SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 10, 1 (2018), 49-54.

# Upper bounds for arithmetic–geometric index of graphs

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**Abstract:** Let $G = (V,E)$, $V = \{1,2,\dots,n\}$, be a simple connected graph with $n$ vertices and $m$ edges and let $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, be the sequence of vertex degrees. With $i \sim j$ we denote the adjacency of the vertices $i$ and $j$ in graph $G$. With $AG = \sum_{i\sim j} \frac{d_i+d_j}{2\sqrt{d_i d_j}}$ we denote arithmetic–geometric topological index. In this paper we give some new upper bounds for this topological index.

**Keywords:** Arithmetic–geometric index, Zagreb indices, multiplicative Zagreb indices.

## 1 Introduction

Let $G = (V,E)$, $V = \{1,2,\dots,n\}$, $E = \{e_1,e_2,\dots,e_m\}$, be a simple connected graph with $n$ vertices and $m$ edges, and let $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, $d_i = d(i)$, and $d(e_1) \ge d(e_2) \ge \cdots \ge d(e_m)$, be sequences of its vertex and edge degrees, respectively. We will use the following notation: $\Delta = d_1$, $\delta = d_n$, $\Delta_{e_1} = d(e_1)+2$ and $\delta_{e_1} = d(e_m)+2$. With $i \sim j$ we denote the adjacency of the vertices $i$ and $j$ in graph $G$.

In [6] and [7] two vertex–degree–based topological indices, the first and the second Zagreb indices, $M_1$ and $M_2$, were defined as

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{i\sim j} d_i d_j.$$

Multiplicative variants of the Zagreb indices, the first and the second multiplicative Zagreb indices, $\Pi_1$ and $\Pi_2$, are defined in [17] as

$$\Pi_1 = \Pi_1(G) = \prod_{i=1}^{n} d_i^2 \quad \text{and} \quad \Pi_2 = \Pi_2(G) = \prod_{i\sim j} d_i d_j.$$

Manuscript received November 12 2017; accepted May 14, 2018.
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The first multiplicative sum Zagreb index, $\Pi_1^*$, was introduced in [5]

$$\Pi_1^* = \Pi_1^*(G) = \prod_{i \sim j} (d_i + d_j).$$

In [12] connectivity index, $R$, later called Randić index, was defined as

$$R = R(G) = \sum_{i \sim j} \frac{1}{\sqrt{d_i d_j}}.$$

Geometric–arithmetic topological index, $GA$, was introduced in [18]

$$GA = GA(G) = \sum_{i \sim j} \frac{2\sqrt{d_i d_j}}{d_i + d_j}.$$

As an inverse variant of this topological index, in [15] arithmetic–geometric vertex–degree–based topological index, $AG$, was defined as

$$AG = AG(G) = \sum_{i \sim j} \frac{d_i + d_j}{2\sqrt{d_i d_j}}.$$

In the literature topological index $GA$ was much more studied than $AG$ index, see [2, 3, 4, 9, 14, 16]. In this paper we are interested in upper bounds on topological index $AG$.

## 2 Preliminary results

In this section we list some analytic inequalities for real number sequences that will be needed in the subsequent considerations.

Let $p = (p_i)$ and $a = (a_i)$, $b = (b_i)$, $i = 1, 2, \ldots, m$, be positive real number sequences with the properties $0 < a \le a_i \le A < +\infty$ and $0 < b \le b_i \le B < +\infty$. In [1] the following inequality was proven

$$\left| \sum_{i=1}^{m} p_i \sum_{i=1}^{m} p_i a_i b_i - \sum_{i=1}^{m} p_i a_i \sum_{i=1}^{m} p_i b_i \right| \le \frac{1}{4}(A-a)(B-b) \left( \sum_{i=1}^{m} p_i \right)^2 . \tag{1}$$

Let $a = (a_i)$, $i = 1, 2, \ldots, m$, be a positive real number sequence. In [8] (see also [19]) the following inequality was proven

$$\left( \sum_{i=1}^{m} \sqrt{a_i} \right)^2 \le (m-1) \sum_{i=1}^{m} a_i + m \left( \prod_{i=1}^{m} a_i \right)^{\frac{1}{m}} . \tag{2}$$

Let $a=(a_i)$, $i=1,2,\ldots,m$, be positive real number sequence. Then, for any real $r$, $r\le 0$ or $r\ge 1$, holds (see for example [10])

$$\sum_{i=1}^{m} a_i^r \ge m^{1-r}\left(\sum_{i=1}^{m} a_i\right)^r . \tag{3}$$

If $0\le r\le 1$, then opposite inequality in (3) is valid.

The inequality (3) in the literature is known as the Jensen's inequality.

Let $p=(p_i)$ and $a=(a_i)$, $i=1,2,\ldots,m$, be two positive real number sequences with the properties

$$p_1+p_2+\cdots+p_m=1 \quad \text{and} \quad 0<a\le a_i\le A<+\infty.$$

In [13] (see also [11]) the next inequality was proven

$$\sum_{i=1}^{m} p_i a_i + aA\sum_{i=1}^{m}\frac{p_i}{a_i} \le a+A. \tag{4}$$

## 3 Upper bounds for $AG$

In the following theorem we establish an upper bound for invariant $AG$ in terms of parameters $m$, $\Delta_{e_1}$, $\delta_{e_1}$ and topological index $R$.

**Theorem 3.1.** *Let G be a simple connected graph with n vertices and $m\ge 2$ edges. Then*

$$AG \le \frac{nm}{2R}+\frac{1}{8}\left(\sqrt{\Delta_{e_1}}-\sqrt{\delta_{e_1}}\right)^2 R. \tag{5}$$

*Equality holds if and only if G is regular or biregular graph.*

**Proof** For $p_i := \frac{1}{\sqrt{d_i d_j}}$, $a_i=b_i:=\sqrt{d_i+d_j}$, $A=B=\sqrt{\Delta_{e_1}}$, $a=b=\sqrt{\delta_{e_1}}$, where summing is performed over all edges in graph $G$, the inequality (1) becomes

$$\sum_{i\sim j}\frac{1}{\sqrt{d_i d_j}}\sum_{i\sim j}\frac{d_i+d_j}{\sqrt{d_i d_j}}-\left(\sum_{i\sim j}\frac{\sqrt{d_i+d_j}}{\sqrt{d_i d_j}}\right)^2 \le \frac{1}{4}\left(\sqrt{\Delta_{e_1}}-\sqrt{\delta_{e_1}}\right)^2\left(\sum_{i\sim j}\frac{1}{\sqrt{d_i d_j}}\right)^2,$$

i.e.

$$2R\cdot AG \le \left(\sum_{i\sim j}\frac{\sqrt{d_i+d_j}}{\sqrt{d_i d_j}}\right)^2+\frac{1}{4}\left(\sqrt{\Delta_{e_1}}-\sqrt{\delta_{e_1}}\right)^2 R^2. \tag{6}$$

For $r=2$, $a_i := \sqrt{\frac{d_i+d_j}{d_id_j}}$, where summing is performed over all edges in graph $G$, the inequality (3) transforms into

$$\left(\sum_{i\sim j}\sqrt{\frac{d_i+d_j}{d_id_j}}\right)^2 \le m\sum_{i\sim j}\frac{d_i+d_j}{d_id_j} = mn.$$

According to this inequality and (6), follows

$$2R\cdot AG \le nm + \frac{1}{4}\left(\sqrt{\Delta_{e_1}} - \sqrt{\delta_{e_1}}\right)^2 R^2,$$

wherefrom (5) is obtained.

□

**Theorem 3.2.** *Let $G$ be a simple connected graph with $n$ vertices and $m \ge 2$ edges. Then*

$$AG \le \frac{1}{2R}\left(m\frac{(\Pi_1^*)^{\frac{1}{m}}}{(\Pi_2)^{\frac{1}{m}}} + n(m-1)\right) + \frac{1}{8}\left(\sqrt{\Delta_{e_1}} - \sqrt{\delta_{e_1}}\right)^2 R. \tag{7}$$

*Equality holds if and only if $G$ is regular or biregular graph.*

**Proof** For $a_i := \frac{d_i+d_j}{d_id_j}$, where summing is performed over all edges in graph $G$, the inequality (2) becomes

$$\left(\sum_{i\sim j}\sqrt{\frac{d_i+d_j}{d_id_j}}\right)^2 \le m\frac{(\Pi_1^*)^{\frac{1}{m}}}{(\Pi_2)^{\frac{1}{m}}} + n(m-1).$$

According to this inequality and (6), we get (7).

□

**Theorem 3.3.** *Let $G$ be a simple connected graph with $m \ge 2$ edges. Then*

$$AG \le \frac{(\Delta_{e_1}+\delta_{e_1})R}{2} - \frac{m\Delta_{e_1}\delta_{e_1}}{2\,(\Pi_1^*)^{\frac{1}{m}}\,(\Pi_2)^{\frac{1}{2m}}}. \tag{8}$$

*Equality holds if and only if $G$ is regular or biregular graph.*

**Proof** For $p_i := \frac{1}{R\sqrt{d_id_j}}$, $a_i := d_i+d_j$, $A=\Delta_{e_1}$, $a=\delta_{e_1}$, where summing is performed over all edges in graph $G$, the inequality (4) becomes

$$\sum_{i\sim j}\frac{d_i+d_j}{\sqrt{d_id_j}} + \Delta_{e_1}\delta_{e_1}\sum_{i\sim j}\frac{1}{\sqrt{d_id_j}(d_i+d_j)} \le (\Delta_{e_1}+\delta_{e_1})R,$$

i.e.

$$2AG + \Delta_{e_1}\delta_{e_1}\sum_{i\sim j}\frac{1}{\sqrt{d_i d_j}(d_i+d_j)} \leq (\Delta_{e_1}+\delta_{e_1})R. \tag{9}$$

Using the arithmetic-geometric mean inequality for real numbers (see e.g. [11]), we get

$$\sum_{i\sim j}\frac{1}{\sqrt{d_i d_j}(d_i+d_j)} \geq \frac{m}{(\Pi_1^*)^{\frac{1}{m}}(\Pi_2)^{\frac{1}{2m}}}.$$

From this inequality and the inequality (9), we arrive at (8).

□

In the following theorem we determine upper bound for invariant $AG$ depending on the parameters $m$, $\Delta$ and $\delta$.

**Theorem 3.4.** *Let G be a simple connected graph with $m \geq 1$ edges. Then*

$$AG \leq \frac{m}{2}\left(\sqrt{\frac{\Delta}{\delta}}+\sqrt{\frac{\delta}{\Delta}}\right).$$

*Equality holds if and only if G is regular or biregular graph.*

**Proof** Since

$$\frac{d_i+d_j}{\sqrt{d_i d_j}} = \sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}} \leq \sqrt{\frac{\Delta}{\delta}}+\sqrt{\frac{\delta}{\Delta}},$$

for any edge in graph $G$, it follows

$$AG = \sum_{i\sim j}\frac{d_i+d_j}{2\sqrt{d_i d_j}} \leq \frac{m}{2}\left(\sqrt{\frac{\Delta}{\delta}}+\sqrt{\frac{\delta}{\Delta}}\right),$$

which completes the proof.

□

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