p,\dots,x_n\rceil^J_n$.

(HC7) Without loss of generality we are going to prove the distributivity on the first argument. Let $\{x^j_1 \mid j \in J_1\} \cup \{x_2,\dots,x_n\} \subseteq L$, let $k \in \{2,\dots,n-1\}$ and let $(i_1,\dots,i_n) \in S_n$ be such that $i_1 \le \cdots \le i_k$ and $i_{k+1} \le \cdots \le i_n$. Then $1 = i_t$ for a $t \in \{1,\dots,n\}$. We analyze the most complicated case $t \in \{1,\dots,k\}$. By the induction hypothesis we have

$$\begin{aligned}&\lceil x_1,\dots,x_{i_t-1},\bigvee_{j\in J_1} x^j_1, x_{i_t+1},\dots,x_k\rceil^J_k\\ &\quad= \bigvee_{j\in J_1}\lceil x_1,\dots,x_{i_t-1},x^j_1,x_{i_t+1},\dots,x_k\rceil^J_k\end{aligned}$$

and therefore

$$\begin{aligned}&\lceil\lceil x_1,\dots,x_{i_t-1},\bigvee_{j\in J_1} x^j_1,x_{i_t+1},\dots,x_k\rceil^J_k,x_{k+1},\dots,x_n\rceil^J_{n-k+1}\\ &\quad=\lceil\bigvee_{j\in J_1}\lceil x_1,\dots,x_{i_t-1},x^j_1,x_{i_t+1},\dots,x_k\rceil^J_k,x_{k+1},\dots,x_n\rceil^J_{n-k+1}\\ &\quad=\bigvee_{j\in J_1}\lceil\lceil x_1,\dots,x_{i_t-1},x^j_1,x_{i_t+1},\dots,x_k\rceil^J_k,x_{k+1},\dots,x_n\rceil^J_{n-k+1}\end{aligned}$$

Also $f^i_n(\bigvee_{j\in J_1} x^j_1,x_2,\dots,x_n) = \bigvee_{j\in J_1} f^i_n(x^j_1,\dots,x_n)$ for all $i \in J$.

(HC2) follows from (HC7) by Proposition 2.1. □

Lemma 3.4. *Let $\mathbb{L}$ be a complete lattice and let $\emptyset \neq J \subseteq I$. Then $(\lceil,\rceil^J_n)_{n\in\mathbb{N}}$ satisfies (HC3).*

Proof. We prove the statement by induction on n. For $n = 1$ the statement is true as a consequence of (HC3) for each f^i_1 and f^i_2 for all $i \in J$. Let $n \ge 2$. Let $x_1,\dots,x_n \in L$. We have to prove that $\lceil x_1,\dots,x_n\rceil^J_n \le \lceil x_2,\dots,x_n\rceil^J_{n-1}$. For every $k \in \{2,\dots,n-1\}$ and every permutation $(i_1,\dots,i_n)$ of the set $\{1,\dots,n\}$ such that $i_1 \le \cdots \le i_k$ and $i_{k+1} \le \cdots \le i_n$ we have two cases:

(1) $1 = i_t, t \in \{1,\dots,k\}$: By the induction hypothesis and (HC4) we know that

$$\begin{aligned}\lceil x_{i_1},\dots,x_{i_k}\rceil^J_k &= \lceil x_{i_t},x_{i_1},\dots,x_{i_{t-1}},x_{i_{t+1}},\dots,x_{i_k}\rceil^J_k\\ &\le \lceil x_{i_1},\dots,x_{i_{t-1}},x_{i_{t+1}},\dots,x_{i_k}\rceil^J_{k-1}.\end{aligned}$$

Using (HC2) and also (HC1) if $k = 2$, we obtain

$$\begin{aligned}\lceil\lceil x_{i_1}, \ldots, x_{i_k}\rceil_k^J, x_{i_{k+1}}, \ldots, x_{i_n}\rceil_{n+1-k}^J \\ \leq \lceil\lceil x_{i_1}, \ldots, x_{i_{t-1}}, x_{i_{t+1}}, \ldots, x_{i_k}\rceil_{k-1}^J, x_{i_{k+1}}, \ldots, x_{i_n}\rceil_{n+1-k}^J \\ \leq \lceil x_2, \ldots, x_n\rceil_{n-1}^J,\end{aligned}$$

by the definition of $\lceil,\rceil_{n-1}^J$.

(2) $1 = i_t, t \in \{k+1, \ldots, n\}$: Using (HC4) we have

$$\begin{aligned}\lceil\lceil x_{i_1}, \ldots, x_{i_k}\rceil_k^J, x_{i_{k+1}}, \ldots, x_{i_n}\rceil_{n+1-k}^J \\ = \lceil x_{i_t}, \lceil x_{i_1}, \ldots, x_{i_k}\rceil_k^J, x_{i_{k+1}}, \ldots, x_{i_{t-1}}, x_{i_{t+1}}, \ldots, x_{i_n}\rceil_{n+1-k}^J.\end{aligned}$$

By the induction hypothesis and the definition of $\lceil,\rceil_{n-1}^J$ we know that

$$\begin{aligned}\lceil x_{i_t}, \lceil x_{i_1}, \ldots, x_{i_k}\rceil_k^J, x_{i_{k+1}}, \ldots, x_{i_{t-1}}, x_{i_{t+1}}, \ldots, x_{i_n}\rceil_{n+1-k}^J \\ \leq \lceil\lceil x_{i_1}, \ldots, x_{i_k}\rceil_k^J, x_{i_{k+1}}, \ldots, x_{i_{t-1}}, x_{i_{t+1}}, \ldots, x_{i_n}\rceil_{n-k}^J \\ \leq \lceil x_2, \ldots, x_n\rceil_{n-1}^J.\end{aligned}$$

Since $(f_n^i)_{n\in\mathbb{N}} \in SeqComm(\mathbb{L})$ we have $f_n^i(x_1, \ldots, x_n) \leq f_{n-1}^i(x_2, \ldots, x_n)$ for all $i \in J$ by (HC3). Therefore, we obtain

$$\bigvee_{i\in J} f_n^i(x_1, \ldots, x_n) \leq \lceil x_2, \ldots, x_n\rceil_{n-1}^J$$

by the definition of $\lceil,\rceil_{n-1}^J$. □

Lemma 3.5. *Let $\mathbb{L}$ be a complete lattice and let $\emptyset \neq J \subseteq I$. Then $(\lceil,\rceil_n^J)_{n\in\mathbb{N}}$ satisfies (HC8).*

Proof. Let $n, m \in \mathbb{N}$ and let $x_1, \ldots, x_n \in L$. If $1 < m < n$ then

$$\lceil\lceil x_1, \ldots, x_m\rceil_m^J, x_{m+1}, \ldots, x_n\rceil_{n-m+1}^J \leq \lceil x_1, \ldots, x_n\rceil_n^J$$

by the definition of $\lceil,\rceil_n^J$. If $m = 1 < n$ then we obtain the inequality by (HC1) and (HC2). Finally, if $m = n$ the inequality follows just from (HC1). □

Proposition 3.6. *Let* $\mathbb{L}$ *be a complete lattice and let* $\emptyset \neq J \subseteq I$*. Then* $(\lceil,\rceil_n^J)_{n\in\mathbb{N}} \in SeqComm(\mathbb{L})$*.*

Proof. We use Lemma 3.3, 3.4, and 3.5. □

Theorem 3.7. *Let* $\mathbb{L}$ *be a complete lattice. Then* $(SeqComm(\mathbb{L}), \leq_s)$ *is complete lattice.*

Proof. Using Proposition 3.1 and [3, Theorem 4.2] it is enough to prove that there is a supremum of arbitrary subset of $SeqComm(\mathbb{L})$. We let $J \subseteq I$. If $J = \emptyset$ then $\bigvee \emptyset$ is the smallest element and it is the zero sequence as we have already stated in Proposition 3.1. Now, let $J \neq \emptyset$. We know that the sequence $(\lceil,\rceil_n^J)_{n\in\mathbb{N}} \in SeqComm(\mathbb{L})$ by Proposition 3.6. Let $j \in J$. Obviously, for each $n \in \mathbb{N}$, $f_n^j \leq_o \bigvee_{i\in J} f_n^i \leq_o \lceil,\rceil_n^J$ by the definition of $\lceil,\rceil_n^J$ and therefore $(f_n^i)_{n\in\mathbb{N}} \leq_s (\lceil,\rceil_n^J)_{n\in\mathbb{N}}$ for all $i \in J$. Let us suppose that $(g_n)_{n\in\mathbb{N}} \in SeqComm(\mathbb{L})$ is such that $(f_n^i)_{n\in\mathbb{N}} \leq_s (g_n)_{n\in\mathbb{N}}$ for all $i \in J$. We prove by induction on n that $\lceil,\rceil_n^J \leq_o g_n$ for all $n \in \mathbb{N}$. For $n \in \{1,2\}$ and for all $i \in J$ we have $f_n^i \leq_o g_n$ by the assumption and hence $\lceil,\rceil_n^J = \bigvee_{i\in J} f_n^i \leq_o g_n$. Therefore, the base is true. Let $n \geq 3$. For every $k \in \{2,\dots,n-1\}$ and every permutation $(i_1,\dots,i_n)$ of the set $\{1,\dots,n\}$ such that $i_1 \leq \dots \leq i_k$ and $i_{k+1} \leq \dots \leq i_n$ we have $\lceil x_{i_1},\dots,x_{i_k}\rceil_k^J \leq g_k(x_{i_1},\dots,x_{i_k})$ and

$$\begin{aligned}&\lceil\lceil x_{i_1},\dots,x_{i_k}\rceil_k^J, x_{i_{k+1}},\dots,x_{i_n}\rceil_{n+1-k}^J\\ &\quad\leq\ g_{n+1-k}(\lceil x_{i_1},\dots,x_{i_k}\rceil_k^J, x_{i_{k+1}},\dots,x_{i_n})\end{aligned}$$

for all $(x_1,\dots,x_n) \in L^n$ by the induction hypothesis. Hence by (HC2), (HC8) and (HC4) we obtain

$$\begin{aligned}&\lceil\lceil x_{i_1},\dots,x_{i_k}\rceil_k^J, x_{i_{k+1}},\dots,x_{i_n}\rceil_{n+1-k}^J\\ &\qquad\leq g_{n+1-k}(g_k(x_{i_1},\dots,x_{i_k}), x_{i_{k+1}},\dots,x_{i_n})\\ &\qquad\qquad\leq g_n(x_{i_1},\dots,x_{i_n}) = g_n(x_1,\dots,x_n)\end{aligned}$$

for all $(x_1,\dots,x_n) \in L^n$. Of course $\bigvee_{i\in J} f_n^i \leq_o g_n$ by the assumption for $(g_n)_{n\in\mathbb{N}}$ and therefore we obtain $\lceil,\rceil_n^J \leq_o g_n$. This finishes the induction proof. Therefore, $(\lceil,\rceil_n^J)_{n\in\mathbb{N}} \leq_s (g_n)_{n\in\mathbb{N}}$. □

Corollary 3.8. *Let* $\mathbb{L}$ *be a complete lattice. Then there is the largest element in* $(SeqComm(\mathbb{L}), \leq_s)$*.*

Proof. The largest element is $(\lceil,\rceil_n)_{n\in\mathbb{N}}$. □

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