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Euler's formula and Platonic solids

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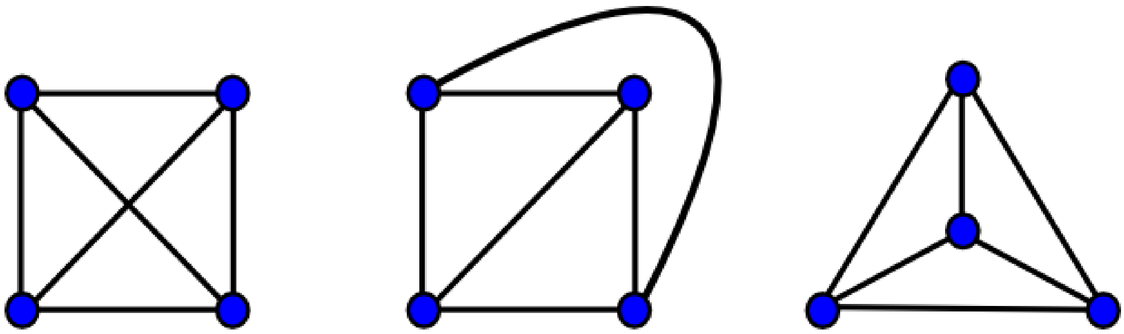
Abstract (I)

In this paper, we will present the concepts of planar graphs, Euler’s characteristic formula, and Platonic solids and show their relationships to one another. After first defining planar graphs, we will prove that Euler’s characteristic holds true for any of them. We will then define Platonic solids, and then using Euler’s formula, prove there exists only five.

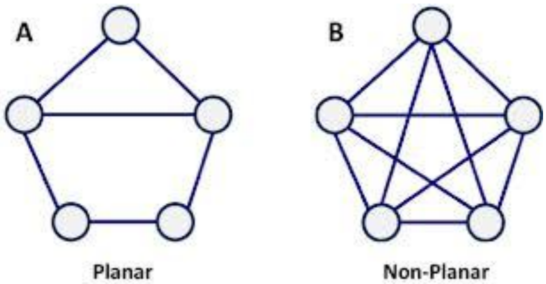
Existence of Planar Graphs (II)

A *planar graph* is one that can be drawn on a plane in such a way that there are no "edge crossings," i.e. edges intersect only at their common vertices. Or informally, a graph is planar if the nodes of the graph can be rearranged (without breaking or adding any edges) so that no edges of the graph cross. The first graph shown below is planar, although you might not think so, the next two graphs are the same graph and confirms that it is planar. This is what I mentioned before about “rearranged” the diagram to eliminate the crossings..

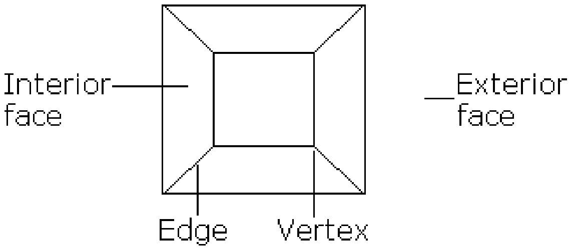
Examples:

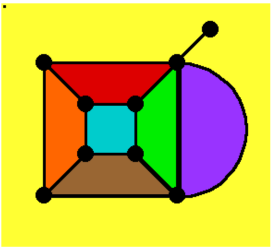


Here are another two examples to show the concept of planar graph:

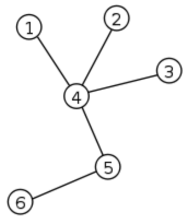


Planar graph consists of vertices/nodes, edges, and faces/regions. When a planar graph is drawn with no crossing edges, it divides the plane into a set of regions, called faces. We count the unbounded area outside the whole graph as an exterior face, and there are interior faces such as the area containing all the edges adjacent to it. Here is the example:





(left: this graph has 7 faces, 9 vertices, and 14 edges)



Also, there is one type of planar graph: trees. Free tree is any connected graph with no cycles. It may look like the real tree with many branches. Yet, free tree doesn't have any interior faces because it doesn't contain cycles, it only has a whole exterior face, and a cycle in a graph means there is a path from an object back to itself.

Euler and his Characteristic Formula (III)

Leonhard Euler was a Swiss Mathematician and Physicist, and is credited with a great many pioneering ideas and theories throughout a wide variety of areas and disciplines. One such area was graph theory. Euler developed his characteristic formula that related the edges (E), faces(F), and vertices(V) of a planar graph, namely that the sum of the vertices and the faces minus the edges is two for any planar graph, and thus for complex polyhedrons. More elegantly, $V - E + F = 2$. We will present two different proofs of this formula.

Proof by Induction on Number of Edges (IV)

Theorem 1: Let G be a connected planar graph with v vertices, e edges, and f faces. Then
$v - e + f = 2$

Proof: Suppose G is a connected planar graph. We will proceed to prove that $v - e + f = 2$ by induction on the number of edges.

Base case: Let G be a single isolated vertex. Then it follows that there is exactly one vertex, on face (the infinite exterior face) and zero edges, thus $v - e + f = 1 - 0 + 1 = 2$, so the formula holds.

Now let us assume that G contains v vertices, e edges, and f faces, and $v - e + f = 2$. No we will show that this graph can be simplified to the base case and maintain a Euler Characteristic value of 2 by removing or contracting edges. There are two cases to consider.

Case 1: the edge connects two vertices, but does not an edge belonging to a closed face. We can then contract this edge to remove it, and combine the vertices to a single vertex. By doing this, we reduce the number of edges by one, the number of vertices by one, and the number of faces remains unchanged since this edge did not bound a face. Also note that the the connectivity of the graph is unchanged because the edge was simply condensed so the paths through the vertices of that edge now lie on a single vertex and are unbroken. We then see that the formula becomes $(v - 1) - (e - 1) + f =$
$v - e + f - 1 + 1 = v - e + f = 2$, by the inductive hypothesis, so it holds. We then repeat this for all such edges. But these are not the only possible edges in our graph, so we have to consider removing the edges correspond to our second case.

Case 2: Suppose e is an edge that is part of a Jordan curve (or polygon) that separates two faces. If the edge is of this type, we simply remove it from the graph. Doing this will obviously reduce the number of edges by one, and since the edge separated two faces, those faces will no longer be separated, so the number of faces will also decrease by one. Note that as a consequence of the Jordan Curve Theorem (or polygon theorem in the case of straight edge segments), there will still be a path around the other side of the edge that was removed, so the graph remains connected. We then have
$v - (e - 1) + (f - 1) = v - e + f + 1 - 1 = v - e + f = 2$, again by our inductive hypothesis. We continue removing edges of this kind till there are no more.

It is clear that there are only two cases to consider since an edge can either belong to a closed face, or does not. So if we continue to remove edges in the two ways described above we will get back to the base case of a single vertex, and as established above, Euler's Characteristic holds for a single vertex. Thus it hold for any connected planar graph.

QED.

We will now give a second, less general proof of Euler’s Characteristic for convex polyhedra projected as planar graphs.

Descartes Vs Euler, the Origin Debate(V)

Although Euler was credited with the formula, there is some debate in the math community on the original origins of $V-E + F =2$, and many believe the true creator to actually be Descartes in 1630. His theorem is said to have stated “The sum of the deficiencies of the solid angles of a polyhedron is always eight right angles”. Deficiencies are defined to be the amount by which the sum of the plane angles at the solid angle, fall short of four right angles, so essentially 360-sum of the angles at the vertex. To visualize this, take a cube which has 8 vertex points, each vertex is made up of 3 right angles. So the deficiency of a cube is 8 right angles because at each vertex the deficiency is 1 right angle.
Using Descartes theorem, the sum of solid angles= $2\pi(V-2)$, he then infers that the number of plane angles is $2F+2V-4$. Since the number of plane angles is always $2E$, using substitution and dividing everything by 2, we get $F+V-2 = E$. Rearranging this equation we arrive at Euler’s characteristic formula $V-E+F=2$, hence it is just a small step from Descartes deficiencies formula to Euler’s characteristic formula, so who is the true creator?

Proof by Summing Interior Angle Measures(VI)

As a tribute to the uncertainty in origin, it is fitting that we will now show a proof of Euler’s characteristic formula using angle sums, as Descartes had.

Suppose we have a polyhedron with E=Edges, V=Vertices, and F=Faces. We can make an embedded planar graph so that all edges are straight lines. Using this planar graph we will formulate two different equations for the sum of the interior angle measures of the polyhedron, and by setting those two equations equal to each other we will be left with the equation V-E+F=2

Sum of the interior angle measures, part I:

- The sum of the interior angle measures can be found by summing the interior angle measures of each face independently, and adding them together.
- Each face of the polyhedron is itself, a n-gon.
- By Corollary 10.22, we know that the interior angle sum of an n-gon is (n-2)180, where n is the number of sides.
- Each face has the same n, so the sum of the interior angle measures of all the faces, would just be F(n-2)180.
- Let us distribute the F: (F*n - 2F)180.
- Does F*n equal E, the total edges in the polyhedron?
- No, since each edge is a member of 2 faces, F*n would be double counting each edge.
- So, F*n = 2E.
- Let us substitute this into our angle sum equation: (2E-2F)180.
- So in conclusion, the total interior angle sum of the polyhedron = (2E-2F)180

Sum of the interior angle measures, part II:

- The vertices of the polyhedron can be classified as either: interior vertices (IntV), or exterior vertices (ExtV). Where interior vertices are those completely surrounded by faces, and exterior vertices are those that are not.
- The total angle sum will be the sum of the angles in conjunction with the interior vertices and the exterior vertices.
- We know the interior vertices will each contribute 360 to the angle sum measures, (IntV)360
- We know that the exterior vertices will contribute (180-theta) where the theta is the measure of the exterior angle. There will be at least two exterior angles. So the total interior angle sum contribution will be ((180-theta(1))+(180-theta(2)))*(ExtV).
- If we distribute, we have: (ExtV)180 -(ExtV)(theta(1))+(ExtV)180 - (ExtV)(theta(2)).
- Group these together: 360(ExtV)-2*(Extv)(theta(1)+theta(2)).
- By Corollary 10.21, we know that the sum of the exterior angles is 360. So(Extv)(theta(1)+theta(2))= 360.

- By substitution: 360(ExtV)-2*360 = the total contribution of the exterior vertices to the total interior angle sum.
- To get the total interior angle sum, we sum the contributions of the exterior and interior vertices and we have: 360(ExtV)-2*360 +360(IntV).
- Pull out 360: 360(ExtV + IntV) + 2*360 = total sum of interior angles.
- And the ExtV+IntV we know to just be V
- So, 360V +2*360 = total sum of the interior angles.

Next we will set these two equations equal to one another:

- 360V+2*360 = (2E-2F)180
- Now if we divide all parts by 360: V+2 = E - F.
- Now rearrange the variables: V-E+F=2

Thus we have proved that V-E+F = 2 QED.

Platonic Solids (VII)

Theorem 2. There are exactly ve Platonic solids

The Platonic Solids are, by definition, three dimensional figures in which all of the faces are congruent regular polygons such that each vertex has the same number of faces meeting at it. There are exactly five of such shapes, all of which are listed below with the number of vertices, edges, and faces of the solid.

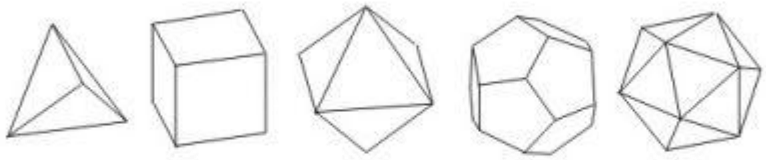
Name	Image	Vertices V	Edges E	Faces F
Tetrahedron		4	6	4
Hexahedron or cube		8	12	6
Octahedron		6	12	8
Dodecahedron		20	30	12
Icosahedron		12	30	20

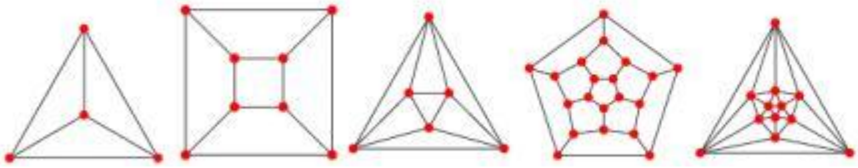
So by for the tetrahedron, cube, octahedron, dodecahedron, and icosahedron respectively $V - E + F = 4 - 6 + 4 = 8 - 12 + 6 = 6 - 12 + 8 = 20 - 30 + 12 = 12 - 30 + 20 = 2$. This fits Euler's Formula which we proved earlier since these are all convex polyhedrons.

People have been discussing these solids for thousands of years, but the ancient Greeks studied platonic solids particularly extensively. In fact, they are named after the very famous Greek philosopher, Plato. At that time, though, they were seen as far more than just the geometric figures that students would treat them to be today. Plato associated the solids with the very classical elements that he believed made up everything in the universe.

The classical elements, air, water, fire, and earth, were each associated with a different Platonic solid. Earth was associated with the cube, air with the octahedron, water with the icosahedron, and fire with the tetrahedron.

To tie the Platonic solids to the planar graphs, each platonic solid has a planar graph as shown below.





Proof that there are 5 Platonic Solids using Euler's Formula

Let's denote the following for a polyhedron:

F: the number of faces
E: the number of edges
V: the number of vertices
n: the number of edges surrounding each face
c: the number of edges that meet at each vertex

Note that for a regular platonic solid, the number of faces surrounding each vertex must be the same for each face and the number of edges that meet at each vertex must be the same for each vertex, so it makes sense to refer to n and c.

Part of being a platonic solid is that each face is a regular polygon. Since the least number of sides of a regular polygon is 3, $n \geq 3$. There must also be at least 3 edges at each vertex, so $c \geq 3$.

Since each edge will appear as the edge of exactly two faces, multiplying the number of faces by the number of edges surrounding each face will double-count each edge, i.e. F * n = 2E.

Since each edge will meet at each at exactly 2 vertices, multiplying the number of edges that meet at each vertex by the number of vertices will double-count each edge, i.e. V * c = 2E

Solving these two equations for F and V respectively, we obtain: $F = \frac{2E}{n} V = \frac{2E}{c}$

Now, by substituting into Euler's Formula (V - E + F = 2), we obtain: $\left(\frac{2E}{c}\right) - E + \left(\frac{2E}{n}\right) = 2$

by factoring $E\left(\frac{2}{c} - 1 + \frac{2}{n}\right) = 2$

E must be positive, since it does not make sense to have a 0 or negative edged polyhedron, so, since a positive times a positive is the only way to get a positive number (like 2) in multiplication,

$$\left(\frac{2}{c} - 1 + \frac{2}{n}\right) > 0$$

So, by algebra,

$$\frac{1}{c} + \frac{1}{n} > \frac{1}{2}$$

First, let's consider

$$\frac{1}{c} > \frac{1}{2} - \frac{1}{n}$$

Since $n \geq 3$, we have that $\frac{1}{n} \leq \frac{1}{3}$, so

$\frac{1}{c} > \frac{1}{2} - \frac{1}{n} > \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, So

$\frac{1}{c} > \frac{1}{6}$, So $c < 6$

Since c only makes sense as an integer and we already have that $c \geq 3$, this implies that c can only be 3, 4, or 5.

By applying the same reasoning n, we obtain that n can only be 3, 4, or 5.

From $\frac{1}{c} > \frac{1}{2} - \frac{1}{n}$, when n = 3, we have $\frac{1}{n} > \frac{1}{6}$, so n < 6, so we still have n = 3, 4, or 5.

From $\frac{1}{c} > \frac{1}{2} - \frac{1}{n}$, when n = 4, we have $\frac{1}{n} > \frac{1}{4}$, so n < 4, so we have that n = 3.

From $\frac{1}{c} > \frac{1}{2} - \frac{1}{n}$, when n = 5, we have $\frac{1}{n} > \frac{3}{10}$, so n < 10/3 < 4, so we have, again, that n = 3.

We have shown that these are the only possibilities and they correspond to the Platonic solids by the following chart:

c	n	V	E	F	**Solid**
3	3	4	6	4	Tetrahedron
3	4	8	12	6	Square
3	5	20	30	12	Dodecahedron
4	3	6	12	8	Octahedron
5	3	12	30	20	Icosahedron

.

Conclusion (VIII)

Clearly, there exist planar graphs and exactly 5 platonic solids. Moreover, there are some interesting applications for planar graphs, such as design problems for circuits, subways, utility lines, and coloring maps of countries: like the exercise we proved in class before (coloring countries sharing borders with 2 different colors). Furthermore, we learned the proofs for Euler characteristic. Yet, Euler characteristic has so many other properties we are not discussed here. But I hope that at the very least this paper give some notes on the importance of Euler's formula and platonic solids.

Works Referenced (IX)

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