Network Coding Theory: A Survey

Riccardo Bassoli, Hugo Marques, Jonathan Rodriguez, Kenneth W. Shum and Rahim Tafazolli

Abstract—This article surveys all known fields of network coding theory and leads the reader through the antecedents of the network coding theory to the most recent results, considering also information theory and matroid theory. By focusing on providing ideas and not formulas, this survey is both fitted for the taste of readers who are mathematically oriented and newcomers to the area. Additionally, this survey also includes an innovative and clear graph representation of the most prominent literature on network coding theory, its relevance and evolution from the very beginning till today.

Index Terms—Network coding theory, information theory, random linear network coding, network error correcting (NEC) codes, capacity region, complexity.

I. Introduction

SINCE the publication of Claude E. Shannon's paper entitled "A Mathematical Theory of Communication" [1], information theory science has born and digital communications have started. The transmission of information through a network would then be interpreted as an exchange of commodities, without the capability of combining or mixing what was sent (*commodity flow*). In 2000, the seminal article [2] changed this perspective by introducing the concept of *information flow* to demonstrate that the combination of information could increase the capacity of a network over the limit achieved by routing. This extension represented the birth of a new promising area of research, regarded as network coding. Prior to that, the family of coding operations was constituted only by: *source coding*, the way to compress the information at the source to increase the efficiency in the transmission, and *channel coding*, the operation of introducing redundant bits in the information sequence to make it reliable by converting the noisy channel into a noiseless one. Network coding opened the way to another coding operation, realised at the packet level: the principal idea is to allow nodes in the network to perform some coding operations. Hence, a node can transmit functions of the messages received earlier on the ingoing edges, onto the outgoing ones.

During the last decade, interest towards network coding theory and its application became increasingly widespread. This

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spurred the use of new mathematical tools (mainly algebra, matroid theory, geometry, graph theory, combinatorics and optimization theory among others) resulting in the incarnation that network coding is today: a broad and complex field rich in mathematical idiom. Moreover, the strong connection with information theory and its important recent results played a main role in leading network coding to its current level.

This survey embraces more than 300 references considered by the authors to be the most relevant literature in the different areas of network coding theory including recent results in information theory. At the best of the authors' knowledge, this is the first work to make such a comprehensive description of network coding theoretic developments including both the pioneering works and the very latest results achieved in this field. The motivation to write such survey is to provide a guide that could serve both the purpose of synchronizing advanced researchers in this area as well as to help initial researchers in approaching the study of the theory. Other relevant surveys on the theoretic areas of network coding are [3]–[6]. If the reader searches surveys more focused on applications then the following are recommended [7], [8]. As for books and tutorials on network coding the following are advised [9]–[15], and the most recent [16]. Finally, network coding website [17] contains an updated list with most of the literature on the field.

A. Network Coding Theory

Figure 1 depicts a visual description of the evolution of network coding theory from the very beginning till today, in which labels are used to identify the articles and to underline the 'major players' in the different areas.

The publication of [2] is the commencement of network coding theory: in fact, this article is the first to refer to this novel research field as 'network coding'. Ahlswede *et al.* revealed a new research topic by studying the problem of characterizing the coding rate region of a multicast scenario. The main result of the authors consisted in a max-flow min-cut theorem, which interpreted the flow of information as something not to be merely routed and replicated anymore. However, that was not only a starting point but also a point of arrival: [2] took advantage of many concepts in information theory, distributed coding and distributed data storage systems, developed during the previous years. First, the geometric framework and the set theoretic one, which were developed to simplify the solution of information theoretic problems, improved the methods to prove converse coding theorems and to calculate coding rate regions. Next, the models used for distributed coding and distributed data storage systems provided special instances to develop the general one deployed by [2].

Subsequently, [18] described an optimal solution achieving the max-flow min-cut bound for directed acyclic graphs: the
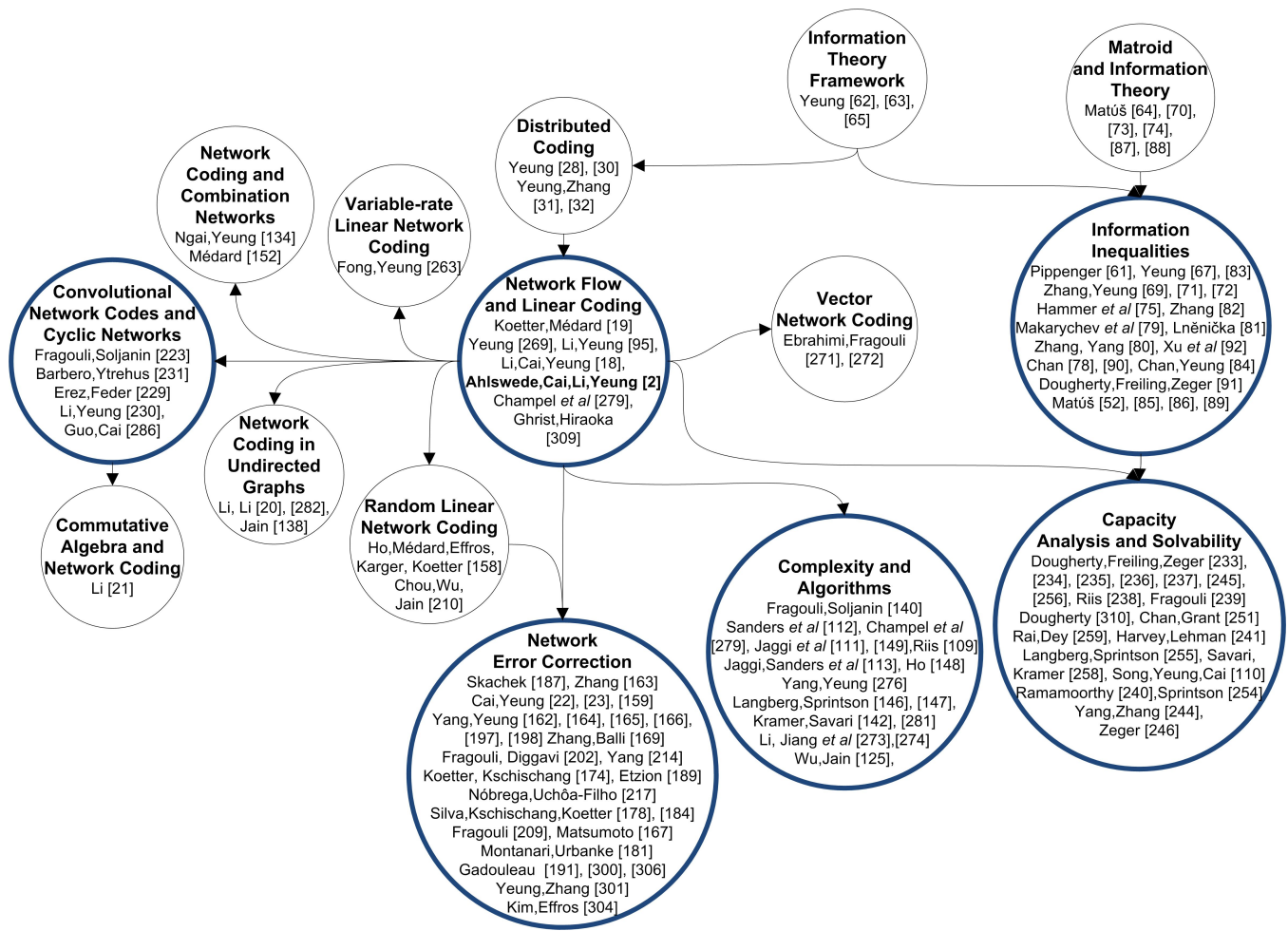
Fig. 1. The graph depicts the evolution of network coding theory from the first related results in information theory to the most recent articles. A selection of the literature is sorted by topic studied and the circles in blue are put in evidence because of the amount of research interested. The labels are chosen in a subjective way, to put in underline the 'major players' and because of clarity (it would have been impossible to put all the name in the references).

optimum was obtained by applying a linear code multicast, a network code obtained by linearly combine information by using coefficients chosen from a finite field. That definition of linear network code mainly deployed tools from graph theory and algebra of vectors. [18] started the formulation of the concepts of the theoretic framework used by deterministic network coding. Among the works involved to build the general theory of deterministic linear network coding, the new approach of [19] opened another way for network coding theory: instead of using some elements from algebra as [18], [19] developed a completely algebraic framework by making connections with algebraic geometry and matrix theory. The fundamental results in [19] prepared the fertile ground for the formulation of random linear network coding. This family of network codes has the main characteristic of randomly choosing the coefficients of linear combinations to get the important benefit of being suitable in dynamic environments. On the other side, a drawback comes: in fact, an error decoding probability was introduced, which depends on the size of the finite field chosen for the code.

The investigation into network flows and linear network coding stated the general theoretic background to apply network coding on directed acyclic scenarios: combination networks raised particular interest because of the various

areas of application. Nevertheless, [20] started the analysis of behaviours and benefits of network coding in undirected networks. Side by side, convolutional network codes were proposed as a better solution than classical block network codes, in directed cyclic graphs: by taking into account cycles, the information in the network experiences delays, that are different from zero. So, the time dependance of the model introduces a trellis structure, which becomes similar to a trellis diagram of a convolutional code. Lately, [21] demonstrated that acyclic linear network coding theory can be extended to cyclic scenarios: precisely, convolutional network codes come to be an instance of network codes defined via commutative algebra.

Network coding theory enlarged its area of interest when [22], [23] showed that network codes represent a generalisation of classical error correcting codes. The attention in network error correction coding was due to its potentials in correcting random errors, erasures and errors introduced by malicious nodes. The main characteristic of the network extension of error correction is that redundancy is in space domain instead of in time domain.

After its foundation, the study of network coding focused on other fundamental theoretic topics to better understand its potentials and limits such as the coding capacity achievable

in different scenarios compared with the one achievable with classical store-and-forward routing, and the conditions under which a network is solvable. The link with matroid theory and the results obtained in information theory about information inequalities provided researchers a strong background to face the capacity and solvability issues of network coding.

Next to the pure theory of network coding, a branch of investigation was focusing on algorithmic implementation of network codes. The complexity of the algorithms is influenced by the size of the finite field required by the network code and depends on the number of sinks in the communication. Other elements affecting the complexity are also the number of edges and the transmission rate of the source. The deployment of many tools from combinatorics and graph theory helped the design of algorithms with reduced complexity.

Recently, vector network coding and variable-rate network coding appeared on the scene. The former is the transmission of vectors, whose elements are on a fixed finite field, combined together at intermediate nodes by using matrices instead of scalar coefficients. This new approach showed particular benefits. First, the complexity of network codes could be reduced because fixed finite fields of small size can be used. Next, in some scenarios, scalar linear network coding is not enough to solve the network and non-linear network coding is necessary; actually, vector network coding represents a way to achieve solvability in those networks by avoiding non-linear network coding. On the other side, the research in variable-rate network coding consisted in the study of the properties of codes with different rates for different sessions in order to enhance fixed rate linear network codes.

B. Structure of the Survey

Even though network coding is considered to be a new research field, the amount of literature on the topic has already achieved significant numbers. This survey will only focus on the theoretic part: applications are deliberately not mentioned. Network coding theory is a complex subject, consisting of several different areas based on different mathematical theories. By this same reason, some characteristics of this work may seem different from the ones owned by a 'classical' survey. At this point, it is important to emphasize that the way chosen to present the theory of network coding is subjective and from own authors perspective: next to the historical approach in showing the developments and the milestones in the research, the presentation is split into main thematic areas to maintain the description coherent, clear and comprehensible.

Section II proceeds with the description of the antecedents that led to the seminal work [2] in order to actively involve the reader in the history of network coding theory. Section III begins with the mathematical preliminaries necessary to understand the fundamentals and recent ramifications of the topic. In this particular field of research, the concepts needed are from multiple subjects: the section is divided into parts, providing the theoretic fundamentals by keeping the number of formulas and the complexity to a minimum. In fact, the aim is to explain the most important basic ideas and to suggest good references to readers interested in delving into. Section IV describes the milestones in information theory that provided results and tools that were important for network coding. Next, Section V shows the developments of network coding theory. The principal branches of network coding theory that were considered are: the investigation of the algorithmic complexity and of the complexity of the codes designed (translated into reducing the size of the alphabet of the code), the research on the behaviours of network coding in undirected graphs and on random linear network coding and network error correcting codes an important generalisation of 'classical' forward error correction; moreover, the analysis of network coding in cyclic networks, the solvability of network coding problems for different scenarios and the definition of the capacity region of the codes are equally important. Finally, Section VI presents the latest results in network coding theory and outlines possible future directions in this research field.

II. Network Coding: the Antecedents

After the year 2000, with the publication of [2], considerable research efforts in network coding theory started. However, that was not an isolated work because it represented the meeting point of years of research in other related fields. In fact, some previous works in distributed and diversity coding played a main role to arrive at that seminal result.

A *diversity coding system* (DCS) is a system in which the data streams of the source are independent and identically distributed, there are different encoders, which encode the information source, and multiple decoders, which should be able to reconstruct perfectly what the source sent; each decoder can only access to a particular subset of the encoders. Moreover, if there is more than a single level of decoders, the system is called multilevel diversity coding (MDC). On the other hand, it is possible to consider a distributed coding system with correlated discrete-alphabet information data sequences with a common decoder: in this case, the behaviours of the system are described by using the Slepian-Wolf results in [24], [25]. The results obtained in these subjects are usually presented in terms of coding rate regions: given a number of sources n, the *admissible coding rate region* is the region represented by the points of the space (i.e. n-dimensional vectors that have the rates as coordinates), in which it is achievable an arbitrarily small decoding error probability with block codes. In Figure 2, examples are depicted to clarify the previous concepts.

After the first fundamental work [26], in 1964, diversity coding theory started to be investigated and applied in different areas, such as distributed information storage, reliability of computer networks, secret sharing. Before [27], in 1992, and later [28], in 1995, introduced the concept of MDC for the first time, respectively, in the case of reliability of storing in distributed disks and in case of satellite networks. In particular, [28] was the first application of MDC considering distortion and elaborating a principle of superposition to calculate the rate region by taking into account independent source streams: the author gave a direct proof of the coding rate region by using the information diagram[1] (*I*-Diagram) and, in the

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[1]Given two random variables X and Y, their *I*-Diagram (see Figure 12) can be obtained by applying a formal substitution of symbols to the Shannon's inequalities and a substitution of the information functions with a function μ. This real function is a signed measure on a field of sets. A *field of sets* is a collection of subsets F of a set A, which is closed under binary unions, intersections and complements.
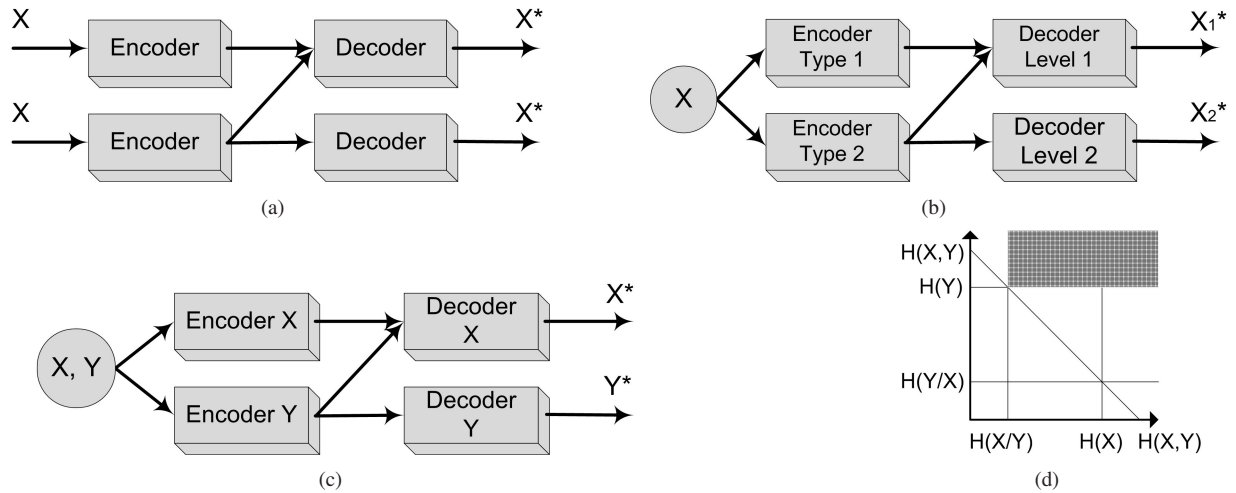
Fig. 2. (a) Single level DCS (b) MDCS, in which there are two levels of decoders: the decoder 'Level 1' can access only to the information of the encoders of 'Type 1', instead of the decoder 'Level 2', which has to access at least to one encoder 'Type 2'. Decoders of the same level recover the source information in the same way (c) A block scheme of a correlated source coding system (d) Bidimensional graphical representation of the admissible coding rate region of the system in (c).

appendix, by using the Slepian-Wolf result. The superposition approach consists of coding individual information sources separately, so that the rate contributing to code one stream does not contribute to code the other ones. This article was the one that inspired the first ideas for the elaboration of network coding.

In 1997, [29] studied the optimality of the principle of superposition by considering two- and three-level diversity coding with three encoders, to find that this method is optimal in the 86 percent of the configurations. The same year, [30] characterized the admissible rate region in a three-level DCS, with symmetrical connections between encoders and decoders; the authors also demonstrated the optimality of superposition coding for this scheme and, with [30, Theorem 2], they tried to generalise the result for a k-level problem, with $k \geq 2$; therefore, they defined a lower bound for the admissible rate region because of the difficulty and the amount of computation needed to reach a complete characterization. Few years later, [31] demonstrated that coding by superposition is optimal in a general symmetrical MDC scenario. Moreover, the authors solved the issue that [30] met before, by specifying the coding rate region by extreme points[2]; they changed the approach and successfully characterize the region for $k \geq 2$, as an intersection of infinite halfspaces. In the hypotheses, the source data streams were mutually independent and the decoders were almost perfectly reconstructing them, rather than in [26], where the condition was to perfectly reconstruct the source data streams. During the same year, [32] investigated the characterization of the admissible rate region in case of a distributed source code in a satellite scenario; the authors calculated an inner [32, Theorem 1] and an outer [32, Theorem 2] bound of that region in terms of the entropy regions Γ_n^* and $\bar{\Gamma}_n^*$ (an explicit evaluation was provided for the special case of linear programming bound). Finally, they proposed a geometrical interpretation of these bounds by using intersections of hyperplanes.

[2]A point **x** in a convex set $X \subset \mathbb{R}^n$ is an *extreme point* if it cannot be represented by the convex combination of two distinct points in the set.
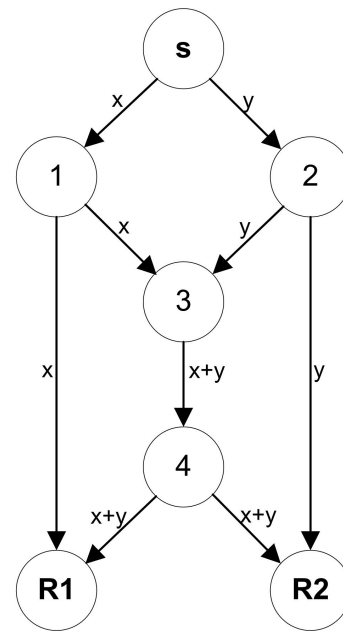

Fig. 3. Representation of network coding, applied to a butterfly network. The edges can only transmit one bit per time unit, so the optimality can be achieved only by linearly combining the information at node 3.

Next, in 2000, [2] faced the issue of defining the admissible coding rate region for MDC without taking into account rate distortion: its scenario was different from a generalisation of the Slepian-Wolf problem because it did not use correlated sources and, the network configuration and the reconstruction requirements on the decoder side were arbitrary. The authors' main result was the elaboration of a new version of max-flow min-cut theorem (Theorem 1) by considering information something that can be coded and combined and not a commodity anymore [2, Conjecture 1]. They demonstrated the capability of the novel 'network coding' to increase throughput provided by classical store-and-forward routing in a 'butterfly network' (Figure 3). In graph theory, a *commodity* is a triple (s_i, t_i, d_i), where s is a source, t is a sink and d is the amount

of flow to be routed, called the *demand*. A *multicommodity flow network* $G = (V, \mathcal{E}, \mathcal{K}, c)$ is a directed graph[3] with vertex set V, edge set $\mathcal{E}$, commodity set $\mathcal{K}$ and a capacity function c. Then, a *multicommodity flow* in a multicommodity flow network $G = (V, \mathcal{E}, \mathcal{K}, c)$ is a set of $k = |\mathcal{K}|$ functions $f_i : \mathcal{E} \to \mathbb{R}^+$, where $i = 1, \ldots, k$, satisfying the following constraints:

- (*joint capacity constraints*) the sum of the values of the flows of the k commodities cannot exceed the value of the capacity of the edge, for each edge $(u, v) \in \mathcal{E}$;
- (*conservation constraints*) the sum of the values of the outgoing flows of the different commodities of a vertex v cannot exceed the sum of the values of the ingoing flows of the different commodities, for each vertex $v \in V - \{s_i, t_i\}$.

In 2002, [33] found an information theoretic upper bound on the information flow for discrete memoryless networks, with multiple sources and multiple sinks: its achievable rate region can reproduce the one shown by [2] if only one source is multicasting the information. Moreover, the author expressed that in a network with noisy links, the bound can be achieved also with the use, before, of channel coding. In 2000, the same result for point-to-point networks was achieved by [34].

Readers can find a good tutorial on the origins of network coding in [35].

III. PRELIMINARIES

The aim of the following subsections will be to provide the initial mathematical principles necessary to understand the results and developments of network coding, shown in the next sections. In particular, the following fields will be addressed:

- Algebra
- Algebraic geometry
- Graph theory
- Combinatorics
- Polytopes
- Projective geometry
- Matroid theory
- Coding theory

The current section will also give interested readers several important references to find full mathematical explanation of the concepts.

A. Algebra and Algebraic Geometry

The following definitions from abstract algebra are chosen from [36]–[40], the ones from algebraic geometry can be found in [41]. The initial focus of this subsection is to expose different algebraic structures with their respective characteristics.

Rings[4], ideals, integral domains, principal ideal domains (PID) and fields are the main structures in algebra and are of vital importance in network coding theory. Figure 4 summarises their hierarchy and their characteristics.

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[3]A *graph* $G(V, \mathcal{E})$ is a mathematical object consisting of two sets: the one called V, whose elements are the vertices or nodes, and the other one called $\mathcal{E}$, whose elements are the edges. If directions are assigned to the edges, the graph is called a *directed graph*.

[4]In the rest of the Survey, all the rings mentioned are commutative rings with unity.
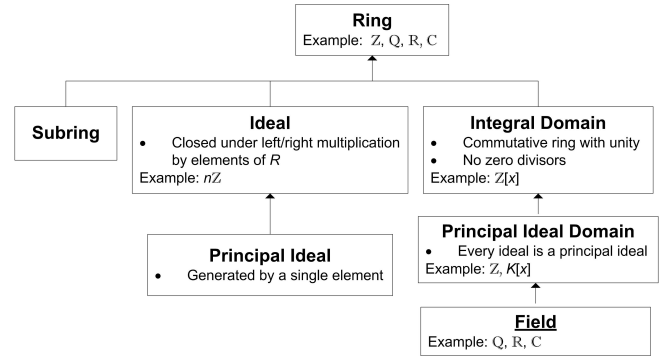

Fig. 4. Representation of the hierarchy of some algebraic structures with their characteristics.

Another main object in algebraic geometry, also important for the study of algebraic sets, is the Grassmannian. The set $\mathcal{G}(m, n)$ is defined as the set of m-dimensional linear subspaces of the vector space $\mathcal{V}$ of dimension n over a field K, and it is called the *Grassmannian* of $\mathcal{V}$. The Grassmannian space allows us to treat subspaces of a vector space without defining any basis for $\mathcal{V}$.

B. Graph Theory and Combinatorics

The study of graphs, networks, with their related combinatorial concepts, is a main issue in theoretic network coding. Consequently, two important theorems taken from [42]–[45] are provided below.

The problem of how to maximize the flow in a network was solved independently by [46] in 1955 and by [47] in 1956. The following theorem shows their result.

Theorem 1 (*Max-Flow Min-Cut Theorem*)**.** In a network, the value of the maximum flow is equal to the capacity of the minimum cut.

A simple example of the application of Theorem 1 and the explanation of some related concepts, are shown in Figure 5.

Next, the following theorem, which connects determinants of matrices and matchings in a graph, is presented to clarify some results of network coding, written in the rest of the article. Before, it is important to state that a *matching* in a graph G is a subset $\mathcal{M}$ of $\mathcal{E}$ such that no two edges in $\mathcal{M}$ have an endpoint in common. A matching in a graph is *perfect* if every vertex is an endpoint of one of the edges.

Theorem 2 (*Edmonds' Theorem* [43])**.** Consider a bipartite graph $G(U, V, \mathcal{E})$, with U and V the two independent sets of vertices with the same cardinality. Let $\mathbf{A}$ be the $n \times n$ matrix obtained from $G(U, V, \mathcal{E})$ as follows:

$$A_{ij} = \begin{cases} x_{ij}, & (u_i, v_j) \in \mathcal{E} \\ 0, & (u_i, v_j) \notin \mathcal{E} \end{cases} \tag{1}$$

Define the multivariate polynomial $Q = \{x_{11}, x_{12}, \ldots, x_{nn}\}$ as being equal to $\det(\mathbf{A})$. Then G has a perfect matching if and only if $Q \neq 0$.

C. Polytopes and Projective Geometry

Polytopes theory studies the properties of convex polytopes, which are important objects in geometry; nevertheless, their
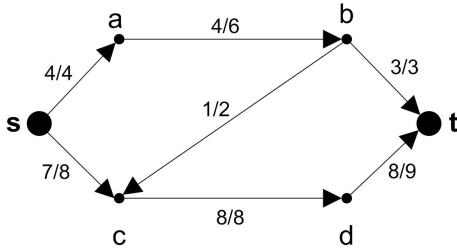
Examples of cut-sets:
{(s,a),(s,c)},
{(a,b),(s,c)},
{(b,t),(b,c),(s,c)},
{(b,t), (c,d)}.

Fig. 5. A directed network with a source *s* and a destination *t*. The values assigned to the edges represent respectively the flow function and the capacity of the edge. A cut-set in the network is a collection of edges, which once deleted divides the network into different parts containing *s* and *t* separated. The *capacity* of a cut is the sum of the capacities of the edges in the cut. Some cut-sets are listed below the graph: in the figure, the *minimum cut-set* is $\{(b, t), (c, d)\}$, whose edges are saturated and their minimum value of the flow is equal to 11. So, the maximum possible flow is the value 11 of the minimum cut-set.

characteristics are very interesting not only in pure geometry but also because of their several applications: information theory and capacity regions analysis of codes are taking advantage of the tools used to study those geometric objects. Projective geometry investigates the behaviours of geometric figures, when geometric transformations are applied. The definitions reported in this subsection can be found in [48] and [49].

A *V-polytope* is the convex hull[5] of a finite set of points in some $\mathbb{R}^n$. An *H-polyhedron* is an intersection of finitely many closed halfspaces in some $\mathbb{R}^n$. An *H-polytope* is an *H*-polyhedron that is *bounded*, i.e. it does not contain a ray $\{\mathbf{x} + t\mathbf{y} : t \geq 0\}$ for any $\mathbf{y} \neq \mathbf{0}$. Thus, a polytope is a point set $P \subset \mathbb{R}^n$, which can be seen either as a *V-polytope* or as an *H*-polytope. In Figure 6, it is shown the difference between the two ways to construct a polytope.

Then, a *cone* is a nonempty set of vectors in $\mathbb{R}^n$, which with any finite set of vectors also contains all their linear combinations with nonnegative coefficients. Every cone contains also the null vector. The forthcoming description of projective geometry is limited to the bidimensional case in order to provide a simpler explanation.

For a given field K, the projective plane $PG(2, K)$ is the same as the Grassmannian $\mathcal{G}(1, 3)$. A *point* in the projective plane can be represented in homogeneous coordinates $(x : y : z)$, with $xyz \neq 0$. Two homogeneous coordinates $(x_1 : y_1 : z_1)$ and $(x_2 : y_2 : z_2)$ are equivalent if there is a nonzero element α in K such that $x_1 = \alpha x_2$, $y_1 = \alpha y_2$, and $z_1 = \alpha z_2$. The projective plane can be regarded as the union of the affine plane and a line at infinity. The affine plane $\{(x, y) : x, y \in K\}$ can be embedded in $PG(2, K)$ by mapping $(x, y) \mapsto (x : y : 1)$. The line at infinity consists of points with homogeneous coordinates $(x : y : 0)$, with x and y not both zero. A *line*

[5]The smallest convex set containing a point set $A \subset \mathbb{R}^n$ is called the *convex hull* of A.
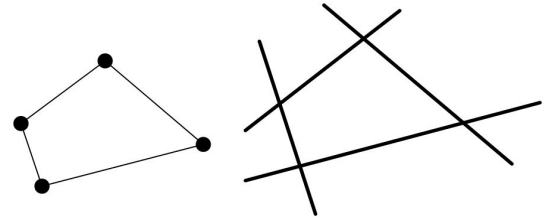

Fig. 6. The representation tries to clarify the difference between the two definitions of polytopes. On the left, the quadrilateral is built by following the definition of *V*-polytope. On the right, the polytope is constructed as the intersection of four halfspaces, bounded by the four lines, by using the definition of *H*-polytope.
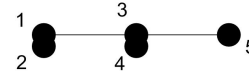

Fig. 7. Geometric representation of the vector matroid $\mathrm{M}(A)$.

in the projective plane is a set of points $(x : y : z)$ satisfying $ax + by + cz = 0$ for some constants a, b and c, with $abc \neq 0$. The basic property of a projective plane is that any pair of two distinct points are contained in a unique line, and any two pair of two distinct lines intersect at a unique point.

D. Matroid Theory

Matroid theory is a discipline which generalises the ideas of linear dependance and linear independence, by linking different fields in mathematics such as linear algebra, graph theory and combinatorial geometry. Because of the scope of this work, the discussion is limited to finite matroids; a reader interested in infinite matroids could start reading , [50, Chapter 3]. Most of the definitions and theorems, which are going to be presented, are taken from [51] and [52].

Before formally describing matroids, we consider two particular examples. The first one is algebraic and the second one is graphic. Consider the columns in the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & 2 \end{bmatrix} \quad (2)$$

with entries in the finite field of size 3, $\mathbb{F}_3$. Note that the first two columns represent the same point in the projective plane over $\mathbb{F}_3$. The third and fourth columns also represent the same point. Column 5 is the sum of columns 1 and 3. These points are collinear in the projective plane. We define a *vector matroid* by defining a *ground set* E, which consists of the columns of the above matrix, and a collection of subsets in E, called the set of *independent sets* $\mathcal{I}$, which consists of linearly independent subsets of E. A subset of E is called *dependent* if it is not in $\mathcal{I}$. A minimal dependent set (with respect to set inclusion) is called a *circuit* and a maximal independent set is called a *basis*. In the above example, if we label the columns by $1, 2, ..., 5$ the ground set is $E = \{1, 2, 3, 4, 5\}$, the circuits are $\{1, 2\}$, $\{3, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 5\}$ and $\{2, 4, 5\}$, and the bases are $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 5\}$, $\{4, 5\}$. Then, the representation of the *vector matroid* of the matrix $\mathbf{A}$ is depicted in Figure 7. Another
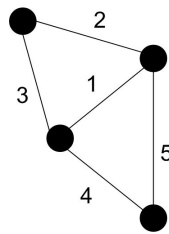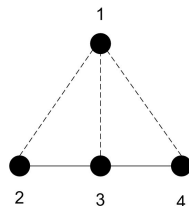
Fig. 8. An undirected graph $G(V, \mathcal{E})$.

Fig. 9. An example of deletion operation in a matroid. The element 1 is deleted and the result is a matroid with $E - X = \{2, 3, 4\}$.

way to define a matroid is to obtain it from a graph. The second example is called a *cycle matroid*, constructed from a graph $G(V, \mathcal{E})$ depicted in Figure 8. In this case, the ground set is identified with the edge set, and the set of circuits in the matroid is the same as the set of cycle in the graph, i.e., $\mathcal{C} = \{\{1, 2, 3\}, \{1, 4, 5\}\}$. The set of bases is $\mathcal{B} = \{\{1, 3, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\}, \{4, 1, 2\}, \{4, 5, 2\}\}$.

After the previous introductory examples, some formal concepts need to be exposed. A matroid M is an ordered pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a collection of subsets of E, satisfying the following conditions:

- $\emptyset \in \mathcal{I}$;
- if $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$;
- (*Independence augmentation axiom*) if I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

It follows immediately from the independence augmentation axiom that if B_1 and B_2 are two bases in a matroid, then B_1 and B_2 have the same cardinality.

If M is representable over a field F, M is said to be F-*representable*. Next, two matroids $(E, \mathcal{I})$ and $(E', \mathcal{I}')$ are said to be *isomorphic* if there exists a bijection $f : E \to E'$ such that $I \in \mathcal{I}$ if and only if $f(I) \in \mathcal{I}'$. A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*. An important characteristic of graphic matroids is that it is possible to determine many properties of such matroids from the pictures of the graph.

An efficient way to specify a matroid is to use its maximal independent sets (bases). Let $\mathrm{M}|X^6$ be a matroid and let the function $r(X)$ be the rank of X, that is, the size of a basis B of $\mathrm{M}|X$. In particular, $r : 2^E \to \mathbb{Z}_+ \cup \{0\}$ is the rank function of a matroid on E if and only if satisfies:

- if $X \subseteq E$, then $0 \leq r(X) \leq |X|$;
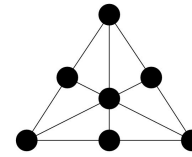- if $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$;

---

[6]The operator '|' represents the *restriction* of M to X or *deletion* of $E - X$ from M, given a subset X of E. An example is shown in Figure 9.
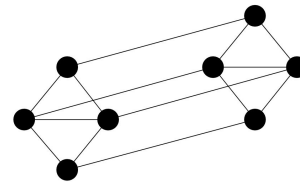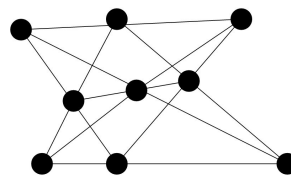
Vamos matroid
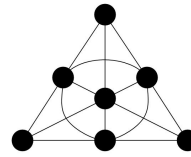
Fano matroid   non Fano matroid
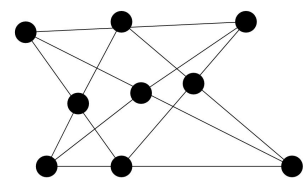
Pappus matroid   non Pappus matroid

Fig. 10. Geometric representations of some important matroids.

- if X and Y are subsets of E, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

In case of representable matroids, the matroid rank function, just shown, coincides with the notion of linear rank of the corresponding subset of vectors, that is, the maximal number of linearly independent vectors contained in the set.

The subsequent theorem is an important relation between rank functions, that needs to be mentioned to make understandable some results that are shown in the next sections in network coding theory. The same inequality can also be rewritten in terms of dimension of subspaces.

Theorem 3 (*Ingleton's Inequality*)**.** Let (E, r) be a representable matroid, then, given the subsets $X_1, X_2, X_3, X_4 \subseteq E$, the following inequality is verified:

$$
\begin{aligned}
r(X_1) + r(X_2) + r(X_3 \cup X_4) + r(X_1 \cup X_2 \cup X_3) + \\
+ r(X_1 \cup X_2 \cup X_4) \leq r(X_1 \cup X_2) + \\
+ r(X_1 \cup X_3) + r(X_1 \cup X_4) + r(X_2 \cup X_3) + \\
+ r(X_2 \cup X_4).
\end{aligned}
\tag{3}
$$

In Figure 10 there are some examples of important matroids: the Vamos matroid (or Vamos cube) is one of the smallest non-representable matroid, the Fano matroid corresponds to the 7-point projective plane (Fano plane) and the Pappus matroid is called in this way because of its relationship with the Pappus configuration in projective geometry.

Furthermore, another important concept to report is the one of polymatroid. Given a partially ordered set S, a real-valued function ρ is called a β-*function*, if it satisfies the following conditions:

- $\rho(a) \geq 0$ for every $a \in A = S - \emptyset$;

- it is non-decreasing, that is, $a \leq b \Rightarrow \rho(a) \leq \rho(b)$;
- it is submodular, that is, it verifies the inequality $\rho(a \vee b) + \rho(a \wedge b) \leq \rho(a) + \rho(b)$ for every $a, b \in S$;
- $\rho(\emptyset) = 0$.

Let E be a nonempty finite set and let $\rho : 2^E \rightarrow \mathbb{R}_+$ be a submodular function such that $\rho(A) + \rho(B) - \rho(A \cup B) - \rho(A \cap B) \geq 0$, with $A, B \subseteq E$. Hence, ρ is a β-function, and the pair (E, ρ) is called *polymatroid*, where E is the *ground set* and ρ is the *ground set rank function* or simply the rank function. Especially, a polymatroid becomes a matroid if $\rho(A) \leq |A|$ and $\rho(A) \in \mathbb{Z}$ (constraints due to the definition of rank function for matroids, written above in this subsection). Next, a polymatroid (E, ρ) is said to be *ingletonian* if it satisfies the Ingleton's inequality (Theorem 3), given all the subsets $A, B, C, D \subseteq E$.

E. Coding Theory

The forthcoming theoretic fundamentals in coding theory can be found reading [53]–[55].

A *q-ary linear code* C is a linear subspace of the n-dimensional vector space $\mathbb{F}_q^n$; if its dimension is k, it is called $C(n, k)$. After that, the *minimum distance* of a code is $\min \{d(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}$, where $d(\mathbf{x}, \mathbf{y})$ is the *Hamming distance* between vector x and y. A *generator matrix* $\mathbf{G}$ for a linear code is a k by n matrix for which the rows are a basis of the code.

After having introduced the previous general definitions, it is necessary to discuss some other properties of codes, such as the bounds on the size of a code. Firstly, let (n, M, d) code be a code of length n consisting of M codewords with minimum distance d. $A_q(n, d)$ is defined as $A_q(n, d) := \max \{M : \text{ an } (n, M, d) \text{ code exists}\}$. Next, a lower bound for these codes is given by the *Gilbert-Varshamov bound*

$$A_q(n, d) \geq \frac{q^n}{V_q(n, d-1)} \qquad (4)$$

where $V_q(n, d-1)$ is the cardinality of the set $S_{d-1}(\mathbf{x})$[7]. Moreover, there are upper bounds for the numbers $A_q(n, d)$, which are interesting to be mentioned in this discourse. Firstly, the one called Singleton bound: by applying the puncturing operation $d - 1$ times on an (n, M, d) code, the output is an $(n - d + 1, M, \geq 1)$ code. Because of that, the number of punctured words becomes $M \leq q^{n-d+1}$, hence, the *Singleton bound* is

$$A_q(n, d) \leq q^{n-d+1}. \qquad (5)$$

By considering a linear code $C(n, k)$, this bound becomes $k \leq n - d + 1$.

A linear code, which has $d = n - k + 1$, is called *maximum distance separable* (MDS) code. The main property of this family of linear codes is that it maximizes the possible distance between the codewords. Given $q, n, e \in \mathbb{N}$, with $q \geq 2$ and $d = 2e + 1$, the well-known *Hamming* (or *sphere-packing*) *bound* is

$$A_q(n, d) \leq \frac{q^n}{V_q(n, e)}. \qquad (6)$$

---

[7]The sphere $S_r(\mathbf{x})$ is the set of words $\mathbf{y} \in \mathbb{F}_q^n$, with centre $\mathbf{x}$ and radius r.

A linear code has to satisfy the condition $d \leq n - k + 1$ because of the Singleton bound.

At this point let discuss a famous subset of the trellis codes family, called *convolutional codes*. The fundamental characteristic of this kind of codes is that the length of the codewords is not fixed but it grows linearly with the length of the messages. Specially, the encoding process is realised by a sequential circuit, described by a state diagram, instead of the temporal evolution of the convolutional code, which is represented by a trellis diagram. In Figure 11, an example of convolutional code is shown for clarification.

Finally, some definitions about rank-metric codes are presented. As in the previous description about classical linear codes, even for array codes it is important to start the discussion by defining a metric. Hence, let $\mathbf{V}, \mathbf{Z}$ be matrices in $F_q^{n \times m}$, then the rank distance between them is defined as

$$d_R(\mathbf{V}, \mathbf{Z}) := \text{rank}(\mathbf{Z} - \mathbf{V}). \qquad (7)$$

A *linear array code* (or *matrix code*) $C(n \times m, k, d)$ is defined as a k-dimensional linear subspace of $F_q^{n \times m}$, with the minimum weight d of any nonzero matrix in C, called the *minimum distance* of the code [56]. Because of the use of metric (7), an array code can also be called a *rank-metric* code and obtaining the minimum rank distance for a rank-metric code will be straightforward. Analogously to classical coding theory, even for rank-metric codes were obtained upper and lower bounds such as Hamming, Singleton and Gilbert-Varshamov bounds. A description of these bounds in the case of rank-metric codes can be read in [57], [58]. A rank-metric code achieving the Singleton bound is called *maximum rank distance* (MRD) code. A subclass of MRD codes were constructed by [59] and [56] respectively in 1985 and 1991, later called Gabidulin codes. A rank-metric code, which have all elements of the same rank, is called *constant-rank code* (CRC).

IV. ENTROPY AND INFORMATION INEQUALITIES

The information theoretic achievements, which delved into the properties of Shannon's information measures, represent fundamental contributions in the progress of network coding theory. The characterization of the coding rate regions, the capacity achievable in different scenarios and, more generally, the solvability of a network coding problem reached the actual solutions mostly due to the theoretic results presented in this section.

In 1948, C. E. Shannon [1] provided the mathematical definition of *entropy* — a fundamental concept to measure the quantity of information we have when a message is received — given a set of n discrete random variables. In the same article, another main concept was explained, called *capacity*, the maximum quantity of information it can be transmitted reliably onto a communication channel. [1] represents the birth of information theory.

Then, in 1978, [60] demonstrated that the entropy function h is a β-function and, hence, the pair (E, h) is a polymatroid, where E is a finite set of random variables; studying entropy functions as rank functions of polymatroids, it became a useful way to analyse information theoretic problems. The main
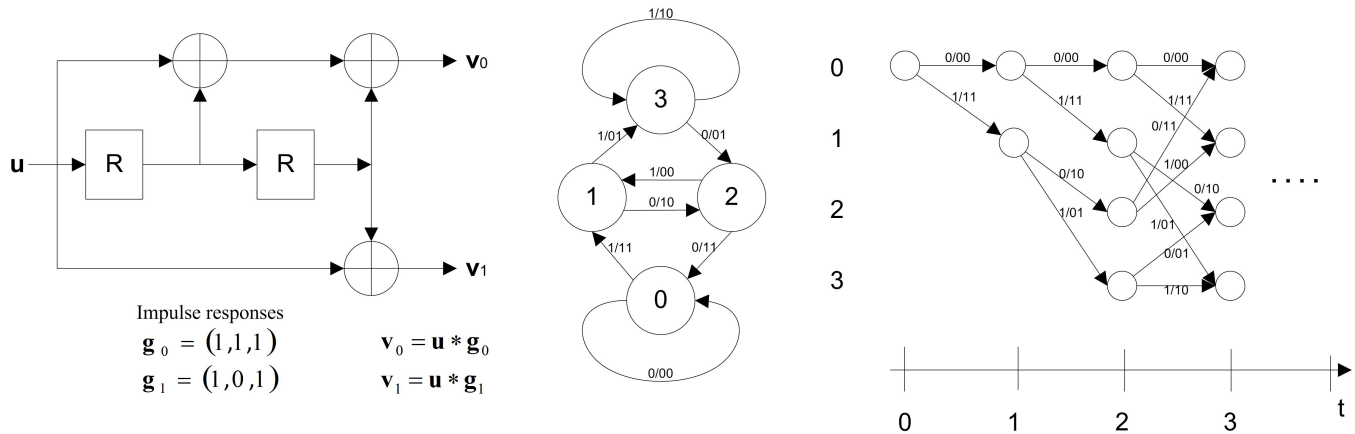
Impulse responses
$$\mathbf{g}_0 = (1,1,1) \qquad \mathbf{v}_0 = \mathbf{u} * \mathbf{g}_0$$
$$\mathbf{g}_1 = (1,0,1) \qquad \mathbf{v}_1 = \mathbf{u} * \mathbf{g}_1$$

Fig. 11. Example of convolutional code $(2,1,2)$. On the right there is the scheme of the encoder, in which the elements of the *information sequence* $\mathbf{u}$ enter sequentially. Then, they pass through the shift registers R, which introduce the delay, and finally, they are encoded into the elements of two codewords $\mathbf{v}_0$ and $\mathbf{v}_1$. As expressed by the figure, $\mathbf{v}_0$, $\mathbf{v}_1$ are calculated as a convolution between the impulse responses (or generator sequences) of the code and the information sequence. This is the reason why this family of codes is called convolutional. In the center it is printed the state diagram of the code, and on the left it is printed its trellis diagram to show the time dependance of the code. In the labels of the arrows, the first value is the input and the other two values are the consequent outputs.

contribution of this work was to bridge matroid theory and information theory.

The quantities entropy, conditional entropy, mutual information and conditional mutual information, are called *Shannon's information measures*. The identities and inequalities, involving only Shannon's information measures, are called respectively *information identities* and *information inequalities*. All the inequalities obtained by combining in some way the basic inequalities, are called *Shannon-type inequalities*. The main role of these inequalities is: they are necessary in the proofs of the converse coding theorems to solve problems in information theory. This is the reason why they were called "laws of information theory" by [61], because they provide the constraints to be fulfilled in information theory. The issue posted by the author was: the entropy functions satisfies the three polymatroids axioms, but are they the only inequalities or there are other conditions?

At the beginning of the 1990s [62], [63] established a direct relation between Shannon's information measures and the general set theory: in this way, the manipulation of Shannon's information measures could be translated into set operations. An example to clarify this result is in Figure 12. In 1994, [64] explained connections between conditional independence structures and matroid theory. Then, few years later, [65] provided for the first time a geometrical framework to investigate information inequalities. According to this framework, the region Γ_n is the set of all basic linear inequalities in the entropy space $\mathcal{H}_n$. This geometrical approach implied a unique description of all the, unconstrained and constrained[8], basic information inequalities. Thanks to that, an unconstrained identity can be seen as a hyperplane in the entropy space and an unconstrained inequality is a halfspace containing the origin; on the other side, a constrained identity is an intersection contained in the hyperplane defined by the equality and the constrained inequality is a linear subspace in $\mathcal{H}_n$. The

new geometrical approach opened also the way to a linear programming approach and, consequently, the possibility of proving all the Shannon-type inequalities through a software package; in fact, a software for MATLAB, called ITIP [66], [67] (Information Theoretic Inequality Prover), and a C-based linear programming solver, called Xitip [68], were developed to prove information theoretic inequalities.

The issue of the full characterization of the region $\bar{\Gamma}_n^{*}$[9] is an important task and, at that time, the main information theoretic problem to solve, became whether or not $\bar{\Gamma}_n^{*} = \Gamma_n$. Therefore, in 1997, the two main results [69, Theorem 1-2] demonstrated, respectively, that $\bar{\Gamma}_n^{*}$ is a convex cone, and, in case of 3-dimensional space, $\Gamma_3^{*} \neq \Gamma_3$ but with $\bar{\Gamma}_3^{*} = \Gamma_3$; in particular, [69, Theorem 2] was a consequence of the main result in [64, Section 5]. Moreover, the results in [64] contributed for finding an inner bound on the entropic region via Ingleton inequalities [70]. By investigating further the previous characterization, the authors of [69] conjectured that $\bar{\Gamma}_n^{*}$ is not sufficient for characterizing all the information inequalities but only to characterize all unconstrained information inequalities, that is, $\bar{\Gamma}_n^{*} \neq \Gamma_n$ for $n > 3$. Because of this, they discovered a new conditional inequality, not implied by the basic information inequalities, so finding an answer to the problem risen before in [61]. The information inequalities, which are not implied by the basic inequalities, are called *non-Shannon-type inequalities*; these authors presented, for the first time, a constrained non-Shannon-type inequality of four random variables [69, Theorem 3]. Later, [71] showed a new inequality in a 4-dimensional space, answering that $\bar{\Gamma}_n^{*} \subset \Gamma_n$; the article shows the first unconstrained non-Shannon-type inequality. In the literature, this unconstrained non-Shannon-type inequality is normally named Zhang-Yeung inequality. The same authors in [72] showed that this unconditional inequality implied a class of 2^{14} non-Shannon-type inequalities. Meanwhile, [73] derived new properties for polymatroids from the inequalities in [69], [71]; moreover, this work presented a constrained

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[8]The geometric region Γ_n can be under a constraint Φ, otherwise, if $\Phi = \mathcal{H}_n$, there are no constraints on the entropies.

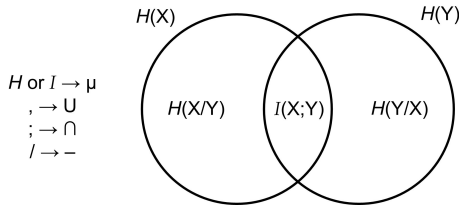[9]$\bar{\Gamma}_n^{*}$ is the closure of the set Γ_n^{*}.

Fig. 12. On the left side of the image, it is represented the correspondence of the information theoretic functions and operators with the set function and operators. On the right side, the *I*-Diagram for two random variables *X* and *Y* is depicted.

non-Shannon-type inequality, which is a variation of the constrained inequality in [69, Theorem 3]. In order to understand the notation of [73] it is useful to read [74].

Next, the work in [75] gave a combinatorial interpretation for a type of linear inequalities, which described the behaviours of both Shannon entropy and Kolmogorov complexity [76], [77]. Afterwards, [78] explained a combinatorial characterization of the class of combinatorially characterizable entropy functions. This work gave a further contribution in the analysis of Γ_n^*.

In 2002, [79] exhibits a new family of unconstrained inequalities for Shannon entropy involving five random variables [79, Theorem 1] and, for the proof, it used an inference rule, which was a refinement and a generalisation of the Zhang-Yeung inequality. Another proof of the Zhang-Yeung inequality was also given later in [80]. Moreover, [81] extended the Zhang-Yeung unconstrained inequality to gaussian variables.

In 2003, [82] gave a further generalisation of the non-Shannon-type inequality of five variables, found before by [79].

Contemporarily, before in 2000 and then in 2002, [83], [84] described an original correspondence between information inequalities and group inequalities.

In 2005, [85] explained some results obtained about information inequalities in the context of polymatroids. In the same context of polymatroids, one year later, [86] showed a constrained non-Shannon-type information inequality in three random variables.

Next, in 2007, [87] considered a ground set of polymatroids $|E| = 4$, to say that if the Zhang-Yeung inequality is violated by a polymatroid, then it is not Shannon-type, and $cl(\boldsymbol{H}_n^{ent}) \subseteq \boldsymbol{H}_n$[10]. [88] obtained inner bounds for the region $cl(\boldsymbol{H}_n^{ent})$[11] in the context of asymptotically entropic polymatroids. The same author, in [89], showed an infinite class of unconstrained non-Shannon-type inequalities.

During the same year, [90] extended the definition of group characterization in [83], [84] to abelian groups and it analysed the behaviours of this algebraic approach in case of vector spaces; finally, the author introduced an 'extension method' (generally speaking with the use of extension and projection processes), through which he was able to obtain the Ingleton's inequality, the Zhang-Yeung inequality and the inequalities in [91]. In fact, the year before, [91] discovered

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[10]The set $\boldsymbol{H}_n$ is a polyhedral cone in the Euclidean space $\mathbb{R}^{2^n}$.

[11]$cl(\boldsymbol{H}_n^{ent})$ is the *closure* of the set $\boldsymbol{H}_n^{ent}$, which is the set of all the entropy functions of random vectors indexed by finite set and taking finite number of values.

six new four-variables unconstrained non-Shannon-type information inequalities; the authors used the software ITIP and the Markov chain construction of [71]. Then, in 2008, an extension of the work in [91] appeared in [92]. The authors proposed a new method to derive non-Shannon-type information inequalities and they obtained also new four-variable non-Shannon-type inequalities by applying the Convex Hull Method (CHM). The application of projection was not limited to [92], because also in [79] the authors applied an inference rule with some requirements, which can be seen as a specific range of a projection.

For a reader interested to get a strong background to face the topics of this subsection, it is very useful to read [93, Chapter 2-3-13-14-15-16] and [94].

V. DEVELOPMENTS IN NETWORK CODING THEORY

The following sections explain the different theoretic developments that came out after the seminal article [2]. In order to provide a clearer and not fragmented presentation, the results are described with a chronological approach, side by side with a separation into thematic areas.

A. The Beginning of the Story

Contemporaneously to the work of [2], in 1998, [95] defined, for the first time, a linear network code for multicasting information in a network and it introduced the *law of information conservation at a node*, which claims that the information going out from a set of nodes of the network has to be the information accumulated at the input of the same set of nodes; so, in the special case of linear codes, the vectors assigned to the outgoing channels of a node of the network are the linear combination of the vectors assigned to the ingoing channels of the same node. Moreover, the authors demonstrated that the max-flow is an upper bound on the information rate received by each nonsource node; next, they showed how to achieve this bound and an optimal solution to multicast by using a generic linear code multicast (LCM). By facing the issue of transmitting information in a cyclic network, the authors proposed a solution, consisting in time-slotted code operations at the nodes and in using time-slotted transmission channels. Another approach to solve the 'cyclic problem' was the implementation of time-invariant LCM, that is, instead of being an assignment of a vector space to the nodes and vectors to the channels, the code is constructed as an assignment, to the nodes, of a module over a PID, with all of its ideals generated by the power z^n, and, to the channels, of vectors. In the acyclic scenario, the authors linked generic LCM and matroid theory: in fact, they took advantage of this relation, obtained by applying a greedy algorithm [52, Chapter 1], to prove the following fundamental result [95, Theorem 5.1]: a generic LCM exists on every acyclic communication network, provided that the base field of the vector space is an infinite or a large enough field. Finally, they showed an application to a combination network to exemplify this result. Later, in 2003, [18] improved the results, stated before in [95]. Then, the theoretic framework, described firstly in [18], was enhanced few years later in [96]: this work defined the concepts of linear multicast, linear broadcast and linear

dispersion for a generic linear network code. An year later, [97] contributed to clarify the previous theoretic concepts of network coding: in fact, the authors explained the relation between a linear dispersion and a generic network coding, finding also a relation on the sizes of the base fields of the code. The complete definition of these theoretic fundamentals can be found in [93, Chapter 19].

By coming back to 2001, [98] proposed an extension of [32] by calculating inner and outer bounds on the admissible rate region in terms of Γ_n^* [98, Theorem 2]. The authors also faced, for the first time, the characterization of the admissibility of a multi-source multi-sink network code. Hence, in 2004, [99] studied the characterization of the achievable rate region for the multi-source multi-sink scenarios, with two sinks.

Later, in 2002, [100] introduced the novel idea of network error correcting (NEC) codes, as a generalisation of classical error coding theory. After the definition of a t-error-correcting network code, they generalised the Hamming bound [100, Theorem 2] and the Gilbert-Varshamov bound [100, Theorem 5] to the context of LCM.

On another context, a year before, [101] presented a different approach to network coding, deriving an algebraic framework to verify the feasibility of the multicast problem: although [2], [95], [18] employed some algebraic concepts (based on vector spaces), the new framework resulted to be completely algebraic (matrices and algebraic sets[12]), with the consequent possibility to apply the mathematical theorems of algebra on network coding. The algebraic framework was developed, firstly, in 2002 by [102], and next, in 2003 by [19]. Their aim was to solve the most general network coding problem, that is, having only a graph of the network G with arbitrary connections (before them, only [32] studied the characterization of the achievable set of connections in an arbitrary case). The main object of this framework is the transfer matrix $\mathbf{M}$ [19, Theorem 3] of the network, which includes all the characteristics of the network itself in its structure. Then, [19, Theorem 2] demonstrates the connection between this pure algebraic concept and the previous approach through Theorem 1. So, the translation of the max-flow min-cut theorem into the new framework modified the network coding problem into the problem of finding a point on an algebraic set. In particular, after having defined the ideal of the linear network coding problem, which have an algebraic set associated to itself, in [19, Theorem 7], the solvability was reduced to find if the algebraic set is empty or not; in order to solve this issue, the authors suggested the application of the Buchberger's algorithm to compute the Groebner basis of the ideal; but, unfortunately, the complexity of the algorithm is not polynomially bounded. Specially, a set $\{g_1, \ldots, g_t\} \subset \mathcal{I}$ is a *Groebner basis* of $\mathcal{I}$ if and only if for every nonzero f

in $\mathcal{I}$, the leading term[13] of f is divisible by the leading term of g_i for some i. Furthermore, given the set of polynomials $\{f_1, \ldots, f_s\}$, $\mathcal{I} = \langle f_1, \ldots, f_s \rangle \neq \{0\}$ is a polynomial ideal. Finally, a Groebner basis for $\mathcal{I}$ can be constructed in a finite number of steps by the Buchberger's algorithm (see [38, Chapter 2]). Figure 13 compares the two main frameworks (the one using vectors and graph theory and the one completely algebraic) to define a network code multicast. Their main characteristics and the differences between them are shown by applying them to a butterfly scenario.

Subsequently, [103] improved the algebraic framework of [19] by providing two results, representing different ways to answer the multicast problem in a linear network coding scenario. The formulation of feasibility of the multicast connection problem in terms of network flows opened the way to the definition of another technique to calculate the determinant of the network transfer matrix $\mathbf{M}$, by using Edmonds' matrices. The simplifications obtained with this formulation allowed to get some characteristics of the transfer matrix determinant polynomials, without using matrix products and inversions. Hence, these deductions [103, Theorem 2] led to the work in [104] about a randomized approach to network coding. Next, with [103, Theorem 3] the authors proposed a new upper bound for the coding field size better than the previous one defined in [102].

In the same year, [105] presented a lower bound required for the alphabet size of the network codes, demonstrating, by using graph colouring theory, that it is computationally hard to determine exactly the alphabet size required; moreover, it described how a smaller alphabet can be achieved through nonlinear network codes instead of linear ones. The authors of this work provided also a taxonomy to classify network information problems, dividing them into: problems with trivial network coding solutions, polynomial time solvable linear coding problems and hard linear network coding problems. Finally, they found [105, Theorem 4.5] that some information flow problems are unsolvable with linear network codes. After that, [106] answered to the issues, raised in [105], with an example, which showed that the failure of linearity in non-multicast networks was only caused by the restrictive prior definition of linearity. In fact, until that time, the coding operations were actuated symbol by symbol, but, by grouping the symbols into vectors before coding, it discovered how to overcome the problem. So, the authors found that linear network coding suffices for network coding on arbitrary networks using linear coding solutions on vector spaces [106, Conjecture]. Finally, they provided a coding theorem to investigate the sufficiency of linear coding in non-multicast networks.

[107] faced the problem of the separation among source coding, channel coding and network coding by demonstrating theorems to allow joint source-channel code, built in a common framework with linear network coding. Next, they showed some scenarios, in which, in order to achieve optimality, it is necessary to employ a joint system between network coding and source coding or between network coding

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[12]Algebraic sets are geometric objects described by polynomial equations. So, consider K a field and $f_1, \ldots, f_s$ polynomials over K in the variables $x_1, \ldots, x_n$: hence, the object defined as $\mathbf{V}(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in K^n : f_i(a_1, \ldots, a_n) = 0\}$ with $1 \leq i \leq s$, is an *algebraic set* defined by the polynomials $f_1, \ldots, f_s$. An example is $\mathbf{V}(x^2 + y^2 - 9)$, which is the circle of radius 3, centered in the origin.

[13]The *leading term* of a polynomial – with respect of lexicographical order – is the monomial with maximum degree in the considered variable. For example, the leading term of the polynomial $f = 8x^4 + 6x^3y + 2xz^3 - y^2z$ with respect to the lexicographical order $x \succ y \succ z$ is $8x^4$.

Linear Code Multicast

Algebraic approach

$$f_{(o,s)} = f_{(s,1)} = f_{(1,3)} = f_{(1,R1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad K_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$f_{(o,s)'} = f_{(s,2)} = f_{(2,3)} = f_{(2,R2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad K_1 = K_2 = K_4 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$f_{(3,4)} = f_{(4,R1)} = f_{(4,R2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad K_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Y(e_1) = \alpha_{1,e_1} X_{(o,s)} + \alpha_{2,e_1} X_{(o,s)'} \qquad Y(e_3) = \beta_{e_1,e_3} Y(e_1)$$
$$Y(e_2) = \alpha_{1,e_2} X_{(o,s)} + \alpha_{2,e_2} X_{(o,s)'} \qquad Y(e_4) = \beta_{e_2,e_4} Y(e_2)$$
$$Z_{(R1,o)} = \varepsilon_{e_7,1} Y(e_7) + \varepsilon_{e_8,1} Y(e_8) \qquad Y(e_7) = \beta_{e_1,e_7} Y(e_1)$$
$$Z_{(R1,o)'} = \varepsilon_{e_7,2} Y(e_7) + \varepsilon_{e_8,2} Y(e_8) \qquad Y(e_6) = \beta_{e_2,e_6} Y(e_2)$$
$$Z_{(R2,o)} = \varepsilon_{e_6,1} Y(e_6) + \varepsilon_{e_9,1} Y(e_9) \qquad Y(e_5) = \beta_{e_3,e_5} Y(e_3) + \beta_{e_4,e_5} Y(e_4)$$
$$Z_{(R2,o)'} = \varepsilon_{e_6,2} Y(e_6) + \varepsilon_{e_9,2} Y(e_9) \qquad Y(e_8) = \beta_{e_5,e_8} Y(e_5)$$
$$Y(e_9) = \beta_{e_5,e_9} Y(e_5)$$

$$\mathbf{A} = \begin{bmatrix} \alpha_{1,e_1} & \alpha_{1,e_2} \\ \alpha_{2,e_1} & \alpha_{2,e_2} \end{bmatrix} \qquad \mathbf{B}_1 = \begin{bmatrix} \varepsilon_{e_7,1} & \varepsilon_{e_7,2} \\ \varepsilon_{e_8,1} & \varepsilon_{e_8,2} \end{bmatrix} \qquad \mathbf{M}_1 = \mathbf{A}(\mathbf{I} - \mathbf{F})^{-1} \mathbf{B}_1^{\mathsf{T}}$$

$$\mathbf{B}_2 = \begin{bmatrix} \varepsilon_{e_6,1} & \varepsilon_{e_6,2} \\ \varepsilon_{e_9,1} & \varepsilon_{e_9,2} \end{bmatrix} \qquad \mathbf{M}_2 = \mathbf{A}(\mathbf{I} - \mathbf{F})^{-1} \mathbf{B}_2^{\mathsf{T}}$$
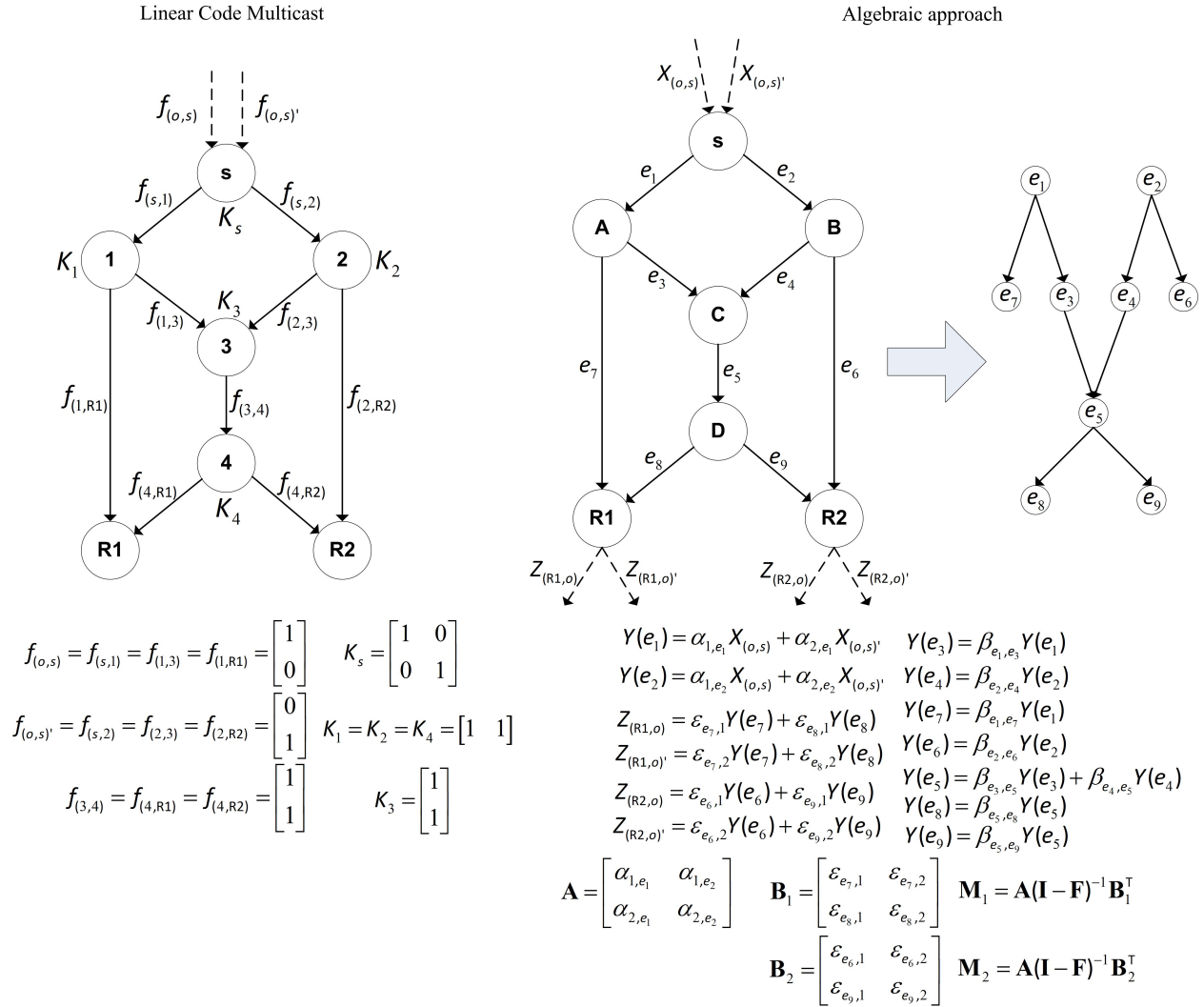
Fig. 13. An example to clarify the differences between the two main frameworks for network coding. On the left, a linear code multicast defined as a local and global encoding mapping: *K* represents the *local encoding kernel* at a node and *f* is the *global encoding kernel* at an edge. Local and global encoding kernels describe the 2-dimensional linear network code for this multicast scenario with a source and two sinks. The ingoing channels of the source are called *imaginary channels* because they have no originating nodes. On the right, a description of the linear network code through the algebraic framework. *X* is the source random process, *Y* is random process at an edge and *Z* is the random process collected by a sink. Next, α, β and ε are constant coefficients in $\mathbb{F}_2$. These coefficients represent the elements of matrices **A**, **B** and **F**. In particular, matrix **F** is the adjacency matrix of the directed labeled line graph, obtained from the graph of the network (see the graph on the extreme right.). Finally, it is possible to see how the transfer matrix **M** is calculated by using the coefficients α, β and ε of the linear combinations.

and channel coding. A summary of that work can be found in [108].

Contemporary but independently to [105], while it was investigating the matrix transposition problem, [109] found that there are network flow problems without scalar solutions but, as it was described in the example in [106], these networks have vectorial solutions only for blocks of dimension greater than 1. Moreover, the author, together with Ahlswede, published the first example of a multicast network with solution over the binary field but which can be solvable only using non-linear Boolean functions; this network was obtained by the use of MDS codes, in particular the non-linear Nordstrom-Robinson code (12, 32, 5).

By studying the results of [32], [110] improved the characterization of the inner and outer bounds in terms of Γ_n^* and $\bar{\Gamma}_n^*$. The authors used the same satellite environments of [32] to define, in this case, the zero-error network coding

problem for acyclic networks; furthermore, they enhanced the previous results in multisource network coding, by extending this scenario to arbitrary acyclic networks.

On the implementation side, [111], [112] independently modified and developed the algorithm firstly proposed to construct a generic LCM in [18, Section 5], by making a new one computationally more efficient. Moreover, the new algorithms, initially proposed to build a linear multicast but also adaptable for a linear broadcast, used a lower threshold on the size of the base field than the previous one. Next, the authors of the two works, realised [113] (published in 2005), providing deterministic polynomial time algorithms and randomized algorithms for linear network coding in directed acyclic graphs.

In 2004, [114] found a connection between linear network coding and linear system theory, especially with the theory of codes on graphs.

B. Algorithms, Alphabet Size, Undirected Graphs and Combination Networks

In 2003, [115] extended the work in [112]. It showed that the alphabet size q is dependent on the graph construction and on the number of the flows from source to different terminals that end into a single edge. By finding that the information transported by edges can be seen as a MDS code, the authors calculated bounds on q through the existing ones on MDS codes: they provided a lower bound on q for some classes of acyclic graphs, lower than the previous bound presented in [112]. Next, they showed a technique to increase the alphabet size available for coding if the amount of memory at a node is smaller than the one needed.

In 2004, [20] studied, for the first time, network coding in undirected networks (bidirectional links) by considering the unicast, broadcast and multicast scenarios. The authors derived upper bounds on the coding advantage[14], discovering that it is always bounded by a constant factor of 2, in opposition with the previous result in [112] for directed scenarios, in which it was not finitely bounded. So, they proved the independency of the achievable throughput by the location of the information source in the network, true only in undirected networks, and they showed how the optimal throughput is simpler to compute in presence of network coding instead of routing. Then, [116] and [117] (respectively in 2004 and 2005) clarified the results in [20]. They proved that Steiner tree packing can be a lower bound on the achievable optimal throughput for network coding and that Steiner strength can be considered an upper bound on it; but these bounds are NP-hard to compute. The authors of these works demonstrated that network coding is mainly useful not to reach the highest optimal throughput but to make it possible in polynomial time: therefore, coding is a way to simplify the design of efficient solutions. Next, [118] calculated a cut-set bound in undirected networks by considering the application of network coding.

In 2004, [119], [120] investigated how to solve the multicast problem: they compared the performance of the optimal routing solution (Packing Steiner tree [121]) with the one obtained through network coding not only theoretically but also through an algorithmic implementation; nevertheless, the Packing Steiner tree is a NP-hard problem so, they implemented this routing method with the help of a greedy tree packing algorithm. The result they discovered was that routing and coding, in terms of throughput, achieve comparably performances; anyway, the implementation of network coding introduced additional enhancements in terms of efficiency, ease management and robustness. In [122], [123], the same authors proved a statement to link the Edmonds' theorem and the Ahlswede et al.'s theorem. Hence, in 2005, [124] described that this unification provided a complexity reduction for network coding systems. Moreover, in [125], the authors enhanced their previous results, by considering both acyclic and cyclic scenarios. In fall 2004, [126] investigated the k source-sink pair problem for directed and undirected graphs. They compared multicommodity flow performance with the one of information flow, in particular, the latter

allowed multiple source-sink pairs to share the bandwidth. In case of directed graphs the authors calculated that network coding provides a higher rate than the best multicommodity flow solution, otherwise, in the undirected scenario, they conjectured that the maximum rate achievable with network coding is equal to the maximum rate reached by fractional multicommodity flow. The same year, [127] showed a link between network coding and linear programming formulations for minimum weight Steiner tree. In particular, it explained the parallel between coding advantage and integrality gap in computer science optimization. By improving the bounds on maximum coding advantage both for directed and undirected networks, The authors enhanced the bound presented before in [116]. In the way of investigating the general information network flow problem introduced firstly by [2], [128] showed a property which affects the alphabet size of network coding: [18, Theorem 5.1] demonstrated that a generic LCM needs an alphabet large enough to exist, but [128] calculated that, given an integer k, the alphabet size needs only to be a perfect kth power to make the network coding problem solvable. So q needs not only to be very large, according to the number of sinks, but it has to match particular values.

[129] applied the method of subtree decomposition[15] to a multicast network to study the average throughput obtained by applying network coding instead of routing. The authors found that network coding was able to reach at most twice the throughput of the respective routing solution. Then, in [130], by using the same method of [129], they proposed four deterministic algorithms to implement network coding in a decentralized manner. Firstly, they studied the case of two sources and n receivers, secondly, the one of h sources and two receivers and, finally, they generalised to h sources and n receivers. In particular, in the last scenario, the algorithms used tools from arcs in projective spaces to handle coding vectors. Moreover, in the same context, [131] provided an alphabet-size bound, in a scenario with two sources and two receivers, together with a decentralized algorithm. Next, [132] connected the theoretic graph colouring problem and network coding: by considering the multicast problem with two sources and by using subtree decomposition, the authors provided a code design algorithm and alphabet size bounds. Concurrently and independently, also [133] used similar concepts to classify the complexity of network coding problems. A graph colouring problem is related with the idea of partitioning a set into parts. A *vertex colouring* of a graph $G = (V, \mathcal{E})$ is a function $\gamma : V \to C$, where C is the set of colours. Then, a *k-colouring* is a proper[16] vertex colouring with at most k colours; this can be seen as partitioning the vertex into k classes, such that the vertices within the same class are not adjacent. A *vertex list assignment* L on a graph G associates a set L_v colours with each vertex v of G. In parallel, the problem of colouring can be also applied to the set of edges.

[134] studied the application of network coding to combination networks, an important kind of network in diversity coding research. The authors calculated the network coding

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[14]The *coding advantage* is the ratio of achievable optimal throughput with network coding and without it.

[15]*Subtree decomposition* is a partition of a line graph into a disjoint union of connected subsets through which the same information is sent.

[16]A colouring function is called *proper* if no two adjacent vertices are assigned the same colour.

gain in case of $\left(\frac{n}{n-1}\right)$ networks and, next, they generalised this result to the $\left(\frac{n}{m}\right)$ networks, showing that choosing n and m appropriately it increases the network coding gain by making it unbounded.

In 2005, by studying the results obtained before by [19], [103], the authors of [135], [136] provided a deterministic algorithm to solve the multicast network coding problem. The previous works found that network coding problem could be reduced to choose the appropriate entries of the transfer matrix of the network and each sink can decode the information received only if it can invert its own transfer matrix. Hence, the transfer matrix needs to be full rank and this request suggested the authors to make a connection with mixed matrices. So, the algorithm they proposed is an algorithm for max-rank completions of mixed matrices. In respect of the algorithm realised before in [113], this new one is slower but, on the other hand, it made a more general approach, opening ways to possible enhancements. [137], [138] continued the investigation of network coding behaviours in undirected networks with multiple source-sink pairs communications. The authors calculated the maximum theoretic information rate for this scenario, confirming the previous results found by [20]. [139], [140] (published in 2006) studied the performances of deterministic network coding in a multicast network with h sources and N receivers: the authors found that the use of network coding increased the throughput proportionally to a factor $\sqrt{N}$ in comparison with directed Steiner tree; moreover, they calculated that the alphabet size required for randomized coding is exponentially larger than the minimum one needed in case of deterministic network coding. In [141], [142] (published in 2006) the authors calculated two upper bounds on network coding rates for directed and undirected graphs, by using the concept of d-separation in Bayesian networks.

[143] proposed new coding schemes for line networks scenarios, by using fountain codes; this work improved and extended the previous work in [144].

[145], [146] (published in 2006) studied the problem related with the amount of encoding nodes needed in the network for a multicast transmission, i.e. finding the minimum number of encoding nodes for a solvable network coding problem. In fact, encoding nodes can be an expense, instead of forwarding nodes, for several reason, such as: the cost is higher, they introduce delay and they increase the overall complexity of the network. The authors demonstrated that the number of encoding nodes only depends on the transmission rate of the source and on the number of receivers. Moreover, they analysed the more general cyclic case: in this scenario the dependence is on the size of the feedback link set of the network[17]; they also calculated a lower bound on the number of encoding nodes for cyclic networks. Next, they established that this issue is practically an NP-hard problem.

In 2006, the same authors presented in [147] (but published in 2009) the first efficient algorithm to realise a network coding multicast, by taking into account the number of encoding nodes. This algorithm also enhanced the time complexity

[17]The minimum number of links that must be removed from the network in order to eliminate cycles.

TABLE I
COMPARISON OF THE COMPLEXITY OF THE ALGORITHMS FOR DETERMINISTIC NETWORK CODING. $\mathcal{M}(\omega) = \mathcal{O}(\omega^{2.3727})$ IS THE TIME REQUIRED TO PERFORM $\omega \times \omega$ MATRIX MULTIPLICATIONS.

Complexity of deterministic algorithms	
$\mathcal{O}(|\mathcal{E}|)$	[9, Algorithm 2.19]
$\mathcal{O}(|\mathcal{E}||T|\omega(\omega + |T|))$	[113, Theorem 3]
$\mathcal{O}(|\mathcal{E}||T|\omega^2)$	[113, Theorem 3]
$\mathcal{O}(|\mathcal{E}||T|\omega + |T|\mathcal{M}(\omega))$	[113, Theorem 9]
$\mathcal{O}(|T|\omega^3 \log \omega)$	[135]
$\mathcal{O}(|\mathcal{E}||T|^2\omega + \omega^4|T|^3(|T| + \omega))$	[147]

of the previous one showed in [113] (in particular they use this last algorithm as a subroutine). Then, they studied the problem of finding integral and fractional network codes with a bounded number of encoding nodes but they found that this issue was NP-hard.

[148] (published in 2009) studied dynamic multisession multicast with the application of intra-session network coding. In particular the authors calculated the capacity region for the intra-session network coding and described some algorithms. Next, [149] analysed the run-time complexity at the single nodes of the network. It provided an algorithm for random linear network coding scenarios called 'permute-add-add network codes', requesting a lower complexity at the intermediate nodes than the one necessary in the previous work in [104]. In particular, the authors proved that their algorithm could achieve the multicast capacity with probability tending to 1. [150] proposed an algorithm to obtain the advantages of network coding and to reduce the number of intermediate coding nodes. They used a 'genetic approach' to decrease the complexity from NP-hard. Then, in 2007, [151] enhanced the algorithm, by adapting it for cyclic scenarios: it increased its performances and it combined the algorithm with the random codes for a decentralized implementation.

The same year, [152] presented a deterministic approach for combination networks using binary coding and achieving the multicast capacity; the authors experimented a quadratic decoding complexity. [153] studied the possibility to approximate the problem of multicasting information successfully with network coding and it demonstrated that it is NP-hard.

Tables I and II respectively summarize the complexity of the algorithms described above and the bounds on the alphabet size of the network codes. Moreover, Figure 14 plots the complexities listed in Table I according to the source rate chosen for transmission and to the number of sinks.

C. Random Linear Network Coding and Network Error Correction

In 2003, [104], together with [154], defined a random network coding approach to multicast, in which the nodes transmit the linear combinations of the incoming information on the outgoing channels, using independently and randomly chosen code coefficients from some finite fields. Nevertheless, on the receiver side, the decoder needs the overall linear combinations of source processes. Thanks to this new approach the

Alphabet size bounds of deterministic network coding		
Lower bounds	$q > \begin{pmatrix} \lvert\mathcal{E}\rvert + \omega - 1 \\ \omega - 1 \end{pmatrix}$	[9, Algorithm 2.19]
	$q \geq \lvert T \rvert$	[113, Theorem 3], [147]
	$q \geq 2\lvert T \rvert$	[113, Theorem 3]
	$q \geq 2\lvert\mathcal{E}\rvert\lvert T \rvert$	[113, Theorem 9]
	$q \geq \sqrt{2\lvert T \rvert}$	[115, Theorem 1]
	$q \geq 2^{\exp(\Omega(\omega^{1/3}))}$	[128, Corollary 4.1]
	$q > \lvert T \rvert$	[135]
Existence bounds	$q \leq \lceil \log_2 \lvert T \rvert \omega + 1 \rceil$	[19, Corollary 1]
	$q \leq \lfloor \sqrt{2\lvert T \rvert - 7/4} + 1/2 \rfloor$	[131, Theorem 2]
	$q \leq \sqrt{\lvert T \rvert}$	[132]

authors allowed network coding to be suitable for networks with unknown or changing topologies. Moreover, by using a randomized approach, a failure probability comes out, which can be arbitrarily reduced by increasing the dimension of the finite field (i.e. the probability decreases exponentially with the increase of the number of bits in the codewords). The authors of these works calculated a lower bound on coding success probability: by considering linearly correlated sources, the randomized codes can be viewed as a distributed compression within the network, hence, the source information is compressed to the capacity of any cut through which it passes. In 2004, [155] demonstrated with [155, Theorem 1] that the error probability for distributed random network codes depends on some error exponents, which are a generalisation of the Slepian-Wolf error exponents for linear coding[18]. The authors considered arbitrarily correlated sources over an arbitrary multicast network, in which the randomized network coding approach was used to perform compression, when necessary. Next, in 2004 with [157] (published in 2006 in [158]), a full description of the theory of random network coding was provided. These articles described a connection with bipartite matching and random network coding (making a link with [103]); then, they generalised the Slepian-Wolf error exponents in case of arbitrary correlated sources, and they showed the benefits of the random network coding application.

In 2002, the first paper about network error correction (NEC) codes [100] was published, but only in 2006 [22], [23] established a full connection between network coding and classical algebraic coding theory. In fact, the authors gave the general definitions of network error-correcting codes and demonstrated that network error correction is a natural generalisation of classical coding theory: especially, they extended the Hamming [22, Theorem 2], Singleton [22, Theorem 4] and Gilbert-Varshamov [23, Theorem 2] bounds to the new t-error-correcting network codes. Classical algebraic error-correcting codes employ correction of errors in point-to-point communications and they introduce redundancy in time dimension.

On the other hand, the other approach to get reliability and efficient communication is using channel coding together with an optimal network code: instead of an error correction link-by-link based, NEC codes realise a distributed error correction in the network and introduce redundancy in space domain.

The relationship between network coding and channel coding was studied firstly, in 2002, by [33] and, in 2003 (published in 2006), by [159]. The former faced the issue, by considering synchronous and independent channels with a unit delay; the latter worked on a point-to-point communication network, with independent and discrete memoryless channels and asynchronous transmissions. In particular, this last paper defined a separation principle for network coding and channel coding, generalising and extending the results obtained before by [160] (about feedback networks) and [33] (about synchronous transmissions).

Next, in 2006, [161] showed some basic definitions about network error correction, it defined some main properties in case of single-source multicast and it proposed some algorithmic implementations. This work gave the important definition of the concept of minimum rank of a network error correction code, which is the NEC respective of the minimum distance in classical coding theory. In the same work, the author defined encoding and decoding processes and he proposed one 'brute force' and one 'fast' decoding algorithm. So, he gave the characterization of the error-correction capability in the cases of global kernel errors and erasure errors. [162] defined Hamming weights for NEC. In general, for a code, Hamming weight and distance are instruments to quantify the ability in detecting and correcting errors. They are not directly applicable in case of NEC because the linear transformations of error vectors affect the reception at the sink: the measurement criteria are kinds of linear transformation of the error vector, the one which can confuse the reception of a message. Hence, the authors presented some theorems about detection and correction capabilities in both unicast and multicast case. Finally, they showed the connection between the concept of minimum rank, defined by [161], and minimum distance: in fact, they demonstrated that the two concepts

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[18]The error exponents for Slepian-Wolf coding were obtained in [156]
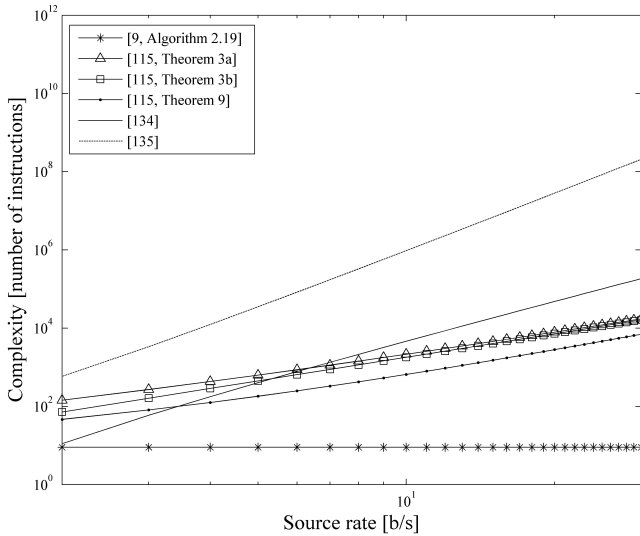
(a)
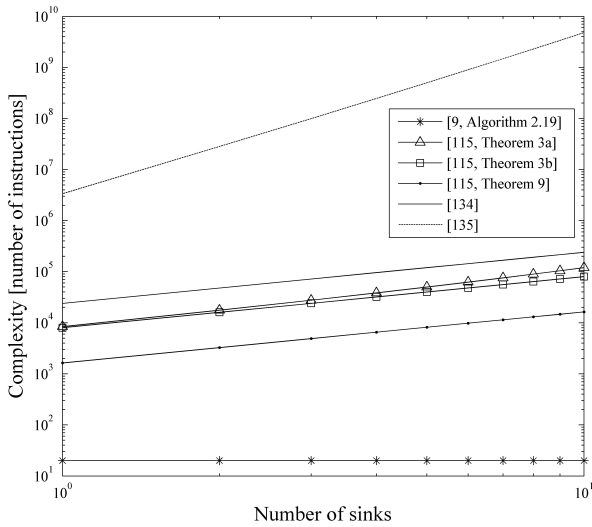
(b)

Fig. 14. Representation of the complexity of the algorithms presented in Table I (a) Complexity as a function of source rate for the multicast scenario of the butterfly network (b) Complexity in terms of number of sinks for a network of 20 edges with a source rate of 20 b/s.

are the same thing [162, Theorem 6]. The previous results were clearly described and improved in [163] (published in 2008): moreover, the author presented another brute force algorithm in case of erasures and global kernel errors [163, Section 5], and defined the condition of existence (involving the cardinality of the alphabet) for a NEC code [163, Section 3] to be a MDS code.

In 2007, [164] gave a refinement of the Singleton bound obtained firstly in [22], through the use of different constraints on the individual sink nodes, when the maximum flows between them and the source are not the same. Then, the authors proposed an algorithm, which constructed a linear network code achieving such a refined Singleton bound from a classical linear block code. After that, [165] obtained a refined version of the Hamming, Singleton and Gilbert-Varshamov

bounds calculated in [22], [23]. These results were proved through simpler proofs than the previous ones, by using a new concepts of Hamming weights. In [166] (published in 2011), all these concepts were deeply explained and enhanced, and two algorithms were shown.

[167] realised a deterministic and centralized algorithm for NEC, attaining the Singleton bound defined in [22]: this algorithm is based on the previous work in [113] and a characteristic is that it did not introduce delay in the transmission. The author gave also a randomized construction, that is, by randomly choosing local encoding vectors: in this way, he tried to remove the time complexity due to a centralized implementation. A year later, [168] found a condition on the size of the alphabet of the code in presence of degradation [168, Corollary 1]: this result showed that the size has to be smaller than the one showed before in [163, Section 3]. In order to achieve it, the authors calculated the probability mass function for a random network code, taking into account degradation, and an improved upper bound for the failure probability. In [169] (published later, in 2009), the same authors improved and organized the results, which they have already obtained before in [168]. Then, [170], [171] summarized the results obtained in the previous works, and, moreover, they described a hybrid NEC code, a code consisting of three different levels of protection to face the issue of deletion of erroneous packets [170, Section 5], [171, Section 7]: a parity check is introduced by the source, an error-correction code is used to protect each packet and, a NEC code is deployed to protect the whole multicast; they said that this method reduced the size of the global kernel. Most of the concepts discussed above can be also found in [172].

A new development arrived with the publication of [173], [174] (this last one was published later in 2008). The authors had the intuition to apply the approach, used before in [175] (Grassmann manifold for calculating the capacity of multiple-antenna fading channels), to implement NEC using random linear network codes to treat errors and erasures: they provided a noncoherent subspace-based formulation of the NEC problem. Firstly, they defined an operator channel to describe the relation between its input subspaces and its output subspaces (the information is interpreted as subspaces of a vector space); secondly, they defined a suitable metric to reformulate the concepts of minimum distance of a code and of maximum dimension of the codewords of a code, in this new geometric context; thirdly, they focused on constant-dimension codes (CDC) and considered the Grassmann graph a suitable way to characterize them; furthermore, the authors gave an interpretation of the metric function as the double of the distance in the Grassmann graph. Grassmann graph is a class of undirected graphs defined as follows. A connected graph G is *distance-regular* if there exist integers b_i, c_i for $i = 0, 1, ..., d$, such that for any two vertices α and β at distance $i = d(\alpha, \beta)$, there are precisely c_i neighbours of β in $G_{i-1}(\alpha)$ and b_i neighbours of β in $G_{i+1}(\alpha)$, where $G_i(\alpha)$ is the set of vertices z with $d(x, z) = i$. The value $i = d(\alpha, \beta)$ represents the length of the shortest path from α to β. Let K be a field and let $\mathcal{V}$ be an n-dimensional vector space over K. The *Grassmann graph* of the m-subspaces of $\mathcal{V}$ has vertex set $[{\mathcal{V} \atop m}]$, the collection of linear subspaces of $\mathcal{V}$ of dimension m. Next,

TABLE III
CONSTRUCTIONS OF NETWORK CODES.

Linear network coding	Coherent/noncoherent
Random linear network coding	Coherent/noncoherent
• Compressed coding vectors	
Subspace coding	Noncoherent
• Lifted rank metric codes	
• Padded codes:	
- Gabidulin codes	
- Spread codes	
• Lifted FD codes	
• Codes obtained by integer linear programming	

the authors provided a reformulation of the classical coding theory bounds for this noncoherent approach and they proved how to construct a Reed-Solomon-like NEC code, providing also a possible efficient decoding algorithm. This kind of noncoherent NEC codes were named KK codes from the authors (Koetter and Kschischang) that firstly proposed them. Concurrently, [176], [177], [178] (the latter published in 2008) investigated the NEC problem through a different approach: the authors chose to develop a kind of rank-metric codes for NEC because of the possibility to take advantage of the powerful tools of classical coding theory. They built optimal rank-metric codes obtained by the nearly optimal subspace codes after a 'lifting'[19] operation: therefore, this family of subspace codes is called lifted rank-metric codes. Then, in [179], the authors proposed a rank-metric approach, by using Gabidulin codes, to priority encoding transmission. [180] studied constant-dimension codes: the authors demonstrated that the combinatorial objectives called Steiner structures are optimal CDC and, hence, they derived Johnson type upper bound, and improved the Wang-Xing-Safavi-Naini bound.

In fall 2007, [181] in line with the work in [178], proposed a code with the structure of a sparse graph code (low-density parity-check code) to be applied in case of a probabilistic channel model, in which the noise is chosen uniformly and randomly from a matrix space. The authors, firstly, calculated the capacity of this channel, next, they showed some characteristics to be satisfied by the code in order to achieve the channel capacity and finally, they provided an algorithm for the encoding and decoding process. A presentation/tutorial about this work can be [182].

By starting from [181], in 2008, [183], [184] (published in 2010) defined both multiplicative and additive matrix channels and calculated their capacities. In particular, on the latter side, the authors found the same results of the previous work [181], also extending its proof [184, Corollary 2]. Then, they defined efficient coding schemes to achieve these capacities. Moreover, they characterized the additive-multiplicative matrix channel and its capacity (they obtained upper and lower bounds for the

general case), presenting a suitable efficient coding scheme.

The same year, [185] developed [173] by generalising its approach with the study of codes of minimum distance 2, $N - 1$ and N, and by calculating a Gilbert upper bound on the code rate. Moreover, it presented together with [186] a kind of padded codes[20] with constant-rank distance, called spread codes. Another family of padded codes was developed by [187]: it modified KK codes by introducing a recursive construction and by increasing their information rate. [188] continued the work started by [174]: it extended the framework of the codes, by considering them in projective spaces. The authors generalised the Singleton, Gilbert-Varshamov and LP bounds and also the two Johnson bounds, but independently from [180]; next, they extended the concepts of linearity and complements to projective spaces and they provided constructions of optimal codes, by using Steiner structures. Concurrently, in [189] (published in 2009), by taking advantage of the concept of 'lifting', expressed before in [178], the authors defined a new way to construct rank-metric codes in projective spaces through Ferrers diagrams. So, they studied constant-dimension codes and their multilevel implementation, providing also decoding algorithms; finally, they described a puncturing process for codes in projective spaces. Due to the growing importance gained by CDC, even other works started studying their properties: hence, [190], [191], [192] worked on the possibility to construct optimal CDCs, by using optimal CRCs. They investigated some properties of CRCs such as upper and lower bounds on the maximum cardinality and asymptotic behaviours of the maximum rate. Next, [193] proposed another approach to design constant-dimension subspace codes by considering this issue as an optimization problem. This novel approach was useful to find several CDCs with more number of codewords than the CDCs already known.

[194], [195] proposed an approach to NEC through block codes in case of directed acyclic networks. They proposed some syndrome-based and maximum likelihood decoding algorithms, and they studied the coding design and complexity. [196] developed the multisource case for the noncoherent

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[19]Given a matrix $\mathbf{X} \in \mathbb{F}_q^{n \times m}$, the subspace $\Pi(\mathbf{X}) = \left\langle \begin{array}{cc} \mathbf{I}_{n \times n} & \mathbf{X} \end{array} \right\rangle \in \mathcal{G}(n+m, n)$ is called the *lifting* of $\mathbf{X}$. Then, given a matrix code $C \subseteq \mathbb{F}_q^{n \times m}$, the lifting of C is a subspace code, whose all the codewords are lifted.

[20]Padded codes are asymptotically good subspace codes designed as a union of lifted product rank-metric codes.

NEC codes, enhancing the framework presented in [174]; moreover, the authors gave two constructions for subspace combinatorial codes of fixed alphabet size. [197], [198] studied and characterized a general transmission system: in this very general context, the authors started giving the definitions of weight measure, minimum weight decoder (MWD) and *t*-error correcting code. Hence, they presented two definitions of minimum distance for a code, in case of error correction and of error detection: these minimum distances are different in the most general case and they only become the same in the most particular case (classical coding theory). By studying the case of non-linear network coding, the authors discovered that the distances are different in a way that implies that the number of correctable errors can be greater than the number of detectable errors [197, Example 3.2]. So, by considering the new definitions of weight, they also described the case of network erasure correction. In order to have a presentation of the results in [162], [164], [165], [197], [198] it can be useful to read [199].

[200] studied a scenario with multiple sources (in this case limited to two) and one receiver, communicating with time-slotted transmissions. The authors found that the channel is similar to the one studied before in [173]. Their main result was to characterize the capacity region in case of subspace coding, moreover showing that this region is a polytope with finite corner points [200, Theorem 1]. In [201], the same authors investigated a time-slotted scenario with a block time-varying channel (it can be considered DMC), in which a source communicates with multiple receivers. They calculated the capacity of this channel [201, Theorem 1], extending the results in the case of erasures [201, Corollary 1]. Next, by giving an expression of the input distribution of the channel, they characterized the capacity to achieve both optimal and approximate solutions (using a large finite field). [202] (published in 2011) explained better and extends the concepts previously presented in [200], [201]. By generalising and enhancing the results published before in [58], [203] investigated the decoder error probability (DEP) of bounded rank-distance decoders for MRD codes. In particular, the authors obtained a conditional DEP to evaluate when a decoder can make a mistake in correcting *t* errors. The results about bounds, they calculated, were special cases of the general ones obtained in [178]. They derived upper and lower bounds on the minimum cardinality of CDCs, given the covering radius. They also proposed a decoding algorithm for these codes, to correct more errors than a bounded subspace distance decoder. Then, in [191] (published in 2010), they studied the properties of 'packing' and 'covering' of subspace codes. They investigated both cases, by considering the subspace metric or the injection metric, to calculate asymptotic rates and comparing the performances of CDCs to those of the general subspace codes.

[204] calculated an enhanced upper bound on the failure probability in a random linear network coding scenario, shown before in [168, Theorem 1], to obtain a tight bound in case of the field size going to infinity. [205] firstly derived the bounds on the code size, presented previously in [174], from the concepts in [206]; secondly, the authors calculated a new linear programming bound on the code size, by defining

a different minimum distance; next, they proposed a code construction for codes correcting single insertion or deletion for the operator channel. [207] studied the noncoherent random linear network coding scenario (single and multiple path subgraphs) and its performances to correct errors and erasures; in particular, it compared these results with the ones obtained in case of random forwarding employed in the nonsource nodes. The results showed that RLNC is better than random forwarding only at high information rates. In [208], the authors characterized the capacity regions in the multiple-sources multicast scenario for coherent [208, Theorem 1] and noncoherent [208, Theorem 2] NEC. In case of multi-source non-multicast network they derived a linear network code from a given error-free network code; in this environment, they also mentioned the application of vector linear network codes. In order to demonstrate this last result they generalised the concept of distance previously obtained in [162]. The concurrent work in [209] introduced a new approach to the problem of random linear network coding: instead of using, as the previous approaches, appended coding vectors [210] or subspace codes [173], [174], it introduced a framework based on compressed coding vectors for the transmission of coding coefficients. By taking some concepts from classical coding theory, the authors designed the encoding process and, on the receiver side, the decoding one, in order to recover the packets received by looking for the non-zero positions of the original coding vectors. The benefit of their novel approach was to reduce the length of coding vectors from n to $r = n - k$ (k is the number of information bits). The performances of compression achieved by this work were improved by [211].

The same year, [212] studied the design of random linear NEC codes and, in particular, how they are affected by changing network topology. By considering the network topology as a probabilistic channel the authors studied the performances in case of a class of unicast networks, the S-networks (and their concatenation): they found that these networks are the lowest bound, in terms of capacity, of the unicast networks.

In 2010, [213] defined generalised erasure matrices for rank-metric codes and applied them in a random linear NEC environment. The authors also provided an algorithm and an example of construction. Later, [214] firstly introduced a vector formulation of the linear operator channel (LOC), a general channel model whose linear network coding is a particular case; it described noncoherent and coherent transmissions of LOCs and a simple method to evaluate the transmission channel to obtain channel information; next, the authors described the subspace coding and rank-metric coding context; so, they analysed linear matrix codes using LOCs and their performances and complexity. The results about subspace coding and LOCs were also presented in [215] and on the other hand, few months later, [216] showed the results about channel training and linear matrix codes. By investigating RLNC networks as symmetric DMCs, [217] obtained a capacity for noncoherent RLNCs in arbitrary networks, which was the same of the one found before by [212], in the particular case of unicast networks. In [217], the same authors started to investigate the multishot code implementation of subspace coding, by generalising the results obtained before in [174] for the special case of one-shot subspace codes. Their construction

method of multi-shot codes was enhanced and better explained for rank-metric codes in [218]. At the end of 2010, [219] studied the upper bounds on the failure probability of random linear network coding, obtained before by [169], by taking advantage of topology information.

Table III summarizes the different ways to construct network codes and it shows if they can be coherent or noncoherent solutions. An interested reader could find a clear tutorial about NEC codes in [220]. Another clear tutorial, which is focused on the issue of the coding vectors transmission and the compressed coding vectors approach, can be found in [221]. A useful survey on subspace and rank-metric codes is [222].

D. Convolutional Network Codes and Cyclic Networks

The theories developed in [130]–[132], were fundamental to obtain [223]; these articles served as base to find a connection between network coding and convolutional codes. So, in 2004, by applying a technique called information flow decomposition, the authors of [223] found a way to reduce the dimensionality of the network coding problem. In particular, this technique consists in replacing the actual code with another code at every node, independently of the global network structure, that has identical rate regions. Moreover, they enhanced the explanation of what they showed in their previous works about the decentralization of the codes and the bounds on the code alphabets. Next, they proposed some algorithms to implement the subtree graph design according to the number of sources and receivers. The final result in [223] was the analysis of the connection between network coding and convolutional codes, discovered thanks to the investigation on how to reduce the decoding complexity. Furthermore, [224] studied deeper the relation between network coding and convolutional codes and it proposed a simplified method to deal with cyclic networks. Nevertheless, it was [225] that started to fully deal with convolutional network codes. It proposed an algorithm for convolutional codes similar to the one presented in [112] for block network codes, and it also provided an analysis of the overhead and of the decoding complexity. This algorithm represented a generalisation of network coding to cyclic networks. Then, [226] analysed the definition of linearity for network coding and it modified this concept into a new one through local reduction, a special method to partition a graph of a network into subgraphs through which the same information flows. In this way, the authors provided a simple construction for convolutional multicast network codes and a new class called filter-bank network codes.

In 2005, [227] provided an algorithm to implement an optimal network code multicast over cyclic networks, in polynomial time. Its result was focused on convolutional codes but with a possible application to block codes. The same authors gave a full description of this algorithm in [228]. Moreover, they enhanced and fully explained it and its performances in [229] (published in 2010).

In 2006, [230] defined convolutional network codes, by using rational power series over finite fields and vector rational power series respectively associated with local encoding kernels and global encoding kernels. Moreover, the authors defined the problem of convolutional multicast and they suggested encoding and decoding algorithms.

In 2008, [231] proposed an algorithm for network coding in cyclic networks in case of multicast, with multiple sources. Indeed this algorithm was developed to work on any kind of network, such as acyclic and cyclic and the authors demonstrated that it ran in polynomial time. In order to delve into this topic it is useful to read [232].

E. Solvability Issues and the Quest for the Capacity Region

In 2004, [233] analysed the important connection between the alphabet size and the solvability of the multicast problem and it found that the solvability becomes easier when the cardinality of the alphabet is large; so, it showed that there is not solution for any large alphabet size but only for a part of them. Moreover, the authors confirmed that a multicast scenario, transmitting at most two messages, has a binary linear solution. However, they also proved that this is not always true by transmitting more than two messages: especially, these last scenarios become solvable over binary fields only considering a non-linear solution.

Then, in 2005, the same authors demonstrated in [234] that the idea that every solvable network has a linear solution over some finite field alphabet or vector dimension was not exact: in fact they showed that some networks have no linear solution. Therefore, linear network codes are not sufficient over finite field alphabets, over commutative ring alphabets, over the most general class of R-modules, and even asymptotically over finite field alphabets. Moreover, they proved the same results in the case of vector linear coding. At this point, they deduced that their conclusions are also valid for the more general convolutional and filter-bank linear codes. [235] defined the concept of routing capacity of a network as the highest possible capacity obtainable from a fractional routing solution and calculated it in case of fractional coding, that is, the dimension of the messages need not to be equal to the capacity of the edges). In this general case, the authors proved that the coding capacity is independent of the alphabet used: hence, they established the computability of this routing capacity. Next, [236] showed that the network coding capacity, in contrast to the routing capacity, in some cases is not achievable. On another way, [237] demonstrated that not all solvable multiple-unicast networks are reversible and later, [238] investigated the reversibility and irreversibility in case of network coding scenarios.

[239] calculated the capacity of an infinite cascade of DMCs and stated upper and lower bounds. [240] showed some results in order to calculate the capacity of network coding in case of fixed and changing topologies. [241] calculated an outer bound on the rate region for network coding problems, by considering general scenarios (even undirected graphs); then, the authors defined the concept of informational dominance, important to derive information theoretic inequalities. The authors also made a comparative study of network coding rates and multicommodity flow rates for directed cycles.

The same year, [242] demonstrated that in single-source multiple-session multicast scenarios, in order to achieve the maximal throughput, it is useful to use group network codes[21], which include linear network codes as a particular subset.

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[21]Network codes over finite algebraic groups.

[243] defined an improved outer bound on the admissible rate region for three-layer networks in a multi-source multi-sink scenario; this bound improved the previous max-flow min-cut bound. Then, in [244], the same authors enhanced the explanation by improving their results.

In 2006, [245] described a method to construct networks from matroid: specifically, it showed, as an example, how to construct a Vamos network from the well known Vamos matroid. Then, the authors proved that Shannon-type inequalities are not in general sufficient to calculate the network coding capacity, hence, they found an upper bound by using the Zhang-Yeung non-Shannon-type inequality. They also proved that Shannon-type inequalities are not sufficient to compute the network coding capacity in the multiple-unicast scenario. In the meantime, [246] investigated the relation between the capability to achieve network coding capacity and the number of coding nodes in the network.

In 2007, [247] improved the previous results about inner and outer bounds of the capacity region shown in [110]: in particular, the authors determined the exact capacity region for general acyclic multi-source multi-sink networks, in terms of constrained regions in the entropy space. [248] proved that the admissible coding rate region can be obtained as the intersection between the convex cone, containing the set of all representable entropy functions, and a collection of hyperplanes, depending on the network topology and on the multicast requirements. This region resulted to be quite different from the inner bound previously calculated in [110]. [249] constructed a network coding multicast problem which relates solvability with polymatroids in case of four random variables. The objective was to create a link between entropy vectors and the capacity region of network codes.

In 2008, [250], [251] showed the presence of a duality between the entropy functions of the entropy space and network codes and the authors enunciated three theorems to describe their results. In these theorems the entropy function h is related with the admissible rate capacity tuple $T(h)$: firstly, the tuple is admissible if h is the entropy function of a set of quasi-uniform random variables; secondly, it is admissible if h is linear group characterizable; thirdly, $T(h)$ satisfies the LP bound if and only if h is a polymatroid. Then, the authors described several implications of these conclusions. In [252], the same authors found that even in case of single-source multicast networks, as in the multi-source scenario, the computation of the region Γ_n^* is complex because Γ_n^* is a non-polyhedral region. Hence, they showed this property for the two cases of single-source networks, with hierarchical sink constraints and with security constraints.

[253] studied the complexity of the approximation of the network coding capacity, by considering a general instance of the problem. The authors found that the problem is NP-hard. In their work they also made a connection between network coding and the index coding problem; this link between the two areas was studied deeply by [254]. Next, they better explained their results in [255] (published in 2011). [256] demonstrated that the problem of determining if a network is scalar-linearly solvable is the same of the one of determining where polynomials have the same roots, given a collection of them. Moreover, the authors showed that for every network

there is a multiple-unicast network which is matroidal, such that the two networks are scalar-linearly solvable over the same fields. [257] described how the linear independence among coding vectors assigned to the edges can be interpreted as a matroid. An edge-disjoint paths network is associated with a network matroid; if the two independent sets of the network matroid and the ones induced by the previous method coincide, the linear network code is generic. Finally, they showed an algorithm for network matroid representation. [258] studied the capacity region of network coding in the special case of undirected ring networks.

In 2009 (published in 2012), [259] analysed the behaviours of network coding in case of sum-networks and, in particular, demonstrated the insufficiency of network coding for this kind of networks and the non-achieving of the network coding capacity. [260] derived a computable outer bound for the multi-source network coding capacity, because the previous linear programming approach had exponential computational complexity, dependant on the network size.

A good tutorial that establishes the link between network coding theory and matroid theory, the capacity region problem and the solvability issue, is [261].

VI. LATEST RESULTS AND POSSIBLE FUTURE DIRECTIONS

This survey presented relevant developments both in information theory — important for understanding network coding — and in network coding theory. The latter is a new field of research, therefore there is still a considerable number of issues that needs further investigation. The next subsections will present the most recent works in network coding theory identifying potential research directions in this area.

A. Variable-Rate Linear Network Coding

In 2006, [262] proposed the application of variable-rate linear network coding in a single-source broadcast scenario. This new technique resulted to have a simpler scheme, requiring less storage space in the non-source nodes of the network. Then, [263] enhanced these concepts by also considering the possibility of link failures. [264] proposed a unified framework for variable-rate linear network codes and variable-rate static linear network codes: in particular, the main idea, used by the authors, was the type-preserving conversion matrix: so, the non-source nodes only need to store one local encoding kernel to save storage space.

B. Theoretic Framework and Algorithmic Developments

Recently, after 2008, a more general theory of network coding was proposed. The first work in this direction was [265], which was enhanced by [21], taking advantage of several results concurrently presented in [266]. They demonstrated that classical field-based network coding theory, and so convolutional network coding, are instances of a new framework based on commutative algebra. Especially, this theoretic extension was possible by seeing the information as belonging to a discrete valuation ring (DVR) and not to a field. A ring is called a DVR if there is a field K and a

discrete valuation ν^{22} on K such that R is the valuation ring of ν. An example of DVR is localization. Then, let define an equivalence relation on the set of pairs (d, r) with $d \in D^{23}$ and $r \in R$, by $(d_1, r_1) \sim (d_2, r_2)$ if and only if $d_1 r_2 - d_2 r_1 = 0$. The equivalence classes form a ring – denoted by $D^{-1}R$ – which is called the *ring of fractions* of R or the *localization* of R at D. We write r/d for the equivalence class containing (d, r). Addition and multiplication are defined naturally by $r_1/d_1 + r_2/d_2 = (r_1 d_2 + r_2 d_1)/(d_1 d_2)$ and $(r_1/d_1)(r_2/d_2) = (r_1 r_2)/(d_1 d_2)$, for $r_1, r_2 \in R$ and $d_1, d_2 \in D$. If $R = K[x]$ is the polynomial ring over a field K and D is the subset of R consisting of polynomials which are not divisible by x, then $D^{-1}R$ is the set of all rational functions in x which can be written in the form $f(x)/(1 + g(x))$, for some polynomials f and g. A good tutorial which explains these new ideas, can be [267].

Another enhancement of the theoretic fundamentals of network coding, in case of directed acyclic networks with one source, was also presented in [268]. The classic definitions of linear dispersion, linear broadcast and linear multicast, given before in [9], used the dimension of the span of the global encoding kernels, associated with the incoming edges, to characterize the different kind of linear network codes. Otherwise the new unified approach took advantage from the concept of regular independent set to describe with coherence the linear independence among set of edges. Next, [269] improved the previous work by applying the new concepts to demonstrate some conditions and relations: thanks to this new theoretic framework, the authors demonstrated the theorems through simplified proofs.

[270] provided a polynomial time algorithm to design binary matrices for deterministic multicast network coding and it enhanced the theoretic algebraic framework defined in [19], [101], [102] by considering operations over matrices. Then, [271], [272] extended [270] and they opened the way to an implementation of a polynomial time algorithm to design vector network coding in a multicast scenario; the new algorithm changed the problem of finding the smallest L ($L \times L$ is the dimension of the coding matrices[24]) into the one of finding the small degree co-prime factor of an algebraic polynomial. This new result suggested an algorithm in case of scalar network coding, operating in polynomial time.

In 2009, [273] developed a binary linear multicast network coding for any acyclic scenario. By fixing the size of the finite field and by extending the multicast dimension, the authors decreased the computation complexity of network coding at the intermediate nodes and achieved a lower implementation cost of the nodes of the network. In particular, this result is useful for the design of wireless scenarios. A year after, [274] presented two ways to construct cooperative systems by using rateless network coding: the two strategies were proposed in case of uplink and downlink communications and considered single-source partial cooperation and multi-user cooperation.

In 2011, [275] showed a technique based on the concept of uncertainty region of vectors of random variables, in order to calculate the converses for the problem of transmission of correlated sources information. The same year, [276], [277] defined a kind of codes – called BATched Sparse (BATS) – which represents an extension of LT codes. The main idea is to use batches, sets of packets combining only the packets from the same subset; moreover, it was demonstrated that the linear transformations performed by the batches can be analysed through the linear operator channel defined in [214]. LT codes are a family of fountain codes, firstly introduced by [278] in 2002. Then, in 2010, [279] firstly used LT codes to reduce the decoding complexity of network coding in a large-scale content dissemination system. The aim of BATS codes is to reduce the complexity of network coding at intermediate nodes for large files transmission through an end-to-end coding scheme.

[280] developed a unified approach for combinatorial coding subgraph selection problems and [281] studied network coding in line and star networks under the condition of node-constraint. In this last article, the authors found a link between network coding and index coding. The performances of network coding in undirected combination networks were studied by [282]. The authors demonstrated that, in this scenario, network coding increases the throughput and reduces the complexity of designing the system.

C. Convolutional Network Codes

An algorithm for basic convolutional network codes with polynomial time complexity, was proposed in 2009 by [283]. This work was based on the new theoretic results made available by [268], [257]. In [284], the authors enhanced that algorithm by extending [283] and they refined and completed a new version of it, later in [285]. [286] reviewed the concepts of global and local encoding kernels in case of cyclic networks: it analysed the conditions to determine the global encoding kernel matrix of a convolutional network code in a cyclic scenario. Next, [287] designed an adaptive random convolutional network code which operates in a small field and adapts to the network topology; the authors showed an example of their method in combination networks. [288] described a new class of convolutional network codes for multicast networks, called delay invariant: this name is due to the fact that the code is not dependent from the delays of the network. [289] proposed a probabilistic polynomial algorithm for directed cyclic networks by calculating the global encoding kernels by using the Mason formula. Hence, the aim of this work was to improve the previous [228], [227], [231]. Next, [290] studied the behaviour of network coding in networks with delay different from one.

D. Network Error Correcting Codes

In 2011 both [291], [292] made a singularity probability analysis of the random transfer matrix (defined in [158]), in noncoherent network coding context (KK codes), by using constant-dimension codes. They found a correspondence between the rank deficiency of the subspace received and the

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[22]A *discrete valuation* on K is a surjective function $\nu : K^\times \to \mathbb{Z}$, satisfying $v(xy) = v(x) + v(y)$ and $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for all x, y in $K^\times$, with $x + y \neq 0$. The subring $\{x \in K^\times : \nu(x) \geq 0\} \cup \{0\}$ is called the *valuation ring* of ν.

[23]D is a subset of the integral domain R.

[24]In vector network coding the coefficients at the non-source nodes are substituted by matrices of coefficients to make operations among vectors.

zero pattern of the transfer matrix. They also derived upper and lower bounds on the decoding failure probability. Then, [293] extended classical error-correcting codes to the context of subspace coding. In particular, the author demonstrated the direct and the converse part of the coding theorem to provide an upper bound on the random network coding capacity. By considering the theoretic concepts about global encoding kernels demonstrated in [163], [294] showed that network MDS codes (coherent scenario) can achieve the refined Singleton bound and require a smaller field size than the known result. The authors described a polynomial time algorithm to implement MDS codes in directed acyclic networks. So, [295] tried to enhance the results in [166] and [294] in order to reduce the complexity of the system and the amount of storage space required in the non-source nodes of the network.

[296] calculated a LP bound for constant-dimension codes which is as strong as the compact Johnson bound. [297] proposed the use of convolutional codes for NEC by showing some advantages in terms of field size and decoding complexity. The same authors investigated convolutional NEC codes also in [298] by showing an upper bound, a bound on the bit error probability and decoding performances in terms of BER. [299] extended the previous work in [272], by applying it in a NEC scenario; moreover, the authors proposed another algorithm to compute the polynomial coprime, achieving less complexity than the original one. By investigating noncoherent network error correction, [300] proposed a matroidal interpretation of KK codes: the authors interpreted RLNC as the communication of flat matroids, because they demonstrated that flat matroids can be seen as generalisations of linear subspaces. Instead of considering the packets as elements of a vector space, they considered the packets as points of affine spaces[25], and provided a new model to investigate noncoherent NEC, called random affine network coding (RANC). [301] extended the generalised Hamming weight for linear block codes to linear network codes, in order to completely characterize the performance of linear network codes on fixed networks. [302]–[304] studied network error correction in case of unequal link capacities for single-source single-sink networks: they analysed the behaviour of network coding to achieve the capacity and they provided upper bounds on the cut-set. [305], based on the previous work in [162], showed unequal error protection applied to static linear network coding. Hence, the authors proposed a new class of network codes called generalised static broadcast codes (GSBC).

E. Capacity Region and Solvability

[306] related the problem of determining the solvability of a network coding problem to the independence number of a guessing graph[26]. [307] continued to research in line with the previous article [235]. It proved that the network routing capacity region is a computable rational polytope, and

provided non-polynomial time algorithms to obtain the region: in case of linear network coding the authors gave an inner bound of the region (a computable rational polytope). The main idea was to reduce the calculation of the capacity region to the problem of a polytope reconstruction. [308] proved that the set of achievable and admissible rate-capacity tuples are the same and that the outer bound in [93] is tight. This results were achieved for network coding in non-multicast scenarios with co-located sources and sinks demanding different subsets of the sources.

F. Possible Future Directions

Despite being a new field of research, the fact that network coding touches many different areas has led to its widespread interest in the research community. In fact there are so many potentials and so many different research areas involved, that its future is difficult to predict. The following is the best tentative the authors could come to forecast possible future research directions in this field; these were based on the most recent results achieved till now:

- The analysis of the complexity of network coding algorithms, the size of the alphabet and the complexity of encoding and decoding processes are areas that need further investigation. It will be important to take advantage of the possibilities of optimizing network coding implementations through the application of new combinatorial and graph theoretic tools; there are still a lot of challenges to solve in combinatorial network coding problems because they are still NP-hard. The investigation of the performances and behaviours of BATS codes is a novel topic that can be interesting to design a real network coding scheme with low complexity, also robust in dynamical network topology with packet loss. It may also be interesting to further study the performance gain of algorithms based on the novel commutative algebraic framework.

- The theoretic framework, despite being well-researched, is still an important topic as shown by the recent results in Subsection VI-B. The application of the new theoretic results to describe static network codes and network error-correcting codes and the evaluation of the potentials of vector network coding have just started. On the commutative algebraic framework side, the research of the consequences of substituting the DVRs with more general ring structures, in which the hierarchy of ideals is not necessarily a chain in order to solve the issues of cyclic networks, may also be important to delve into. Next, another main way of research is the gain of network coding in undirected networks: Li and Li conjectured that the capacity of single-source multicast with network coding is the same as the capacity with routing only. Hence, network coding seems to have no advantage over classical routing schemes for single-source multicast in any undirected network. At the moment, no counter-example to this conjecture have been provided yet.

- As shown in Subsection VI-A, variable-rate network coding have many potentials: the analysis of the performances of its application to random linear network

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[25]An *n*-dimensional *affine space* over the field K is the set $K^n = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in K\}$. Considering the field $K = \mathbb{R}$, the corresponding affine space is $\mathbb{R}^n$. Then, points, lines, planes, vector subspaces, etc. are called *affine subspaces*.

[26]Given a graph G with n vertices and an integer $s \geq 2$, the *s-guessing* graph of G has $[s]^n$ as vertex set.

codes, the efficient construction schemes for the type-preserving conversion matrix and static type-preserving conversion matrix and the complexity of the algorithms, are all interesting topics that also need further research.

- Convolutional network coding for cyclic networks is a well-known subject but some issues still require attention such as the decoding problem in case of rational power series, the applications in multi-source networks, the developments of algorithms over cyclic networks and the analysis of the behaviours in case of random linear network coding.

- A lot of interest has recently been put on network error correction theory. Every year new fundamental trends appear and the relevance of this topic increases. Some developments in the near future may be related with: the definition of product network error-correcting codes, the use of convolutional codes for NEC, the extension of the rank distribution analysis for non-square network transfer matrices by investigating the decoding failure probability and, hence, the discovery of coding and decoding methods to reduce the complexity of these processes. Another main area is the one that is analysing the performances of systems which use network coding and channel coding together.

- The complete definition of the capacity region and the limits of network coding were studied mainly with the help of the information theoretic geometric framework and matroid theory. Several concepts were discovered by researchers but there are still open problems: the problem of determining the network coding capacity region in the multiple sources non-multicast case and the explicit evaluation of the capacity region in multi-source multicast scenarios are some of them. [309] described a novel approach to network coding theory through sheaf cohomology and homological algebra. In the main result of the paper, the authors proved the max-flow bound for single-source multicast by using tools from homological algebra. Their promising work may give some insights to the hard capacity problem, with the possibility to overtake the Γ_n^* geometric framework.

As [310] has recently shown, the network coding problem seems to be undecidable. The demonstration provided in this work has two 'holes' so, it will represent a main issue to determine whether the network coding problem is decidable.

VII. CONCLUSIONS

This survey is, at the authors' best knowledge, the first survey to embrace the most relevant literature in the different research areas related with network coding theory. The initial goal was to directly involve the reader in network coding theory by explaining its antecedents. Next, a discussion on the main fundamental mathematical preliminaries to provide the mathematics necessary to understand the theoretic developments and results of network coding: in fact, the description of Section III passed from the basic definitions of algebra and abstract algebra, through matroid and graph theory, to reach the description of the basics of coding theory. Following the preliminaries, a historical overview of the results in information theory, that were fundamentals to understand network

coding theory. More specifically, the description started with the seminal work of C. E. Shannon, passing by the novel geometric framework for entropies and ending with the poly-matroidal approach to information theoretic problems. Section V presented the path of network coding theory from its very beginning to the actual results — the chronological approach was in parallel with thematic subsections in order to avoid a description fragmented and confused. Finally, Section VI outlined the most recent theoretic results and offered a list of possible future research directions in network coding theory.

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