On robust and dynamic identifying codes

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Abstract

A subset C of vertices in an undirected graph $G = (V, E)$ is called a 1-identifying code if the sets $I(v) = \{u \in C : d(u, v) \leq 1\}$, $v \in V$, are nonempty and no two of them are the same set. It is natural to consider classes of codes that retain the identification property under various conditions, e.g., when the sets $I(v)$ are possibly slightly corrupted. We consider two such classes of robust codes. We also consider dynamic identifying codes, i.e., walks in G whose vertices form an identifying code in G.

1 Introduction

Let $G = (V, E)$ be an undirected graph, and denote by $d(u, v)$ the graphic distance in G, i.e., $d(u, v)$ is the number of edges in any shortest path from u to v (or ∞ if no such path exists). For all $v \in V$,

$$B_r(v) = \{u \in V : d(u, v) \leq r\}.$$

Assume that $C \subseteq V$, which we call a *code*, is given. The elements of C are called *codewords*. For all $v \in V$ we denote

$$I_r(v) = B_r(v) \cap C.$$

The set $I_r(v)$ is called the identifying set of v. When $r = 1$, we drop the subscript. A code $C \subseteq V$ is called r-identifying if all the sets $I_r(v)$, $v \in V$, are nonempty, and no two of them are the same set.

Definition 1 *A subset $C \subseteq V$ is called a t-edge-robust r-identifying code (in G) if the code C is r-identifying in every graph $G^1 = (V, E^1)$, where $E^1 = E \triangle E'$ and $E' \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$ has size at most t. Here $X \triangle Y$ denotes the symmetric difference, i.e., $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.*

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In other words, C must remain r-identifying in all graphs obtained by adding or deleting at most t edges (i.e., the total number of additions and deletions together must be at most t).

Theorem 1 *A subset $C \subseteq V$ is a t-edge-robust 1-identifying code in G if and only if the following three conditions hold:*

i) $|I(u) \triangle I(v)| \geq t+1$ for every two different vertices $u \in V$ and $v \in V$,

ii) $|I(u) \triangle I(v)| \geq t+2$ for every two different codewords $u \in C$ and $v \in C$ which are not adjacent in G, and

iii) $|I(u)| \geq t+1$ for every $u \notin C$.

Proof. This is easy to check. If in G we add or delete the edge between u and v, and u and v are not both codewords, say $u \notin C$, then $I(u)$ remains the same, or v is added to $I(u)$, or v is removed from $I(u)$; and $I(v)$ remains the same. If both u and v are codewords, and they are not adjacent, then adding the edge between u and v removes u and v from $I(u) \triangle I(v)$; and if u and v are adjacent codewords and we delete the edge between them, then u and v are inserted to $I(u) \triangle I(v)$. $\square$

Definition 2 *A subset $C \subseteq V$ is called a t-vertex-robust r-identifying code (in G) if $|I_r(v)| \geq t+1$ for all $v \in V$ and if for all $u, v \in V$, $u \neq v$, and $A, B \subseteq V$ with $|A|, |B| \leq t$, we have $I_r(u) \triangle A \neq I_r(v) \triangle B$.*

The condition $|I_r(v)| \geq t+1$ was not included in [46].

Clearly, the definition remains the same if we require that $A, B \subseteq C$ instead of $A, B \subseteq V$.

Equivalently [46], C is a t-vertex-robust r-identifying code, if $|I_r(v)| \geq t+1$ for all $v \in V$ and if for all $u, v \in V$, $u \neq v$, we have $|I_r(u) \triangle I_r(v)| \geq 2t+1$.

If $t = 0$, both definitions reduce to the definition of an r-identifying code.

If C is a t-vertex-robust 1-identifying code then it is also a t-edge-robust 1-identifying code.

It is important to notice the differences between the two definitions. Consider, for instance, the case $t = r = 1$. Informally, in the second variant, the identifying set is (possibly) corrupted by adding or deleting one codeword; likewise in the first variant, except that if c is a codeword, it cannot be deleted from its own identifying set (because we only delete edges). Another difference is that in the second variant we are given the corrupted identifying set $I(v)$ (where v is unknown) without any information how it has been corrupted, and we should be able to uniquely determine v. In the first variant the situation is different: we work separately in each given G^1.

Identifying codes were introduced in Karpovsky, Chakrabarty and Levitin [37]. The motivation for identifying codes comes from maintenance of multiprocessor architectures. Assume that each vertex of the graph G contains a processor, and each edge corresponds to a dedicated link between two processors. We

	square	king	triangular	hexagonal
1-identifying	7/20	2/9	1/4	$\leq 3/7$
1-edge-robust 1-identifying	1/2	$\leq 3/8$	3/7	2/3
1-vertex-robust 1-identifying	$\leq 5/8$	1/2	$\leq 3/5$	$\leq 41/50$
dynamic 1-identifying	2/5		1/3	1/2

Table 1: Minimum possible densities in the four grids and meshes.

ask some of the vertices (the codewords) to test their r-neighbourhoods, and each of them reports back to us by sending a 0, if it did not detect any problems, and a 1, otherwise. If the codewords form an r-identifying code, then based on these answers we can tell the location of the malfunctioning processor or tell that there is none, under the assumption that the number of malfunctioning processors is at most one. More generally, we wish to identify the malfunctioning processors, provided that there are at most l of them.

Many results have been obtained about identifying codes in hypercubes [6], [7], [25], [30], [34], [35], [38], [39], [40], [41], [44], [45], the square and king grids and triangular and hexagonal meshes [2], [8], [9], [10], [13], [15], [16], [17], [18], [26], [27], [28], [29], [32], [33], paths, cycles and trees [3], [4], directed graphs [12], and identifying codes and complexity [11], [12], [18], [34], [35].

In the final section of the paper we are interested in performing the above testing process (for $l = r = 1$) using a **dynamic agent**. For **dynamic location detection** the sensor (or software for testing) is migrating within a network according to a pre-computed walk in such a way that the set of vertices on this walk is the corresponding identifying code. For the case of diagnosis in multiprocessors, at every vertex v of the walk the dynamic agent reports to the host if there is a faulty vertex in its neighbourhood $B_1(v)$, and the host can identify the (at most one) faulty vertex by the end of the process. The approach offers great flexibility and portability, and it is efficient for handling dynamically changing environments. Dynamic agents implemented as mobile software objects for monitoring and control of computer networks have been described in [20], [21] and [22]. The new approach results in a drastic reduction of the number of sensors (which is a major limitation for many applications of identifying codes) at the expense of the time required for identification.

Table 1 summarizes some of our results in the four grids and meshes: with the dynamic codes we are working with finite graphs, and the values in the table are certain limits (cf. Section 4), but they are anyway included for comparison. The results of the first row can be found in [13], [2]; [17], [10]; [37]; and [16].

In a closely related problem of *r-locating-dominating sets*, the sets $I_r(v)$ are required to be nonempty and pairwise different, but only for $v \notin C$. In this case, each processor c corresponding to a codeword sends a 2, if c itself is malfunctioning, a 0, if it did not detect any problems, and a 1, otherwise. For results on this problem, see, e.g., [3], [23], [43], [47], [48], [49]. The class of *fault-*

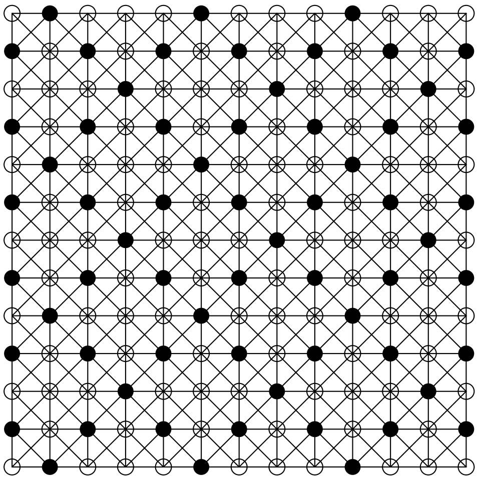

Figure 1: A 1-edge-robust 1-identifying code with density 3/8.

tolerant locating-dominating sets [49] can cope with the situation in which at most one codeword incorrectly transmits a 0 instead of a 1 or 2. Fault-tolerant identifying codes will be studied in [1].

The following two somewhat different versions have also been considered. A code $C \subseteq V$ in $G = (V, E)$ is called *strongly r-identifying* (see [31], [39], [41]), if the sets $\{I_r(v), I_r(v) \setminus \{v\}\}$ are disjoint for $v \in V$, and none of them contains the empty set: the idea being that if v is a codeword that corresponds to a malfunctioning processor, then v may fail to report about problems. Here we again think that each codeword transmits either a 0 or a 1 (i.e., 2's are not used). In another variant (see [31]), we assume that v in this case always fails to report about problems.

Vertex-robust identifying codes have been used in [46] for location detection by emergency networks of sensors in the case of unpredictable changes of topologies. A greedy algorithm for construction of irreducible vertex-robust identifying codes has also been proposed in [46].

2 On edge-robust codes

Theorem 2 *If $G = (V, E)$ is a finite d-regular graph and $d \geq 2$, and $C \subseteq V$ is a 1-edge-robust 1-identifying code in G, then*

$$|C| \geq \frac{2|V|}{d}.$$

Proof. We denote $N = V \setminus C$, and

$$C_i = \{c \in C : |I(c)| = i\},$$

$$N_i = \{x \in N : |I(x)| = i\}.$$

We further denote

$$C_{\geq i} = \bigcup_{j \geq i} C_j$$

and

$$N_{\geq i} = \bigcup_{j \geq i} N_j.$$

By definition, every $v \in N$ has at least two codeword neighbours.

There can be a codeword that belongs to C_1, but not more than one: adding an edge between any two such codewords would result in a graph in which C is no longer identifying. If $v \in C_1$, then $d \geq 2$ implies that v has at least two neighbours, say $u \in N$ and $u' \in N$. If $u \in N_2$ and the other codeword neighbour of u were w, then adding an edge between v and w would result in a graph in which C is not identifying. Hence $u \in N_{\geq 3}$, and similarly, $u' \in N_{\geq 3}$. We have proved that

$$|N_{\geq 3}| \geq 2|C_1|. \tag{1}$$

Assume that $v \in C_2$ and that its unique codeword neighbour is c. Then $c \in C_{\geq 4}$: if $c \in C_3$ and $I(c) = \{v, c, u\}$, then adding an edge between v and u would give a graph where C is no longer identifying. Moreover, any codeword in $C_{\geq 4}$ can have at most one neighbour that belongs to C_2: otherwise adding an edge between any two such neighbours would again give a graph where C is not identifying. Hence we have

$$|C_{\geq 4}| \geq |C_2|.$$

We can now prove the theorem by counting in two ways the number of pairs (v, c), where $v \in V$, $c \in C$ and $d(v, c) \leq 1$. Using the two inequalities from above we get

$$\begin{aligned}(d+1)|C| &= \sum_{i=2}^{d} i|N_i| + \sum_{i=1}^{d+1} i|C_i| \\ &\geq |C_{\geq 4}| + 3|C| - |C_2| - 2|C_1| + 2|N| + |N_{\geq 3}| \\ &\geq 3|C| + 2|N| = 3|C| + 2(|V| - |C|),\end{aligned}$$

i.e.,

$$d|C| \geq 2|V|$$

as claimed. □

Example 1 Consider a finite square grid (with wrapping around), in which the vertex set is $\mathbb{Z}_n \times \mathbb{Z}_m$, and two vertices (i,j) and (i',j') are neighbours if either $i = i'$ and $j - j' = \pm 1 \pmod{m}$; or $i - i' = \pm 1 \pmod{n}$ and $j = j'$. Assume that $n \geq 6$ is even and $m \geq 5$. If we take as codewords all the points (i,j) with i even, then it is easy to check that we obtain a 1-edge-robust 1-identifying code. This code has cardinality $|V|/2$, and is therefore the smallest possible by the previous theorem.

We are often interested in infinite graphs. Consider, for instance, the infinite square grid whose vertex set is $\mathbb{Z}^2$ and in which two vertices are adjacent if their Euclidean distance equals 1. In this graph we denote

$$Q_n = \{(i,j) \mid |i| \leq n, |j| \leq n\}$$

and the density D of a code $C \subseteq \mathbb{Z}^2$ is

$$D = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|}.$$

Theorem 3 *The smallest possible density of a* 1*-edge-robust* 1*-identifying code in the infinite square grid is* 1/2.

Proof. It is easy to check that the code that consists of all points $(i,j) \in \mathbb{Z}^2$ such that i is even is a 1-edge-robust 1-identifying code and has density 1/2.

Assume now that C is a 1-edge-robust 1-identifying code in the infinite square grid. Using the same argument as in the proof of Theorem 2 we get the inequality $|C_2 \cap Q_n| \leq |C_{\geq 4} \cap Q_{n+1}| \leq 8n + 8 + |C_{\geq 4} \cap Q_n|$ — as some of the codeword neighbours of the points in $C_2 \cap Q_n$ may not be in Q_n, but are anyway in Q_{n+1}, and $|Q_{n+1} \setminus Q_n| = 8n + 8$. Instead of the first inequality (1), we just use the fact that $|C_1| \leq 1$. Using these we get (where now $d = 4$)

$$\begin{aligned}
&(d+1)|C \cap Q_n| + (d+1)(8n+8) \\
&\quad \geq (d+1)|C \cap Q_{n+1}| \\
&\quad \geq \sum_{i=2}^{d} i|N_i \cap Q_n| + \sum_{i=1}^{d+1} i|C_i \cap Q_n| \\
&\quad \geq |C_{\geq 4} \cap Q_n| + 3|C \cap Q_n| - |C_2 \cap Q_n| - 2 + 2|N \cap Q_n| \\
&\quad \geq 3|C \cap Q_n| + 2|N \cap Q_n| - (8n+8) - 2,
\end{aligned}$$

and

$$\frac{|C \cap Q_n|}{|Q_n|} \geq \frac{2}{d} - \frac{(d+2)(8n+8)+2}{d|Q_n|},$$

resulting in the lower bound $2/d = 1/2$ on the density. □

The next theorem is about the infinite king grid, where the vertex set is $\mathbb{Z}^2$, and two vertices are adjacent if their Euclidean distance equals 1 or $\sqrt{2}$.

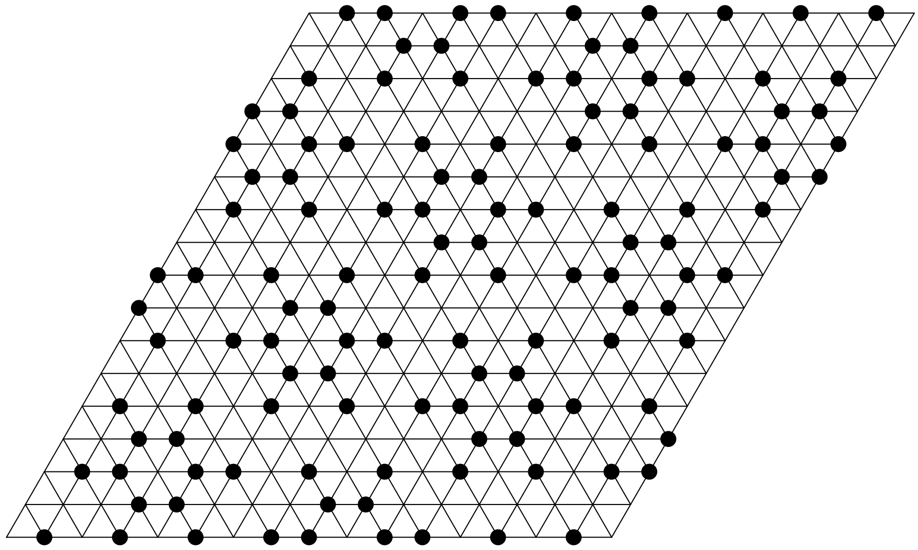

Figure 2: An optimal 1-edge-robust 1-identifying code with density 3/7.

Theorem 4 *In the infinite king grid there is a* 1*-edge-robust* 1*-identifying code with density* 3/8*.*

Proof. See Figure 1. □

Consider the infinite triangular mesh T for which

$$V = \{i(1,0) + j(\frac{1}{2}, \frac{\sqrt{3}}{2}) \mid i, j \in \mathbb{Z}\}$$

and two edges are adjacent if their Euclidean distance is 1. In T we denote

$$T_n = \{i(1,0) + j(\frac{1}{2}, \frac{\sqrt{3}}{2}) \mid |i| \leq n, |j| \leq n\}.$$

The density D of a code $C \subseteq T$ is defined by

$$D = \limsup_{n \to \infty} \frac{|C \cap T_n|}{|T_n|}.$$

Theorem 5 *([24]) The minimal density of a* 1*-edge-robust* 1*-identifying code in the infinite triangular mesh is* 3/7*.*

Proof. The proof can be found in [24]. For completeness, the construction from [24] is given in Figure 2. □

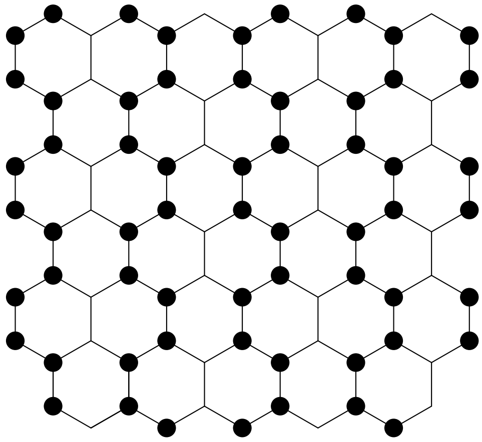

Figure 3: An optimal 1-edge-robust 1-identifying code with density 2/3.

Theorem 6 *The minimal density of a* 1*-edge-robust* 1*-identifying code in the infinite hexagonal mesh is* $2/3$*.*

Proof. It is easy to check that the code in Figure 3 is a 1-edge-robust 1-identifying code and has density 2/3. The lower bound on the density follows from the proof of Theorem 2 in the same way as in the proof of Theorem 3. For the exact definition of density in this case, see, e.g., [16]. □

Denote by $K(n, R)$ the minimum cardinality of a binary code of length n with covering radius R, and

$$V(n, r) = \sum_{i=0}^{r} \binom{n}{i}.$$

Theorem 7 *Assume that* $t \geq 1$*. In the binary* n*-dimensional cube* $\mathbb{Z}_2^n$ *with* $n \geq t + 3$ *there is a* t*-edge-robust* 1*-identifying code with cardinality at most* $2V(n-1, t)K(n-1, t+1)$*.*

Proof. We use a small modification of a construction from [37].

Denote by B_t the set of all binary words of length $n - 1$ and weight at most t. Let A be a code attaining the bound $K(n - 1, t + 1)$.

We define

$$C = \{a + x : a \in A, x \in B_t\} \oplus \mathbb{Z}_2.$$

Obviously, the cardinality of C is at most $2V(n-1, t)K(n-1, t+1)$, and we claim that it is a t-edge-robust 1-identifying code in $\mathbb{Z}_2^n$.

We denote

$$F(a) = \{a + x : x \in B_t\} \oplus \mathbb{Z}_2.$$

We now consider any fixed graph G^1 which is as in Definition 1. Let $I(v)$ denote the identifying set of v in G and $J(v)$ in G^1.

Let $v = (x, y)$, where $x \in \mathbb{Z}_2^{n-1}$ and $y \in \mathbb{Z}_2$.

By definition of A, there is a codeword $a \in A$ such that $d(a, x) \leq t + 1$. One immediately checks that

$$|I(v) \cap F(a)| = \begin{cases} n+1 & \text{if } d(x,a) < t \\ t+2 & \text{if } d(x,a) = t \\ t+1 & \text{if } d(x,a) = t+1. \end{cases} \tag{2}$$

In particular, the sets $J(v)$ are nonempty for all v. We prove that no two of them are the same (in the fixed G^1).

Let $v' = (x', y') \in \mathbb{Z}_2^n$, $v' \neq v$, where $x' \in \mathbb{Z}_2^{n-1}$ and $y' \in \mathbb{Z}_2$, be another vertex.

Assume first that v' is a vertex such that $I(v') \cap F(a) = \emptyset$. If G^1 was obtained by deleting i edges and adding j edges, then by (2),

$$|J(v) \cap F(a)| \geq t + 1 - i > t - i \geq j \geq |J(v') \cap F(a)|,$$

and we are done.

It therefore suffices to consider the vertices v' for which $I(v') \cap F(a) \neq \emptyset$.

We claim that $J(v) \cap F(a) \neq J(v') \cap F(a)$. Consider the size of $(I(v) \cap F(a)) \triangle (I(v') \cap F(a))$. Adding or deleting one edge when changing from G to G^1 can decrease it by at most one — except when both v and v' are in $F(a)$, and the edge to be added is $\{v, v'\}$ and $d(v, v') \geq 2$ (i.e., v and v' were not adjacent), in which case the quantity decreases by two. All in all, when moving from G to G^1, the quantity may decrease by at most $t + 1$, and by at most t if we do not add $\{v, v'\}$ as an edge.

Assume first that $x = x'$. Then $d(v, v') = 1$ and it suffices to show that $|(I(v) \cap F(a)) \triangle (I(v') \cap F(a))| > t$. If $w(x) = t+1$, then $I(v) \cap I(v') \cap F(a) = \emptyset$, and

$$\begin{aligned} & |(I(v) \cap F(a)) \triangle (I(v') \cap F(a))| \\ &= |I(v) \cap F(a)| + |I(v') \cap F(a)| - 2|I(v) \cap I(v') \cap F(a)| \\ &\geq (t+1) + (t+1) - 0 = 2t + 2 > t. \end{aligned}$$

If $w(x) \leq t$, then

$$|(I(v) \cap F(a)) \triangle (I(v') \cap F(a))| \geq (t+2) + (t+2) - 4 = 2t > t.$$

Assume then that $x \neq x'$.

If $d(x, a) < t$ or $d(x', a) < t$, then

$$|(I(v) \cap F(a)) \triangle (I(v') \cap F(a))| \geq (n+1) + (t+1) - 4 > t + 1,$$

and we are done; so assume that $d(x,a), d(x',a) \in \{t, t+1\}$. Then $|I(v) \cap I(v') \cap F(a)| \leq 1$.

If $d(x,a) = d(x',a) = t+1$, then $v \notin F(a)$ and $v' \notin F(a)$, and

$$|(I(v) \cap F(a)) \bigtriangleup (I(v') \cap F(a))| \geq (t+1) + (t+1) - 2 = 2t > t,$$

and we are done. In all other cases,

$$|(I(v) \cap F(a)) \bigtriangleup (I(v') \cap F(a))| \geq (t+2) + (t+1) - 2 = 2t+1 > t+1,$$

completing the proof. $\square$

Denote by $M_E(n,t)$ the minimum cardinality of any binary t-edge-robust 1-identifying code $C \subseteq \mathbb{Z}_2^n$.

If C is a t-edge-robust 1-identifying code in $G = (V, E)$, then for every $x \notin C$ we have $|I(x)| \geq t+1$. Hence

$$(d+1)|C| \geq (t+1)(|V| - |C|)$$

and

$$|C| \geq \frac{(t+1)|V|}{d+t+2}.$$

Applying this to $\mathbb{Z}_2^n$ we get

$$M_E(n,t) \geq \frac{(t+1)2^n}{n+t+2}.$$

If the conjecture

$$\lim_{n \to \infty} \frac{K(n,R)}{2^n / \sum_{i=0}^{R} \binom{n}{i}} = 1 \tag{3}$$

holds for $R = t+1$, then using Theorem 7 we get

$$M_E(n,t) = \frac{(t+1)2^n}{n}(1 + f(n)),$$

where $f(n) \to 0$, when $n \to \infty$. Anyway, we know from [50] (the construction is also explained in [14, Section 4.5]) that there is a family of values n_i for which $n_i \to \infty$ and

$$\lim_{i \to \infty} \frac{K(n_i, 2)}{2^{n_i} / \sum_{j=0}^{2} \binom{n_i}{j}} = 1.$$

As we shall see in Section 3, the minimum cardinality $M_V(n,t)$ of any 1-vertex-robust 1-identifying code $C \subseteq \mathbb{Z}_2^n$ satisfies the equation

$$M_V(n,t) = \frac{(t+2)2^n}{n}(1 + g(n)), \tag{4}$$

where $g(n) \to 0$, when $n \to \infty$, provided that the conjecture (3) holds for $R = t+2$.

Theorem 8 *Assume that $G = (V, E)$ is any connected graph with at least two vertices, and that C is a $(1, \leq 2)$-identifying code in G, i.e., the sets $I(F) := \cup_{x \in F} I(x)$ are different for all F of size at most two ($I(\emptyset) = \emptyset$). Then C is 1-edge-robust 1-identifying.*

Proof. Assume that C is $(1, \leq 2)$-identifying. Denote by $I(v)$ the identifying set of a vertex v in G and by $J(v)$ in G^1.

Because the graph is connected and has at least two vertices, we know that for all $v \in V$ we have $|I(v)| \geq 2$ (and hence $J(v) \neq \emptyset$). Indeed, assume that $I(v) = \{c\}$. If $c = v$, and u is any neighbour of v, then $I(u, c) = I(u)$; if $c \neq v$, then $I(c, v) = I(c)$, both contrary to our assumption.

Let $u, v \in V$, $u \neq v$. Clearly, $I(u) \neq I(v)$, so C is identifying in G.

Assume that G^1 is obtained by deleting an edge e from G. If e is incident to neither u nor v, then $J(u) = I(u) \neq I(v) = J(v)$. If e is the edge between u and v, then the fact that $I(u) \neq I(v)$ implies that there is a vertex $w \neq u, v$ such that $w \in I(u) \setminus I(v)$ or $w \in I(v) \setminus I(u)$, and the existence of such a vertex w shows that $J(u) \neq J(v)$. If e is incident to exactly one of the vertices u and v, say u, and the other endpoint of e is w, then $J(u) = J(v)$ would imply $I(u, w) = I(v, w)$ against our assumption.

Assume then that G^1 is obtained by adding an edge e to G. If e is incident to neither u nor v, there is again nothing to prove. If e is incident to exactly one of them, say u, and w is the other endpoint of e, then $J(u) = J(v)$ would imply $I(u, w) = I(v, w)$. Assume finally that e is the edge between u and v (and hence that there was no edge between them in G) and that $J(u) = J(v)$. Because $|I(u)| \geq 2$, there is a vertex w which is neither u nor v but is adjacent to both of them. But then $I(u, w) = I(v, w)$, contrary to our assumption. $\square$

Theorem 9 *i) $M_E(3, 1) = 6$,*
ii) $M_E(4, 1) = 9$, and
iii) $M_E(5, 1) \leq 16$.

Proof. i) It is easy to check that the code $\mathbb{Z}_2^3 \setminus \{000, 111\}$ is 1-edge-robust 1-identifying. The lower bound follows from Theorem 2.

ii) It is not difficult to check that the 9-element code

$$0000, 1010, 1001, 0110, 0101, 1110, 1101, 1011, 0111$$

is a 1-edge-robust 1-identifying code. By Theorem 2, the cardinality of a 1-edge-robust 1-identifying code of length four is at least eight, and if there is such a code, then equality holds in Theorem 2. Assume that C is a 1-edge-robust 1-identifying code of length four with eight codewords. From the proof of Theorem 2 we see that this implies that $|C_{\geq 5}| = 0$ and $|N_{\geq 3}| \leq 2$. We first observe that $C_1 = \emptyset$: if $c \in C_1$, then (as we saw in the proof of Theorem 2) its four non-codeword neighbours would all belong to $N_{\geq 3}$. Since $C_1 = \emptyset$, the argument of Theorem 2 shows that $N = N_2$.

We consider two possibilities: either there is a codeword in C_2 or $C = C_{\geq 3}$.

If $c \in C_2$, then without loss of generality $c = 0000$, $1000 \in C$, $0100 \notin C$, $0010 \notin C$ and $0001 \notin C$. From the proof of Theorem 2 we know that $1000 \in C_{\geq 4}$. But since we know that $C_{\geq 5} = \emptyset$, we can without loss of generality assume that $1100 \in C$, $1010 \in C$ and $1001 \notin C$. Since $0100 \in N = N_2$, we know that $0110 \notin C$ and $0101 \notin C$. Since $0010 \in N = N_2$, we know that $0011 \notin C$. But then $0001 \in N_1$, which is a contradiction.

Assume then that $C = C_3$ (and $N = N_2$). Without loss of generality, $0000 \notin C$, $1000 \in C$, $0100 \in C$, $0010 \notin C$, $0001 \notin C$.

We now complete C in two ways. First, if $0011 \notin C$, then $0010 \in N_2$ implies that $0110 \in C$ and $1010 \in C$; $0001 \in N_2$ implies that $0101 \in C$ and $1001 \in C$; $1000 \in C_3$ implies that $1100 \notin C$; $0011 \in N_2$ implies that $0111 \in C$ and $1011 \in C$; $1100 \in N_2$ implies that $1110 \notin C$ and $1101 \notin C$; and finally $1101 \in N_2$ implies that $1111 \notin C$. The resulting code, however, is not 1-edge-robust 1-identifying, because $I(1010) = \{1010, 1000, 1011\}$ and $I(1001) = \{1001, 1000, 1011\}$, and adding an edge between 1010 and 1001 violates the identification property.

Assume instead that $0011 \in C$. By symmetry we can assume that 1010 is the other codeword in $I(0010)$. Then $0110 \notin C$; $0100 \in C_3$ implies that $0101 \in C$ and $1100 \in C$; $1000 \in C_3$ implies that $1001 \notin C$; $1100 \in C_3$ implies that $1110 \notin C$ and $1101 \notin C$; $0011 \in C_3$ implies that $0111 \in C$ and $1011 \in C$; and finally $1110 \in N_2$ implies that $1111 \notin C$. Again, the resulting code is not 1-edge-robust 1-identifying, because $I(1011) = \{1011, 0011, 1010\}$ and $I(0010) = \{0011, 1010\}$, and adding an edge between 1011 and 0010 violates the identification property.

iii) We know from [45] that there is a $(1, \leq 2)$-identifying code of cardinality 16 in $\mathbb{Z}_2^5$, and the result immediately follows from Theorem 8. $\square$

Consider next the q-ary cube whose vertex set is $\mathbb{Z}_q^n$ and two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if $x_i = y_i$ for all indices except one, say j, and $x_j - y_j \equiv \pm 1 \pmod{q}$.

Theorem 10 *Assume that $A \subseteq \mathbb{Z}_2^{n-1}$ is a binary code with covering radius 1. Let $q > 4$ and*

$$E(0) = \{0, 2, 4, \ldots, q-2\}$$
$$E(1) = \{1, 3, 5, \ldots, q-1\}$$

if q is even, and

$$E(0) = \{0, 2, 4, \ldots, q-1\}$$
$$E(1) = \{0, 1, 3, 5, \ldots, q-2\}$$

if q is odd. Then

$$C = \mathbb{Z}_q \oplus \bigcup_{(x_1,\ldots,x_{n-1})\in A} E(x_1) \oplus \ldots \oplus E(x_{n-1})$$

is a 1-edge-robust 1-identifying code in the q-ary cube.

Proof. In the nonbinary hypercube the code C is clearly 1-identifying: if $v = (v_1, \ldots, v_n)$, and $v \in C$, then the set $I(v)$ contains three elements with the same last $n-1$ coordinates, namely, $(v_1-1, v_2, \ldots, v_n)$, v and $(v_1+1, v_2, \ldots, v_n)$, and, if $v \notin C$, then $I(v)$ contains the words $(v_1, \ldots, v_{i-1}, v_i - 1, v_{i+1}, \ldots, v_n)$ and $(v_1, \ldots, v_{i-1}, v_i + 1, v_{i+1}, \ldots, v_n)$ for some $i \geq 2$, and in both cases there is a unique vertex within distance one from these words (because $q > 4$).

Assume that we delete one edge — and let $I(v)$ now refer to the new graph. If the set $I(v)$ contains more than one vertex with the same last $n-1$ coordinates, we know that v is a codeword. If the deleted vertex does not connect two codewords, then the sets $I(v)$ for all $v \in C$ are the same as in the original graph, and we are done. If the edge does connect two codewords u and u', then $I(u)$ and $I(u')$ may only contain two codewords with the same last $n-1$ coordinates, but then $u \notin I(u) \setminus I(u')$, and we can still uniquely identify v. If $I(v)$ does not contain any two elements with the same last $n-1$ coordinates, then v is a non-codeword, and since there is at most one non-codeword v for which $I(v)$ does not contain the words $(v_1, \ldots, v_{i-1}, v_i - 1, v_{i+1}, \ldots, v_n)$ and $(v_1, \ldots, v_{i-1}, v_i + 1, v_{i+1}, \ldots, v_n)$ for some $i \geq 2$, we are again done.

Assume that we add one edge to the original hypercube. If $I(v)$ in this new graph contains at least three codewords with the same last $n-1$ coordinates, then v is a codeword. If the added edge does not connect two codewords, we are again done. If the edge indeed connects two codewords u and u', then $I(u)$ and $I(u')$ may contain more than three codewords with the same last $n-1$ coordinates. If $u = (u_1, u_2, \ldots, u_n)$, then $(u_1 - 1, u_2, \ldots, u_n) \in I(u)$ and $(u_1 + 1, u_2, \ldots, u_n) \in I(u)$, but they cannot both belong to $I(u')$. Hence the case when v is a codeword is clear. If, finally, $I(v)$ contains at most two codewords with the same last $n-1$ coordinates, then v is a non-codeword, and we know that in $I(v)$ there are some two codewords of the form $(v_1, \ldots, v_{i-1}, v_i - 1, v_{i+1}, \ldots, v_n)$ and $(v_1, \ldots, v_{i-1}, v_i+1, v_{i+1}, \ldots, v_n)$ for some $i \geq 2$. Consequently, we conclude that given $I(v)$, there is a unique vertex within distance one from all the codewords in $I(v)$ — except that there may be one (but only one) v such that for $I(v)$ there is no such vertex, but then we can identify v, because it is the unique non-codeword with that property. $\square$

Corollary 1 *If $q > 4$ is even and $n = 2^s$, $s \geq 2$, then the minimum possible size of a 1-edge-robust 1-identifying code in the q-ary cube equals q^n/n.*

Proof. Take in the previous theorem A to be a perfect binary code of length $2^s - 1$; the lower bound follows from Theorem 2. $\square$

Corollary 2 *If $q > 4$ is even, then for this fixed q the minimum possible cardinality of a 1-edge-robust 1-identifying code in the q-ary cube equals*

$$\frac{q^n}{n}(1 + g(n)),$$

where $g(n) \to 0$ when $n \to \infty$.

Proof. By Theorem 2, the minimum possible cardinality is at least q^n/n. On the other hand, by Theorem 10 there is such a code with cardinality at most $qK(n-1,1)(q/2)^{n-1}$. According to [36],

$$\frac{K(n-1,1)n}{2^{n-1}} \to 1$$

when $n \to \infty$, from which the claim follows. □

3 On vertex-robust codes

We begin with a general lower bound.

Theorem 11 *Assume that $G = (V, E)$ is a d-regular graph, $t > 0$, and that C is a t-vertex-robust 1-identifying code with cardinality K in G. Then*

$$K \geq \frac{(t+2)|V|}{d + 1 + \frac{2-t^2}{(t+1)(2t+3)}}.$$

Proof. We use the same notations C_i, N_i and $C_{\geq i}$ as in the proof of Theorem 2.

We first observe that no codeword can have two neighbours u_1 and u_2 that both belong to $C_{t+1} \cup N_{t+1}$: otherwise $|I(u_1) \bigtriangleup I(u_2)| \leq 2t$.

By definition, $C_0 = C_1 = \ldots = C_t = \emptyset$. If $c \in C_{t+1}$, and u is a codeword neighbour of c, then $u \in C_{\geq t+4}$, because $|I(u) \bigtriangleup I(c)| \geq 2t+1$. By our preliminary observation, we therefore have

$$t|C_{t+1}| \leq |C_{\geq t+4}|. \tag{5}$$

In the same way we see that vertices from C_{t+2} can have codeword neighbours only from $C_{\geq t+3}$. Counting in two ways the number of edges between the vertices in $C_{t+1} \cup C_{t+2}$ and the vertices in C we see that

$$t|C_{t+1}| + (t+1)|C_{t+2}| \leq (t+2)|C_{t+3}| + (t+3)|C_{\geq t+4}| + \sum_{i=5}^{d+1-t} (i-4)|C_{t+i}|. \tag{6}$$

By definition, $N_0 = N_1 = \ldots = N_t = \emptyset$. Vertices from N_{t+1} cannot have neighbours from C_{t+1}. Counting the number of codeword neighbours of $N_{t+1} \cup C_{t+1}$ and again using our preliminary observation we get the inequality

$$K - |C_{t+1}| \geq (t+1)|N_{t+1}| + t|C_{t+1}|,$$

i.e.,

$$|N_{t+1}| \leq \frac{K}{t+1} - |C_{t+1}|. \tag{7}$$

Counting the pairs (c, x) such that $c \in C$, $x \in V$, $d(c, x) = 1$, we get

$$\begin{aligned} Kd \quad \geq \quad & t|C_{t+1}| + (t+1)|C_{t+2}| + (t+2)|C_{t+3}| + (t+3)|C_{\geq t+4}| \\ & + \sum_{i=5}^{d+1-t} (i-4)|C_{t+i}| + (t+1)|N_{t+1}| + (t+2)(|V| - K - |N_{t+1}|). \end{aligned} \tag{8}$$

Combining (7) and (8) we obtain

$$\begin{aligned} Kd \quad \geq \quad & t|C_{t+1}| + (t+1)|C_{t+2}| + (t+2)|C_{t+3}| + (t+3)|C_{\geq t+4}| \\ & +(t+2)(|V| - K) - \frac{K}{t+1} + |C_{t+1}| + \sum_{i=5}^{d+1-t} (i-4)|C_{t+i}|. \end{aligned} \tag{9}$$

Now, multiply (5) by $-z$, and (6) by $-y$ ($y, z \geq 0$), and add with (9) to obtain

$$\begin{aligned} Kd \quad \geq \quad & a_1|C_{t+1}| + a_2|C_{t+2}| + a_3|C_{t+3}| + a_4|C_{\geq t+4}| \\ & +(1-y) \sum_{i=5}^{d+1-t} (i-4)|C_{t+i}| - (t+2)K - \frac{K}{t+1} + (t+2)|V| \\ \geq \quad & K\left(\min\{a_1, a_2, a_3, a_4\} - (t+2) - \frac{1}{t+1}\right) + (t+2)|V|, \end{aligned}$$

provided that $y \leq 1$. Here

$$\begin{array}{ll} a_1 = (1+y+z)t + 1, & a_2 = (1+y)(t+1), \\ a_3 = (1-y)(t+2), & a_4 = (1-y)(t+3) - z. \end{array}$$

It is readily seen that

$$\max_{y,z} \min\{a_1, a_2, a_3, a_4\} = (t+1)(1 + \frac{1}{2t+3}),$$

and the maximum is achieved for $y = 1/(2t+3)$, $y/t \leq z \leq 1 - y$. In fact, a_2 or a_3 is smaller than $(t+1)(1 + 1/(2t+3))$ unless $y = 1/(2t+3)$. We have therefore obtained the bound

$$Kd \geq K\left(-1 - \frac{1}{t+1} + \frac{t+1}{2t+3}\right) + (t+2)|V|$$

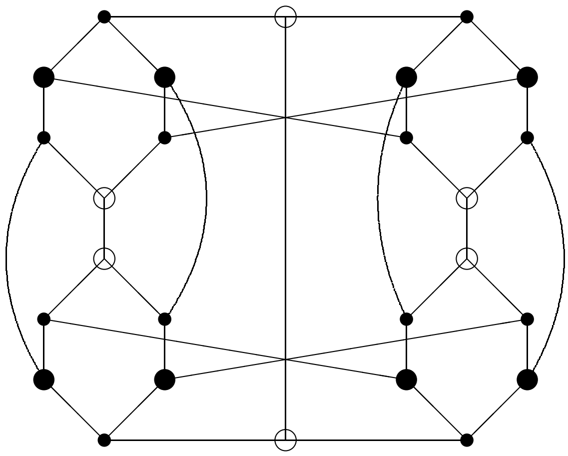

Figure 4: A 3-regular graph with 26 vertices and a 1-vertex-robust 1-identifying code with 20 codewords. The big solid circles denote codewords in C_4, the small solid circles codewords in C_3.

as claimed. $\square$

In particular, for $t = 1$,

$$K \geq \frac{30|V|}{10d + 11},$$

and for $t = 2$,

$$K \geq \frac{84|V|}{21d + 19}.$$

So, for $d = 3$, $t = 1$ we get $K/|V| \geq 30/41$, for $d = 4$, $t = 1$ we get $K/|V| \geq 10/17$ and for $d = 6$, $t = 1$ we get $K/|V| \geq 30/71$.

The case $d = 3$, $t = 1$, can be improved. Indeed, then $|C_2| = 0$ because $|C_{\geq 5}| = 0$, and $2|N_2| + 3|N_3| = |C_3|$, and $2|C_3| \leq 3|C_4|$. It follows that $|N_2| \leq |C_3|/2 \leq 3K/10$. Then, from the inequality

$$3K \geq 2|C_3| + 3|C_4| + 3(|V| - K) - |N_2|$$

it follows that $K/|V| \geq 10/13$. This bound is tight: there exists a 3-regular graph that has a code attaining this bound, as shown in Figure 4.

Theorem 12 *For all $n \geq t + 3$ there is a t-vertex-robust 1-identifying code $C \subseteq \mathbb{Z}_2^n$ whose cardinality is at most $K(n, t + 2)V(n, t + 1)$.*

Proof. Let A be a code attaining the bound $K(n, t + 2)$ and take

$$C = \bigcup_{a \in A} (a + B_{t+1}),$$

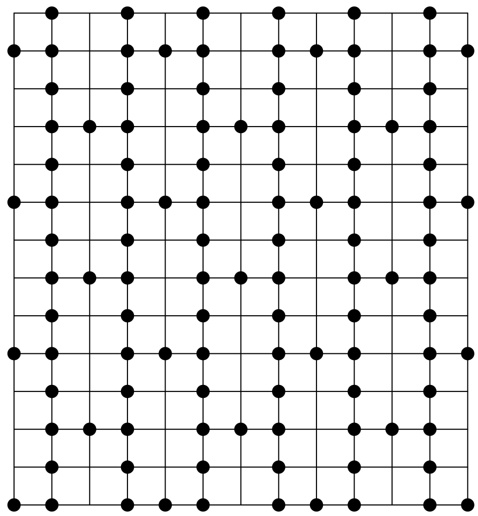

Figure 5: A 1-vertex-robust 1-identifying code with density 5/8.

where B_{t+1} denotes the set of all binary words of length n and weight at most $t+1$. It is not difficult to check that $|I(x) \triangle I(y)| \geq 2t+2$ whenever $x, y \in \mathbb{Z}_2^n$ and $x \neq y$. □

Theorem 13 *Let $t \geq 1$ be fixed. If the conjecture (3) holds for $R = t+2$, we have*

$$M_V(n,t) = \frac{(t+2)2^n}{n}(1+g(n)),$$

where $g(n) \to 0$ when $n \to \infty$.

Proof. The upper bound follows from the previous theorem, and the lower bound from Theorem 11. □

Theorem 14 *In the infinite square grid there is a* 1*-vertex-robust* 1*-identifying code with density* 5/8*.*

Proof. See Figure 5. □

Theorem 15 *The smallest possible density of a* 1*-vertex-robust* 1*-identifying code in the infinite king grid is* 1/2*.*

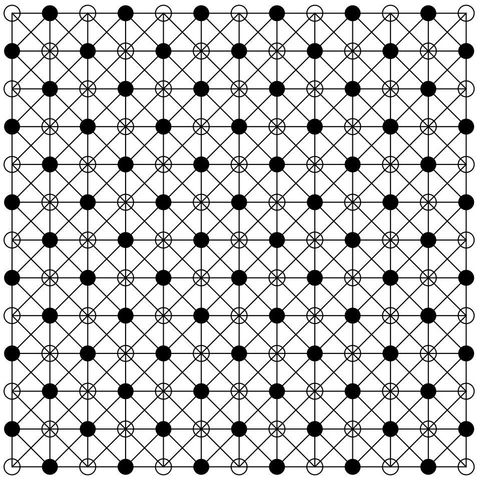

Figure 6: A 1-vertex-robust 1-identifying code with density 1/2.

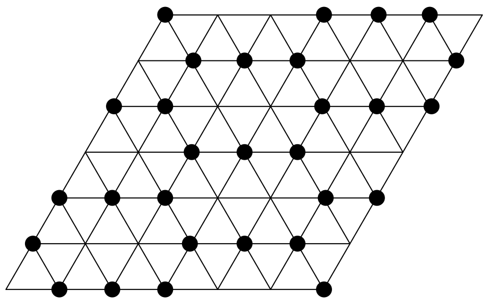

Figure 7: A 1-vertex-robust 1-identifying code with density 3/5.

Proof. For a construction, see Figure 6. Conversely, assume that C is a 1-vertex-robust 1-identifying code in the infinite king grid. For all i and j, the set $B_1((i,j)) \triangle B_1((i+1,j))$ has size 6 and contains at least 3 codewords of C, and the lower bound on the density follows. □

Theorem 16 *In the infinite triangular mesh there is a* 1*-vertex-robust* 1*-identifying code with density* $3/5$.

Proof. See Figure 7. □

Theorem 17 *Consider the infinite hexagonal mesh* H.

(i) A code C *is a* 1*-vertex-robust* 1*-identifying in* H *if and only if there are no two non-codewords whose graphic distance equals two or three.*

(ii) In H *there is a* 1*-vertex-robust* 1*-identifying code with density* $41/50$.

Proof. (i) Assume first that $C \subseteq V$ is a 1-vertex-robust 1-identifying code in H, and let $u \notin C$ and $v \notin C$ be any two non-codewords. If $d(u,v) = 2$, then there is a unique vertex x which is adjacent to both. Let w be the third neighbour of x. Now $|I(x) \triangle I(w)| \leq 2$. If $d(u,v) = 3$, let x and w be the two middle vertices on a shortest path connecting u and v. Then $|I(x) \triangle I(w)| \leq 2$.

Conversely, assume that $C \subseteq V$ and that there are no two non-codewords whose graphic distance would be two or three. First, for every $v \in V$, we have $|I(v)| \geq 2$: of the three neighbours of v, at least two must be in C, because the pairwise distance between any two of its neighbours equals two.

Let $u, v \in V$, $u \neq v$, be arbitrary. If $d(u,v) \geq 3$, then $|I(u) \triangle I(v)| \geq 2+2 = 4$ by what we have already proved.

Assume that $d(u,v) = 2$. If $u \notin C$, then by our distance requirement, $v \in C$, and the two of its neighbours that are at distance three from u are also in C and hence $|I(u) \triangle I(v)| \geq 3$. Assume then that $u \in C$. Out of the two neighbours of u that are at distance three from v, at least one is in C (because their pairwise distance is two). For the same reason, out of the two neighbours of v that are at distance three from u, at least one is in C. Again $|I(u) \triangle I(v)| \geq 3$.

Assume finally that $d(u,v) = 1$. If $|I(u) \triangle I(v)| \leq 2$, then u (resp. v) would have two non-codeword neighbours — which cannot be since their distance is two — or u has a neighbour other than v which is a non-codeword and v has a neighbour other than u which is a non-codeword — but their distance equals three.

Hence C is a 1-vertex-robust 1-identifying code.

(ii) This follows from Figure 8. □

We next determine the smallest possible densities for the square and king grids and triangular and hexagonal meshes when $r = 1$ and $t > 1$.

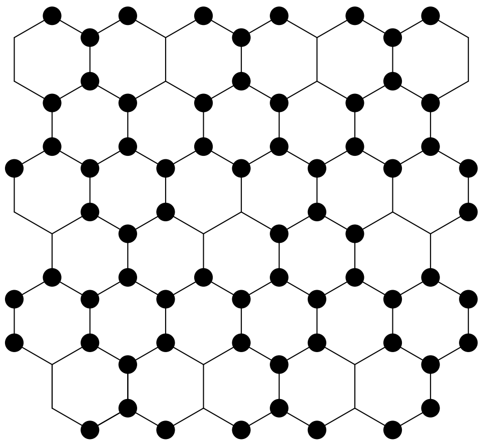

Figure 8: A 1-vertex-robust 1-identifying code with density 41/50.

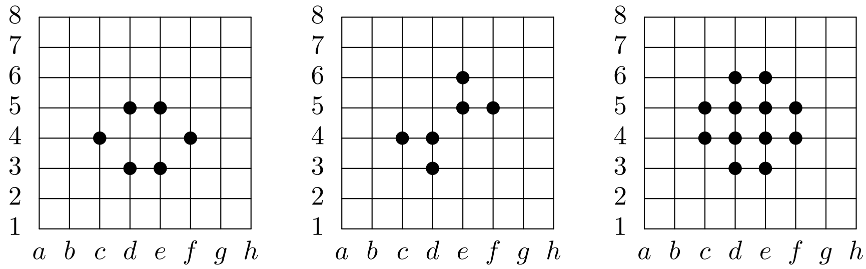

Figure 9: Patterns that each contain at most one non-codeword.

Theorem 18 *In the infinite square grid the smallest* 2*-vertex-robust* 1*-identifying code has density* $11/12$ *— and no* t*-vertex-robust* 1*-identifying codes exist if* $t > 2$.

Proof. Let u and v be any two neighbouring points in $\mathbb{Z}^2$. Then $|B_1(u) \triangle B_1(v)| = 6$ and the points in $B_1(u) \triangle B_1(v)$ form the pattern given in the left-hand figure of Figure 9 (here u and v are $d4$ and $e4$) — up to rotation. We see that it is not possible that $|I(u) \triangle I(v)| \geq 7$, which proves the second claim.

Assume that C is a 2-vertex-robust 1-identifying code. Then $|I(u) \triangle I(v)| \geq 5$ for all $u, v \in \mathbb{Z}^2$, $u \neq v$, and hence each of the 6-element patterns discussed above must contain at most one non-codeword.

The same must be true for all the 6-element patterns given in the middle figure of Figure 9 (and the ones obtained from it by rotation), because they can

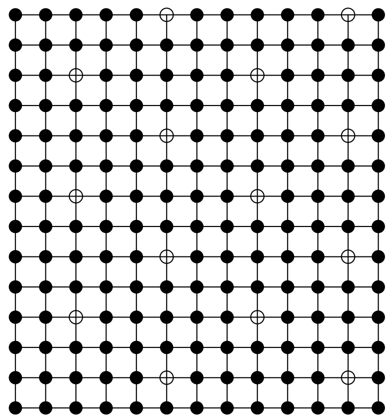

Figure 10: An optimal 2-vertex-robust 1-identifying code with density 11/12.

be obtained as sets $B_1(u) \triangle B_1(v)$, when the Euclidean distance between u and v is $\sqrt{2}$.

These two facts together prove that at most one of the points in the third pattern S given in the rightmost figure of Figure 9 can be a non-codeword: given any two of its points, we can always choose one of our 6-element patterns that contains both of them. By considering the translates $(x, y) + S$, we see that the density of C is at least 11/12 (cf. [8]). The bound is exact, because it is easy to check that the code in Figure 10 has density 11/12, and that it is a 2-vertex-robust 1-identifying code. □

Theorem 19 *(i) The smallest possible density of a* 2*-vertex-robust* 1*-identifying code in the king grid is* 5/6*.*

(ii) No t-vertex-robust 1*-identifying codes with $t > 2$ exist in the king grid.*

Proof. (i) It is easy to check that the code

$$C = \{(i, j) : i - j \not\equiv 0 \bmod 6\}$$

in Figure 11 has density 5/6 and that it is 2-vertex-robust 1-identifying code.

Assume conversely that we have any 2-vertex-robust 1-identifying code A. Then for all $(i, j) \in \mathbb{Z}^2$, the set

$$\begin{aligned} &B_1((i,j)) \triangle B_1((i+1,j)) \\ &\quad = \{(i-1,j-1),(i-1,j),(i-1,j+1),(i+2,j-1),(i+2,j),(i+2,j+1)\} \end{aligned} \tag{10}$$

must contain at least five codewords. The same argument as in the proof of Theorem 18 shows that the density of A must be at least 5/6.

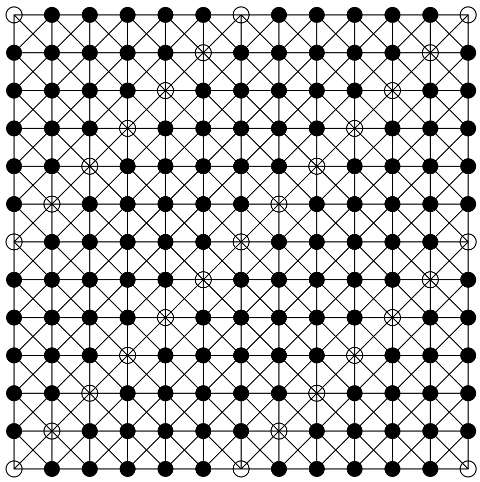

Figure 11: An optimal 2-vertex-robust 1-identifying code with density 5/6.

(ii) This immediately follows from (10). □

Theorem 20 *Consider the infinite triangular mesh T.*

(i) A code $C \subseteq T$ is a 2-vertex-robust 1-identifying code if and only if the graphic distance between any two different non-codewords is at least four.

(ii) The smallest possible density of a 2-vertex-robust 1-identifying code in T is 11/12.

(iii) No t-vertex-robust 1-identifying codes with $t > 2$ exist in T.

Proof. Let u and v be any two neighbouring points in $T = (V, E)$. Then $|B_1(u) \triangle B_1(v)| = 6$ and the points in $B_1(u) \triangle B_1(v)$ form the pattern given in the left-hand figure of Figure 12 (here u and v are $d4$ and $e4$) — up to rotation. We see that — even if all the vertices in T were codewords — it is not possible that $|I(u) \triangle I(v)| \geq 7$, which proves (iii).

Consider the case $t = 2$, and assume that $C \subseteq V$ is a 2-vertex-robust 1-identifying code in T. Then $|I(u) \triangle I(v)| \geq 5$ for all $u, v \in V$, $u \neq v$, and hence each of the 6-element patterns discussed above must contain at most one non-codeword. Then the graphic distance between any two non-codewords must be at least four: given any two points with graphic distance three or less, we can choose such a 6-element pattern that contains both of them. Assume conversely that $C \subseteq V$ has the property that the graphic distance between any two non-codewords is at least four. Let $u, v \in V$ be arbitrary. By the

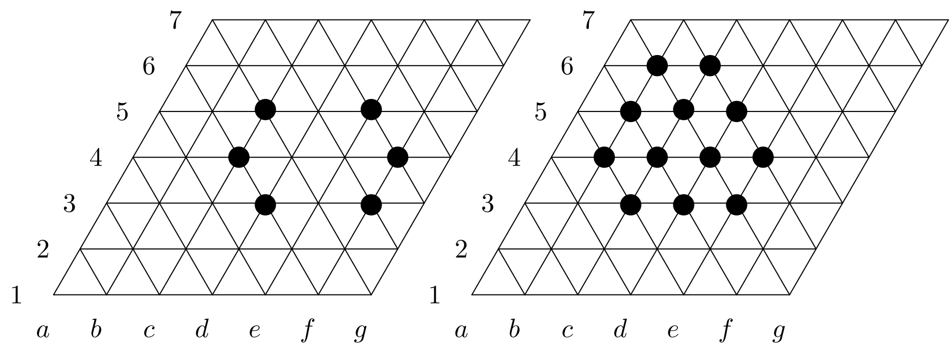

Figure 12: Two patterns both containing at most one non-codeword.

assumption, $B_1(u)$ and $B_1(v)$ can contain at most one non-codeword each, and hence $|I(u)| \geq 6$ and $|I(v)| \geq 6$. If the graphic distance $d(u,v)$ is at least two, then $|B_1(u) \cap B_1(v)| \leq 2$, and consequently, $|I(u) \triangle I(v)| \geq 8$. If $d(u,v) = 1$, then $|B_1(u) \cap B_1(v)| = 4$, and $|I(u) \triangle I(v)| \geq 4$, but equality can only hold, if there is exactly one non-codeword in $I(u) \setminus I(v)$ and one in $I(v) \setminus I(u)$, which is not possible, because the graphic distance between any two non-codewords is bigger than three. This proves (i).

Assume that C is a 2-vertex-robust 1-identifying code in T. By (i), the pattern S of 12 points given in the right-hand figure of Figure 12 can contain at most one non-codeword. But then by considering the translates $(x,y)+S$ we see that the density of C is at least 11/12 (cf. [8]). The bound is exact, because the code in Figure 13 clearly has density 11/12, and the graphic distance between any two non-codewords is at least four. □

In the hexagonal mesh, no t-vertex-robust 1-identifying codes exist for $t \geq 2$, because for any two neighbouring points u and v the symmetric difference $B_1(u) \triangle B_1(v)$ only contains four points.

If G is finite, and $C \subseteq V$ has K elements, and $C = \{c_1, c_2, \ldots, c_K\}$, it is sometimes convenient to think of the sets $I_r(v)$ as incidence vectors

$$\mathbf{a}(v) = (a_1(v), a_2(v), \ldots, a_K(v)) \in \mathbb{Z}_2^K$$

where $a_i(v) = 1$ if $c_i \in I_r(v)$, and 0, otherwise. Then $|I_r(u) \triangle I_r(v)|$ equals the Hamming distance between the vectors $\mathbf{a}(u)$ and $\mathbf{a}(v)$. We denote the set $\{\mathbf{a}(v) : v \in V\}$ by C^+.

Consider finally the following problem: given the number of vertices, how should we choose the edges so that there would be as small a t-vertex-robust 1-identifying code as possible. This is an important problem if codewords are expensive, and we can choose the graph topology.

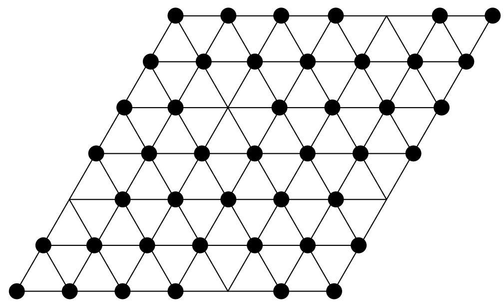

Figure 13: An optimal 2-vertex-robust 1-identifying code with density 11/12.

Theorem 21 *Assume that $r \geq 3$ and N satisfies the inequalities*

$$\frac{2^{2^r-2}}{2^r-1} \leq N \leq 2^{2^r-r-1}-1.$$

Then in any graph with N vertices all 1-vertex-robust 1-identifying codes have at least 2^r-1 codewords; and there is a graph G with N vertices such that in G there is a 1-vertex-robust 1-identifying code with exactly 2^r-1 codewords.

Proof. Assume that G is any graph with N vertices, and that C is a 1-vertex-robust 1-identifying code with K codewords in G. The code C^+ has length K, cardinality N and minimum distance at least 3. By the sphere-packing bound, $N \leq 2^K/(K+1)$ and therefore $K \geq 2^r-1$ as claimed.

We now construct such a graph and code. Denote $k = 2^r-r-1$, and let $\mathbf{H}$ be an $r \times k$ matrix whose columns are all the binary vectors in $\mathbb{Z}_2^r$ with at least two 1's. Denote

$$\mathbf{B} = \begin{pmatrix} \mathbf{I}_k & \mathbf{H}^T \\ \mathbf{H} & \mathbf{I}_r \end{pmatrix},$$

where $\mathbf{I}_s$ denotes the $s \times s$ identity matrix. The first k columns of $\mathbf{B}$ generate the binary Hamming code of length 2^r-1. Moreover, it is easy to check that the last r columns belong to this Hamming code. Namely, consider the i-th column of $\mathbf{B}$, where $k < i$. To get it as a linear combination of the first k columns of $\mathbf{B}$, we have to add the columns where the i-th *row* of $\mathbf{B}$ has 1's (because $\mathbf{B}$ is symmetric). But this linear combination gives exactly the right r last components, because every row in $\mathbf{H}$ has an odd number of 1's, and any two different rows of $\mathbf{H}$ both have 1's in an even number of places.

Because $\mathbf{B}$ is symmetric, we can define that the codeword vertices in our graph G are v_i, $i = 1, 2, \ldots, 2^r-1$, and that v_i and v_j $(i \neq j)$ are adjacent in G if and only if the (i,j)-entry in $\mathbf{B}$ equals 1. Next, we take any $N-(2^r-1)$ different non-zero linear combinations of the first k columns of $\mathbf{B}$ that do not occur as a column in $\mathbf{B}$: and let them correspond to the words $\dashv(u)$, $u \notin C$ of C^+; i.e., they tell us the edges between codewords and non-codewords in G. The edges between non-codewords can be chosen freely.

By the construction, the resulting code C in G has the property that all the codewords in C^+ are non-zero codewords in the Hamming code of length $2^r - 1$, and therefore C is clearly a 1-vertex-robust 1-identifying code. □

We can also take $t > 1$ and use the same construction. We then assume that $(\mathbf{I}_k, \mathbf{H}^T)$ is a generator matrix for a code with $|V| + 1$ codewords and minimum distance at least $2t + 1$ and that $\mathbf{H}$ has the property that $\mathbf{H}\mathbf{H}^T = \mathbf{I}_r$, i.e., the number of 1's in each row is odd, and the number of 1's in common for any two different rows is even.

Example 2 Assume that we wish to construct a graph with $2^{12} - 1$ vertices with as small a 3-vertex-robust 1-identifying code as possible.

If C is a 3-vertex-robust 1-identifying code in a graph with $2^{12} - 1$ vertices, then $|C^+| = 2^{12} - 1$ and the minimum distance of C^+ is at least 7 (and all codewords have weight at least four). By the sphere-packing bound, the length n of C^+ must satisfy $2^n / (\binom{n}{3} + \binom{n}{2} + n + 1) \geq 2^{12} - 1$, i.e., $n \geq 23$. In other words, $|C| \geq 23$.

To construct a graph G with $2^{12} - 1$ vertices and a 3-vertex-robust 1-identifying code C with 23 codewords in G we use the binary Golay code. It is easy to check that we can proceed in the same way as in the previous proof but now using the 11×12 matrix $\mathbf{H}$ whose rows are the last eleven columns of the matrix in [42, Figure 2.13].

4 Dynamic identifying codes

Definition 3 *Assume that G is finite. A sequence $\mathbf{c}_1$, $\mathbf{c}_2$, ..., $\mathbf{c}_M$ of vertices such that*

> *i) $d(\mathbf{c}_i, \mathbf{c}_{i+1}) = 1$ for all $i = 1, 2, \ldots, M - 1$, i.e., $\mathbf{c}_1$, $\mathbf{c}_2$, ..., $\mathbf{c}_M$ form a* **walk** *in G,*
>
> *ii) $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_M\}$ is an r-identifying code in C,*

is called a **dynamic r-identifying code** *in G.*

The elements $\mathbf{c}_1$, $\mathbf{c}_2$, ..., $\mathbf{c}_M$ are called **codewords**. If $r = 1$, we just speak of a dynamic identifying code.

For a similar study of covering codes instead of identifying codes using dynamic agents we refer to [5] and [19].

From now on, we will call the usual codes **static codes**, i.e., a static code is just a subset of vertices, whereas a dynamic code is a walk in G. We say that the

density of the dynamic code in Definition 3 is M/N, where N is the number of vertices in G. If A is a dynamic identifying code, we denote its density by $D(A)$. Given G, we would like to determine the smallest possible density of a dynamic identifying code in G.

The integer M in Definition 3 is called the **length** of the dynamic code. The corresponding static r-identifying code $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_M\}$ may have fewer than M codewords, since our walk through the graph may take us through the same vertex more than once.

Example 3 Assume that we wish to find a graph with $N > 3$ vertices, and as small a dynamic identifying code as possible. Clearly, the length of a dynamic identifying code must be at least $\log_2(N+1)$. Take $M = \lceil \log_2(N+1) \rceil$.

First take M vertices and connect them so that they form a path. We choose this path $\mathbf{c}_1$, $\mathbf{c}_2$, $\ldots$, $\mathbf{c}_M$ as our dynamic code. The I-sets of the codewords are certain 2- and 3-element sets, and no two are the same. Take $N - M$ other nonempty subsets of the set $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_M\}$, and corresponding to each of them create a new vertex and connect it to the elements of the subset. Clearly the resulting graph G has N vertices and the code is a dynamic identifying code of length M in G. $\square$

Denote the degree of a vertex v in G by $d_G(v)$.

Theorem 22 *Assume that $G = (V, E)$ is a graph with N vertices and that $d_G(v) \leq d$ for all $v \in V$. If there is a dynamic identifying code of length M in G, then $M \geq 2(N-1)/(d+1)$.*

Proof. Assume that $\mathbf{c}_1$, $\mathbf{c}_2$, $\ldots$, $\mathbf{c}_M$ is such a dynamic code. Let C denote the subgraph of G consisting of the vertices $\mathbf{c}_i$ $(i = 1, 2, \ldots, M)$ and the edges between them (in G). Let v_1, v_2, $\ldots$, v_K $(K \leq M)$ denote the different vertices in C.

Count the number of pairs (u, v), where $u, v \in V$, $u \notin C$, $v \in C$, and $d(u, v) = 1$. Clearly, this number is at most

$$\sum_{i=1}^{K} (d - d_C(v_i)).$$

On the other hand, corresponding to each i, there can be at most one vertex $u \notin C$ such that $I(u) = \{v_i\}$: hence there are at most K vertices $u \notin C$ whose I-set is a singleton set, and all the others contain at least two codewords. We obtain the inequality

$$\sum_{i=1}^{K} (d - d_C(v_i)) \geq K \cdot 1 + (N - 2K) \cdot 2. \tag{11}$$

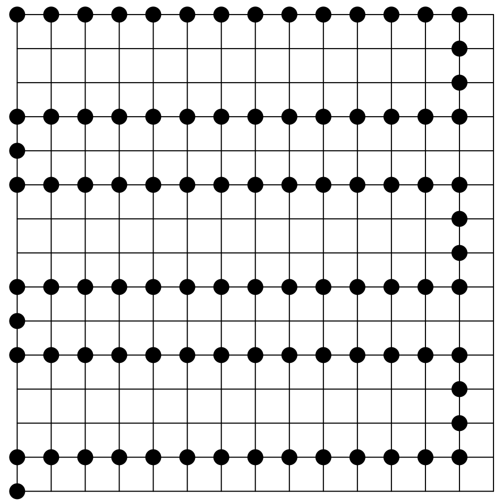

Figure 14: A dynamic identifying code in $\hat{Q}_7$.

Since C is connected, it has a spanning tree C', and

$$\sum_{i=1}^{K} d_C(v_i) \geq \sum_{i=1}^{K} d_{C'}(v_i) = 2(K-1).$$

Substituting this in (11) we get

$$Kd - 2(K-1) \geq K + 2(N - 2K),$$

i.e., $K(d+1) \geq 2(N-1)$, as claimed. $\square$

It is known that the minimum possible density of a static identifying code in the infinite square grid is 7/20; see [13], [2].

In the context of dynamic codes we consider the subgraphs Q_n, and the graphs $\hat{Q}_n$ ($n \geq 1$) with the same vertex set but where now coordinates are modulo $2n+2$, so that each vertex (i,j) has exactly four neighbours $(i-1,j)$, $(i+1,j)$, $(i,j-1)$, $(i,j+1)$.

Example 4 It is easy to check that the code in Figure 14 is a dynamic identifying code in $\hat{Q}_7$.

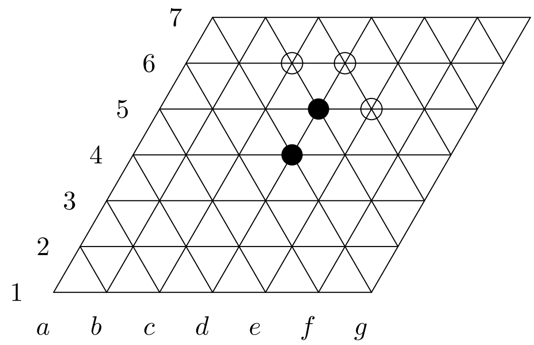

Figure 15: Proof of Theorem 3.

Theorem 23 *If A_n is a dynamic identifying code in Q_n with the smallest possible density, then $D(A_n) \to 2/5$, when $n \to \infty$. If $\hat{A}_n$ is a dynamic identifying code in $\hat{Q}_n$ with the smallest possible density, then $D(\hat{A}_n) \to 2/5$, when $n \to \infty$.*

Proof. In both cases the lower bound follows from Theorem 22.

The set

$$C = \{(x, y) \in \mathbb{Z}^2 : x \equiv 0, 2 \pmod 5\}$$

is clearly a static identifying code in the infinite square grid (cf. Figure 14, where we have formed a "snake"; in this asymptotical proof a very simple argument will suffice). Assume that $n \geq 2$. It is not difficult to check that the union of $C \cap Q_{n-1}$ and $Q_n \setminus Q_{n-1}$ is a static identifying code both in Q_n and $\hat{Q}_n$, and it is easy to form a walk that goes through all the codewords and has length at most $\frac{2}{5}(2n+1)^2 + 8(2n+1)$. $\square$

Consider next the infinite triangular mesh T and its subgraphs T_n. The smallest possible density of a static identifying code in T is $1/4$; see [37].

Let $\hat{T}_n$ ($n \geq 1$) be the graph with the same vertex set as T_n in which every vertex $v(i, j)$ has the six neighbours

$$v(i-1, j+1), v(i, j+1), v(i+1, j), v(i+1, j-1), v(i, j-1), v(i-1, j),$$

but now the indices are modulo $2n + 2$. We say that $\hat{T}_n$ is obtained from T_n by wrapping around.

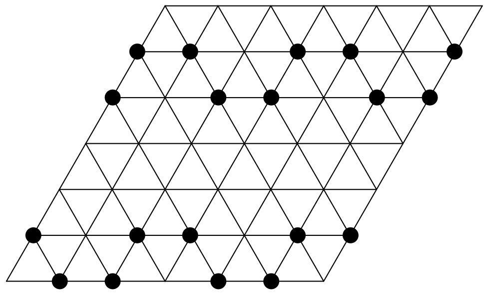

Figure 16: A static identifying code with density 1/3 in the infinite triangular mesh.

Theorem 24 *Assume that* $\mathbf{c}_1$, $\mathbf{c}_2$, $\dots$, $\mathbf{c}_M$ *is a sequence of vertices in* T *(or* T_n*, or* $\hat{T}_n$*) and that* $d(\mathbf{c}_i, \mathbf{c}_{i+1}) = 1$ *for all* $i = 1, 2, \dots, M-1$*. Then*

$$\left| \bigcup_{i=1}^{M} B_1(\mathbf{c}_i) \right| \leq 3M + 4.$$

Proof. The claim is clear for $M = 1$. Assume that $M \geq 2$ and that the claim is true for $M - 1$, and let $\mathbf{c}_1$, $\mathbf{c}_2$, $\dots$, $\mathbf{c}_M$ be any sequence of vertices such that $d(\mathbf{c}_i, \mathbf{c}_{i+1}) = 1$ for all $i = 1, 2, \dots, M-1$. By the induction hypothesis $| \cup_{i=1}^{M-1} B_1(\mathbf{c}_i)| \leq 3M + 1$. Without loss of generality, $\mathbf{c}_M$ is the vertex d5 in Figure 15 and $\mathbf{c}_{M-1}$ is the vertex d4. But then $B_1(\mathbf{c}_M) \setminus \cup_{i=1}^{M-1} B_1(\mathbf{c}_i)$ consists of at most three vertices, namely the ones denoted by open circles, and hence $| \cup_{i=1}^{M} B_1(\mathbf{c}_i)| \leq 3M + 4$. □

Of course, for the vertices $v(1, 0)$, $v(2, 0)$, $\dots$, $v(M, 0)$ of T equality holds in the formula of the previous theorem.

Theorem 25 *If* $n \to \infty$*, and* A_n *is the dynamic identifying code in* T_n *with the smallest possible density, then* $D(A_n) \to 1/3$*. If* $\hat{A}_n$ *is the dynamic identifying code in* $\hat{T}_n$ *with the smallest possible density, then* $D(\hat{A}_n) \to 1/3$*.*

Proof. Assume that $n \geq 3$.

Consider the graph T_n. If A is the code in Figure 16, we claim that

$$C = (T_n \setminus T_{n-2}) \cup (A \cap T_{n-2})$$

is a static identifying code in T_n. Clearly, $I(v) \neq \emptyset$ for all $v \in T_n$ (here $I(v)$ always refers to C). Whenever $I(v)$ contains two codewords whose Euclidean

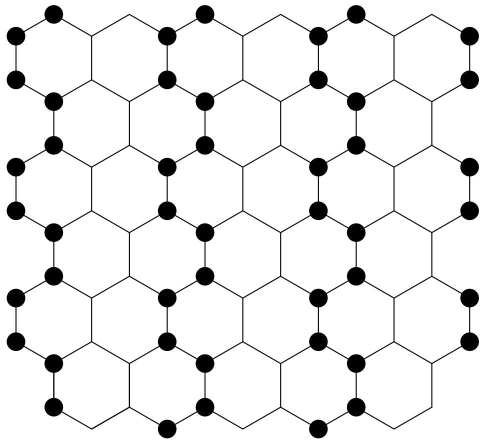

Figure 17: A static identifying code in the infinite hexagonal mesh.

distance equals 2, we can immediately conclude that v must be the middle point of the line segment connecting these two codewords. In particular, if $v \in T_n \setminus T_{n-2}$, this is always the case, except when v is one of the four corners of T_n, but then it is otherwise immediately clear from the set $I(v)$ what v is. So it suffices to check that $I(u) \neq I(v)$ whenever $u, v \in T_{n-2}$, $u \neq v$. Because the original code A is identifying, there is a codeword $c \in A$ which is within graphic distance one from u or from v but not from both. But the same codeword $c \in T_{n-1}$ is also in C and ensures that $I(u) \neq I(v)$.

It is clear that using this static identifying code we can construct a dynamic identifying code in T_n whose density tends to 1/3, when $n \to \infty$.

The lower bound immediately follows from Theorem 24.

The claim for $\hat{T}_n$ is proved in exactly the same way by using the code $(T_n \setminus T_{n-1}) \cup (A \cap T_{n-1})$. □

Consider next hexagonal meshes. Figure 17 shows a certain static identifying code in the infinite hexagonal mesh H. Define a sequence H_2, H_3, ... of finite subgraphs of H. The graphs H_2, H_3 and H_4 are shown in Figure 18 (where we have labeled one vertex of H by 0). The graph H_m consists of $2m^2 + 4m$ vertices, and the number of vertices in the perimeter of H_m is $8m-2$. Moreover, $H_2 \subseteq H_3 \subseteq H_4 \subseteq \ldots$, and the union of the graphs H_m, $m \geq 2$, is H.

Theorem 26 *If $n \to \infty$, and A_n is a dynamic identifying code in H_n with the smallest possible density, then $D(A_n) \to 1/2$.*

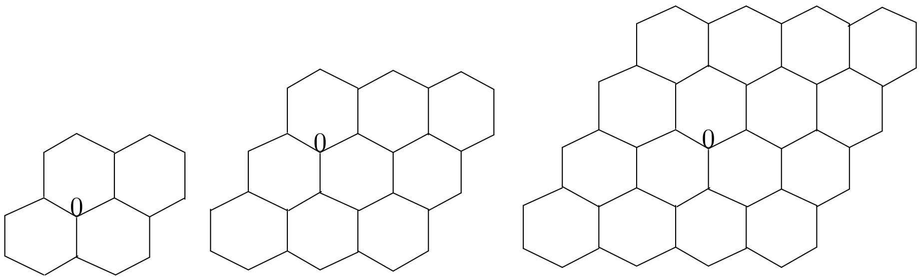

Figure 18: The graphs H_2, H_3 and H_4.

Proof. The lower bound 1/2 follows from Theorem 22. When we superpose Figure 17 on H_n, and also take as codewords all the vertices in the perimeter of H_n that were not already codewords, we obtain a static identifying code in H_n. From these codes we easily construct dynamic identifying codes C_n with $D(C_n) \to 1/2$ when $n \to \infty$. □

It is known that the density of a static identifying code in the infinite hexagonal mesh must be at least 16/39, and that there are static identifying codes with density 3/7; see [16].

From [6] we know that if C is a static identifying code in $\mathbb{Z}_2^n$, then $C \oplus \mathbb{Z}_2^m$ is a static identifying code in $\mathbb{Z}_2^{n+m}$ for all $m \geq 2$. Using the known static identifying codes from [37] and the exactly same technique as in [5], we obtain the following result.

Theorem 27 *If the conjecture (3) holds for $R = 2$, then the minimum possible length $M(n)$ of a dynamic identifying code in the binary n-dimensional cube $\mathbb{Z}_2^n$ satisfies*

$$M(n) = \frac{2^{n+1}}{n}(1 + f(n)),$$

where $f(n) \to 0$, when $n \to \infty$.

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