The Mathematics of Origami

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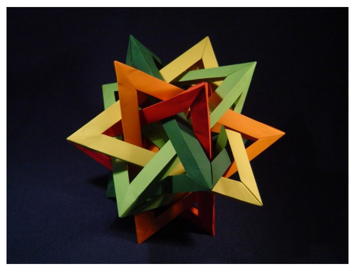
1 Introduction

Origami is a type of art first originated from Japan. It is possible to fold many beautiful shapes in origami. Most amazingly, many astonishing pieces of origami are produced from a single piece of paper, with no cuttings. Just like constructions using straight edge and compass, constructions through paper folding is both mathematically interesting and aesthetic, particularly in origami. Some of the different categories of origami are presented below:

- Modular Origami



- Origami Tessellation

- Origami Animal

There are many other beautiful shapes that can be constructed through paper folding. Surprisingly, it turns out that origami is much more powerful than straight-edge and compass creations, because many things that cannot be created using straight-edge and compass, such as the doubling of a cube and trisection of an angle, can be created through paper folding [3]. This result turns out to be quite unexpected, because we can only fold straight lines in origami due to the fact that curves are completely arbitrary in folding. Since the study of origami is fairly recent, there is no limit yet to the type of constructions that can be formed through paper folding.

The focus of this paper will be on deciding what kind of shapes are possible to construct using origami, and what kind of shapes are not. It will be mainly based on David Auckly and John Cleveland's article, "Totally Real Origami and Impossible Paper Folding". Since we have yet to discover a boundary in the creation of origami, Auckly and Cleveland gave a limited definition of origami in their paper. Although this "limited" definition excludes those properties of origami that made them exceptionally powerful, Auckly and Cleveland managed to find a way of determining which constructions are possible from the given points and lines using this new definition. In this paper, we will first take a look at what is constructible under the definition of origami given by Auckly and Cleveland, and then inspect those other axioms of origami that make them powerful.

2 Some Basics in Abstract Algebra

Before getting into origami, we need to develop a set of definitions needed to understand the algebra in Auckly and Cleveland's paper.

2.1 Groups

Definition 2.1. *A* ***group*** *is a set G together with a multiplication on G which satisfies three axioms:*

a) The multiplication is associative, that is to say $(xy)z = x(yz)$ for any three (not necessarily distinct) elements from G.

b) There is an element e in G, called an identity element, such that $xe = x = ex$ for every x in G.

c) Each element x of G has a (so-called) inverse x^{-1} which belongs to the set G and satisfies $x^{-1}x = e = xx^{-1}$.

Definition 2.2. *An* ***abelian*** *group is a group G such that for all $x, y \in G$, $xy = yx$. (In this case, xy has an invisible operator, which could either be $x + y$ or $x \times y$, but not both at the same time).*

Definition 2.3. *A* ***symmetric group****, S_n, is the set of the permutations of n elements $\{x_1, \ldots, x_n\}$.*

Let us look at an example of symmetric group. List notation is used to describe a set, $\{[s_1], [s_2], \ldots, [s_n]\}$, where $s_1, \ldots, s_n$ are the elements of the set. Using list notation, we can express the symmetric group of the elements $\{x_1, x_2, x_3\}$:

$$S_3 = \{[x_1, x_2, x_3], [x_1, x_3, x_2], [x_2, x_1, x_3], [x_2, x_3, x_1], [x_3, x_1, x_2], [x_3, x_2, x_1]\}$$

This set is consisting of all permutations of $\{x_1, x_2, x_3\}$.

2.2 Ring

Definition 2.4. *A* ***ring*** $\mathcal{R}$ *is a set, whose objects can be added and multiplied, (i.e. we are given associations* $(x, y) \to x + y$ *and* $(x, y) \to xy$ *from pairs of elements of* $\mathcal{R}$*, into* $\mathcal{R}$*), satisfies the following conditions: [4]*

a) Under addition, $\mathcal{R}$ *is an additive, and abelian group.*

b) For all $x, y, z \in \mathcal{R}$*, we have*

$$x(y + z) = xy + xz \quad \text{and} \quad (y + z)x = yx + zx$$

c) For all $x, y, z \in R$*, we have associativity* $(xy)z = x(yz)$*.*

d) There exists an element $e \in \mathcal{R}$ *such that* $ex = xe = x$ *for all* $x \in \mathcal{R}$*, where* e *is the identity element.*

An example of a ring is the set of integers, $\mathbb{Z}$, because addition is commutative and associative, and multiplication is associative. For any three integers x, y, z, we have $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$. In addition, let the multiplicative identity $e = 1$, then $1 \cdot x = x \cdot 1 = x \ \forall \ x \in R$. Therefore, the set of integers form a ring.

2.3 Field

Definition 2.5. *A commutative ring such that the subset of nonzero elements form a group under multiplication is called a* ***field****. [4]*

A field is essentially a ring that allows multiplication to be commutative, after removing the zero element. For a field, everything other than the zero element must have an inverse. Otherwise it is a ring. An example would be that the set of integers, $\mathbb{Z}$, is not a field, but the set of rationals, $\mathbb{Q}$, is a field. The reason is that integers do not have multiplicative inverse: $2 \cdot \frac{1}{2} = 1$ but $\frac{1}{2} \notin \mathbb{Z}$. Matrices are not a ring or a field, because it is not commutative under multiplication.

2.4 Polynomials

Definition 2.6. *A number* α *is an* ***algebraic number*** *if it is a root of a polynomial with rational coefficients. [2]*

Definition 2.7. *A polynomial $p(x)$ in any field, $\mathbb{F}[x]$, is called* ***irreducible*** *over $\mathbb{F}$ if it is of degree ≥ 1, and given a factorization, $p(x) = f(x)g(x)$, with $f, g \in \mathbb{F}[x]$, then $\deg f$ or $\deg g = 0$. [4]*

For example, consider the following polynomials:

$$p_1(x) = x^2 - 4 = (x+2)(x-2)$$

$$p_2(x) = x^2 - \frac{1}{4} = (x + \frac{1}{2})(x - \frac{1}{2})$$

p_1 is reducible over $\mathbb{Z}[x]$ because both $x + 2$ and $x - 2$ are polynomials over integers as well. However, $p_2(x)$ is not reducible over $\mathbb{Z}[x]$. It is, nonetheless, reducible over $\mathbb{Q}[x]$, because both of the factored polynomials are polynomials over $\mathbb{Q}$.

Remark: Any algebraic number, α, could be expressed as a root of a unique irreducible polynomial in $\mathbb{Q}[x]$, denoted by $p_\alpha(x)$. This polynomial $p_\alpha(x)$ will divide any polynomial in $\mathbb{Q}[x]$ that has α as a root.

Definition 2.8. *The* ***conjugates*** *of α are the roots of the polynomial $p_\alpha(x)$. An algebraic number is* ***totally real*** *if all of its conjugates are real. We denote the set of totally real numbers by $\mathbb{F}_{TR}$. [2]*

To make sense of the previous section, consider the number, $\sqrt{5 + 2\sqrt{2}}$. It is an algebraic number, since it could be expressed as a unique irreducible polynomial in $\mathbb{Q}[x]$. The other roots of this polynomials are its conjugates: $\pm\sqrt{5 \pm 2\sqrt{2}}$. We can see that these numbers form a unique polynomial in $\mathbb{Q}$:

$$\left(x + \sqrt{5 + 2\sqrt{2}}\right)\left(x - \sqrt{5 + 2\sqrt{2}}\right)\left(x + \sqrt{5 - 2\sqrt{2}}\right)\left(x - \sqrt{5 - 2\sqrt{2}}\right)$$

$$= x^4 - 10x^2 + 17$$

which is a polynomial with rational coefficients. Additionally, since all the conjugates of $\sqrt{5 + 2\sqrt{2}}$ are real, $\sqrt{5 + 2\sqrt{2}}$ is totally real. Additionally, the polynomial above, $p = x^4 - 10x^2 + 17$, is the unique irreducible polynomial, which will divide any polynomials in $\mathbb{Q}[x]$ that contains $\sqrt{5 + 2\sqrt{2}}$ as its root. On the other hand, the number $\sqrt{1 + \sqrt{2}}$ is not totally real, because two of its conjugates, $\pm\sqrt{1 - \sqrt{2}}$, are imaginary.

The last topic we will go over is symmetric polynomials. We have already defined symmetric group. The following is definition for symmetric polynomials:

Definition 2.9. *Let $\mathcal{R}$ be a ring and let $t_1, \ldots, t_n$ be algebraically independent elements over $\mathcal{R}$. Let x be a variable, and let G be a symmetric group on n*

letters. Let σ be a permutation of integers $(t_1, \ldots, t_n)$. Given a polynomial $f(t) = \mathcal{R}[t_1, \ldots, t_n]$, we define σf to be:

$$\sigma f(t_1, \ldots, t_n) = f(t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(n)})$$

A polynomial is called ***symmetric*** *if $\sigma f = f$ for all $\sigma \in G$. [4]*

For example, let $f(t_1, t_2) = t_1^2 - t_2^2$. This is not a symmetric polynomial, because we can let $\sigma : t_1 \mapsto t_2$, $\sigma : t_2 \mapsto t_1$:

$$t_2^2 - t_1^2 \neq t_1^2 - t_2^2 \quad \Longrightarrow \quad \sigma f \neq f.$$

However, $t_1^2 + t_2^2$ is a symmetric polynomial, because $t_1^2 + t_2^2 = t_2^2 + t_1^2$.

Knowing the definition for symmetric polynomial, let's take a look at the following polynomial:

$$\begin{aligned} p(x) &= (x - t_1)(x - t_2) \cdots (x - t_n) \\ &= x^n - s_1 x^{n-1} + \cdots + (-1)^n s_n \end{aligned}$$

where each s_j is given as:

$$\begin{aligned} s_1 &= t_1 + t_2 + \cdots + t_n \\ s_j &= \text{the sum of all products of j distinct } t_k\text{'s} \\ s_n &= t_1 \cdot t_2 \cdots t_n. \end{aligned}$$

The polynomials, $s_j(t_1, \ldots, t_n)$, $1 \leq j \leq n$ are called the **elementary symmetric polynomials** of $t_1, \ldots, t_n$. [4]

Another way to express that is:

$$\prod_{k=1}^{n} (x - t_k) = \sum_{j=0}^{n} (-1)^j s_j(t_1, \ldots, t_n) x^{n-j} \tag{1}$$

For example, expanding $(x - t_1)(x - t_2)(x - t_3)$, we have:

$$(x - t_1)(x - t_2)(x - t_3) = x^3 - (t_1 + t_2 + t_3)x^2 + (t_1t_2 + t_2t_3 + t_1t_3) - t_1t_2t_3$$

All the following polynomials,

$$\begin{aligned} s_1 &= t_1 + t_2 + t_3 \\ s_2 &= t_1t_2 + t_2t_3 + t_1t_3 \\ s_3 &= t_1t_2t_3 \end{aligned}$$

are elementary symmetric polynomials.

Now we are done introducing the definitions in abstract algebra that would occur in this discussion of the paper folding. We can now start looking at some properties of origami.

3 Properties of Origami

3.1 Basic Constructions

In order to understand origami construction, we will need to understand some of the most basic folds that can be created. The following is the definition given by Auckly and Cleveland of origami pair. This definition is the basis of what we mean by "origami" in this paper:

Definition 3.1. $\{\mathcal{P}, \mathcal{L}\}$ *is an* ***origami pair*** *if* $\mathcal{P}$ *is a set of points in* $\mathbb{R}^2$ *and* $\mathcal{L}$ *is a collections of lines in* $\mathbb{R}^2$ *satisfying:*

a) The point of intersection of any two non-parallel lines in $\mathcal{L}$ *is a point in* $\mathcal{P}$*.*

b) Given any two distinct points in $\mathcal{P}$*, there is a line* $\mathcal{L}$ *going through them.*

c) Given any two distinct points in $\mathcal{P}$*, the perpendicular bisector of the line segment with given end points is a line in* $\mathcal{L}$*.*

d) If L_1 *and* L_2 *are lines in* $\mathcal{L}$*, then the line which is equidistant from* L_1 *and* L_2 *is in* $\mathcal{L}$*.*

e) If L_1 *and* L_2 *are lines in* $\mathcal{L}$*, then there exists a line* L_3 *in* $\mathcal{L}$ *such that* L_3 *is the mirror reflection of* L_2 *about* L_1*.*

Some diagrams to illustrate the above five constructions are shown in Figure 1 and 2. The dashed lines are the lines that we are constructing using origami. They represent creases on our sheet of paper. To see some detailed illustrations of how to obtain constructions (a)-(e), see [2].

To show that we can construct many things in origami just like construction using straight-edge and compass, let us look at the following lemma that describes how to construct parallel lines via origami, using the definitions above:

Lemma. *It is possible to construct a line parallel to a given line through any given point using origami.*

Proof. Refer to Figure 3. The line given is L, and the point given is p. We are trying to find the line parallel to L through p. To do that, pick two points P_1, P_2 on L. By property (b) in the definition of origami pair, we could construct lines L_1 and L_2, which goes through P_1p, P_2p respectively. By property (e), we could reflect L_1 and L_2 across L to obtain L_3 and L_4. By property (a), the intersection of L_3 and L_4 is constructible, so we could find the line, L_5, that goes between that point and p, by property (b). Moreover, the line L_5 will intersect L at a point, according to property (a), so call this point P_3. Then use property (c) to construct a perpendicular bisector to $\overline{pP_3}$. Reflect L across this bisector with property (e) would give us the resulting line. [2] $\square$

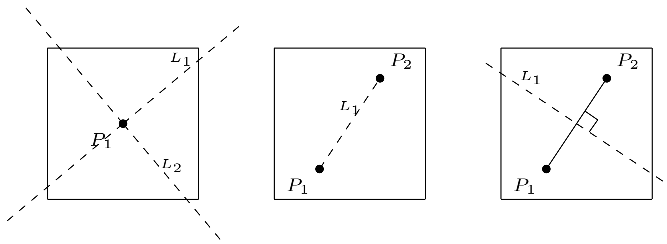


Figure 1: Construction (a), (b), (c), from left to right

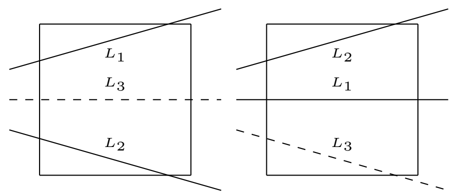


Figure 2: Construction (d), (e), from left to right

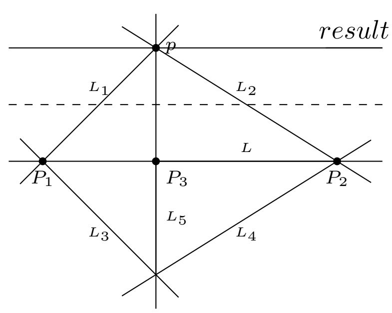


Figure 3: Construction of the line parallel to L through point p

3.2 Origami Numbers

The objective of this paper, as stated before, is to answer which constructions are possible using only the five axioms described before. But before getting to that, let's look at some definitions:

Definition 3.2. *A subset of $\mathbb{R}^2$, $\mathcal{P}$, is closed under origami construction if there exists a collection of lines, $\mathcal{L}$, such that $(\mathcal{P}, \mathcal{L})$ is an origami pair. [2]*

Definition 3.3. *The set of* ***origami constructible points*** *$\mathcal{P}_0$ is defined as: $\mathcal{P}_0 = \{P \mid (0, 0),\ (0, 1) \in \mathcal{P}$ and $\mathcal{P}$ is closed under origami constructions$\}$. [2]*

Definition 3.4. *$\mathbb{F}_0 = \{\alpha \in \mathbb{R} \mid \exists\, v_1, v_2 \in \mathcal{P}$ such that $|\alpha| = dist(v_1, v_2)\}$ is the* ***set of origami numbers****. [2]*

It is easy to see that $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ are origami numbers, because we can always find the midpoint of a line through folding. But how about numbers like $\frac{1}{3}, \frac{1}{5}, \ldots$? We can show through Figure 4 that these are also origami numbers. This is done by constructing a line parallel to the line created by points $(0, \frac{3}{4})$, $(1, 0)$ through the point $(0, \frac{1}{4})$. It turns out that the sum, difference, product and

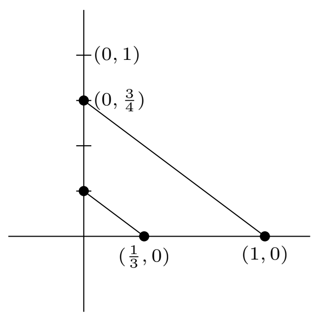


Figure 4: Showing that $\frac{1}{3}$ is also an origami number

quotient of origami numbers are origami numbers. Another interesting fact is that, one class of origami numbers could be expressed using right triangles. It turns out, this class of origami number is the most important one that we will be discussing. It is as follows:

$$\sqrt{1 + \alpha^2} \text{ is an origami number whenever } \alpha \text{ is an origami number.}$$

This origami number can be expressed as a right triangle, with 1 and α as the legs of the triangle, and $\sqrt{1 + \alpha^2}$ as the hypotenuse. From this class of origami number, we obtain the following theorem:

Theorem 3.1. *The collection of origami numbers, $\mathbb{F}_0$, is a field closed under the operation $\alpha \mapsto \sqrt{1 + \alpha^2}$. (This means that take any number α, the number $\sqrt{1 + \alpha^2}$ would give us another number $\alpha' \in \mathbb{F}_0$). [2]*

For example, if $\alpha = 2$, then we can construct the number $\sqrt{5}$ by making a right triangle with legs 2 and 1. Therefore, $\sqrt{5}$ is an origami number.

For a proof of the previous theorem, please refer to [2].

Now we have the operation, $\alpha \mapsto \sqrt{1+\alpha^2}$, that would produce origami numbers. But in order to know which geometric shapes are constructible and which are not, we will need to find out if there are more operations that would produce origami numbers. It turns out that there are not, as we shall see.

Definition 3.5. $\mathbb{F}_{\sqrt{1+\alpha^2}}$ *is the smallest subfield of* $\mathbb{C}$ *closed under the operation* $\alpha \mapsto \sqrt{1+\alpha^2}$.

By Theorem 3.1, it is true that $\mathbb{F}_{\sqrt{1+\alpha^2}} \in \mathbb{F}_0$, since the smallest subfield closed under the operation $\alpha \mapsto \sqrt{1+\alpha^2}$ obviously should give us a collection of origami numbers. However, as the below theorem shows, it is actually true that $\mathbb{F}_{\sqrt{1+\alpha^2}} = \mathbb{F}_0$. This means that the collection of origami number is a closed field under the operation $\alpha \mapsto \sqrt{1+\alpha^2}$.

Theorem 3.2. $\mathbb{F}_0 = \mathbb{F}_{\sqrt{1+\alpha^2}}$. *[2]*

Proof. We have already know that $\mathbb{F}_{\sqrt{1+\alpha^2}} \in \mathbb{F}_0$, so all we need to show is that $\mathbb{F}_0 \in \mathbb{F}_{\sqrt{1+\alpha^2}}$. This means that we need to show any origami number may be expressed using the usual field operations and the operation $\alpha \mapsto \sqrt{1+\alpha^2}$. Using the five axioms for origami construction in the section before, we can limit down to four distinct ways of constructing new origami points from old points or lines:

a) Fold a line between two existing points, p_1 and p_2, which intersect the given line L at point (x, y).

b) Fold a perpendicular bisector to two points, p_1 and p_2, which intersect the given line L at point (x, y).

c) Reflect a line L_2 across line L_1 to obtain L_3, which intersects the line L at point (x, y).

d) Form the angle bisector, L_1, which intersect the line L at the point (x, y).

Since the fourth case is already proven in [2], we will prove only the first case instead. Refer to Figure 5 for a diagram of the first case.

To prove that the case (a) would give us $\mathbb{F}_0 \in \mathbb{F}_{\sqrt{1+\alpha^2}}$, we may assume that point p_1 is at the origin, because the point (x, y) can be found by adding (p_{1x}, p_{1y}) to the translated point. Additionally, adding and substracting origami numbers will give us origami numbers, so this operation still preserves the closure. Furthermore, we may assume that point p_2 is on a unit circle, because

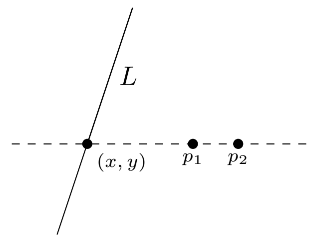


Figure 5

multiplying the scaled point by $\sqrt{p_{2x}^2 + p_{2y}^2} = |p_{2x}|\sqrt{1 + (p_{2y}/p_{2x})^2}$ would reverse the scaling. Additionally, we can assume that p_2 is the point $(1, 0)$ because we can just apply the rotation matrix on the coordinates of p_2 to give us the desired location. These operations would allow us to simplify the problem, and assume the slope of the line $\overrightarrow{p_1p_2}$ is zero. Let the equation of the line L be, $L(x) = ax + b$, then the line L and $\overrightarrow{p_1p_2}$ will intersect at $(x, y) = (-\frac{b}{a}, 0)$. We can then apply the rotation, scaling, and translation needed to bring (x, y) back to its correct location. Within this process of finding (x, y), we have only used operations of the form addition, multiplication, and the operation $\alpha \mapsto \sqrt{1 + \alpha^2}$. Therefore, $\mathbb{F}_0 = \mathbb{F}_{\sqrt{1+\alpha^2}}$. □

Now we are ready to prove the final results of origami construction with the five axioms described before, that is, all origami numbers are totally real. To obtain this result, we will combine what we know of origami construction in addition to some abstract algebra, especially the section on polynomials.

4 Possible Origami Constructions

In this section, our goal is to show that the set of totally real numbers form a field under the operation $\alpha \mapsto \sqrt{1 + \alpha^2}$. Obtaining this result would allow us to define exactly what we could construct using origami. The following theorem will be essential to the derivation of the final results in this paper.

Theorem 4.1.

$$\mathbb{F}_{\sqrt{1+\alpha^2}} \subset \mathbb{F}_{TR}$$

Proof. Recall our definition, that $\mathbb{F}_{\sqrt{1+\alpha^2}}$ is the smallest subfield of $\mathbb{C}$ closed under the operation $\alpha \mapsto \sqrt{1 + \alpha^2}$, and $\mathbb{F}_{TR}$ is the set of algebraic numbers whose conjugates are real. Let $\alpha, \beta \in \mathbb{F}_{TR}$. We can show this theorem is true simply by showing that $-\alpha$, α^{-1}, $\sqrt{1 + \alpha^2}$, $\alpha + \beta$, $\alpha \cdot \beta \in \mathbb{F}_{TR}$. Showing $-\alpha$, α^{-1}, $\alpha + \beta$, $\alpha \cdot \beta \in \mathbb{F}_{TR}$ is essential because in order to form a field, we need additive and multiplicative inverses, and commutativity, associativity in the operations "+" and "×". Otherwise we do not have a field. And if we could show at the same time that $\sqrt{1 + \alpha^2} \in \mathbb{F}_{TR}$, then $\mathbb{F}_{\sqrt{1+\alpha^2}}$, being the smallest subfield closed under the operation $\alpha \mapsto \sqrt{1+\alpha^2}$, must be in $\mathbb{F}_{TR}$.

Let $\{\alpha_i\}_{i=1}^n$ be the conjugates of α and $\{\beta_i\}_{i=1}^n$ be the conjugates of β. We will prove the theorem by considering the following five polynomials:

$$
\begin{aligned}
q_{-\alpha}(t) &= \prod_{i=1}^{n}(t+\alpha_i) && (2)\\
q_{\alpha^{-1}}(t) &= \left(\prod_{i=1}^{n}(t-\alpha_i^{-1})\right)\left(\prod_{i=1}^{n}\alpha_i\right) && (3)\\
q_{\sqrt{1+\alpha^2}}(t) &= \prod_{i=1}^{n}(t^2-1-\alpha_i^2) && (4)\\
q_{\alpha+\beta}(t) &= \prod_{i=1}^{n}\prod_{j=1}^{m}(t-\alpha_i-\beta_j) && (5)\\
q_{\alpha\beta}(t) &= \prod_{i=1}^{n}\prod_{j=1}^{m}(t-\alpha_i\beta_j) && (6)
\end{aligned}
$$

Our goal is to show that for each of the five polynomial $q(t)$, all the roots of its minimal polynomial, $p(t)$, must be real. Let us first consider equation (4), because this is the most important equation of the five listed above. Expanding out equation (4) using polynomial notations mentioned before, we have:

$$
\begin{aligned}
q_{\sqrt{1+\alpha^2}}(t) &= \prod_{i=1}^{n}(t^2-1-\alpha_i^2)\\
&= \sum_{j=0}^{n}(-1)^j s_j(1+\alpha_1,\,\ldots\,,1+\alpha_n)t^{n-j}
\end{aligned}
$$

It is clear that the coefficients of the t^k will be symmetric polynomials in the α_j. Therefore, by Definition 2.6 and 2.8, the polynomial $q(t)$ may be expressed as some rational polynomial with the elementary symmetric polynomials of the α_j as its coefficients. Let $p(t)$ be the minimal polynomial of this particular polynomial $q(t)$. Since $(-1)^j s_j(\alpha)$ will be the coefficients of $q(t)$ for α, we may conclude that $q_{\sqrt{1+\alpha^2}}(t) \in \mathbb{Q}[t]$. It is obvious that $\sqrt{1+\alpha^2}$ is a root of $q_{\sqrt{1+\alpha^2}}(t)$, therefore $p_{\sqrt{1+\alpha^2}}(t)$ divides $q_{\sqrt{1+\alpha^2}}(t)$. This also means that all the roots in $p_{\sqrt{1+\alpha^2}}(t)$ will be the roots of $q_{\sqrt{1+\alpha^2}}(t)$, while the converse is not necessarily true. Since α is totally real, this mens all its conjugates, α_i, must be real. Thus, $1+\alpha_i^2$ are all real and positive, so $\pm\sqrt{1+\alpha_i^2}$ are all real as well $\Longrightarrow$ all of the roots of $q_{\sqrt{1+\alpha^2}}(t)$ are all real. But out of all the roots for $q(t)$, only some of them will be roots of $q_{\sqrt{1+\alpha^2}}(t)$, and these roots are precisely the conjugates of $\sqrt{1+\alpha^2}$. Since all of the roots of $p(t)$ are real as well, we now conclude that $\sqrt{1+\alpha^2}$ is totally real.

We have proved that $\mathbb{F}_{\sqrt{1+\alpha^2}} \subset \mathbb{F}_{TR}$ is a true statement for the operation $\alpha \mapsto \sqrt{1+\alpha^2}$. Proving the other four equations would be similar. After showing that all five of the following operations, $\alpha \mapsto -\alpha$, $\alpha \mapsto \alpha^{-1}$, $\alpha \mapsto \sqrt{1+\alpha^2}$, $(\alpha,\beta) \mapsto \alpha+\beta$, $(\alpha,\beta) \mapsto \alpha \cdot \beta \in \mathbb{F}_{TR}$ by considering the five polynomials, we would have finished proving this theorem. $\square$

The result above gives us a way of deciding which shapes are constructible using origami, and which are not. We have concluded that all those points that could be constructed have to be totally real. For example, it is not possible to construct a cube that has twice the volume as the given cube, because if this construction was possible, then $2^{1/3}$ must be an origami number, therefore totally real. However, the conjugates of $2^{1/3}$ are $2^{1/3}(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)$ and $2^{1/3}$, and the first two conjugates are not real. Since $2^{1/3}$ is not an origami number, it is not constructible under origami operations, which makes doubling the volume of a cube impossible to construct through origami.

It turns out that using the five axioms of origami defined previously, we can derive an interesting connection between this kind of origami construction and the construction using straight-edge and compass. That is:

Corollary. *Everything which is constructible with origami is constructible with a compass and straight edge, but the converse is not true.*

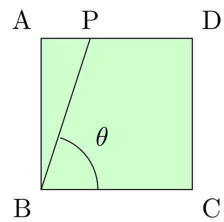
To see a proof of the previous Corollary, see [2].

It is the case that under the definition of origami using only the axioms defined in Definition 3.1, origami seems a lot less powerful compared to straight-edge and compass methods. There are three famous problems that occurred in straight-edge and compass construction: doubling the area of a circle, trisection of an angle, and doubling the volume of a cube. All three problems have been proven to be unsolvable using straight-edge and compass centuries after people first attempted to solve them. Nonetheless, under that particular definition of origami, none of these three problems could be solved either.

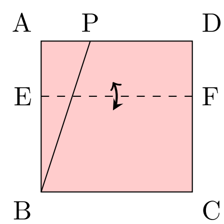
5 The Complete Axioms of Origami

It turns out that origami construction are actually a lot more powerful than straight-edge and compass construction. This, however, does not contradict the result we derived previously, since we had only defined origami construction using the set of operations allowed in Definition 3.1. This set of operations, on the other hand, do not cover every possible construction in origami. As we shall see, although we can only fold straight lines (since curves are more arbitrary), some constructions that are impossible using the traditional straight-edge and compass methods, could be achieved through origami. Two of which are the trisection of an angle, and the doubling of a cube.

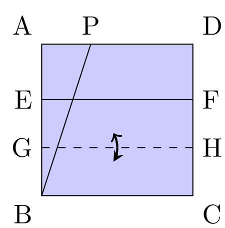
The following shows a visual representation of how to trisect an acute angle using origami:



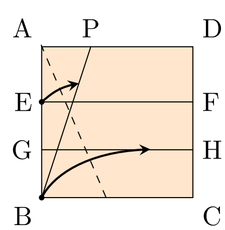
(a)



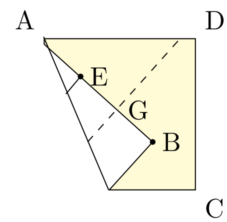
(b)



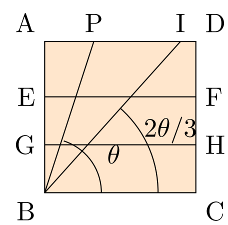
(c)



(d)



(e)



(f)

The reason that origami is so much more powerful than given in Definition 3.1, was that it allows the construction of folding two points onto two different lines. The following list is the complete axioms of origami, developed by Huzita in 1992 (except for the last one, which was discovered by Hatori in 2002): [1]

a) Given two points, p_1 and p_2, we can fold a line connecting them.

b) Given two points, p_1 and p_2, we can fold p_1 onto p_2.

c) Given two lines, l_1 and l_2, we can fold l_1 onto l_2.

d) Given a point p_1 and a line l_1, we can make a fold perpendicular to l_1 passing through the point p_1.

e) Given two points p_1 and p_2 and a line l_1, we can make a fold that places p_1 onto l_1 and passes through the point p_2.

f) Given two points p_1 and p_2 and two lines l_1 and l_2, we can make a fold that places p_1 onto line l_1 and places p_2 onto line l_2.

g) Given a point p_1 and two lines l_1 and l_2, we can make a fold perpendicular to l_2 that places p_1 onto line l_1.

The five axioms that were mentioned in [2], which were the same axioms as in Definition 3.1, covered only the top five of these seven axioms above. The last two, which were not discussed by Auckly and Cleveland, are what made origami powerful, and enabled paper folding to create what straight-edge and compass cannot create. Looking at the previous figure, we can see that one of the steps require axiom (f), which is not a legal operation defined in Definition 3.1.

The reason for such result is that straight-edge and compass method can only create things that are solutions to quadratic equations, or other equations with the exponent of the unknown no larger than two. For example, given a set of lines with certain lengths, we can use straight-edge and compass to construct any linear combination of those lines, multiply, or square root of those lengths. Thus, the quadratic equation is the highest order of equation that straight-edge and compass is able to solve. As a result, we cannot trisect an angle or double the volume of a cube, since those would require solving a cubic equation, something that straight-edge and compass construction cannot do. On top of that, doubling the area of a circle would require the construction of length π, which cannot even be written as a root of a polynomial with finite number of terms. [3]

However, the cubic equation can be solved using origami, through the axiom (f). By allowing the simultaneous alignment of two different points onto two different lines, we can solve the trisection of an angle and the doubling of a cube problem. For details on the mathematical reasoning behind this, refer to [3]. As a result, origami construction is more powerful in comparison to straight-edge compass construction.

6 Conclusion

Considering that origami construction is so powerful, one might wonder why would the authors, Auckly and Cleveland, define origami using only five of the seven axioms, which made it seem so much less powerful? One possible explanation is that we have yet to find a limit to what could be constructed through origami. Since we have the definition of origami number, which is closed under $\alpha \mapsto \sqrt{1+\alpha^2}$, we can draw some similarities between that and straight-edge and compass construction, which can carry through the operation $\alpha \mapsto \sqrt{\alpha}$. Both of these involve only solving quadratic equations. As a result, an "upper bound" in origami construction is found. Since origami is a fairly new topic, people have yet to find its limit in construction. For example, as mentioned before, Humiaki Huzita, a Japanese-Italian mathematician and origami artist, discovered six of the seven axioms of origami in 1992, but ten years later, Hatori summarized another axiom. It is possible that the seven axioms mentioned previously are not complete. New properties could always be unearthed.

In addition to these intriguing constructional properties, origami is worth studying and exploring in other math related fields. For example, there is a connection between origami and topology, even to graph theory, something that we don't usually assume origami would associate with. Even beyond its mathematical properties, they are practically useful and artistically pleasing.

References

[1] Wolfram mathworld. http://mathworld.wolfram.com/Origami.html.

[2] Auckly, D., and Cleveland, J. Totally real origami and impossible paper folding. *American Mathematics Monthly 102* (1995), 215 – 226.

[3] Lang, R. J. Origami and geometric constructions. *Self Published* (1996 2003).

[4] Lang, S. Undergraduate algebra. *Springer Publishing Company* (2004 - 2005).